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The geometry of Hida families II: Λ-adic (φ,Γ)-modules and Λ-adic Hodge theory

Bryden Cais

To Haruzo Hida on the occasion of his 60th birthday

Abstract

We construct the Λ-adic crystalline and Dieudonné analogues of Hida's ordinary Λ-adic étale cohomology, and employ integral p-adic Hodge theory to prove Λ-adic comparison isomorphisms between these cohomologies and the Λ-adic de Rham cohomology studied in Cais [*The geometry of Hida families I: Λ-adic de Rham cohomology*, Math. Ann. (2017), doi:10.1007/s00208-017-1608-1] as well as Hida's Λ-adic étale cohomology. As applications of our work, we provide a 'cohomological' construction of the family of (φ,Γ)-modules attached to Hida's ordinary Λ-adic étale cohomology by Dee [*Φ–Γ modules for families of Galois representations*, J. Algebra **235** (2001), 636–664], and we give a new and purely geometric proof of Hida's finiteness and control theorems. We also prove suitable Λ-adic duality theorems for each of the cohomologies we construct.

Contents

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1. Introduction

1.1 Motivation

In a series of ground-breaking papers [Hid86a] and [Hid86b], Hida constructed p-adic analytic families of p-ordinary Galois representations interpolating the Galois representations attached to p-ordinary cuspidal Hecke eigenforms in integer weights $k \geqslant 2$ by Deligne [Del71, Car86]. At the heart of Hida's construction is the p-adic étale cohomology $H^1_{\text{ét}} := \varprojlim_r H^1_{\text{ét}}(X_1(Np^r)_{\overline{\mathbf{Q}}}, \mathbf{Z}_p)$ of the tower of modular curves over $\mathbf{Q}$, which is naturally a module for the 'big' p-adic Hecke algebra $\mathfrak{H}^* := \varprojlim_r \mathfrak{H}^*_r$, which is itself an algebra over the completed group ring $\Lambda := \mathbf{Z}_p[\![\Delta_1]\!] \simeq \mathbf{Z}_p[\![T]\!]$ via the diamond operators $\Delta_r := 1 + p^r\mathbf{Z}_p$. Writing $e^* \in \mathfrak{H}^*$ for the idempotent attached to the (adjoint) Atkin operator U_p^*, Hida proved (via explicit computations in group cohomology) that the ordinary part $e^*H^1_{\text{ét}}$ of $H^1_{\text{ét}}$ is finite and free as a module over Λ, and that the resulting Galois representation $\rho : \mathscr{G}_{\mathbf{Q}} \longrightarrow \mathrm{Aut}_\Lambda(e^*H^1_{\text{ét}})$ p-adically interpolates those attached to ordinary cuspidal eigenforms.

By analyzing the geometry of the tower of modular curves, Mazur and Wiles [MW86] showed that both the inertial invariants and covariants of the local (at p) representation ρ_p are free of the same finite rank over Λ, and hence that the ordinary filtration of the Galois representations attached to ordinary cuspidal eigenforms interpolates in Hida's p-adic family. As an application, they gave examples of cusp forms f and primes p for which the specialization of the associated Hida family of Galois representations to weight $k = 1$ is not Hodge–Tate, and so does not arise from a weight-one cusp form via the construction of Deligne and Serre [DS74]. Shortly thereafter, Tilouine [Til87] clarified the geometric underpinnings of [Hid86a] and [MW86].

In [Oht95, Oht99] and [Oht00], Ohta initiated the study of the p-adic Hodge theory of Hida's ordinary Λ-adic (local) Galois representation ρ_p. Using the invariant differentials on the tower of p-divisible groups over $R_\infty := \mathbf{Z}_p[\mu_{p^\infty}]$ attached to the 'good quotient' modular abelian varieties introduced in [MW84] and studied in [MW86] and [Til87], Ohta constructed a certain $\Lambda_{R_\infty} := R_\infty[\![\Delta_1]\!]$-module $e^*H^1_{\mathrm{Hdg}}$, which is the Hodge cohomology analogue of $e^*H^1_{\text{ét}}$. Via an integral version of the Hodge–Tate comparison isomorphism [Tat67] for ordinary p-divisible groups, Ohta established a Λ-adic Hodge–Tate comparison isomorphism relating $e^*H^1_{\mathrm{Hdg}}$ and the semisimplification of the 'semilinear representation' $\rho_p \mathbin{\widehat{\otimes}} \mathscr{O}_{\mathbf{C}_p}$. Using Hida's results, Ohta showed that $e^*H^1_{\mathrm{Hdg}}$ is free of finite rank over Λ_{R_∞} and specializes to finite level exactly as one expects. As applications of his theory, Ohta provided a construction of two-variable p-adic L-functions attached to families of ordinary cusp forms differing from that of Kitagawa [Kit94] and, in a subsequent paper [Oht00], provided a new and streamlined proof of the theorem of Mazur and Wiles [MW84] (Iwasawa's main conjecture for $\mathbf{Q}$; see also [Wil90]). We remark that Ohta's Λ-adic Hodge–Tate isomorphism is a crucial ingredient in the forthcoming partial proof of Sharifi's conjectures [Sha11, Sha07] due to Fukaya and Kato [FK12].

In [Cai17], we continued the trajectory begun by Ohta by constructing the de Rham analogue of $e^*H^1_{\text{ét}}$. Using the canonical integral structures in de Rham cohomology studied in [Cai09] and certain Katz and Mazur [KM85] integral models $\mathcal{X}_r$ of $X_1(Np^r)$ over $R_r := \mathbf{Z}_p[\mu_{p^r}]$, for each $r > 0$ we constructed a canonical short exact sequences of free R_r-modules

$$0 \longrightarrow H^0(\mathcal{X}_r, \omega_{\mathcal{X}_r/R_r}) \longrightarrow H^1(\mathcal{X}_r/R_r) \longrightarrow H^1(\mathcal{X}_r, \mathscr{O}_{\mathcal{X}_r}) \longrightarrow 0 \tag{1.1.1}$$

whose scalar extension to $K_r := \mathrm{Frac}(R_r)$ recovers the Hodge filtration of the de Rham cohomology of $X_1(Np^r)$ over K_r. Extending scalars to R_∞, taking projective limits, and passing to ordinary parts gives a sequence of Λ_{R_∞}-modules with semilinear $\Gamma := \mathrm{Gal}(K_\infty/K_0)$-action

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and commuting $\mathfrak{H}^*$-action

$$0 \longrightarrow e^*H^0(\omega) \longrightarrow e^*H^1_{\mathrm{dR}} \longrightarrow e^*H^1(\mathscr{O}) \longrightarrow 0. \tag{1.1.2}$$

The main result of [Cai17] is that (1.1.2) is the correct de Rham analogue of Hida's ordinary Λ-adic étale cohomology and Ohta's ordinary Λ-adic Hodge cohomology (see [Cai17, Theorem 3.7]).

THEOREM 1.1.1. *Let $d = \sum_{k=3}^{p+1} d_k$ for $d_k := \dim_{\mathbf{F}_p} S_k(\Gamma_1(N); \mathbf{F}_p)^{\mathrm{ord}}$ the $\mathbf{F}_p$-dimension of the space of mod p weight-k ordinary cusp forms for $\Gamma_1(N)$. Then (1.1.2) is a short exact sequence of free Λ_{R_∞}-modules of ranks d, $2d$, and d, respectively. Applying $\otimes_{\Lambda_{R_\infty}} R_\infty[\Delta_1/\Delta_r]$ to (1.1.2) recovers the ordinary part of the scalar extension of (1.1.1) to R_∞.*

The natural cup-product auto-duality of (1.1.1) over $R'_r := R_r[\mu_N]$ induces a canonical $\Lambda_{R'_\infty}$-linear and $\mathfrak{H}^*$-equivariant auto-duality of (1.1.2) which intertwines the dual semilinear action of $\Gamma \times \mathrm{Gal}(K'_0/K_0) \simeq \mathrm{Gal}(K'_\infty/K_0)$ with a certain $\mathfrak{H}^*$-valued twist of its standard action; see [Cai17, Proposition 3.8] for the precise statement. We moreover proved that, as one would expect, the Λ_{R_∞}-module $e^*H^0(\omega)$ is canonically isomorphic to the module $eS(N, \Lambda_{R_\infty})$ of ordinary Λ_{R_∞}-adic cusp forms of tame level N; see [Cai17, Corollary 3.14].

1.2 Results

In this paper, we complete our study of the geometry and Λ-adic Hodge theory of Hida families begun in [Cai17] by constructing the crystalline counterpart to Hida's ordinary Λ-adic étale cohomology, Ohta's Λ-adic Hodge cohomology, and our Λ-adic de Rham cohomology. Via a careful study of the geometry of modular curves and abelian varieties and comparison isomorphisms in integral p-adic cohomology, we prove the appropriate control and finiteness theorems, and a suitable Λ-adic version of every integral comparison isomorphism one could hope for. In particular, we are able to recover the entire family of p-adic Galois representations ρ_p (and not just its semisimplification) from our Λ-adic crystalline cohomology. A remarkable byproduct of our work is a *cohomological* construction of the family of étale (φ, Γ)-modules attached to $e^*H^1_{\text{ét}}$ by Dee [Dee01]. As an application of our theory, we give a new and purely geometric proof of Hida's freeness and control theorems for $e^*H^1_{\text{ét}}$.

In order to survey our main results more precisely, we introduce some notation. Throughout this paper, we fix a prime $p > 2$ and a positive integer N with $Np > 4$. Fix an algebraic closure $\overline{\mathbf{Q}}_p$ of $\mathbf{Q}_p$ as well as a p-power compatible sequence $\{\varepsilon^{(r)}\}_{r \geqslant 0}$ of primitive p^rth roots of unity in $\overline{\mathbf{Q}}_p$. As above, we set $K_r := \mathbf{Q}_p(\mu_{p^r})$ and $K'_r := K_r(\mu_N)$, and we write R_r and R'_r for the rings of integers in K_r and K'_r, respectively. Denote by $\mathscr{G}_{\mathbf{Q}_p} := \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ the absolute Galois group and by $\mathscr{H}$ the kernel of the p-adic cyclotomic character $\chi : \mathscr{G}_{\mathbf{Q}_p} \to \mathbf{Z}_p^\times$. Using that $K'_0/\mathbf{Q}_p$ is unramified, we canonically identify $\Gamma = \mathscr{G}_{\mathbf{Q}_p}/\mathscr{H}$ with $\mathrm{Gal}(K'_\infty/K'_0)$. We will denote by $\langle u \rangle$ (respectively $\langle v \rangle_N$) the diamond operator[1] in $\mathfrak{H}^*$ attached to $u^{-1} \in \mathbf{Z}_p^\times$ (respectively $v^{-1} \in (\mathbf{Z}/N\mathbf{Z})^\times$) and write Δ_r for the image of the restriction of $\langle \cdot \rangle : \mathbf{Z}_p^\times \hookrightarrow \mathfrak{H}^*$ to $1 + p^r\mathbf{Z}_p \subseteq \mathbf{Z}_p^\times$. For convenience, we put $\Delta := \Delta_1$ and, for any ring A, we write $\Lambda_A := \varprojlim_r A[\Delta/\Delta_r]$ for the completed group ring on Δ over A; if φ is an endomorphism of A, we again write φ for the induced endomorphism of Λ_A that acts as the identity on Δ. For any ring homomorphism $A \to B$, we will write $(\cdot)_B := (\cdot) \otimes_A B$ and $(\cdot)_B^\vee := \mathrm{Hom}_B((\cdot) \otimes_A B, B)$ for these functors from A-modules to B-modules.[2] If G is any group

[1] Note that $\langle u^{-1} \rangle = \langle u \rangle^*$ and $\langle v^{-1} \rangle_N = \langle v \rangle_N^*$, where $\langle \cdot \rangle^*$ and $\langle \cdot \rangle_N^*$ are the adjoint diamond operators; see the discussion below [Cai17, Definition A.15 and Appendix B].

[2] This convention is somewhat at odds with our notation Λ_A, which is generally not isomorphic to the tensor product $\Lambda \otimes_{\mathbf{Z}_p} A$; we hope that this abuse causes no confusion.

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of automorphisms of A and M is an A-module with a semilinear action of G, for any 1-cocycle $\psi: G \to A^\times$ we will write $M(\psi)$ for the A-module M with 'twisted' semilinear G-action given by $g\cdot m := \psi(g)gm$. Finally, we denote by $X_r := X_1(Np^r)$ the usual modular curve over $\mathbf{Q}$ classifying (generalized) elliptic curves with a $[\mu_{Np^r}]$-structure, and by $J_r := J_1(Np^r)$ its Jacobian.

We analyze the tower of p-divisible groups attached to the 'good quotient' modular abelian varieties introduced by Mazur and Wiles [MW84]. To avoid technical complications with logarithmic p-divisible groups, following [MW86] and [Oht95], we will henceforth remove the trivial tame character by working with the subidempotent $e^{*\prime}$ of e^* corresponding to projection to the part where $\mu_{p-1} \subseteq \mathbf{Z}_p^\times$ acts *non*trivially via the diamond operators. As is well known (e.g. [Hid86a, § 9] and [MW84, ch. 3, § 2]), the p-divisible group $G_r := e^{*\prime}J_r[p^\infty]$ over $\mathbf{Q}$ extends to a p-divisible group $\mathcal{G}_r$ over R_r, and we write $\overline{\mathcal{G}}_r := \mathcal{G}_r \times_{R_r} \mathbf{F}_p$ for its special fiber. Denoting by $\mathbf{D}(\cdot)$ the contravariant Dieudonné module functor on p-divisible groups over $\mathbf{F}_p$, we form the projective limits

$$\mathbf{D}_\infty^\star := \varprojlim_r \mathbf{D}(\overline{\mathcal{G}}_r^\star) \quad \text{for } \star \in \{\text{ét}, \text{m}, \text{null}\}, \tag{1.2.1}$$

taken along the mappings induced by $\overline{\mathcal{G}}_r \to \overline{\mathcal{G}}_{r+1}$. Each of these is naturally a Λ-module equipped with linear Frobenius F and Verschiebung V morphisms satisfying $FV = VF = p$, as well as a linear action of $\mathfrak{H}^*$ and a 'geometric inertia' action of Γ that reflects the fact that the generic fiber of $\mathcal{G}_r$ descends to $\mathbf{Q}_p$. The Λ-modules (1.2.1) have the expected structure.

Theorem 1.2.1. *There is a canonical split short exact sequence of finite and free Λ-modules*

$$\begin{tikzcd} 0 \arrow[r] & \mathbf{D}_\infty^{\text{ét}} \arrow[r] & \mathbf{D}_\infty \arrow[r] & \mathbf{D}_\infty^{\text{m}} \arrow[r] & 0 \end{tikzcd} \tag{1.2.2}$$

with linear $\mathfrak{H}^*$- and Γ-actions. As a Λ-module, $\mathbf{D}_\infty$ is free of rank $2d'$, while $\mathbf{D}_\infty^{\text{ét}}$ and $\mathbf{D}_\infty^{\text{m}}$ are free of rank d', where $d' := \sum_{k=3}^{p} \dim_{\mathbf{F}_p} S_k(\Gamma_1(N); \mathbf{F}_p)^{\text{ord}}$. For $\star \in \{\text{m}, \text{ét}, \text{null}\}$, there are canonical isomorphisms

$$\mathbf{D}_\infty^\star \underset{\Lambda}{\otimes} \mathbf{Z}_p[\Delta/\Delta_r] \simeq \mathbf{D}(\overline{\mathcal{G}}_r^\star) \tag{1.2.3}$$

which are compatible with the extra structures. Via the canonical splitting of (1.2.2), $\mathbf{D}_\infty^\star$ for $\star = \text{ét}$ (respectively $\star = \text{m}$) is identified with the maximal subspace of $\mathbf{D}_\infty$ on which F (respectively V) acts invertibly. The Hecke operator $U_p^* \in \mathfrak{H}^*$ acts as F on $\mathbf{D}_\infty^{\text{ét}}$ and as $\langle p\rangle_N V$ on $\mathbf{D}_\infty^{\text{m}}$, while Γ acts trivially on $\mathbf{D}_\infty^{\text{ét}}$ and via $\langle\chi(\cdot)\rangle^{-1}$ on $\mathbf{D}_\infty^{\text{m}}$.

The short exact sequence (1.2.2) is very nearly Λ-adically auto-dual.

Theorem 1.2.2. *There is a canonical $\mathfrak{H}^*$-equivariant isomorphism of exact sequences of $\Lambda_{R_0'}$-modules*

$$\begin{tikzcd} 0 \arrow[r] & \mathbf{D}_\infty^{\text{ét}}(\langle\chi\rangle\langle a\rangle_N)_{\Lambda_{R_0'}} \arrow[r] \arrow[d, "\simeq"] & \mathbf{D}_\infty(\langle\chi\rangle\langle a\rangle_N)_{\Lambda_{R_0'}} \arrow[r] \arrow[d, "\simeq"] & \mathbf{D}_\infty^{\text{m}}(\langle\chi\rangle\langle a\rangle_N)_{\Lambda_{R_0'}} \arrow[r] \arrow[d, "\simeq"] & 0 \\ 0 \arrow[r] & (\mathbf{D}_\infty^{\text{m}})^\vee_{\Lambda_{R_0'}} \arrow[r] & (\mathbf{D}_\infty)^\vee_{\Lambda_{R_0'}} \arrow[r] & (\mathbf{D}_\infty^{\text{ét}})^\vee_{\Lambda_{R_0'}} \arrow[r] & 0 \end{tikzcd} \tag{1.2.4}$$

that is $\Gamma \times \mathrm{Gal}(K_0'/K_0)$-equivariant and intertwines F (respectively V) on the top row with $V^\vee$ (respectively $F^\vee$) on the bottom.[3]

[3] Here $F^\vee$ (respectively $V^\vee$) is the map taking a linear functional f to $\varphi^{-1} \circ f \circ F$ (respectively $\varphi \circ f \circ V$), where φ is the Frobenius automorphism of $R_0' = \mathbf{Z}_p[\mu_N]$.

