Ergod. Th. & Dynam. Sys. (1982), **2**, 45–48
Printed in Great Britain

An ε-free Rohlin lemma

EHUD LEHRER AND BENJAMIN WEISS
Department of Mathematics, Hebrew University, Jerusalem, Israel

(*Received* 24 *December* 1981)

Abstract. For an ergodic transformation T, necessary and sufficient conditions are given for a set A and an integer n so that the existence of a set B with $\{T^iB\}_{i=0}^{n-1}$ pairwise disjoint and $\bigcup_{i=0}^{n-1} T^iB \supset A$ is ensured.

The classic Rohlin lemma states that if T is an invertible aperiodic transformation of a probability space (X, B, μ) then for any $n \geq 2$ and any $\varepsilon > 0$ there is a set $B \subset X$ such that

(i) $B, TB, \dots T^{n-1}B$ are disjoint,

(ii) $\mu(\bigcup_0^{n-1} T^iB) > 1-\varepsilon$.

This elementary lemma has turned out to be a very basic tool in ergodic theory and is most often applied in the case when μ is non-atomic and T is ergodic (in which case T is necessarily aperiodic). There is a very simple proof in this case, one simply chooses a set E with $0 < \mu(E) < \varepsilon/n$, builds the Kakutani skyscraper over E, i.e. sets $E_0 = E$ and inductively puts

$$E_{n+1} = T^{n+1}E_0 \backslash \bigcup_0^n E_i,$$

and sets

$$B = \bigcup_{k=1}^{\infty} T^{-(n-1)}E_{kn-1}.$$

Property (i) is then immediate while (ii) follows, since the ergodicity of T implies that $\bigcup_0^\infty E_k$ has measure 1.

In many applications, one needs to build a nested sequence of such R-towers, i.e. sets B_i with $\{T^jB_i\}_{j=0}^{n_i-1}$ disjoint and with

$$\cdots \subset \bigcup_{j=0}^{n_i-1} T^jB_i \subset \bigcup_{j=0}^{n_{i+1}-1} T^jB_{i+1} \subset \cdots.$$

This is usually done by an infinite process in which even the first tower gets defined only at the conclusion of an infinite procedure. In this connection it is quite natural to raise the following problem:

PROBLEM. *Given a set $A \subset X$, and an $n \geq 2$, is there a set B, with $\{T^jB\}_0^{n-1}$ disjoint and $\bigcup_0^{n-1} T^iB \supset A$?*

https://doi.org/10.1017/S0143385700009561 Published online by Cambridge University Press

Here, and in the sequel, all set inclusions and equalities are to be taken modulo sets of measure zero. It is the purpose of this note to give the solution to this problem. To begin with, if $\mu(A)=1$ then clearly a solution exists if and only if T has for an eigenvalue a primitive root of unity of order n. Since this situation is readily understood we assume henceforth $\mu(A)<1$. It turns out that if T^n is ergodic then there are no further obstructions to a positive solution of the problem, while if T^n is not ergodic (but T is) then we can completely describe the exceptional cases. We proceed to these results and leave some further comments and extensions to the conclusion of this note. Since no use is made of the fact that T is measure preserving we shall deal henceforth with invertible *non-singular* ergodic mappings of a non-atomic measure space (possibly with infinite measure) (X, B, μ).

THEOREM 1. *If T is an invertible non-singular ergodic mapping of (X, B, μ) and $A \subset X$ is given, then for any n for which $\bigcup_{k=0}^{\infty} T^{-kn}(X\setminus A)$ equals X, there is a set B such that $\{T^iB\}_0^{n-1}$ are disjoint, and $\bigcup_0^{n-1} T^iB \supset A$. In particular such sets B exist for any A with $\mu(X\setminus A)>0$ if T^n is ergodic.*

Proof. By the assumptions of the theorem T is conservative and hence the hypotheses imply

$$\bigcup_{k\geq K} T^{-kn-L}E = X$$

for any K, L where $E=X\setminus A$. Since

$$E \subset \bigcup_{k\geq 1} T^{-kn-1}E,$$

there is some $E_1 \subset E$, and $0<l_1=k_1n+1$ such that $\mu(E_1)>0$ and $T^{l_1}E_1 \subset E$.

Again since

$$E_1 \subset \bigcup_{k\geq k_1} T^{-kn-2}E,$$

there is some $E_2 \subset E$ with $\mu(E_2)>0$, and $l_2=k_2n+2>l_1$ with

$$T^{l_2}E_2 \subset E.$$

Continuing inductively till $n-2$, we finally obtain a subset $E_{n-2} \subset E$ of positive measure, and

$$l_i = k_in+i, \quad 0<l_1<l_2<\cdots<l_{n-2}$$

with

$$T^{l_i}E_{n-2} \subset E \quad \text{for } 1\leq i\leq n-2.$$

Now since T is aperiodic we can find a subset $F\subset E_{n-2}$ of positive measure with

$$F, TF, \ldots, T^{l_{n-2}}F$$

all pairwise disjoint. Since T is ergodic, X can be represented as a skyscraper over F, where the minimal height of any *pure column* in the skyscraper is at least $l_{n-2}+1$. Here by a *pure column* of height l we refer to

$$\{\bar{F}, T\bar{F}, \ldots, T^{l-1}\bar{F}\}$$

where $\bar{F}\subset F$ is the set of x such that l is the least positive integer for which $T^lx\in F$.

https://doi.org/10.1017/S0143385700009561 Published online by Cambridge University Press