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In [MW86], Mazur and Wiles related the ordinary filtration of $e^{*\prime}H^1_{\text{ét}}$ to the étale cohomology of the Igusa tower studied in [MW83]. We can likewise interpret the slope filtration (1.2.2) in terms of the crystalline cohomology of the Igusa tower as follows. For each r, let I_r^∞ and I_r^0 be the two 'good' irreducible components of $\mathcal{X}_r \times_{R_r} \mathbf{F}_p$ (see the discussion preceding Proposition 2.3.3), each of which is isomorphic to the Igusa curve $\mathrm{Ig}(p^r)$ of tame level N and p-level p^r. For $\star \in \{0, \infty\}$, define

$$H^1_{\text{cris}}(I^\star) := \varprojlim_r H^1_{\text{cris}}(I^\star_r/\mathbf{Z}_p)$$

with the projective limit taken along the trace mappings on crystalline cohomology (see [Ber74, VII, § 2.2]) induced by the canonical degeneracy maps on Igusa curves. Then $H^1_{\text{cris}}(I^\star)$ is naturally a Λ-module (via the diamond operators) with commuting linear Frobenius F and Verschiebung V endomorphisms satisfying $FV = VF = p$, and we write $H^1_{\text{cris}}(I^\star)^{V_{\text{ord}}}$ (respectively $H^1_{\text{cris}}(I^\star)^{F_{\text{ord}}}$) for the maximal V- (respectively F-) stable submodule on which V (respectively F) acts invertibly. Letting U_p^* act as F (respectively $\langle p\rangle_N V$) on $H^1_{\text{cris}}(I^\star)$ for $\star = \infty$ (respectively $\star = 0$) and the Hecke operators outside p (viewed as correspondences on the Igusa curves) act via pullback and trace at each level r, we obtain an action of $\mathfrak{H}^*$ on $H^1_{\text{cris}}(I^\star)$. Finally, we let Γ act trivially on $H^1_{\text{cris}}(I^\star)$ for $\star = \infty$ and via $\langle \chi^{-1}\rangle$ for $\star = 0$, and we denote by f' the idempotent of Λ corresponding to projection to the part where $\mu_{p-1} \subseteq \mathbf{Z}_p^\times$ acts nontrivially via the diamond operators.

Theorem 1.2.3. *There is a canonical $\mathfrak{H}^*$- and Γ-equivariant isomorphism of Λ-modules*

$$\mathbf{D}_\infty = \mathbf{D}_\infty^{\mathrm{m}} \oplus \mathbf{D}_\infty^{\text{ét}} \simeq f' H^1_{\text{cris}}(I^0)^{V_{\text{ord}}} \oplus f' H^1_{\text{cris}}(I^\infty)^{F_{\text{ord}}} \tag{1.2.5}$$

which respects the given direct sum decompositions and is compatible with F and V.

We note that our 'Dieudonné module' analogue (1.2.5) is a significant sharpening of its étale counterpart [MW86, § 4], which is formulated only up to isogeny (i.e. after inverting p). From $\mathbf{D}_\infty$, we can recover the Λ-adic Hodge filtration (1.1.2), so the latter is canonically split.

Theorem 1.2.4. *There is a canonical Γ- and $\mathfrak{H}^*$-equivariant isomorphism of exact sequences*

$$\begin{tikzcd} 0 \arrow[r] & e^{*\prime}H^0(\omega) \arrow[r] \arrow[d, "\simeq"] & e^{*\prime}H^1_{\mathrm{dR}} \arrow[r] \arrow[d, "\simeq"] & e^{*\prime}H^1(\mathscr{O}) \arrow[r] \arrow[d, "\simeq"] & 0 \\ 0 \arrow[r] & \mathbf{D}_\infty^{\mathrm{m}} \otimes_\Lambda \Lambda_{R_\infty} \arrow[r] & \mathbf{D}_\infty \otimes_\Lambda \Lambda_{R_\infty} \arrow[r] & \mathbf{D}_\infty^{\text{ét}} \otimes_\Lambda \Lambda_{R_\infty} \arrow[r] & 0 \end{tikzcd} \tag{1.2.6}$$

where the mappings on the bottom row are the canonical inclusion and projection morphisms corresponding to the direct sum decomposition $\mathbf{D}_\infty = \mathbf{D}_\infty^{\mathrm{m}} \oplus \mathbf{D}_\infty^{\text{ét}}$. In particular, the Hodge filtration exact sequence (1.1.2) is canonically split and admits a canonical descent to Λ.

For any subfield K of $\mathbf{C}_p$ with ring of integers R, writing $eS(N;\Lambda_R)$ for the module of ordinary Λ_R-adic cusp forms of level N in the sense of [Oht95, 2.5.5], we deduce from Theorem 1.2.4 the following result.

Corollary 1.2.5. *There is a canonical isomorphism of finite free Λ (respectively $\Lambda_{R_0'}$)-modules*

$$e'S(N,\Lambda) \simeq \mathbf{D}_\infty^{\mathrm{m}} \quad \textit{respectively} \quad e'\mathfrak{H} \otimes_\Lambda \Lambda_{R_0'} \simeq \mathbf{D}_\infty^{\text{ét}}(\langle a\rangle_N) \otimes_\Lambda \Lambda_{R_0'}$$

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that intertwines $T \in \mathfrak{H} := \varprojlim \mathfrak{H}_r$ *with* $T^* \in \mathfrak{H}^*$ *(respectively* $T \otimes 1$ *with* $T^* \otimes 1$*), where* U_p^* *acts as* $\langle p \rangle_N V$ *on* $\mathbf{D}_\infty^{\mathrm{m}}$ *and as* F *on* $\mathbf{D}_\infty^{\text{ét}}$*. The second of these isomorphisms is in addition* $\mathrm{Gal}(K_0'/K_0)$*-equivariant.*

We are also able to recover the semisimplification of $e^{*\prime} H^1_{\text{ét}}$ from $\mathbf{D}_\infty$. Writing $\mathscr{I} \subseteq \mathscr{G}_{\mathbf{Q}_p}$ for the inertia subgroup at p, for any $\mathbf{Z}_p[\mathscr{G}_{\mathbf{Q}_p}]$-module M, let $M^{\mathscr{I}}$ (respectively $M_{\mathscr{I}} := M/M^{\mathscr{I}}$) be the sub (respectively quotient) module of invariants (respectively covariants) under $\mathscr{I}$.

THEOREM 1.2.6. *There are canonical isomorphisms of* $\Lambda_{W(\overline{\mathbf{F}}_p)}$*-modules with linear* $\mathfrak{H}^*$*-action and semilinear actions of* F*,* V*, and* $\mathscr{G}_{\mathbf{Q}_p}$

$$\mathbf{D}_\infty^{\text{ét}} \mathop{\otimes}_{\Lambda} \Lambda_{W(\overline{\mathbf{F}}_p)} \simeq (e^{*\prime} H^1_{\text{ét}})^{\mathscr{I}} \mathop{\otimes}_{\Lambda} \Lambda_{W(\overline{\mathbf{F}}_p)}, \tag{1.2.7a}$$

$$\mathbf{D}_\infty^{\mathrm{m}}(-1) \mathop{\otimes}_{\Lambda} \Lambda_{W(\overline{\mathbf{F}}_p)} \simeq (e^{*\prime} H^1_{\text{ét}})_{\mathscr{I}} \mathop{\otimes}_{\Lambda} \Lambda_{W(\overline{\mathbf{F}}_p)}. \tag{1.2.7b}$$

Writing φ *for the Frobenius automorphism of* $W(\overline{\mathbf{F}}_p)$*, the isomorphism (1.2.7a) intertwines* $F \otimes \varphi$ *with* $\mathrm{id} \otimes \varphi$ *and* $\mathrm{id} \otimes g$ *with* $g \otimes g$ *for* $g \in \mathscr{G}_{\mathbf{Q}_p}$*, whereas (1.2.7b) intertwines* $V \otimes \varphi^{-1}$ *with* $\mathrm{id} \otimes \varphi^{-1}$ *and* $g \otimes g$ *with* $g \otimes g$*, where* $g \in \mathscr{G}_{\mathbf{Q}_p}$ *acts on the Tate twist* $\mathbf{D}_\infty^{\mathrm{m}}(-1) := \mathbf{D}_\infty^{\mathrm{m}} \otimes_{\mathbf{Z}_p} \mathbf{Z}_p(-1)$ *as* $\langle \chi(g)^{-1} \rangle \otimes \chi(g)^{-1}$*.*

Theorem 1.2.6 gives the following 'explicit' description of the semisimplification of $e^{*\prime} H^1_{\text{ét}}$.

COROLLARY 1.2.7. *For any* $T \in (e^* \mathfrak{H}^*)^\times$*, let* $\lambda(T) : \mathscr{G}_{\mathbf{Q}_p} \to e^* \mathfrak{H}^*$ *be the unique continuous (for the* p*-adic topology on* $e^* \mathfrak{H}^*$*) unramified character whose value on (any lift of)* Frob_p *is* T*. Then* $\mathscr{G}_{\mathbf{Q}_p}$ *acts on* $(e^{*\prime} H^1_{\text{ét}})^{\mathscr{I}}$ *(respectively* $(e^{*\prime} H^1_{\text{ét}})_{\mathscr{I}}$*) through the character* $\lambda(U_p^{*-1})$ *(respectively* $\chi^{-1} \cdot \langle \chi^{-1} \rangle \lambda(\langle p \rangle_N^{-1} U_p^*)$*).*

Together, Corollary 1.2.5 and Theorem 1.2.6 provide a refinement of the main result of [Oht95]. We are also able to recover the main theorem of [MW86] (the ordinary filtration of $e^{*\prime} H^1_{\text{ét}}$ interpolates).

COROLLARY 1.2.8. *Let* d' *be as in Theorem 1.2.1. Each of* $(e^{*\prime} H^1_{\text{ét}})^{\mathscr{I}}$ *and* $(e^{*\prime} H^1_{\text{ét}})_{\mathscr{I}}$ *is a free* Λ*-module of rank* d' *and, for* $r \geqslant 1$*, there are canonical* $\mathfrak{H}^*$*- and* $\mathscr{G}_{\mathbf{Q}_p}$*-equivariant isomorphisms of* $\mathbf{Z}_p[\Delta/\Delta_r]$*-modules*

$$(e^{*\prime} H^1_{\text{ét}})^{\mathscr{I}} \mathop{\otimes}_{\Lambda} \mathbf{Z}_p[\Delta/\Delta_r] \simeq e^{*\prime} H^1_{\text{ét}}(X_{r\overline{\mathbf{Q}}_p}, \mathbf{Z}_p)^{\mathscr{I}}, \tag{1.2.8a}$$

$$(e^{*\prime} H^1_{\text{ét}})_{\mathscr{I}} \mathop{\otimes}_{\Lambda} \mathbf{Z}_p[\Delta/\Delta_r] \simeq e^{*\prime} H^1_{\text{ét}}(X_{r\overline{\mathbf{Q}}_p}, \mathbf{Z}_p)_{\mathscr{I}}. \tag{1.2.8b}$$

To recover the full Λ-adic local Galois representation $e^{*\prime} H^1_{\text{ét}}$, rather than just its semisimplification, we study the $\mathbf{Z}_p$-valued $\mathscr{G}_{\mathbf{Q}_p}$-representations $L_r = (T_p G_r)^\vee$ and the (φ, Γ)-modules associated to them by Fontaine [Fon90]. Since G_r acquires good reduction over R_r, the $\mathscr{G}_{\mathbf{Q}_p}$-representation $L_r \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ is 'crystabelline' in the sense that its restriction to $\mathscr{G}_{K_r}$ is crystalline. As such, the theory of *Wach modules*, initiated by Fontaine [Fon90, § B] and Wach [Wac96] and refined by Berger [Ber04] and Berger and Breuil [BB10], associates to each L_r a certain (φ, Γ)-module $\mathfrak{N}(L_r)$ over the power series ring $\mathbf{Z}_p[\![u]\!]$. Our aim is to *p-adically interpolate* the Wach modules $\mathfrak{N}(L_r)$ for $r \geqslant 1$ to construct a 'crystalline avatar' from which we will be able to recover *all* other ordinary Λ-adic cohomologies, as well as the Λ-adic family of (φ, Γ)-modules associated to $e^{*\prime} H^1_{\text{ét}}$ by [Dee01].

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There are two obstructions to doing this. The first is that the Wach modules $\mathfrak{N}(L_r)$ of Berger and Breuil do not enjoy the kind of functoriality properties needed to carry out such an interpolation. The second, and related, problem is that $\mathfrak{N}(L_r)$ is not 'of geometric origin' in the sense that it does not enjoy good p-integral comparison isomorphisms with the crystalline or de Rham cohomology groups of the p-divisible group $\mathcal{G}_r$.

In order to address these problems, we will appeal to the theory of [CL17], which uses Dieudonné crystals and windows à la Lau and Zink to provide a *geometric* theory of 'Wach modules' that will allow us to build the desired Λ-adic crystalline realization. To describe this theory, set $\mathfrak{S}_r := \mathbf{Z}_p[\![u_r]\!]$, and view $\mathfrak{S}_r$ as a $\mathbf{Z}_p$-subalgebra of $\mathfrak{S}_{r+1}$ via the map sending u_r to $\varphi(u_{r+1}) := (1+u_{r+1})^p - 1$. We write $\mathfrak{S}_\infty := \varinjlim \mathfrak{S}_r$ for the rising union[4] of the $\mathfrak{S}_r$, equipped with its Frobenius *automorphism* φ and commuting action of Γ determined by $\gamma u_r := (1+u_r)^{\chi(\gamma)} - 1$. The main result of [CL17] provides an exact anti-equivalence $\mathcal{G} \rightsquigarrow \mathfrak{M}_r(\mathcal{G})$ between the category of p-divisible groups over R_r with a descent G of their generic fiber to $\mathbf{Q}_p$ and a certain subcategory of the category of (φ, Γ)-modules over $\mathfrak{S}_r$. By the very construction of $\mathfrak{M}_r$, one has a canonical isomorphism $\mathfrak{M}_r(\mathcal{G}) \otimes_{\mathfrak{S}_r, \varphi} S_r \simeq \mathbf{D}(\mathcal{G}_0)_{S_r}$, where S_r is the p-adic completion of the PD envelope of the surjection $\mathfrak{S}_r \twoheadrightarrow \mathscr{O}_{K_r}$ taking u_r to $\varepsilon^{(r)} - 1$ and $\mathbf{D}(\mathcal{G}_0)_\star$ is the *Dieudonné crystal* of the p-divisible group $\mathcal{G}_0 := \mathcal{G} \times_{\mathscr{O}_{K_r}} \mathscr{O}_{K_r}/(p)$; this isomorphism provides the crucial link with geometry that we need to carry out our constructions. Moreover, the full $\mathscr{G}_{\mathbf{Q}_p}$-representation $(T_p G)^\vee$ can be recovered from $\mathfrak{M}_r(\mathcal{G})$ by extending scalars to an appropriate period ring and passing to Frobenius invariants.

To construct our crystalline analogue of Hida's ordinary Λ-adic étale cohomology, we form

$$\mathfrak{M}_\infty^\star := \varprojlim \Big(\mathfrak{M}_r(\mathcal{G}_r^\star) \underset{\mathfrak{S}_r}{\otimes} \mathfrak{S}_\infty \Big) \quad \text{for } \star \in \{\text{ét}, \text{m}, \text{null}\}$$

with the projective limits taken along the mappings induced by $\mathcal{G}_r \times_{R_r} R_{r+1} \to \mathcal{G}_{r+1}$ via the functoriality of $\mathfrak{M}_r(\cdot)$ and its compatibility with base change. These are $\Lambda_{\mathfrak{S}_\infty}$-modules equipped with a semilinear action of Γ, a linear and commuting action of $\mathfrak{H}^*$, and a φ (respectively φ^{-1}) semilinear endomorphism F (respectively V) satisfying $FV = \omega$ and $VF = \varphi^{-1}(\omega)$, for $\omega := \varphi(u_1)/u_1 = u_0/\varphi^{-1}(u_0) \in \mathfrak{S}_\infty$.

THEOREM 1.2.9. *There is a canonical short exact sequence of finite free $\Lambda_{\mathfrak{S}_\infty}$-modules with linear $\mathfrak{H}^*$-action, semilinear Γ-action, and semilinear endomorphisms F, V satisfying $FV = \omega$, $VF = \varphi^{-1}(\omega)$*

$$0 \longrightarrow \mathfrak{M}_\infty^{\text{ét}} \longrightarrow \mathfrak{M}_\infty \longrightarrow \mathfrak{M}_\infty^{\text{m}} \longrightarrow 0. \tag{1.2.9}$$

Each of $\mathfrak{M}_\infty^\star$ for $\star \in \{\text{ét}, \text{m}\}$ is free of rank d' over $\Lambda_{\mathfrak{S}_\infty}$, while $\mathfrak{M}_\infty$ is free of rank $2d'$, where d' is as in Theorem 1.2.1. Extending scalars on (1.2.9) along the canonical surjection $\Lambda_{\mathfrak{S}_\infty} \twoheadrightarrow \mathfrak{S}_\infty[\Delta/\Delta_r]$ yields the short exact sequence

$$0 \longrightarrow \mathfrak{M}_r(\mathcal{G}_r^{\text{ét}}) \underset{\mathfrak{S}_r}{\otimes} \mathfrak{S}_\infty \longrightarrow \mathfrak{M}_r(\mathcal{G}_r) \underset{\mathfrak{S}_r}{\otimes} \mathfrak{S}_\infty \longrightarrow \mathfrak{M}_r(\mathcal{G}_r^{\text{m}}) \underset{\mathfrak{S}_r}{\otimes} \mathfrak{S}_\infty \longrightarrow 0$$

compatibly with $\mathfrak{H}^*$, Γ, F, and V. The Frobenius endomorphism F commutes with $\mathfrak{H}^*$ and Γ, whereas the Verschiebung V commutes with $\mathfrak{H}^*$ and satisfies $V\gamma = \varphi^{-1}(\omega/\gamma\omega) \cdot \gamma V$ for all $\gamma \in \Gamma$.

[4] The p-adic completion of $\mathfrak{S}_\infty$ is actually a very nice ring: it is canonically and Frobenius equivariantly isomorphic to $W(\mathbf{F}_p[\![u_0]\!]^{\text{rad}})$, for $\mathbf{F}_p[\![u_0]\!]^{\text{rad}}$ the perfect closure of the $\mathbf{F}_p$-algebra $\mathbf{F}_p[\![u_0]\!]$.

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Again, in the spirit of Theorem 1.2.2 and [Cai17, Proposition 3.8], there is a corresponding 'auto-duality' result for $\mathfrak{M}_\infty$. To state it, we must work over $\mathfrak{S}'_\infty := \mathfrak{S}_\infty[\mu_N]$.

THEOREM 1.2.10. *Let $\mu : \Gamma \to \Lambda_{\mathfrak{S}_\infty}^\times$ be the 1-cocycle given by $\mu(\gamma) := (u_1/\gamma u_1)\chi(\gamma)\langle\chi(\gamma)\rangle$. There is a canonical $\mathfrak{H}^*$- and $\mathrm{Gal}(K'_\infty/K_0)$-compatible isomorphism of short exact sequences*

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{M}_\infty^{\text{ét}}(\mu\langle a\rangle_N)_{\Lambda_{\mathfrak{S}'_\infty}} \arrow[r] \arrow[d, "\simeq"'] & \mathfrak{M}_\infty(\mu\langle a\rangle_N)_{\Lambda_{\mathfrak{S}'_\infty}} \arrow[r] \arrow[d, "\simeq"'] & \mathfrak{M}_\infty^{\text{m}}(\mu\langle a\rangle_N)_{\Lambda_{\mathfrak{S}'_\infty}} \arrow[r] \arrow[d, "\simeq"'] & 0 \\
0 \arrow[r] & (\mathfrak{M}_\infty^{\text{m}})^\vee_{\Lambda_{\mathfrak{S}'_\infty}} \arrow[r] & (\mathfrak{M}_\infty)^\vee_{\Lambda_{\mathfrak{S}'_\infty}} \arrow[r] & (\mathfrak{M}_\infty^{\text{ét}})^\vee_{\Lambda_{\mathfrak{S}'_\infty}} \arrow[r] & 0
\end{tikzcd} \tag{1.2.10}$$

intertwining F and V on the top row with $V^\vee$ and $F^\vee$, respectively, on the bottom. The action of $\mathrm{Gal}(K'_\infty/K_0)$ on the bottom row is the standard one $\gamma \cdot f := \gamma f \gamma^{-1}$ on linear duals.

The $\Lambda_{\mathfrak{S}_\infty}$-modules $\mathfrak{M}_\infty^{\text{ét}}$ and $\mathfrak{M}_\infty^{\text{m}}$ admit canonical descents to Λ.

THEOREM 1.2.11. *There are canonical $\mathfrak{H}^*$-, Γ-, F-, and V-equivariant isomorphisms of $\Lambda_{\mathfrak{S}_\infty}$-modules*

$$\mathfrak{M}_\infty^{\text{ét}} \simeq \mathbf{D}_\infty^{\text{ét}} \otimes_\Lambda \Lambda_{\mathfrak{S}_\infty}, \tag{1.2.11a}$$

intertwining F and V with $F \otimes \varphi$ and $F^{-1} \otimes \varphi^{-1}(\omega) \cdot \varphi^{-1}$, respectively, and $\gamma \in \Gamma$ with $\gamma \otimes \gamma$,

$$\mathfrak{M}_\infty^{\text{m}} \simeq \mathbf{D}_\infty^{\text{m}} \otimes_\Lambda \Lambda_{\mathfrak{S}_\infty}, \tag{1.2.11b}$$

intertwining F and V with $V^{-1} \otimes \omega \cdot \varphi$ and $V \otimes \varphi^{-1}$, respectively, and γ with $\gamma \otimes \chi(\gamma)^{-1}\gamma u_1/u_1$. In particular, F (respectively V) acts invertibly on $\mathfrak{M}_\infty^{\text{ét}}$ (respectively $\mathfrak{M}_\infty^{\text{m}}$).

From $\mathfrak{M}_\infty$, we can recover $\mathbf{D}_\infty$ and $e^{*\prime}H^1_{\text{dR}}$, with their additional structures.

THEOREM 1.2.12. *Let $\tau : \Lambda_{\mathfrak{S}_\infty} \twoheadrightarrow \Lambda$ be the Λ-algebra surjection induced by $u_r \mapsto 0$. There is a canonical Γ- and $\mathfrak{H}^*$-equivariant isomorphism of split exact sequences of finite free Λ-modules*

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{M}_\infty^{\text{ét}} \otimes_{\Lambda_{\mathfrak{S}_\infty}, \tau\circ\varphi} \Lambda \arrow[r] \arrow[d, "\simeq"'] & \mathfrak{M}_\infty \otimes_{\Lambda_{\mathfrak{S}_\infty}, \tau\circ\varphi} \Lambda \arrow[r] \arrow[d, "\simeq"'] & \mathfrak{M}_\infty^{\text{m}} \otimes_{\Lambda_{\mathfrak{S}_\infty}, \tau\circ\varphi} \Lambda \arrow[r] \arrow[d, "\simeq"'] & 0 \\
0 \arrow[r] & \mathbf{D}_\infty^{\text{ét}} \arrow[r] & \mathbf{D}_\infty \arrow[r] & \mathbf{D}_\infty^{\text{m}} \arrow[r] & 0
\end{tikzcd} \tag{1.2.12}$$

which carries $F \otimes 1$ to F and $V \otimes 1$ to V.

Let $\theta : \Lambda_{\mathfrak{S}_\infty} \to \Lambda_{R_\infty}$ be the Λ-algebra surjection induced by $u_r \mapsto \varepsilon^{(r)} - 1$. There is a canonical Γ- and $\mathfrak{H}^*$-equivariant isomorphism of split exact sequences of finite free Λ_{R_∞}-modules

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{M}_\infty^{\text{ét}} \otimes_{\Lambda_{\mathfrak{S}_\infty}, \theta\circ\varphi} \Lambda_{R_\infty} \arrow[r] \arrow[d, "\simeq"'] & \mathfrak{M}_\infty \otimes_{\Lambda_{\mathfrak{S}_\infty}, \theta\circ\varphi} \Lambda_{R_\infty} \arrow[r] \arrow[d, "\simeq"'] & \mathfrak{M}_\infty^{\text{m}} \otimes_{\Lambda_{\mathfrak{S}_\infty}, \theta\circ\varphi} \Lambda_{R_\infty} \arrow[r] \arrow[d, "\simeq"'] & 0 \\
0 \arrow[r] & e^{*\prime}H^1(\mathscr{O}) \arrow[r, "i"'] & e^{*\prime}H^1_{\text{dR}} \arrow[r, "j"'] & e^{*\prime}H^0(\omega) \arrow[r] & 0
\end{tikzcd} \tag{1.2.13}$$

where i and j are the canonical sections given by the splitting in Theorem 1.2.4.

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To recover Hida's ordinary étale cohomology from $\mathfrak{M}_\infty$, we require the period ring of Fontaine[5] $\widetilde{\mathbf{E}}^+ := \varprojlim \mathscr{O}_{\mathbf{C}_p}/(p)$, with the projective limit taken along the p-power mapping; this is a perfect valuation ring of characteristic p equipped with a canonical action of $\mathscr{G}_{\mathbf{Q}_p}$ via 'coordinates'. We write $\widetilde{\mathbf{E}}$ for the fraction field of $\widetilde{\mathbf{E}}^+$ and $\widetilde{\mathbf{A}} := W(\widetilde{\mathbf{E}})$ for its ring of Witt vectors, equipped with its canonical Frobenius automorphism φ and $\mathscr{G}_{\mathbf{Q}_p}$-action induced by Witt functoriality. Our fixed choice of p-power compatible sequence $\{\varepsilon^{(r)}\}_{r\geqslant 0}$ determines an element $\underline{\varepsilon} := (\varepsilon^{(r)} \bmod p)_{r\geqslant 0}$ of $\widetilde{\mathbf{E}}^+$, and we $\mathbf{Z}_p$-linearly embed $\mathfrak{S}_\infty$ in $\widetilde{\mathbf{A}}$ via $u_r \mapsto \varphi^{-r}([\underline{\varepsilon}]-1)$, where $[\cdot]$ is the Teichmüller section. This embedding is φ- and $\mathscr{G}_{\mathbf{Q}_p}$-compatible, with $\mathscr{G}_{\mathbf{Q}_p}$ acting on $\mathfrak{S}_\infty$ through the quotient $\mathscr{G}_{\mathbf{Q}_p} \twoheadrightarrow \Gamma$.

THEOREM 1.2.13. *Twisting the structure map $\mathfrak{S}_\infty \to \widetilde{\mathbf{A}}$ by the Frobenius automorphism φ, there is a canonical isomorphism of short exact sequences of $\Lambda_{\widetilde{\mathbf{A}}}$-modules with $\mathfrak{H}^*$-action*

$$\begin{CD}
0 @>>> \mathfrak{M}_\infty^{\text{ét}} \underset{\Lambda_{\mathfrak{S}_\infty},\varphi}{\otimes} \Lambda_{\widetilde{\mathbf{A}}} @>>> \mathfrak{M}_\infty \underset{\Lambda_{\mathfrak{S}_\infty},\varphi}{\otimes} \Lambda_{\widetilde{\mathbf{A}}} @>>> \mathfrak{M}_\infty^{\text{m}} \underset{\Lambda_{\mathfrak{S}_\infty},\varphi}{\otimes} \Lambda_{\widetilde{\mathbf{A}}} @>>> 0\\
@. @V{\simeq}VV @V{\simeq}VV @V{\simeq}VV @.\\
0 @>>> (e^{*\prime}H^1_{\text{ét}})^{\mathscr{I}} \underset{\Lambda}{\otimes} \Lambda_{\widetilde{\mathbf{A}}} @>>> e^{*\prime}H^1_{\text{ét}} \underset{\Lambda}{\otimes} \Lambda_{\widetilde{\mathbf{A}}} @>>> (e^{*\prime}H^1_{\text{ét}})_{\mathscr{I}} \underset{\Lambda}{\otimes} \Lambda_{\widetilde{\mathbf{A}}} @>>> 0
\end{CD} \tag{1.2.14}$$

that is $\mathscr{G}_{\mathbf{Q}_p}$-equivariant for the 'diagonal' action of $\mathscr{G}_{\mathbf{Q}_p}$ (with $\mathscr{G}_{\mathbf{Q}_p}$ acting on $\mathfrak{M}_\infty$ through Γ) and intertwines $F\otimes\varphi$ with $\mathrm{id}\otimes\varphi$ and $V\otimes\varphi^{-1}$ with $\mathrm{id}\otimes\omega\cdot\varphi^{-1}$. In particular, there is a canonical isomorphism of Λ-modules, compatible with the actions of $\mathfrak{H}^*$ and $\mathscr{G}_{\mathbf{Q}_p}$,

$$e^{*\prime}H^1_{\text{ét}} \simeq \Big(\mathfrak{M}_\infty \underset{\Lambda_{\mathfrak{S}_\infty},\varphi}{\otimes} \Lambda_{\widetilde{\mathbf{A}}}\Big)^{F\otimes\varphi=1}. \tag{1.2.15}$$

Theorem 1.2.13 allows us to give a new proof of Hida's finiteness and control theorems for $e^{*\prime}H^1_{\text{ét}}$.

COROLLARY 1.2.14 (Hida). *Let d' be as in Theorem 1.2.1. Then $e^{*\prime}H^1_{\text{ét}}$ is a free Λ-module of rank $2d'$. For each $r\geqslant 1$, there is a canonical isomorphism of $\mathbf{Z}_p[\Delta/\Delta_r]$-modules with linear $\mathfrak{H}^*$- and $\mathscr{G}_{\mathbf{Q}_p}$-actions*

$$e^{*\prime}H^1_{\text{ét}} \underset{\Lambda}{\otimes} \mathbf{Z}_p[\Delta/\Delta_r] \simeq e^{*\prime}H^1_{\text{ét}}(X_{r\overline{\mathbf{Q}}_p}, \mathbf{Z}_p),$$

which is moreover compatible with the isomorphisms (1.2.8a) and (1.2.8b) in the evident manner.

We also deduce a new proof of Ohta's duality theorem [Oht95, Theorem 4.3.1] (cf. [MW86, § 6]).

COROLLARY 1.2.15 (Ohta). *Let $\nu : \mathscr{G}_{\mathbf{Q}_p} \to \mathfrak{H}^*$ be the character $\nu := \chi\langle\chi\rangle\lambda(\langle p\rangle_N)$. There is a canonical $\mathfrak{H}^*$- and $\mathscr{G}_{\mathbf{Q}_p}$-equivariant isomorphism of short exact sequences of Λ-modules*

$$\begin{CD}
0 @>>> (e^{*\prime}H^1_{\text{ét}})^{\mathscr{I}}(\nu) @>>> e^{*\prime}H^1_{\text{ét}}(\nu) @>>> (e^{*\prime}H^1_{\text{ét}})_{\mathscr{I}}(\nu) @>>> 0\\
@. @V{\simeq}VV @V{\simeq}VV @V{\simeq}VV @.\\
0 @>>> \mathrm{Hom}_\Lambda((e^{*\prime}H^1_{\text{ét}})_{\mathscr{I}}, \Lambda) @>>> \mathrm{Hom}_\Lambda(e^{*\prime}H^1_{\text{ét}}, \Lambda) @>>> \mathrm{Hom}_\Lambda((e^{*\prime}H^1_{\text{ét}})^{\mathscr{I}}, \Lambda) @>>> 0
\end{CD} \tag{1.2.16}$$

[5] Though we use the notation introduced by Berger and Colmez.

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1.3 Overview of the article

In § 2, we introduce and study the tower of p-divisible groups whose cohomology allows us to construct our Λ-adic Dieudonné and crystalline analogues of Hida's étale cohomology in § 3 and § 4.2, respectively. We establish Λ-adic comparison isomorphisms between each of these cohomologies using the integral comparison isomorphisms of [Cai10] and [CL17], the latter of which is recalled (and specialized to the case of ordinary p-divisible groups) in § 4.1. This enables us to give a new proof of Hida's freeness and control theorems and of Ohta's duality theorem in § 4.3. A key technical ingredient in our proofs is the commutative algebra formalism developed in [Cai17, § 3.1] for dealing with projective limits of cohomology and establishing appropriate 'freeness and control' theorems by reduction to characteristic p.

As remarked in § 1.2, and following [Oht95] and [MW86], our construction of the Λ-adic Dieudonné and crystalline counterparts to Hida's étale cohomology excludes the trivial eigenspace for the action of $\mu_{p-1} \subseteq \mathbf{Z}_p^\times$ so as to avoid technical complications with logarithmic p-divisible groups. In [Oht00], Ohta used the 'fixed part' (in the sense of Grothendieck [Gro72, 2.2.3]) of Néron models with semiabelian reduction to extend his results on Λ-adic Hodge cohomology to allow trivial tame nebentype character. We are confident that by using Kato's logarithmic Dieudonné theory [Kat89] one can appropriately generalize our results in §§ 3–4 to include the missing eigenspace for the action of μ_{p-1}.

1.4 Notation

If $\varphi : A \to B$ is any map of rings, we will often write $M_B := M \otimes_A B$ for the B-module induced from an A-module M by extension of scalars. When we wish to specify φ, we will write $M \otimes_{A,\varphi} B$ or simply $M \otimes_\varphi B$. If $\varphi : T' \to T$ is any morphism of schemes, for any T-scheme X we denote by $X_{T'}$ the base change of X along φ. If $f : X \to Y$ is any morphism of T-schemes, we will write $f_{T'} : X_{T'} \to Y_{T'}$ for the morphism of T'-schemes obtained from f by base change along φ. When $T = \mathrm{Spec}(R)$ and $T' = \mathrm{Spec}(R')$ are affine, we abuse notation and write $X_{R'}$ or $X \times_R R'$ for $X_{T'}$. We frequently work with schemes over a discrete valuation ring R, and will write $\mathcal{X}, \mathcal{Y}, \ldots$ for schemes over $\mathrm{Spec}(R)$, reserving $X, Y, \ldots$ (respectively $\overline{\mathcal{X}}, \overline{\mathcal{Y}}, \ldots$) for their generic (respectively special) fibers. As this article is a continuation of [Cai17], we will freely use the notation and conventions therein.