On each pure column B will be defined separately. For the columns whose height is a multiple of n we simply take every nth level beginning with the first. The reason for constructing F the way we did is to ensure that for any pure column of height l where $l \equiv j \pmod n$, we can erase j-levels, all of which belong to E, so that the remaining j-columns all have heights that are multiples of n, where once again B is defined by taking every nth level beginning with the first. The j-levels that are erased are the

$$l_0+1, l_1+1, l_2+1, \ldots, l_{j-1}+1$$

(where $l_0 = 0$). We give as illustration the definition for $l \equiv 3 \pmod n$. Form the intervals

$$(1, \ldots, l_1 - 1), \quad (l_1+1, \ldots, l_2-1), \quad (l_2+1, \ldots, l-1),$$

and set

$$\begin{aligned}\bar{B} = (T\bar{F} \cup T^{n+1}\bar{F} \cup \cdots \cup T^{l_1-n}\bar{F}) \\ \cup (T^{l_1+1}\bar{F} \cup T^{l_1+n+1}\bar{F} \cup \cdots \cup T^{l_2-n}\bar{F}) \\ \cup (T^{l_2+1}\bar{F} \cup T^{l_2+n+1}\bar{F} \cup \cdots \cup T^{l-n}\bar{F}).\end{aligned}$$

Since $l_1 \equiv 1$, $l_2 \equiv 2$, $l_3 \equiv 3 \pmod n$ each of the intervals has a length which is a multiple of n, and thus

$$\bigcup_0^{n-1} T^i\bar{B} = \left(\bigcup_0^{l-1} T^i\bar{F}\right) \setminus (\bar{F} \cup T^{l_1}\bar{F} \cup T^{l_2}\bar{F}).$$

The set B constructed in this fashion evidently satisfies the requirements of the theorem. □

Before we continue with the general description let us look at an example of the possible obstructions in case T^n is not ergodic. Suppose that T^2 is not ergodic, and that $X = A_1 \cup A_2$ with $\mu(A_i) > 0$, $i = 1, 2$ and $T^2 A_i = A_i$, but that T^4, when restricted to A_1 is ergodic. Then we claim that A_1 cannot be included in a tower of height 4, say $A_1 \subset \bigcup_0^3 T^i B$, because then

$$D = A_1 \cap (B \cup TB)$$

will satisfy: D, $T^2 D$ disjoint and $D \cup T^2 D = A_1$ which will contradict the ergodicity of T^4 on A_1. In general, the possible obstructions will be that A contains one of a finite collection of sets that behaves like A_1 did above.

It is well known that if T is ergodic but T^n is not then the field of T^n-invariant sets I_n is atomic, and consists of k-atoms, for some k that divides n, each actually invariant under T^k. Let us say, in this situation, that I_n is of *order k*.

THEOREM 2. *If T is an invertible, non-singular ergodic mapping of a measure space* (X, B, μ), *and* $A \subset X$ *with* $\mu(X \setminus A) > 0$, *and* $n \geq 2$ *is given, then there is a set B with* $\{T^i B\}_0^{n-1}$ *disjoint, and* $A \subset \bigcup_0^{n-1} T^i B$ *unless all of the following conditions obtain:*

(*a*) T^n *is not ergodic,*

(*b*) *the field of* T^n*-invariant sets* I_n *is of type k for some* $1 < k < n$,

(*c*) *A contains some atom of* I_n,

in which case a set B with the listed properties cannot exist.

https://doi.org/10.1017/S0143385700009561 Published online by Cambridge University Press

Proof. We form $Y=\bigcup_{i=0}^{\infty} T^{in}(X\backslash A)$. If $Y=X$ then a set B with the desired properties exists by theorem 1. In the contrary case, $X\backslash Y\subset A$ is invariant under T^n and thus (a), (c) hold. Next if I_n is of order n then for B we can take any atom of I_n, and finally if (a), (b), (c) hold then no such set B can exist since if an atom of I_n, C, is contained in $\bigcup_0^{n-1} T^iB$ with $\{T^iB\}_0^{n-1}$ pairwise disjoint then each $(T^iB\cap C)$ is also T^n-invariant which implies that C coincides with some level and thus I_n would be of order n. □

As an immediate consequence of theorem 2 we formulate one special case separately:

COROLLARY: *If T is ergodic and $\mu(X\backslash A)>0$, then for any prime p, there is a set B with $\{T^iB\}_0^{p-1}$ pairwise disjoint and $\bigcup_0^{p-1} T^iB\supset A$.*

For the finite measure preserving case there is a stronger form of the Rohlin lemma [1] in which the base B is made independent of a preassigned partition $\mathscr{P}$. There is clearly no hope in general for such a result in our form of the lemma since the partition could consist of $\{A, X\backslash A\}$, for example. We can prove an approximate version – i.e. we can, in the favourable cases make the base ε-independent of a preassigned $\mathscr{P}$ – but since the paper is to be ε-free we leave these results as exercises for the reader.

One final comment is in order. There are by now many extensions of the Rohlin lemma to more general groups (cf. [2]), but the nature of the proof given here for theorem 1 makes it unlikely that it can be generalized even to $\mathbb{Z}^2$.

E.L.'s work is part of an MSc thesis written at the Hebrew University. B.W.'s research was supported in part by BNSF Grant no. 1927/80.

REFERENCES

[1] Y. Katznelson & B. Weiss. Commuting measure preserving transformations. *Israel J. Math.* **12** (1972), 161–173.

[2] D. Ornstein & B. Weiss. Ergodic theory of amenable group actions, I: The Rohlin lemma. *Bull. Amer. Math. Soc.* NS **2** (1980), 161–164.