2. Λ-adic Barsotti–Tate groups

To construct the crystalline analogue of Hida's ordinary Λ-adic étale cohomology, we will study the crystalline cohomology of a certain 'tower' $\{\mathcal{G}_r\}_{r\geqslant 1}$ of p-divisible groups (a Λ-adic Barsotti–Tate group in the sense of Hida [Hid14, Hid05a, Hid05b]) whose construction involves artfully cutting out certain p-divisible subgroups of the modular Jacobians $J_r[p^\infty]$ over $\mathbf{Q}$ and the 'good reduction' theorems of Langlands–Carayol–Saito. The construction of $\{\mathcal{G}_r\}_{r\geqslant 1}$ is well known (e.g. [MW86, § 1], [MW84, ch. 3, § 1], and [Til87, Definition 1.2]) but as we shall need finer information about the $\mathcal{G}_r$ than is available in the literature, we devote this section to recalling their construction and properties.

We will use the notation of § 1.2 and of [Cai17, Appendix B] throughout, which we first briefly recall. For $r \geqslant 1$, we write $X_r := X_1(Np^r)$ for the canonical model over $\mathbf{Q}$ with rational cusp at $i\infty$ of the modular curve arising as the quotient of the extended upper-halfplane by the congruence subgroup $\Gamma_1(Np^r)$ (cf. [Cai17, Remark B.7]). There are two natural degeneracy mappings $\rho, \sigma : X_{r+1} \rightrightarrows X_r$ of curves over $\mathbf{Q}$ induced by the self-maps of the upper-halfplane $\rho : \tau \mapsto \tau$ and $\sigma : \tau \mapsto p\tau$; see [Cai17, Remark B.8]. Denote by $J_r := \mathrm{Pic}^0_{X_r/\mathbf{Q}}$ the Jacobian of X_r over $\mathbf{Q}$

and write $\mathfrak{H}_r(\mathbf{Z})$ for the $\mathbf{Z}$-subalgebra of $\mathrm{End}_{\mathbf{Q}}(J_r)$ generated by the Hecke operators $\{T_\ell\}_{\ell\nmid Np}$, $\{U_\ell\}_{\ell\mid Np}$ and the Diamond operators $\{\langle u\rangle\}_{u\in\mathbf{Z}_p^\times}$. We define $\mathfrak{H}_r(\mathbf{Z})^*$ similarly, using instead the 'transpose' Hecke and diamond operators, and set $\mathfrak{H}_r := \mathfrak{H}_r(\mathbf{Z})\otimes_{\mathbf{Z}}\mathbf{Z}_p$ and $\mathfrak{H}_r^* := \mathfrak{H}_r(\mathbf{Z})^*\otimes_{\mathbf{Z}}\mathbf{Z}_p$; see [Cai17, § 1.4 as well as Definitions A.15 and B.26]. As usual, we write $e_r\in\mathfrak{H}_r$ and $e_r^*\in\mathfrak{H}_r^*$ for the idempotents of these semilocal $\mathbf{Z}_p$-algebras corresponding to the Atkin operators U_p and U_p^*, respectively. We put $e := (e_r)_r$ and $e^* := (e_r^*)_r$ for the induced idempotents of the 'big' p-adic Hecke algebras $\mathfrak{H} := \varprojlim_r \mathfrak{H}_r$ and $\mathfrak{H}^* := \varprojlim_r \mathfrak{H}_r^*$, with the projective limits formed using the transition mappings induced by the maps on Jacobians $J_r \rightrightarrows J_{r'}$ for $r'\geqslant r$ arising (via Picard functoriality) from σ and ρ, respectively. Let w_r be the Atkin–Lehner 'involution' of X_r over $\mathbf{Q}(\mu_{Np^r})$ corresponding to a choice of primitive Np^rth root of unity as in the discussion preceding [Cai17, Proposition B.9]; we simply write w_r for the automorphism $\mathrm{Alb}(w_r)$ of J_r over $\mathbf{Q}(\mu_{Np^r})$ induced by Albanese functoriality. We note that for any Hecke operator $T\in\mathfrak{H}_r(\mathbf{Z})$, one has the relation $w_rT = T^*w_r$ as endomorphisms of J_r over $\mathbf{Q}(\mu_{Np^r})$.

2.1 Spaces of ordinary modular forms

As in [Cai17, § 3.3], for a ring A, a nonnegative integer k, and a congruence subgroup Γ of $\mathrm{SL}_2(\mathbf{Z})$, we write $S_k(\Gamma;A)$ for the space of weight-k cusp forms for Γ over A, and for ease of notation we put $S_k(\Gamma) := S_k(\Gamma;\overline{\mathbf{Q}})$. If Γ', Γ are congruence subgroups, then associated to any $\gamma\in\mathrm{GL}_2(\mathbf{Q})$ with $\gamma^{-1}\Gamma'\gamma\subseteq\Gamma$ is an injective pullback mapping $\iota_\gamma : S_k(\Gamma)\hookrightarrow S_k(\Gamma')$ given by $\iota_\gamma(f) := f|_{\gamma^{-1}}$, as well as a surjective 'trace' mapping

$$\mathrm{tr}_\gamma : S_k(\Gamma')\twoheadrightarrow S_k(\Gamma)\quad\text{given by}\ \ \mathrm{tr}_\gamma(f) := \sum_{\delta\in\gamma^{-1}\Gamma'\gamma\backslash\Gamma}(f|_\gamma)|_\delta \tag{2.1.1}$$

with $\mathrm{tr}_\gamma\circ\iota_\gamma$ multiplication by $[\Gamma:\gamma^{-1}\Gamma'\gamma]$ on $S_k(\Gamma)$. If $\Gamma'\subseteq\Gamma$, then, *unless specified to the contrary*, we will always view $S_k(\Gamma)$ as a subspace of $S_k(\Gamma')$ via ι_{id}.

For nonnegative integers $i\leqslant r$, we set $\Gamma_r^i := \Gamma_1(Np^i)\cap\Gamma_0(p^r)$ for the intersection (taken inside $\mathrm{SL}_2(\mathbf{Z})$) and put $\Gamma_r := \Gamma_r^r$. We will need the following fact concerning the trace mapping (2.1.1) attached to the canonical inclusion $\Gamma_r\subseteq\Gamma_i$ for $r\geqslant i$; for notational clarity, we will write $\mathrm{tr}_{r,i} : S_k(\Gamma_r)\to S_k(\Gamma_i)$ for this map.

Lemma 2.1.1. *Fix integers $i\leqslant r$ and let $\mathrm{tr}_{r,i} : S_k(\Gamma_r)\to S_k(\Gamma_i)$ be the trace mapping (2.1.1) attached to the inclusion $\Gamma_r\subseteq\Gamma_i$. For $\alpha := \left(\begin{smallmatrix}1&0\\0&p\end{smallmatrix}\right)$, we have an equality of $\overline{\mathbf{Q}}$-endomorphisms of $S_k(\Gamma_r)$*

$$\iota_{\alpha^{r-i}}\circ\mathrm{tr}_{r,i} = (U_p^*)^{r-i}\sum_{\delta\in\Delta_i/\Delta_r}\langle\delta\rangle. \tag{2.1.2}$$

Proof. This follows immediately from [Oht95, 2.3.3], using the equalities of operators $(\cdot)|_{\sigma_\delta} = \langle\delta\rangle$ and $U_p^* = w_rU_pw_r^{-1}$ on $S_k(\Gamma_r)$; cf. also [Til87, p. 339]. ☐

Perhaps the most essential 'classical' fact for our purposes is that the Hecke operator U_p acting on spaces of modular forms 'contracts' the p-level, as is made precise by the following.

Lemma 2.1.2. *If $f\in S_k(\Gamma_r^i)$, then U_p^df is in the image of the map $\iota_{\mathrm{id}} : S_k(\Gamma_{r-d}^i)\hookrightarrow S_k(\Gamma_r^i)$ for each integer $d\leqslant r-i$. In particular, $U_p^{r-i}f$ is in the image of $S_k(\Gamma_i)\hookrightarrow S_k(\Gamma_r^i)$.*

Proof of Lemma 2.1.2. As in the proof of [Oht99, 1.2.10], this follows easily from the decomposition

$$\Gamma_r^i \alpha^d \Gamma_{r-d}^i = \coprod_{j=0}^{p^d-1} \Gamma_r^i \begin{pmatrix} 1 & j \\ 0 & p^d \end{pmatrix},$$

where $\alpha := \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)$; see [Hid86a, Lemma 4.3 and p. 570] and cf. [Hid14, § 2]. □

For each integer i and any character $\varepsilon : (\mathbf{Z}/Np^i\mathbf{Z})^\times \to \overline{\mathbf{Q}}^\times$, we denote by $S_2(\Gamma_i, \varepsilon)$ the $\mathfrak{H}_i$-stable subspace of weight-2 cusp forms for Γ_i over $\overline{\mathbf{Q}}$ on which the diamond operators act through $\varepsilon(\cdot)$. Define

$$\overline{V}_r := \bigoplus_{i=1}^{r} \bigoplus_{\varepsilon} S_2(\Gamma_i, \varepsilon), \tag{2.1.3}$$

where the inner sum is over all Dirichlet characters defined modulo Np^i whose p-parts are *primitive* (i.e. whose conductor has p-part exactly p^i). We view $\overline{V}_r$ as a $\overline{\mathbf{Q}}$-subspace of $S_2(\Gamma_r)$ in the usual way (i.e. via the embeddings ι_{id}). We define $\overline{V}_r^*$ as the direct sum (2.1.3), but viewed as a subspace of $S_2(\Gamma_r)$ via the 'nonstandard' embeddings $\iota_{\alpha^{r-i}} : S_2(\Gamma_i) \to S_2(\Gamma_r)$.

As in [Cai17, 2.33], we write f' for the idempotent of $\mathbf{Z}_{(p)}[\mathbf{F}_p^\times]$ corresponding to 'projection away from the trivial $\mathbf{F}_p^\times$-eigenspace'; explicitly, we have

$$f' := 1 - \frac{1}{p-1} \sum_{g \in \mathbf{F}_p^\times} g. \tag{2.1.4}$$

We set $h' := (p-1)f'$, so that $h'^2 = (p-1)h'$, and define endomorphisms of $S_2(\Gamma_r)$:

$$U_r^* := h' \circ (U_p^*)^{r+1} = (U_p^*)^{r+1} \circ h' \quad \text{and} \quad U_r := h' \circ (U_p)^{r+1} = (U_p)^{r+1} \circ h'. \tag{2.1.5}$$

Corollary 2.1.3. *As subspaces of $S_2(\Gamma_r)$, we have $w_r(\overline{V}_r^*) = \overline{V}_r$. The space $\overline{V}_r$ (respectively $\overline{V}_r^*$) is naturally an $\mathfrak{H}_r$ (respectively $\mathfrak{H}_r^*$)-stable subspace of $S_2(\Gamma_r)$ and admits a canonical descent to $\mathbf{Q}$. Furthermore, the endomorphisms U_r and U_r^* of $S_2(\Gamma_r)$ factor through $\overline{V}_r$ and $\overline{V}_r^*$, respectively.*

Proof. The first assertion follows from the relation $w_r \circ \iota_{\alpha^{r-i}} = \iota_{\mathrm{id}} \circ w_i$ as maps $S_2(\Gamma_i) \to S_2(\Gamma_r)$, together with the fact that w_i on $S_2(\Gamma_i)$ carries $S_2(\Gamma_i, \varepsilon)$ isomorphically onto $S_2(\Gamma_i, \varepsilon^{-1})$. The $\mathfrak{H}_r$-stability of $\overline{V}_r$ is clear as each of $S_2(\Gamma_i, \varepsilon)$ is an $\mathfrak{H}_r$-stable subspace of $S_2(\Gamma_r)$; that $\overline{V}_r^*$ is $\mathfrak{H}_r^*$-stable follows from this and the commutation relation $T^* w_r = w_r T$. That $\overline{V}_r$ and $\overline{V}_r^*$ admit canonical descents to $\mathbf{Q}$ is clear, as $\mathscr{G}_{\mathbf{Q}}$-conjugate Dirichlet characters have equal conductors. The final assertion concerning the endomorphisms U_r and U_r^* follows easily from Lemma 2.1.2, using the fact that $h' : S_2(\Gamma_r) \to S_2(\Gamma_r)$ has image contained in $\bigoplus_{i=1}^{r} S_k(\Gamma_r^i)$. □

Definition 2.1.4. We denote by V_r and V_r^* the canonical descents to $\mathbf{Q}$ of $\overline{V}_r$ and $\overline{V}_r^*$, respectively.

2.2 Good quotient abelian varieties

Following [MW84, ch. III, § 1] and [Til87, § 2], we recall the construction of certain 'good' quotient abelian varieties of J_r whose cotangent spaces are naturally identified with V_r and V_r^*. In what follows, we will make frequent use of the following elementary result.

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Lemma 2.2.1. *Let $f : A \to B$ be a homomorphism of commutative group varieties over a field K of characteristic 0. Then:*

(i) *the formation of* Lie *and* Cot *commutes with the formation of kernels and images; in particular, if A is connected and* $\mathrm{Lie}(f) = 0$ *(respectively* $\mathrm{Cot}(f) = 0$*), then $f = 0$;*

(ii) *let $i : B' \hookrightarrow B$ be a closed immersion of commutative group varieties over K with B' connected. If* $\mathrm{Lie}(f)$ *factors through* $\mathrm{Lie}(i)$*, then f factors (necessarily uniquely) through i;*

(iii) *let $j : A \twoheadrightarrow A''$ be a surjection of commutative group varieties over K with connected kernel. If* $\mathrm{Cot}(f)$ *factors through* $\mathrm{Cot}(j)$*, then f factors (necessarily uniquely) through j.*

Proof. This follows easily from the fact that objects in the category of commutative group varieties over a field of characteristic zero are automatically *smooth*, so the functors $\mathrm{Lie}(\cdot)$ and $\mathrm{Cot}(\cdot)$ on this category are *exact*. $\square$

To proceed with the construction of good quotients of J_r, let $Y_r := X_1(Np^{r+1}; Np^r)$ be the canonical model over $\mathbf{Q}$ with rational cusp at $i\infty$ of the modular curve corresponding to the congruence subgroup Γ_{r+1}^r (cf. [Cai17, Remark B.22]), and consider the 'degeneracy mappings' of curves over $\mathbf{Q}$ for $i = 1, 2$

$$X_r \xrightarrow{\pi} Y_{r-1} \xrightarrow{\pi_i} X_{r-1}, \tag{2.2.1$_i$}$$

where π and π_2 are induced by the canonical inclusions of subgroups $\Gamma_r \subseteq \Gamma_r^{r-1} \subseteq \Gamma_{r-1}$ via the upper-halfplane self-map $\tau \mapsto \tau$, and π_1 is induced by the inclusion $\alpha^{-1}\Gamma_r^{r-1}\alpha \subseteq \Gamma_{r-1}$ via the mapping $\tau \mapsto p\tau$, where α is as in Lemma 2.1.1; see [Cai17, B.8–B.9 and Remark B.23] for a moduli-theoretic description of these maps. Note that the composites $\pi \circ \pi_2$ and $\pi \circ \pi_1$ coincide with ρ and σ, respectively.

These mappings covariantly (respectively contravariantly) induce mappings on the associated Jacobians via Albanese (respectively Picard) functoriality. Writing $JY_r := \mathrm{Pic}^0_{Y_r/\mathbf{Q}}$ and setting $K_0^i := JY_0$ for $i = 1, 2$, we inductively define abelian subvarieties $\iota_r^i : K_r^i \hookrightarrow JY_r$ and abelian variety quotients $\alpha_r^i : J_r \twoheadrightarrow B_r^i$ as follows:

$$B_r^i := J_r / \mathrm{Pic}^0(\pi)(K_{r-1}^i) \quad \text{and} \quad K_r^i := \ker\Big(JY_r \xrightarrow{\alpha_r^i \circ \mathrm{Alb}(\pi_i)} B_r^i\Big)^0 \tag{2.2.2$_i$}$$

for $r \geqslant 1$, $i = 1, 2$, with α_r^i and ι_r^i the obvious mappings; here $(\cdot)^0$ denotes the connected component of the identity of $(\cdot)$. As in [Oht95, § 3.2], we have modified Tilouine's construction [Til87, § 2] so that the kernel of α_r^i is connected; i.e. is an abelian subvariety of J_r (cf. Remark 2.2.4). Note that we have a commutative diagram of abelian varieties over $\mathbf{Q}$ for $i = 1, 2$

$$\begin{tikzcd}
& J_r \arrow[r, two heads, "\alpha_r^i"] & B_r^i \arrow[d, equal] \\
K_r^i \arrow[r, hook, "\iota_r^i"] \arrow[d, equal] & JY_r \arrow[u, "\mathrm{Alb}(\pi_i)"] \arrow[r, "\alpha_r^i \circ \mathrm{Alb}(\pi_i)"] \arrow[d, "\mathrm{Pic}^0(\pi)"] & B_r^i \\
K_r^i \arrow[r, "\mathrm{Pic}^0(\pi)\circ\iota_r^i"'] & J_{r+1} \arrow[r, two heads, "\alpha_{r+1}^i"'] & B_{r+1}^i
\end{tikzcd} \tag{2.2.3$_i$}$$

with bottom two horizontal rows that are complexes.

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Warning 2.2.2. While the bottom row of ($2.2.3_i$) is exact in the middle *by definition* of α^i_{r+1}, the central row is *not* exact in the middle: it follows from the fact that $\mathrm{Alb}(\pi_i)\circ\mathrm{Pic}^0(\pi_i)$ is multiplication by p on J_r that the component group of the kernel of $\alpha^i_r\circ\mathrm{Alb}(\pi_i): JY_r\to B^i_r$ is nontrivial with order divisible by p. Moreover, there is no mapping $B^i_r\to B^i_{r+1}$ which makes the diagram ($2.2.3_i$) commute.

In order to be consistent with the literature, we adopt the following convention.

DEFINITION 2.2.3. We set $B_r := B^2_r$ and $B^*_r := B^1_r$, with B^i_r defined inductively by ($2.2.2_{\mathrm{i}}$). We likewise set $\alpha_r := \alpha^2_r$ and $\alpha^*_r := \alpha^1_r$.

Remark 2.2.4. One checks that our quotient $\alpha_r: J_r \twoheadrightarrow B_r$ coincides with that of Ohta [Oht95, § 3.2]. On the other hand, Tilouine constructed[6] an abelian variety quotient $\alpha'_r: J_r\twoheadrightarrow B'_r$ which factors through α_r via an isogeny $B_r\twoheadrightarrow B'_r$ which has degree divisible by p, as one sees using Warning 2.2.2. Due to this fact, it is *essential* for our purposes to work with B_r rather than B'_r. On the other hand, our B_r is naturally a quotient of the 'good' quotient $J_r\twoheadrightarrow A_r$ constructed by Mazur and Wiles in [MW84, ch. III, § 1], and the kernel of the corresponding surjection $A_r\twoheadrightarrow B_r$ is isogenous to $J_0\times J_0$.

PROPOSITION 2.2.5. *Over $F:=\mathbf{Q}(\mu_{Np^r})$, the automorphism w_r of J_{rF} induces an isomorphism of quotients $B_{rF}\simeq B^*_{rF}$. The abelian variety B_r (respectively B^*_r) is the unique quotient of J_r by a $\mathbf{Q}$-rational abelian subvariety with the property that the induced map on cotangent spaces*

$$\mathrm{Cot}(B_r)\xhookrightarrow[\mathrm{Cot}(\alpha_r)]{}\mathrm{Cot}(J_r)\simeq S_2(\Gamma_r;\mathbf{Q})\quad \textit{respectively}\quad \mathrm{Cot}(B^*_r)\xhookrightarrow[\mathrm{Cot}(\alpha^*_r)]{}\mathrm{Cot}(J_r)\simeq S_2(\Gamma_r;\mathbf{Q})$$

has image precisely V_r (respectively V^*_r). In particular, there are canonical actions of the Hecke algebras[7] *$\mathfrak{H}_r(\mathbf{Z})$ on B_r and $\mathfrak{H}^*_r(\mathbf{Z})$ on B^*_r for which α_r and α^*_r are equivariant.*

Proof. By the construction of B^i_r and the fact that $\rho w_r = \langle p\rangle_N w_{r-1}\sigma$ as maps $X_{rF}\to X_{r-1F}$ [Cai17, Proposition B.9], the automorphism w_r of J_{rF} carries $\ker(\alpha_r)$ to $\ker(\alpha^*_r)$ and induces an isomorphism $B_{rF}\simeq B^*_{rF}$ over F that intertwines the action of $\mathfrak{H}_r$ on B_r with $\mathfrak{H}^*_r$ on B^*_r. The isogeny $B_r\twoheadrightarrow B'_r$ of Remark 2.2.4 induces an isomorphism on cotangent spaces, compatibly with the inclusions into $\mathrm{Cot}(J_r)$. Thus, the claimed identification of the image of $\mathrm{Cot}(B_r)$ with V_r follows from [Til87, Proposition 2.1] (using [Til87, Definition 2.1]). The claimed uniqueness of $J_r\twoheadrightarrow B_r$ follows easily from Lemma 2.2.1(iii). Similarly, since the subspace V_r of $S_2(\Gamma_r)$ is stable under $\mathfrak{H}_r$, we conclude from Lemma 2.2.1(iii) that for any $T\in\mathfrak{H}_r(\mathbf{Z})$, the induced morphism $J_r\xrightarrow{T}J_r\twoheadrightarrow B_r$ factors through α_r and hence that $\mathfrak{H}_r(\mathbf{Z})$ acts on B_r compatibly (via α_r) with its action on J_r. □

[6] The notation Tilouine uses for his quotient is the same as the notation we have used for our (slightly modified) quotient. To avoid conflict, we have therefore chosen to alter his notation.

[7] We must warn the reader that Tilouine [Til87] writes $\mathfrak{H}_r(\mathbf{Z})$ for the $\mathbf{Z}$-subalgebra of $\mathrm{End}(J_r)$ generated by the Hecke operators acting via the $(\cdot)^*$-action (i.e. by 'Picard' functoriality) whereas our $\mathfrak{H}_r(\mathbf{Z})$ is defined using the $(\cdot)_*$-action. This discrepancy is due primarily to the fact that Tilouine identifies *tangent* spaces of modular abelian varieties with spaces of modular forms, rather than cotangent spaces as is our convention. Our notation for regarding Hecke algebras as subalgebras of $\mathrm{End}(J_r)$ agrees with that of Mazur and Wiles [MW84, ch. II, § 5], [MW86, § 7], and Ohta [Oht95, 3.1.5].

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LEMMA 2.2.6. *There exist unique morphisms $B_r^* \leftrightarrows B_{r-1}^*$ of abelian varieties over $\mathbf{Q}$ making*

$$\begin{tikzcd} J_r \arrow[r, "\alpha_r^*"] \arrow[d, "\mathrm{Alb}(\sigma)"'] & B_r^* \arrow[d] \\ J_{r-1} \arrow[r, "\alpha_{r-1}^*"'] & B_{r-1}^* \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} J_r \arrow[r, "\alpha_r^*"] & B_r^* \\ J_{r-1} \arrow[r, "\alpha_{r-1}^*"'] \arrow[u, "\mathrm{Pic}^0(\rho)"] & B_{r-1}^* \arrow[u] \end{tikzcd}$$

commute; these maps are moreover $\mathfrak{H}_r^*(\mathbf{Z})$-equivariant. By a slight abuse of notation, we will simply write $\mathrm{Alb}(\sigma)$ and $\mathrm{Pic}^0(\rho)$ for the induced maps on B_r^* and B_{r-1}^*, respectively.

Proof. Under the canonical identification of $\mathrm{Cot}(J_r) \otimes_{\mathbf{Q}} \overline{\mathbf{Q}}$ with $S_2(\Gamma_r)$, the mapping on cotangent spaces induced by $\mathrm{Alb}(\sigma)$ (respectively $\mathrm{Pic}^0(\rho)$) coincides with $\iota_\alpha : S_2(\Gamma_{r-1}) \to S_2(\Gamma_r)$ (respectively $\mathrm{tr}_{r,r-1} : S_2(\Gamma_r) \to S_2(\Gamma_{r-1})$). As the kernel of $\alpha_r^* : J_r \twoheadrightarrow B_r^*$ is connected by definition, thanks to Lemma 2.2.1(iii) it suffices to prove that ι_α (respectively $\mathrm{tr}_{r,r-1}$) carries V_{r-1}^* to V_r^* (respectively V_r^* to V_{r-1}^*). On one hand, the composite $\iota_\alpha \circ \iota_{\alpha^{r-1-i}} : S_2(\Gamma_i, \varepsilon) \to S_2(\Gamma_r)$ coincides with the embedding $\iota_{\alpha^{r-i}}$, and it follows immediately from the definition of V_r^* that ι_α carries V_{r-1}^* into V_r^*. On the other hand, an easy calculation using (2.1.2) shows that one has equalities of maps $S_2(\Gamma_i, \varepsilon) \to S_2(\Gamma_r)$

$$\iota_\alpha \circ \mathrm{tr}_{r,r-1} \circ \iota_{\alpha^{(r-i)}} = \begin{cases} \iota_{\alpha^{(r-i)}} p U_p^* & \text{if } i < r, \\ 0 & \text{if } i = r. \end{cases}$$

Thus, the image of $\iota_\alpha \circ \mathrm{tr}_{r,r-1} : V_r^* \to S_2(\Gamma_r)$ is contained in the image of $\iota_\alpha : V_{r-1}^* \to S_2(\Gamma_r)$; since ι_α is injective, we conclude that the image of $\mathrm{tr}_{r,r-1} : V_r^* \to S_2(\Gamma_{r-1})$ is contained in V_{r-1}^*, as desired. $\square$

For f' as in (2.1.4), we write $e^{*\prime} := f' e^* \in \mathfrak{H}^*$ and $e' := f' e \in \mathfrak{H}$ for the subidempotents of e^* and e, respectively, corresponding to projection away from the trivial eigenspace of μ_{p-1}.

PROPOSITION 2.2.7. *The maps α_r and α_r^* induce isomorphisms of p-divisible groups over $\mathbf{Q}$*

$$e^{*\prime} J_r[p^\infty] \simeq e^{*\prime} B_r^*[p^\infty] \quad \text{and} \quad e' J_r[p^\infty] \simeq e' B_r[p^\infty], \tag{2.2.4}$$

respectively, that are $\mathfrak{H}^*$- (respectively $\mathfrak{H}$)-equivariant and compatible with change in r via $\mathrm{Alb}(\sigma)$ and $\mathrm{Pic}^0(\rho)$ (respectively $\mathrm{Alb}(\rho)$ and $\mathrm{Pic}^0(\sigma)$).

We view the maps (2.1.5) as endomorphisms of J_r in the obvious way, and again write U_r^* and U_r for the induced endomorphisms of B_r^* and B_r, respectively. To prove Proposition 2.2.7, we need the following geometric incarnation of Corollary 2.1.3.

LEMMA 2.2.8. *There exists a unique $\mathfrak{H}_r^*(\mathbf{Z})$ (respectively $\mathfrak{H}_r(\mathbf{Z})$)-equivariant map $W_r^* : B_r^* \to J_r$ (respectively $W_r : B_r \to J_r$) of abelian varieties over $\mathbf{Q}$ such that the diagram*

$$\begin{tikzcd} J_r \arrow[r, two heads, "\alpha_r^*"] \arrow[d, "U_r^*"'] & B_r^* \arrow[d, "U_r^*"] \arrow[dl, "W_r^*"'] \\ J_r \arrow[r, two heads, "\alpha_r^*"'] & B_r^* \end{tikzcd} \quad \text{respectively} \quad \begin{tikzcd} J_r \arrow[r, two heads, "\alpha_r"] \arrow[d, "U_r"'] & B_r \arrow[d, "U_r"] \arrow[dl, "W_r"'] \\ J_r \arrow[r, two heads, "\alpha_r"'] & B_r \end{tikzcd} \tag{2.2.5}$$

commutes.

Proof. This follows at once from Corollary 2.1.3 and Lemma 2.2.1(iii), using the fact that the kernels of α_r and α_r^* are by construction connected. □

Proof of Proposition 2.2.7. From (2.2.5), we get commutative diagrams of p-divisible groups over $\mathbf{Q}$

$$\begin{tikzcd} e^{*\prime}J_r[p^\infty] \arrow[r, "\alpha_r^*"] \arrow[d, "U_r^*"', "\simeq"] & e^{*\prime}B_r^*[p^\infty] \arrow[d, "\simeq"', "U_r^*"] \arrow[dl, "W_r^*"'] \\ e^{*\prime}J_r[p^\infty] \arrow[r, "\alpha_r^*"'] & e^{*\prime}B_r^*[p^\infty] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} e'J_r[p^\infty] \arrow[r, "\alpha_r"] \arrow[d, "U_r"', "\simeq"] & e'B_r[p^\infty] \arrow[d, "\simeq"', "U_r"] \arrow[dl, "W_r"'] \\ e'J_r[p^\infty] \arrow[r, "\alpha_r"'] & e'B_r[p^\infty] \end{tikzcd} \tag{2.2.6}$$

in which all vertical arrows are isomorphisms due to the very definition of the idempotents $e^{*\prime}$ and e'. An easy diagram chase then shows that *all* arrows must be isomorphisms. □

As in the introduction, we put $K_r = \mathbf{Q}_p(\mu_{p^r})$, $K_r' := K_r(\mu_N)$, and write R_r and R_r' for the valuation rings of K_r and K_r', respectively. We set $\Gamma := \mathrm{Gal}(K_\infty/K_0)$, and write $a : \mathrm{Gal}(K_0'/K_0) \to (\mathbf{Z}/N\mathbf{Z})^\times$ for the character giving the tautological action of $\mathrm{Gal}(K_0'/K_0)$ on μ_N.

PROPOSITION 2.2.9. *The abelian varieties B_r and B_r^* acquire good reduction over K_r.*

Proof. See [MW84, ch. III, § 2, Proposition 2] and cf. [Hid86a, § 9, Lemma 9]. □

We will write $\mathcal{B}_r$, $\mathcal{B}_r^*$, and $\mathcal{J}_r$, respectively, for the Néron models of the base changes $(B_r)_{K_r}$, $(B_r^*)_{K_r}$, and $(J_r)_{K_r}$ over $T_r := \mathrm{Spec}(R_r)$; due to Proposition 2.2.7, both $\mathcal{B}_r$ and $\mathcal{B}_r^*$ are abelian schemes. The Néron mapping property gives canonical actions of $\mathfrak{H}_r(\mathbf{Z})$ on $\mathcal{B}_r$, $\mathcal{J}_r$ and of $\mathfrak{H}_r^*(\mathbf{Z})$ on $\mathcal{B}_r^*$, $\mathcal{J}_r$ over R_r extending the actions on generic fibers as well as 'semilinear' actions of Γ encoding the descent of the generic fibers to $\mathbf{Q}_p$. For each r, the Néron mapping property further provides diagrams

$$\begin{tikzcd} \mathcal{J}_r \times_{T_r} T_{r+1} \arrow[r, "\alpha_r^*"] \arrow[d, shift right, "\mathrm{Pic}^0(\rho)"'] & \mathcal{B}_r^* \times_{T_r} T_{r+1} \arrow[d, shift left, "\mathrm{Pic}^0(\rho)"] \\ \mathcal{J}_{r+1} \arrow[r, "\alpha_{r+1}^*"'] \arrow[u, shift right, "\mathrm{Alb}(\sigma)"'] & \mathcal{B}_{r+1}^* \arrow[u, shift left, "\mathrm{Alb}(\sigma)"] \end{tikzcd} \quad \text{respectively} \quad \begin{tikzcd} \mathcal{J}_r \times_{T_r} T_{r+1} \arrow[r, "\alpha_r"] \arrow[d, shift right, "\mathrm{Pic}^0(\sigma)"'] & \mathcal{B}_r \times_{T_r} T_{r+1} \arrow[d, shift left, "\mathrm{Pic}^0(\sigma)"] \\ \mathcal{J}_{r+1} \arrow[r, "\alpha_{r+1}"'] \arrow[u, shift right, "\mathrm{Alb}(\rho)"'] & \mathcal{B}_{r+1} \arrow[u, shift left, "\mathrm{Alb}(\rho)"] \end{tikzcd} \tag{2.2.7}$$

of smooth commutative group schemes over T_{r+1} in which the inner and outer rectangles commute, and all maps are $\mathfrak{H}_{r+1}^*(\mathbf{Z})$ (respectively $\mathfrak{H}_{r+1}(\mathbf{Z})$) and Γ-equivariant.

DEFINITION 2.2.10. We define $\mathcal{G}_r := e^{*\prime}(\mathcal{B}_r^*[p^\infty])$ and we write $\mathcal{G}_r' := \mathcal{G}_r^\vee$ for its Cartier dual. For each $r \geqslant s$, noting that U_p^* is an automorphism of $\mathcal{G}_r$, we obtain from (2.2.7) canonical morphisms

$$\rho_{r,s} : \mathcal{G}_s \times_{T_s} T_r \xrightarrow{\mathrm{Pic}^0(\rho)^{r-s}} \mathcal{G}_r \quad \text{and} \quad \rho_{r,s}' : \mathcal{G}_s' \times_{T_s} T_r \xrightarrow{(U_p^{*-1}\,\mathrm{Alb}(\sigma))^{\vee\, r-s}} \mathcal{G}_r', \tag{2.2.8}$$

where $(\cdot)^i$ denotes the i-fold composition, formed in the obvious manner. In this way, we get towers of p-divisible groups $\{\mathcal{G}_r, \rho_{r,s}\}$ and $\{\mathcal{G}_r', \rho_{r,s}'\}$; we will write G_r and G_r' for the unique descents of the generic fibers of $\mathcal{G}_r$ and $\mathcal{G}_r'$ to $\mathbf{Q}_p$, respectively.[8] We let $T^* \in \mathfrak{H}_r^*$ act on $\mathcal{G}_r$ through the action of $\mathfrak{H}_r^*(\mathbf{Z})$ on $\mathcal{B}_r^*$, and on $\mathcal{G}_r' = \mathcal{G}_r^\vee$ by duality (i.e. as $(T^*)^\vee$). The maps (2.2.8) are then $\mathfrak{H}_r^*$-equivariant.

[8] Of course, $G_r' = G_r^\vee$. Our nonstandard notation $\mathcal{G}_r'$ for the Cartier dual of $\mathcal{G}_r$ is preferable, due to the fact that $\rho_{r,s}'$ is *not* simply the dual of $\rho_{r,s}$; indeed, these two mappings go in opposite directions.

Remark 2.2.11. If $\mathcal{G}$ is *any* p-divisible group over R_r, the data of a descent G of the generic fiber of $\mathcal{G}$ to $\mathbf{Q}_p$ is, by Tate's theorem, equivalent to the data of isomorphisms $\mathcal{G} \simeq \overline{\gamma}^*(\mathcal{G})$ for each $\overline{\gamma}$ in Γ/Γ_r satisfying the obvious cocycle condition. Since the formation of the maximal étale quotient of $\mathcal{G}$ and of the maximal connected and multiplicative-type sub-p-divisible groups of $\mathcal{G}$ is functorial in $\mathcal{G}$, it follows that for $\star \in \{\text{ét}, 0, \text{m}\}$, the p-divisible group $\mathcal{G}^\star$ is likewise equipped with a descent of its generic fiber to $\mathbf{Q}_p$, which, by a slight abuse of notation, we simply denote by $G^\star$.

By Proposition 2.2.7, G_r is canonically isomorphic to $e^{*\prime} J_r[p^\infty]$, compatibly with the action of $\mathfrak{H}_r^*$. Since J_r is a Jacobian, and hence principally polarized, one might expect that $\mathcal{G}_r$ is isomorphic to its dual. However, this is *not quite* the case as the canonical isomorphism $J_r \simeq J_r^\vee$ intertwines the actions of $\mathfrak{H}_r$ and $\mathfrak{H}_r^*$, thus interchanging the idempotents $e^{*\prime}$ and e'. To describe the precise relationship, we proceed as follows. For each $\gamma \in \mathrm{Gal}(K'_\infty/K_0) \simeq \Gamma \times \mathrm{Gal}(K'_0/K_0)$, let us write $\phi_\gamma : G_{r K'_r} \xrightarrow{\simeq} \gamma^*(G_{r K'_r})$ for the descent data isomorphisms encoding the unique $\mathbf{Q}_p = K_0$-descent of $G_{r K'_r}$ furnished by G_r. We 'twist' this descent data by the $\mathrm{Aut}_{\mathbf{Q}_p}(G_r)$-valued character $\langle\chi\rangle\langle a\rangle_N$ of $\mathrm{Gal}(K'_\infty/K_0)$: explicitly, for $\gamma \in \mathrm{Gal}(K'_r/K_0)$, we set $\psi_\gamma := \phi_\gamma \circ \langle\chi(\gamma)\rangle\langle a(\gamma)\rangle_N$ and note that since $\langle\chi(\gamma)\rangle\langle a(\gamma)\rangle_N$ is defined over $\mathbf{Q}_p$, the map $\gamma \rightsquigarrow \psi_\gamma$ really does satisfy the cocycle condition. We denote by $G_r(\langle\chi\rangle\langle a\rangle_N)$ the unique p-divisible group over $\mathbf{Q}_p$ corresponding to this twisted descent datum. Since the diamond operators commute with the Hecke operators, there is a canonical induced action of $\mathfrak{H}_r^*$ on $G_r(\langle\chi\rangle\langle a\rangle_N)$. By construction, there is a canonical K'_r-isomorphism $G_r(\langle\chi\rangle\langle a\rangle_N)_{K'_r} \simeq G_{r K'_r}$. Since G_r acquires good reduction over K_r and the $\mathscr{G}_{K_r}$-representation afforded by the Tate module of $G_r(\langle\chi\rangle\langle a\rangle_N)$ is the twist of $T_p G_r$ by the *unramified* character $\langle a\rangle_N$, we conclude that $G_r(\langle\chi\rangle\langle a\rangle_N)$ also acquires good reduction over K_r, and we denote its corresponding prolongation to R_r by $\mathcal{G}_r(\langle\chi\rangle\langle a\rangle_N)$.

PROPOSITION 2.2.12. *There is a natural $\mathfrak{H}_r^*$-equivariant isomorphism of p-divisible groups over R_r*

$$\mathcal{G}'_r \simeq \mathcal{G}_r(\langle\chi\rangle\langle a\rangle_N) \tag{2.2.9}$$

which is compatible with change in r using $\rho'_{r,s}$ on $\mathcal{G}'_r$ and $\rho_{r,s}$ on $\mathcal{G}_r$.

Proof. Let $\varphi_r : J_r \to J_r^\vee$ be the canonical principal polarization over $\mathbf{Q}_p$; one then has the relation $\varphi_r \circ T = (T^*)^\vee \circ \varphi_r$ for each $T \in \mathfrak{H}_r(\mathbf{Z})$. On the other hand, the K'_r-automorphism $w_r : J_{r K'_r} \to J_{r K'_r}$ intertwines $T \in \mathfrak{H}_r(\mathbf{Z})$ with $T^* \in \mathfrak{H}_r^*(\mathbf{Z})$. Thus, the K'_r-morphism

$$\psi_r : J^\vee_{r K'_r} \xrightarrow{(U_p^{*r})^\vee} J^\vee_{r K'_r} \xrightarrow[\simeq]{\varphi_r^{-1}} J_{r K'_r} \xrightarrow[\simeq]{w_r} J_{r K'_r}$$

is $\mathfrak{H}_r^*(\mathbf{Z})$-equivariant. Passing to the induced map on p-divisible groups and applying $e^{*\prime}$, we obtain from Proposition 2.2.7 an $\mathfrak{H}_r^*$-equivariant isomorphism $\psi_r : G'_{r K'_r} \simeq G_{r K'_r}$ of p-divisible groups. As

$$\begin{tikzcd}
J_{r K'_r} \arrow[r, "\langle\chi(\gamma)\rangle\langle a\rangle_N w_r"] \arrow[d, "1\times\gamma"'] & J_{r K'_r} \arrow[d, "1\times\gamma"] \\
(J_{r K'_r})_\gamma \arrow[r, "\gamma^*(w_r)"'] & (J_{r K'_r})_\gamma
\end{tikzcd}$$

commutes for all $\gamma \in \mathrm{Gal}(K'_r/K_0)$ [Cai17, Proposition B.9], the K'_r-isomorphism ψ_r uniquely descends to an $\mathfrak{H}_r^*$-equivariant isomorphism $G'_r \simeq G_r(\langle\chi\rangle\langle a\rangle_N)$ of p-divisible groups over $\mathbf{Q}_p$.

By Tate's theorem, this identification uniquely extends to the desired isomorphism (2.2.9). The asserted compatibility with change in r boils down to the commutativity of the diagrams

$$\begin{tikzcd} e^{*\prime}J_s[p^\infty]^\vee \arrow[r, "(U_p^{*s})^\vee"] \arrow[d, "(U_p^{*-1}\,\mathrm{Alb}(\sigma))^{\vee r-s}"'] & e^{*\prime}J_s[p^\infty]^\vee \arrow[d, "\mathrm{Alb}(\sigma)^{\vee r-s}"] \\ e^{*\prime}J_r[p^\infty]^\vee \arrow[r, "(U_p^{*r})^\vee"'] & e^{*\prime}J_r[p^\infty]^\vee \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} J_{sK_r'}^\vee \arrow[r, "\varphi_s^{-1}"] \arrow[d, "\mathrm{Alb}(\sigma)^{\vee r-s}"'] & J_{sK_r'} \arrow[r, "w_s"] \arrow[d, "\mathrm{Pic}^0(\sigma)^{r-s}"] & J_{sK_r'} \arrow[d, "\mathrm{Pic}^0(\rho)^{r-s}"] \\ J_{rK_r'}^\vee \arrow[r, "\varphi_r^{-1}"'] & J_{rK_r'} \arrow[r, "w_r"'] & J_{rK_r'} \end{tikzcd}$$

for all $s \leqslant r$. The commutativity of the first diagram is clear, while that of the second follows from [Cai17, Proposition B.9] and the fact that for *any* finite morphism $f : Y \to X$ of smooth curves over a field K, one has $\varphi_Y \circ \mathrm{Pic}^0(f) = \mathrm{Alb}(f)^\vee \circ \varphi_X$, where $\varphi_\star : J_\star \to J_\star^\vee$ is the canonical principal polarization on Jacobians for $\star = X, Y$ (see, for example, [Cai10, proof of Lemma 5.5]). $\square$

2.3 The special fiber of $\mathcal{G}_r$

We now wish to study the special fiber of $\mathcal{G}_r$, and relate it to the special fibers of the integral models of modular curves studied in [Cai17, Appendix B]. To that end, let $\mathcal{X}_r$ be the Katz–Mazur integral model of X_r over R_r defined in [Cai17, Definition B.6]; it is a regular scheme that is proper and flat of pure relative dimension 1 over Spec R_r with smooth generic fiber naturally isomorphic to X_{rK_r}. According to [Cai17, Proposition B.14], the special fiber $\overline{\mathcal{X}}_r := \mathcal{X}_r \times_{R_r} \mathbf{F}_p$ is the 'disjoint union with crossings at the supersingular points' [KM85, 13.1.5] of smooth and proper Igusa curves $I_{(a,b,u)} := \mathrm{Ig}_{\max(a,b)}$ indexed by triples (a, b, u) with a, b running over nonnegative integers that sum to r and $u \in (\mathbf{Z}/p^{\min(a,b)}\mathbf{Z})^\times$; in particular, $\overline{\mathcal{X}}_r$ is geometrically *reduced*. We write $\overline{\mathcal{X}}_r^{\mathrm{n}}$ for the normalization of $\overline{\mathcal{X}}_r$, which is a disjoint union of Igusa curves $I_{(a,b,u)}$. The canonical semilinear action of Γ on $\mathcal{X}_r$ that encodes the descent data of the generic fiber to $\mathbf{Q}_p$ [Cai17, B.2] induces, by base change, an $\mathbf{F}_p$-linear 'geometric inertia action' of Γ on $\overline{\mathcal{X}}_r^{\mathrm{n}}$; in this way the p-divisible group $\mathrm{Pic}^0_{\overline{\mathcal{X}}_r^{\mathrm{n}}/\mathbf{F}_p}[p^\infty]$ of the Jacobian of $\overline{\mathcal{X}}_r^{\mathrm{n}}$ over $\mathbf{F}_p$ is equipped with an action of Γ over $\mathbf{F}_p$ and (via the Hecke correspondences [Cai17, Definitions A.15 and B.26]) canonical actions of $\mathfrak{H}_r$ and $\mathfrak{H}_r^*$.

Definition 2.3.1. Define $\Sigma_r := e_r^{*\prime}\,\mathrm{Pic}^0_{\overline{\mathcal{X}}_r^{\mathrm{n}}/\mathbf{F}_p}[p^\infty]$, equipped with the induced actions of $\mathfrak{H}_r^*$ and Γ.

Since $\mathcal{X}_r$ is regular, and proper flat over R_r with (geometrically) reduced special fiber, $\mathrm{Pic}^0_{\mathcal{X}_r/R_r}$ is a smooth R_r-scheme by [BLR90, § 8.4, Proposition 2 and § 9.4, Theorem 2]. By the Néron mapping property, we thus have a natural mapping $\mathrm{Pic}^0_{\mathcal{X}_r/R_r} \to \mathcal{J}_r^0$ that recovers the canonical identification on generic fibers, and is an isomorphism by [BLR90, § 9.7, Theorem 1]. Composing with $\alpha_r^* : \mathcal{J}_r \to \mathcal{B}_r^*$ and passing to special fibers yields a homomorphism of smooth commutative algebraic groups over $\mathbf{F}_p$

$$\mathrm{Pic}^0_{\overline{\mathcal{X}}_r/\mathbf{F}_p} \xrightarrow{\simeq} \overline{\mathcal{J}}_r^0 \longrightarrow \overline{\mathcal{B}}_r^*. \tag{2.3.1}$$

Due to [BLR90, § 9.3, Corollary 11], the normalization map $\overline{\mathcal{X}}_r^{\mathrm{n}} \to \overline{\mathcal{X}}$ induces a surjective homomorphism $\mathrm{Pic}^0_{\overline{\mathcal{X}}_r/\mathbf{F}_p} \to \mathrm{Pic}^0_{\overline{\mathcal{X}}_r^{\mathrm{n}}/\mathbf{F}_p}$ with kernel that is a smooth, connected *linear* algebraic group over $\mathbf{F}_p$. As any homomorphism from an affine group variety to an abelian variety is zero, we conclude that (2.3.1) uniquely factors through this quotient, and we obtain a natural map of abelian varieties:

$$\mathrm{Pic}^0_{\overline{\mathcal{X}}_r^{\mathrm{n}}/\mathbf{F}_p} \longrightarrow \overline{\mathcal{B}}_r^* \tag{2.3.2}$$

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that is necessarily equivariant for the actions of $\mathfrak{H}_r^*(\mathbf{Z})$ and Γ. The following proposition relates the special fiber of $\mathcal{G}_r$ to the p-divisible group Σ_r of Definition 2.3.1, and will allow us in Corollary 2.3.5 to give an explicit description of the special fiber of $\mathcal{G}_r$.

PROPOSITION 2.3.2. *The mapping (2.3.2) induces an $\mathfrak{H}_r^*$- and Γ-equivariant isomorphism*

$$\overline{\mathcal{G}}_r := e^{*\prime}\overline{\mathcal{B}}_r^*[p^\infty] \simeq e^{*\prime}\operatorname{Pic}^0_{\overline{\mathcal{X}}_r^{\mathrm{n}}/\mathbf{F}_p}[p^\infty] =: \Sigma_r \tag{2.3.3}$$

that is compatible with change in r via the maps $\rho_{r,s}$ on $\overline{\mathcal{G}}_r$ and the maps $\operatorname{Pic}^0(\rho)^{r-s}$ on Σ_r.

Proof. The diagram (2.2.5) induces a corresponding diagram of Néron models over R_r and hence of special fibers over $\mathbf{F}_p$. Arguing as above, we obtain a commutative diagram of abelian varieties

$$\begin{tikzcd}
\operatorname{Pic}^0_{\overline{\mathcal{X}}_r^{\mathrm{n}}/\mathbf{F}_p} \arrow[r, "\overline{\alpha}_r^*"] \arrow[d, "U_r^*"'] & \overline{\mathcal{B}}_r^* \arrow[d, "U_r^*"] \arrow[dl, "W_r^*"] \\
\operatorname{Pic}^0_{\overline{\mathcal{X}}_r^{\mathrm{n}}/\mathbf{F}_p} \arrow[r, "\overline{\alpha}_r^*"'] & \overline{\mathcal{B}}_r^*
\end{tikzcd} \tag{2.3.4}$$

over $\mathbf{F}_p$. The proof of Proposition 2.2.7 now goes through *mutatis mutandis* to give the isomorphism (2.3.3). □

Via the (absolute) Frobenius map and the Cartier operator [Oda69, Definition 5.5], the de Rham cohomology of any smooth and proper curve over $\mathbf{F}_p$ is naturally a module for the (commutative) Dieudonné ring $A := \mathbf{Z}_p[F,V]/(FV-p)$; see [Oda69, § 5] and cf. [Cai17, § 2.1] We now apply Oda's description [Oda69, Theorem 5.10] of Dieudonné modules in terms of de Rham cohomology and our analysis of $H^1_{\mathrm{dR}}(\overline{\mathcal{X}}_r^{\mathrm{n}}/\mathbf{F}_p)$ from [Cai17, § 2.3] to better understand the p-divisible group Σ_r.

For each r, as in [Cai17, Remark B.16] we write $I_r^\infty := I_{(r,0,1)}$ and $I_r^0 := I_{(0,r,1)}$ for the two 'good' irreducible components of $\overline{\mathcal{X}}_r$. By [Cai17, Proposition 2.21], the ordinary part of the de Rham cohomology of $\overline{\mathcal{X}}_r^{\mathrm{n}}$ is entirely captured by the de Rham cohomology of these two good components. We reinterpret this fact in the language of Dieudonné modules as follows.

PROPOSITION 2.3.3. *For each r, there is a natural isomorphism of A-modules*

$$\mathbf{D}(\Sigma_r[p]) \simeq e_r^{*\prime}H^1_{\mathrm{dR}}(\overline{\mathcal{X}}_r^{\mathrm{n}}/\mathbf{F}_p) \simeq f'H^0(I_r^\infty,\Omega^1)^{V_{\mathrm{ord}}} \oplus f'H^1(I_r^0\mathscr{O})^{F_{\mathrm{ord}}} \tag{2.3.5}$$

which is compatible with $\mathfrak{H}_r^*$, Γ, and change in r and which carries $\mathbf{D}(\Sigma_r^{\mathrm{m}}[p])$ (respectively $\mathbf{D}(\Sigma_r^{\text{ét}}[p])$) isomorphically onto the maximal subspace $f'H^0(I_r^0,\Omega^1)^{V_{\mathrm{ord}}}$ (respectively $f'H^1(I_r^\infty,\mathscr{O})^{F_{\mathrm{ord}}}$) of cohomology on which V (respectively F) acts invertibly. Here $\mathbf{D}(\cdot)$ is the contravariant *Dieudonné module on the category of finite flat commutative p-power order group schemes over $\mathbf{F}_p$. In particular, Σ_r is ordinary.*

Proof. The identifications of [Cai17, Proposition 2.21] are induced by the canonical closed immersions $i_r^\star : I_r^\star \hookrightarrow \overline{\mathcal{X}}_r^{\mathrm{n}}$ and are therefore compatible with the natural actions of Frobenius and the Cartier operator. The isomorphism (2.3.5) then follows from [Oda69, § 5] and [Cai17, Proposition 2.21]. Since this isomorphism is compatible with F and V, we have

$$\mathbf{D}(\Sigma_r^{\mathrm{m}}[p]) \simeq \mathbf{D}(\Sigma_r[p])^{V_{\mathrm{ord}}} \simeq f'H^0(I_r^0,\Omega^1)^{V_{\mathrm{ord}}} \tag{2.3.6a}$$

and

$$\mathbf{D}(\Sigma_r^{\text{ét}}[p]) \simeq \mathbf{D}(\Sigma_r[p])^{F_{\text{ord}}} \simeq f' H^1(I_r^\infty, \mathscr{O})^{F_{\text{ord}}} \tag{2.3.6b}$$

and, extending $\mathbf{D}(\cdot)$ to the category of p-divisible groups as per usual, we conclude that the canonical inclusion $\mathbf{D}(\Sigma_r^{\text{m}}) \oplus \mathbf{D}(\Sigma_r^{\text{ét}}) \hookrightarrow \mathbf{D}(\Sigma_r)$ is surjective, whence Σ_r is ordinary by Dieudonné theory. $\square$

With Proposition 2.3.3 as a starting point, we can now completely describe the structure of Σ_r in terms of the two good components $I_r^\star$. Since $\overline{\mathcal{X}}_r^{\text{n}}$ is the disjoint union of proper smooth and irreducible Igusa curves $I_{(a,b,u)}$, we have a canonical identification of abelian varieties over $\mathbf{F}_p$

$$\operatorname{Pic}^0_{\overline{\mathcal{X}}_r^{\text{n}}/\mathbf{F}_p} = \prod_{(a,b,u)} \operatorname{Pic}^0_{I_{(a,b,u)}/\mathbf{F}_p}. \tag{2.3.7}$$

For $\star = 0, \infty$, let us write $j_r^\star := \operatorname{Pic}^0_{I_r^\star/\mathbf{F}_p}$ for the Jacobian of $I_r^\star$ over $\mathbf{F}_p$. The canonical closed immersions $i_r^\star : I_r^\star \hookrightarrow \overline{\mathcal{X}}_r^{\text{n}}$ yield (by Picard and Albanese functoriality) homomorphisms of abelian varieties over $\mathbf{F}_p$

$$\operatorname{Alb}(i_r^\star) : j_r^\star \longrightarrow \operatorname{Pic}^0_{\overline{\mathcal{X}}_r^{\text{n}}/\mathbf{F}_p} \quad \text{and} \quad \operatorname{Pic}^0(i_r^\star) : \operatorname{Pic}^0_{\overline{\mathcal{X}}_r^{\text{n}}/\mathbf{F}_p} \longrightarrow j_r^\star. \tag{2.3.8}$$

Via the identification (2.3.7), we know that $j_r^\star$ is a direct factor of $\operatorname{Pic}^0_{\overline{\mathcal{X}}_r^{\text{n}}/\mathbf{F}_p}$; in these terms $\operatorname{Alb}(i_r^\star)$ is the unique mapping which projects to the identity on $j_r^\star$ and to the zero map on all other factors, while $\operatorname{Pic}^0(i_r^\star)$ is simply projection onto the factor $j_r^\star$. As Σ_r is a direct factor of $f' \operatorname{Pic}^0_{\overline{\mathcal{X}}_r^{\text{n}}/\mathbf{F}_p}[p^\infty]$, these mappings induce homomorphisms of p-divisible groups over $\mathbf{F}_p$

$$f' j_r^0[p^\infty]^{\text{m}} \xrightarrow{\operatorname{Alb}(i_r^0)} f' \operatorname{Pic}^0_{\overline{\mathcal{X}}_r^{\text{n}}/\mathbf{F}_p}[p^\infty]^{\text{m}} \xrightarrow{\text{proj}} \Sigma_r^{\text{m}}, \tag{2.3.9a}$$

$$\Sigma_r^{\text{ét}} \xrightarrow{\text{incl}} f' \operatorname{Pic}^0_{\overline{\mathcal{X}}_r^{\text{n}}/\mathbf{F}_p}[p^\infty]^{\text{ét}} \xrightarrow{\operatorname{Pic}^0(i_r^\infty)} f' j_r^\infty[p^\infty]^{\text{ét}}, \tag{2.3.9b}$$

which we (somewhat abusively) again denote by $\operatorname{Alb}(i_r^0)$ and $\operatorname{Pic}^0(i_r^\infty)$, respectively. The following is a sharpening of [MW84, ch. 3, § 3, Proposition 3] (see also [Til87, Proposition 3.2]).

Proposition 2.3.4. *The mappings (2.3.9a) and (2.3.9b) are isomorphisms. They induce a canonical split short exact sequence of p-divisible groups over $\mathbf{F}_p$*

$$0 \longrightarrow f' j_r^0[p^\infty]^{\text{m}} \xrightarrow{\operatorname{Alb}(i_r^0)\circ V^r} \Sigma_r \xrightarrow{\operatorname{Pic}^0(i_r^\infty)} f' j_r^\infty[p^\infty]^{\text{ét}} \longrightarrow 0 \tag{2.3.10}$$

which is:

(i) *Γ-equivariant for the geometric inertia action on Σ_r, the trivial action on $f' j_r^\infty[p^\infty]^{\text{ét}}$, and the action via $\langle \chi(\cdot)\rangle^{-1}$ on $f' j_r^0[p^\infty]^{\text{m}}$;*

(ii) *$\mathfrak{H}_r^*$-equivariant with U_p^* acting on $f' j_r^\infty[p^\infty]^{\text{ét}}$ as F and on $f' j_r^0[p^\infty]^{\text{m}}$ as $\langle p\rangle_N V$;*

(iii) *compatible with change in r via the mappings $\operatorname{Pic}^0(\rho)$ on $j_r^\star$ and Σ_r.*

Proof. It is clearly enough to prove that the sequence (2.3.10) induced by (2.3.9a) and (2.3.9b) is exact. By Dieudonné theory, it suffices to prove such exactness after applying $\mathbf{D}(\cdot)$. As the resulting sequence consists of finite free $\mathbf{Z}_p$-modules, exactness may be checked modulo p, where it follows immediately from Proposition 2.3.3 by using [Cai17, Proposition 2.21]. The claimed compatibility with Γ, $\mathfrak{H}_r^*$, and change in r follows easily from [Cai17, Propositions B.17, B.18, and B.25]. $\square$

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Together, Propositions 2.3.2 and 2.3.4 give the desired description of the special fiber of $\mathcal{G}_r$ (cf. [MW86, §§ 3 and 4, Proposition 1] and [MW84, pp. 267–274]).

Corollary 2.3.5. *For each r, the p-divisible group $\mathcal{G}_r/R_r$ is ordinary and there is a canonical exact sequence compatible with change in r via $\rho_{r,s}$ on $\overline{\mathcal{G}}_r$ and $\mathrm{Pic}^0(\rho)^{r-s}$ on $j_r^\star[p^\infty]$*

$$0 \longrightarrow f' j_r^0[p^\infty]^{\mathrm{m}} \xrightarrow{\mathrm{Alb}(i_r^0)\circ V^r} \overline{\mathcal{G}}_r \xrightarrow{\mathrm{Pic}^0(i_r^\infty)} f' j_r^\infty[p^\infty]^{\text{ét}} \longrightarrow 0, \tag{2.3.11}$$

where $i_r^\star : I_r^\star \hookrightarrow \overline{\mathcal{X}}_r^{\mathrm{n}}$ are the canonical closed immersions for $\star = 0, \infty$. Moreover, (2.3.11) is compatible with the actions of $\mathfrak{H}^*$ and Γ, with U_p^* (respectively $\gamma \in \Gamma$) acting on $f' j_r^0[p^\infty]^{\mathrm{m}}$ as $\langle p\rangle_N V$ (respectively $\langle \chi(\gamma)\rangle^{-1}$) and on $f' j_r^\infty[p^\infty]^{\text{ét}}$ as F (respectively id).

3. Λ-adic Dieudonné modules

3.1 Ordinary families of Dieudonné modules

Let $\{\mathcal{G}_r/R_r\}_{r\geqslant 1}$ be the tower of p-divisible groups given by Definition 2.2.10. From the canonical morphisms $\rho_{r,s} : \mathcal{G}_s \times_{T_s} T_r \to \mathcal{G}_r$, we obtain a map on special fibers $\overline{\mathcal{G}}_s \to \overline{\mathcal{G}}_r$ over $\mathbf{F}_p$ for each $r \geqslant s$; applying the contravariant Dieudonné module functor yields a projective system of finite free $\mathbf{Z}_p$-modules $\{\mathbf{D}(\overline{\mathcal{G}}_r)\}_{r\geqslant 1}$ with compatible linear endomorphisms F, V satisfying $FV = VF = p$.

Definition 3.1.1. We write $\mathbf{D}_\infty := \varprojlim_r \mathbf{D}(\overline{\mathcal{G}}_r)$ for the projective limit of the system $\{\mathbf{D}(\overline{\mathcal{G}}_r)\}_r$. For $\star \in \{\text{ét}, \mathrm{m}\}$, we write $\mathbf{D}_\infty^\star := \varprojlim_r \mathbf{D}(\overline{\mathcal{G}}_r^\star)$ for the corresponding projective limit.

Since $\mathfrak{H}_r^*$ acts by endomorphisms on $\overline{\mathcal{G}}_r$, compatibly with change in r, we obtain an action of $\mathfrak{H}^*$ on $\mathbf{D}_\infty$ and on $\mathbf{D}_\infty^\star$. Likewise, the 'geometric inertia action' of Γ on $\overline{\mathcal{G}}_r$ gives an action of Γ on $\mathbf{D}_\infty$ and $\mathbf{D}_\infty^\star$. As $\overline{\mathcal{G}}_r$ is ordinary thanks to Corollary 2.3.5, applying $\mathbf{D}(\cdot)$ to the (split) connected-étale sequence of $\overline{\mathcal{G}}_r$ gives, for each r, a functorially split exact sequence

$$0 \longrightarrow \mathbf{D}(\overline{\mathcal{G}}_r^{\text{ét}}) \longrightarrow \mathbf{D}(\overline{\mathcal{G}}_r) \longrightarrow \mathbf{D}(\overline{\mathcal{G}}_r^{\mathrm{m}}) \longrightarrow 0 \tag{3.1.1}$$

with $\mathbf{Z}_p$-linear actions of Γ, F, V, and $\mathfrak{H}_r^*$. Since projective limits commute with finite direct sums, we obtain a split short *exact* sequence of Λ-modules with linear $\mathfrak{H}^*$- and Γ-actions and commuting linear endomorphisms F, V satisfying $FV = VF = p$:

$$0 \longrightarrow \mathbf{D}_\infty^{\text{ét}} \longrightarrow \mathbf{D}_\infty \longrightarrow \mathbf{D}_\infty^{\mathrm{m}} \longrightarrow 0. \tag{3.1.2}$$

We can now prove Theorem 1.2.1, which asserts that each term in (3.1.2) is finite and free over Λ, and elucidates the structure of each as a Hecke module with Γ-action.

Proof of Theorem 1.2.1. In [Cai17, § 3.1], we established a commutative algebra formalism for dealing with projective limits of modules and proving structural and control theorems as in the statement of Theorem 1.2.1. In order to apply the main result of our formalism to the present situation, we take (in the notation of [Cai17, Lemma 3.2]) $A_r = \mathbf{Z}_p$, $I_r = (p)$, and M_r each one of the terms in (3.1.1), and we must check that the hypotheses

(i) $\overline{M}_r := M_r/pM_r$ is a free $\mathbf{F}_p[\Delta/\Delta_r]$-module of rank d' (respectively $2d'$, d');

(ii) for all $s \leqslant r$, the induced transition maps $\overline{\rho}_{r,s} : \overline{M}_r \longrightarrow \overline{M}_s$ are surjective

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hold. By Propositions 2.3.2 and 2.3.3, there is a natural isomorphism of split short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{D}(\overline{\mathcal{G}}_r^{\text{ét}})_{\mathbf{F}_p} \arrow[r] \arrow[d, "\simeq"] & \mathbf{D}(\overline{\mathcal{G}}_r)_{\mathbf{F}_p} \arrow[r] \arrow[d, "\simeq"] & \mathbf{D}(\overline{\mathcal{G}}_r^{\mathrm{m}})_{\mathbf{F}_p} \arrow[r] \arrow[d, "\simeq"] & 0 \\
0 \arrow[r] & f'H^1(I_r^0,\mathscr{O})^{F_{\mathrm{ord}}} \arrow[r] & f'H^0(I_r^\infty,\Omega^1)^{V_{\mathrm{ord}}} \oplus f'H^1(I_r^0,\mathscr{O})^{F_{\mathrm{ord}}} \arrow[r] & f'H^0(I_r^\infty,\Omega^1)^{V_{\mathrm{ord}}} \arrow[r] & 0
\end{tikzcd}$$

that is compatible with change in r using the trace mappings attached to $\rho : I_r^\star \to I_s^\star$ and the maps on Dieudonné modules induced by $\overline{\rho}_{r,s} : \overline{\mathcal{G}}_s \to \overline{\mathcal{G}}_r$. The hypotheses (i) and (ii) are therefore satisfied thanks to [Cai17, Proposition 2.8 and Lemma 2.20]. It follows that the conclusions of [Cai17, Lemma 3.2] hold in the present situation, which gives the finite freeness over Λ of each term in (3.1.2), as well as the fact that this sequence specializes as in (1.2.3). As F (respectively V) acts invertibly on $\mathbf{D}(\overline{\mathcal{G}}_r^{\text{ét}})$ (respectively $\mathbf{D}(\overline{\mathcal{G}}_r^{\mathrm{m}})$) for all r, the asserted characterization of $\mathbf{D}_\infty^\star$ as a submodule of $\mathbf{D}_\infty$ for $\star = \mathrm{m}, \text{ét}$ is clear, while Γ and U_p^* act as claimed thanks to Corollary 2.3.5. □

The short exact sequence (3.1.2) is very nearly 'auto dual', in a sense made precise by Theorem 1.2.2, which we now prove.

Proof of Theorem 1.2.2. As in the proof of Theorem 1.2.1, we apply the formalism of [Cai17, § 3.1]. Let us write $\rho'_{r,s} : \overline{\mathcal{G}}'_r \to \overline{\mathcal{G}}'_s$ for the maps on special fibers induced by (2.2.8). Thanks to Proposition 2.2.12, the definition 2.2.10 of $\overline{\mathcal{G}}'_r := \overline{\mathcal{G}}_r^\vee$, the identifications

$$\mathcal{G}_r \times_{R_r} R'_r \simeq \mathcal{G}_r(\langle\chi\rangle\langle a\rangle_N) \times_{R_r} R'_r,$$

and the compatibility of the Dieudonné module functor with duality, there are isomorphisms of R'_0-modules

$$\mathbf{D}(\overline{\mathcal{G}}_r)(\langle\chi\rangle\langle a\rangle_N) \underset{\mathbf{Z}_p}{\otimes} R'_0 \simeq \mathbf{D}(\overline{\mathcal{G}_r(\langle\chi\rangle\langle a\rangle_N)}) \underset{\mathbf{Z}_p}{\otimes} R'_0 \simeq \mathbf{D}(\overline{\mathcal{G}}'_r) \underset{\mathbf{Z}_p}{\otimes} R'_0 = \mathbf{D}(\overline{\mathcal{G}}_r^\vee) \underset{\mathbf{Z}_p}{\otimes} R'_0 \simeq (\mathbf{D}(\overline{\mathcal{G}}_r))^\vee_{R'_0} \tag{3.1.3}$$

that are $\mathfrak{H}_r^*$-equivariant, $\mathrm{Gal}(K'_r/K_0)$-compatible for the standard action $\sigma \cdot f(m) := \sigma f(\sigma^{-1}m)$ on the R'_0-linear dual of $\mathbf{D}(\overline{\mathcal{G}}_r) \otimes_{\mathbf{Z}_p} R'_0$, and compatible with change in r using $\rho_{r,s}$ on $\mathbf{D}(\overline{\mathcal{G}}_r)$ and $\rho'_{r,s}$ on $\mathbf{D}(\overline{\mathcal{G}}'_r)$. We claim that the resulting perfect 'evaluation' pairings

$$\langle\cdot,\cdot\rangle_r : \mathbf{D}(\overline{\mathcal{G}}_r)(\langle\chi\rangle\langle a\rangle_N) \underset{\mathbf{Z}_p}{\otimes} R'_0 \times \mathbf{D}(\overline{\mathcal{G}}_r) \underset{\mathbf{Z}_p}{\otimes} R'_0 \longrightarrow R'_0 \tag{3.1.4}$$

are compatible with change in r via the maps $\rho_{r,s}$ and $\rho'_{r,s}$ in the sense of [Cai17, 3.3]; i.e. that

$$\langle \rho_{r,s}x, \rho'_{r,s}y\rangle_s = \sum_{\delta\in\Delta_s/\Delta_r} \langle x, \delta^{-1}y\rangle_r \tag{3.1.5}$$

holds for all x, y. Indeed, the compatibility of (3.1.3) with change in r and the very definition (2.2.8) of the transition maps $\rho'_{r,s}$ imply that for $r \geqslant s$

$$\langle \mathbf{D}(\mathrm{Pic}^0(\rho)^{r-s})x, y\rangle_s = \langle x, \mathbf{D}(U_p^{*\,s-r}\,\mathrm{Alb}(\sigma)^{r-s})y\rangle_r; \tag{3.1.6}$$

on the other hand, it follows from Lemma 2.1.1 (using Lemma 2.2.1) that we have

$$\mathrm{Pic}(\rho) \circ \mathrm{Alb}(\sigma) = U_p^* \sum_{\delta\in\Delta_r/\Delta_{r+1}} \langle\delta^{-1}\rangle, \tag{3.1.7}$$

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in $\mathrm{End}_{\mathbf{Q}_p}(J_{r+1})$, and together (3.1.6)–(3.1.7) imply the desired compatibility (3.1.5). It follows that the hypotheses of [Cai17, Lemma 3.4] are verified, and we conclude that the pairings (3.1.4) give rise to a perfect $\mathrm{Gal}(K'_\infty/K_0)$-compatible duality pairing

$$\langle\cdot,\cdot\rangle : \mathbf{D}_\infty(\langle\chi\rangle\langle a\rangle_N)\otimes_\Lambda \Lambda_{R'_0}\times \mathbf{D}_\infty\otimes_\Lambda\Lambda_{R'_0}\to\Lambda_{R'_0}$$

with respect to which T^* is self-adjoint for all $T^*\in\mathfrak{H}^*$, as this is true at each finite level r thanks to the $\mathfrak{H}_r^*$-compatibility of (3.1.3). That the resulting isomorphism (1.2.4) intertwines F with $V^\vee$ and V with $F^\vee$ is an easy consequence of the compatibility of the Dieudonné module functor with duality. □

Next, we prove Theorem 1.2.3, which relates $\mathbf{D}_\infty^\star$ to the crystalline cohomology of the Igusa tower.

Proof of Theorem 1.2.3. From the exact sequence (2.3.11), we obtain for each r isomorphisms

$$\mathbf{D}(\overline{\mathcal{G}}_r^{\mathrm{m}})\xrightarrow[V^r\circ\mathbf{D}(\mathrm{Alb}(i_r^0))]{\simeq} f'\mathbf{D}(j_r^0[p^\infty])^{V_{\mathrm{ord}}}\quad\text{and}\quad f'\mathbf{D}(j_r^\infty[p^\infty])^{F_{\mathrm{ord}}}\xrightarrow[\mathbf{D}(\mathrm{Pic}^0(i_r^\infty))]{\simeq}\mathbf{D}(\overline{\mathcal{G}}_r^{\text{ét}}) \tag{3.1.8}$$

that are $\mathfrak{H}^*$- and Γ-equivariant (with respect to the actions specified in Corollary 2.3.5), and compatible with change in r via the mappings $\mathbf{D}(\rho_{r,s})$ on $\mathbf{D}(\overline{\mathcal{G}}_r^\star)$ and $\mathbf{D}(\rho)$ on $\mathbf{D}(j_r^\star[p^\infty])$. On the other hand, for *any* smooth and proper curve X over a perfect field k of characteristic p, by [MM74] and [Ill79, II, § 3C, Remarque 3.11.2] there are natural isomorphisms of left modules over the Dieudonné ring for k

$$\mathbf{D}(J_X[p^\infty])\simeq H^1_{\mathrm{cris}}(J_X/W(k))\simeq H^1_{\mathrm{cris}}(X/W(k)) \tag{3.1.9}$$

that for any finite map of smooth proper curves $f:Y\to X$ over k intertwine $\mathbf{D}(\mathrm{Pic}(f))$ and $\mathbf{D}(\mathrm{Alb}(f))$ with trace and pullback by f on crystalline cohomology, respectively. Applying this to $X=I_r^\star$ for $\star=0,\infty$, appealing to (3.1.8), and passing to inverse limits completes the proof. □

3.2 Λ-adic Hodge comparison isomorphism

We now wish to relate the 'slope filtration' (3.1.2) to the Hodge filtration (1.1.2) of the ordinary Λ-adic de Rham cohomology studied in [Cai17]. In order to do this, we will relate both to the Dieudonné crystals of the p-divisible groups $\mathcal{G}_r$.

Let R be a complete discrete valuation ring with perfect residue field k of characteristic p and fraction field K of characteristic 0, and fix a p-divisible group $\mathcal{G}$ over R. For $n\geqslant 0$, set $\mathcal{G}_n:=\mathcal{G}\times_R R/p^{n+1}R$ and write $\mathbf{D}(\mathcal{G}_0)_\star$ for the (contravariant) Dieudonné crystal of $\mathcal{G}_0$ as defined in [MM74]; for simplicity, if $S\twoheadrightarrow R/pR$ is a divided power thickening with S a p-adically complete ring on which p is topologically nilpotent, we write $\mathbf{D}(\mathcal{G}_0)_S:=\varprojlim_n\mathbf{D}(\mathcal{G}_0)_{S/p^nS}$ for the locally free S-module obtained by 'evaluating' $\mathbf{D}(\mathcal{G}_0)_\star$ on the (pro-)object $S\twoheadrightarrow R/pR$ of the (big) crystalline site of R/pR relative to[9] $\mathbf{Z}_p$. On the other hand, for $n\geqslant 0$ the universal extension $\mathscr{E}(\mathcal{G}_n)$ of $\mathcal{G}_n^\vee$ by a vector group (in the category of *fppf* sheaves of abelian groups) exists and is unique up to canonical isomorphism and compatible with any base change thanks to [MM74, I, § 1.8]. It follows from the very construction of $\mathbf{D}(\mathcal{G}_0)_\star$ given in [MM74] that one has a natural isomorphism of free R-modules $\mathbf{D}(\mathcal{G}_0)_R\simeq\varprojlim_n\mathrm{Lie}\,\mathscr{E}(\mathcal{G}_n)$, which provides $\mathbf{D}(\mathcal{G}_0)_R$ with a canonical Hodge filtration

$$0\longrightarrow\omega_{\mathcal{G}}\longrightarrow\mathbf{D}(\mathcal{G}_0)_R\longrightarrow\mathrm{Lie}(\mathcal{G}^\vee)\longrightarrow 0. \tag{3.2.1}$$

[9] With the canonical divided powers on $p\mathbf{Z}_p$; see [BM79, § 2.2] for generalities on the crystalline site and crystals.

When $\mathcal{G}$ is *ordinary*, the sequence (3.2.1) is functorially split. More precisely, the connected-étale sequence of $\mathcal{G}_0$ and the exactness of $\mathbf{D}(\cdot)_R$ provide $\mathbf{D}(\mathcal{G}_0)_R$ with a canonical slope filtration

$$0 \longrightarrow \mathbf{D}(\mathcal{G}_0^{\text{ét}})_R \longrightarrow \mathbf{D}(\mathcal{G}_0)_R \longrightarrow \mathbf{D}(\mathcal{G}_0^{\text{m}})_R \longrightarrow 0 \tag{3.2.2}$$

and we have (cf. [Kat81]) the following result.

LEMMA 3.2.1. *The compositions deduced from the Hodge and slope filtrations of* $\mathbf{D}(G_0)_R$ *over* R

$$\omega_{\mathcal{G}} \hookrightarrow \mathbf{D}(\mathcal{G}_0)_R \twoheadrightarrow \mathbf{D}(\mathcal{G}_0^{\text{m}})_R \quad \text{and} \quad \mathbf{D}(\mathcal{G}_0^{\text{ét}})_R \hookrightarrow \mathbf{D}(\mathcal{G}_0)_R \twoheadrightarrow \operatorname{Lie}(\mathcal{G}^{\vee}) \tag{3.2.3}$$

are isomorphisms. In particular, the Hodge and slope filtrations of $\mathbf{D}(\mathcal{G}_0)_R$ *are functorially split.*

Proof. Applying $\mathbf{D}(\cdot)_R$ to the connected-étale sequence of $\mathcal{G}_0$ and using the functoriality of (3.2.1) yields a functorial commutative diagram with exact columns and rows

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & 0 \arrow[r] \arrow[d] & \omega_{\mathcal{G}} \arrow[r] \arrow[d] & \omega_{\mathcal{G}^{\text{m}}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbf{D}(\mathcal{G}_0^{\text{ét}})_R \arrow[r] \arrow[d] & \mathbf{D}(\mathcal{G}_0)_R \arrow[r] \arrow[d] & \mathbf{D}(\mathcal{G}_0^{\text{m}})_R \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \operatorname{Lie}(\mathcal{G}^{\text{ét}\vee}) \arrow[r] \arrow[d] & \operatorname{Lie}(\mathcal{G}^{\vee}) \arrow[r] \arrow[d] & 0 & \\
 & 0 & 0 & &
\end{tikzcd} \tag{3.2.4}$$

where we have used the fact that the invariant differentials and Lie algebra of an étale p-divisible group (such as $\mathcal{G}^{\text{ét}}$ and $\mathcal{G}^{\text{m}\vee} \simeq \mathcal{G}^{\vee\text{ét}}$) are both zero. The lemma follows. □

The structure of $\mathbf{D}(\mathcal{G}_0^{\star})_R$ is particularly simple for $\star = \text{ét}, \text{m}$. Indeed, writing e for the ramification index of R over $W := W(k)$, the Frobenius and Verschiebung morphisms of $\mathcal{G}_0$ induce isomorphisms

$$\mathcal{G}_0^{\text{ét}} \xrightarrow[\simeq]{F^r} (\mathcal{G}_0^{\text{ét}})^{(p^r)} \simeq \varphi^{r*}\overline{\mathcal{G}}^{\text{ét}} \times_k R/pR, \tag{3.2.5a}$$

$$\mathcal{G}_0^{\text{m}} \xleftarrow[\simeq]{V^r} (\mathcal{G}_0^{\text{m}})^{(p^r)} \simeq \varphi^{r*}\overline{\mathcal{G}}^{\text{m}} \times_k R/pR \tag{3.2.5b}$$

for each integer r with $p^r \geqslant e$; here we have used the fact that for such r, the map $x \mapsto x^{p^r}$ of R/pR factors through $R/pR \twoheadrightarrow k$. We have the following result.

LEMMA 3.2.2. *With notation as above, for each integer* r *with* $p^r \geqslant e$*, the* r*th iterate of the relative Frobenius (respectively Verschiebung) morphism of* $\mathcal{G}_0$ *induces a natural isomorphism of* R*-modules*

$$\mathbf{D}(\mathcal{G}_0^{\text{ét}})_R \simeq \mathbf{D}(\overline{\mathcal{G}}^{\text{ét}})_W \otimes_{W,\varphi^r} R \quad \text{respectively} \quad \mathbf{D}(\mathcal{G}_0^{\text{m}})_R \simeq \mathbf{D}(\overline{\mathcal{G}}^{\text{m}})_W \otimes_{W,\varphi^r} R.$$

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Proof. As the Dieudonné crystal is compatible with base change and the maps $\varphi^r : W \to W$ and $W \hookrightarrow R$ extend to PD morphisms $\varphi^r : (W,p) \to (W,p)$ and $(W,p) \to (R,p)$ over $\varphi^r : k \to k$ and $k \to R/pR$, respectively, the asserted isomorphisms follow after applying $\mathbf{D}(\cdot)_R$ to (3.2.5a)–(3.2.5b). $\square$

The key to relating the Hodge (1.1.2) and slope (3.1.2) filtrations is the following comparison.

Proposition 3.2.3. *For each positive integer r, there is a natural $\mathfrak{H}_r^*$- and Γ-equivariant isomorphism*

$$\begin{tikzcd}
0 \arrow[r] & \omega_{\mathcal{G}_r} \arrow[r] \arrow[d, "\simeq"'] & \mathbf{D}(\mathcal{G}_{r,0})_{R_r} \arrow[r] \arrow[d, "\simeq"] & \mathrm{Lie}(\mathcal{G}_r^\vee) \arrow[r] \arrow[d, "\simeq"] & 0 \\
0 \arrow[r] & e^{*\prime}H^0(\omega_r) \arrow[r] & e^{*\prime}H^1_{\mathrm{dR},r} \arrow[r] & e^{*\prime}H^1(\mathscr{O}_r) \arrow[r] & 0
\end{tikzcd} \tag{3.2.6}$$

compatible with change in r using the mappings (2.2.8) on the top row and the maps ρ_* on the bottom. Here the bottom row, with obvious abbreviated notation, is obtained from (1.1.1) by applying $e^{*\prime}$.

Proof. We will construct (3.2.6) in two steps. To begin with, for any base scheme T and any commutative T-group scheme F, denote by $\mathscr{E}xtrig_T(F, \mathbf{G}_m)$ the fppf sheaf of abelian groups associated to the functor which to each T-scheme T' assigns the group of equivalence classes of pairs (E,σ), where E is an extension of $F_{T'}$ by $\mathbf{G}_m$ over T' (in the category of fppf abelian sheaves) and $\sigma : \mathrm{Inf}^1(F_{T'}) \to E$ is a morphism of T'-pointed T'-schemes projecting to the canonical closed immersion $\mathrm{Inf}^1(F_{T'}) \to F_{T'}$. Specializing to the case that $T = \mathrm{Spec}\, R$ is the spectrum of a discrete valuation ring R with fraction field K of characteristic zero and perfect residue field of characteristic p, let A be an abelian variety over K with Néron model $\mathcal{A}$ over T. Then $\mathscr{E}xtrig_T(\mathcal{A}, \mathbf{G}_m)$ is represented on the category of smooth T-schemes by a smooth and separated T-group scheme, and one has a short exact sequence

$$0 \longrightarrow \omega_{\mathcal{A}} \longrightarrow \mathscr{E}xtrig_T(\mathcal{A}, \mathbf{G}_m) \longrightarrow \mathcal{A}^{\vee 0} \longrightarrow 0 \tag{3.2.7}$$

of smooth group schemes over T, where $\omega_{\mathcal{A}}$ is the vector group attached to the sheaf of invariant differentials on $\mathcal{A}$ and $\mathcal{A}^{\vee 0}$ is the identity component of the Néron model over T of the dual abelian variety $A^\vee/K$; see Proposition 2.6 and the discussion following [Cai10, Remark 2.9].

We first claim that the map $\alpha_r^* : \mathcal{J}_r \twoheadrightarrow \mathcal{B}_r^*$ arising via the Néron mapping property from Definition 2.2.3 induces a canonical isomorphism of short exact sequences of free R_r-modules

$$\begin{tikzcd}
0 \arrow[r] & \omega_{\mathcal{G}_r} \arrow[r] \arrow[d, "\simeq"'] & \mathbf{D}(\mathcal{G}_{r,0})_{R_r} \arrow[r] \arrow[d, "\simeq"'] & \mathrm{Lie}(\mathcal{G}_r^\vee) \arrow[r] \arrow[d, "\simeq"] & 0 \\
0 \arrow[r] & e^{*\prime}\omega_{\mathcal{J}_r} \arrow[r] & e^{*\prime}\,\mathrm{Lie}\,\mathscr{E}xtrig(\mathcal{J}_r, \mathbf{G}_m) \arrow[r] & e^{*\prime}\,\mathrm{Lie}(\mathcal{J}_r^{\vee 0}) \arrow[r] & 0
\end{tikzcd} \tag{3.2.8}$$

that is $\mathfrak{H}_r^*$- and Γ-equivariant and compatible with change in r using the map on Néron models induced by $\mathrm{Pic}^0(\rho)$ and the maps (2.2.8) on $\mathcal{G}_r$. To prove this, we introduce the following notation: set $V := \mathrm{Spec}(R_r)$ and, for $n \geqslant 0$, put $V_n := \mathrm{Spec}(R_r/p^{n+1}R_r)$. For any scheme (or p-divisible group) T over V, we put $T_n := T \times_V V_n$. If $\mathcal{A}$ is a Néron model over V, we write $H(\mathcal{A})$ for the short exact sequence of free R_r-modules obtained by applying Lie to the 'canonical extension' (3.2.7). If G is a p-divisible group or an abelian scheme over V, then for each n we likewise write

$H(G_n)$ for the short exact sequence obtained by applying Lie to the universal extension of $G_n^\vee$ by a vector group. We claim that when $\mathcal{A}$ is an abelian scheme over V, then there are natural and compatible (with change in n) isomorphisms of short exact sequences

$$H(\mathcal{A}_n[p^\infty]) \simeq H(\mathcal{A}_n) \simeq H(\mathcal{A})/p^{n+1}, \tag{3.2.9}$$

which justifies our slight abuse of notation. Indeed, in this case the short exact sequence (3.2.7) is naturally isomorphic to the *universal extension* of $\mathcal{A}^\vee = \mathcal{A}^{\vee 0}$ by a vector group thanks to [MM74, I, § 2.6 and Proposition 2.6.7]. It then follows easily from the universal mapping property of the universal extension and [MM74, I, 1.12 and II, § 13] that the natural map of fppf abelian sheaves $\mathcal{A}_n[p^\infty] \to \mathcal{A}_n$ induces the first isomorphism $H(\mathcal{A}_n[p^\infty]) \simeq H(\mathcal{A}_n)$ above. The second isomorphism follows from the fact that the universal extension of $\mathcal{A}^\vee$ is compatible with arbitrary base change; see [MM74, I, 1.9 and the proof of Proposition 2.6.7].

Applying the contravariant functor $e^{*\prime}H(\cdot)$ to the diagram of Néron models over V induced by (2.2.5) yields a commutative diagram of short exact sequences of free R_r-modules

$$\begin{tikzcd} e^{*\prime}H(\mathcal{J}_r) & e^{*\prime}H(\mathcal{B}_r^*) \arrow[l] \\ e^{*\prime}H(\mathcal{J}_r) \arrow[u, "U_r^*"] \arrow[ur] & e^{*\prime}H(\mathcal{B}_r) \arrow[l] \arrow[u, "U_r^*"'] \end{tikzcd} \tag{3.2.10}$$

in which both vertical arrows are isomorphisms by definition of $e^{*\prime}$. As in the proofs of Propositions 2.2.7 and 2.3.2, it follows that the horizontal maps must be isomorphisms as well:

$$e^{*\prime}H(\mathcal{J}_r) \simeq e^{*\prime}H(\mathcal{B}_r^*). \tag{3.2.11}$$

Since these isomorphisms are induced via the Néron mapping property and the functoriality of $H(\cdot)$ by the $\mathfrak{H}_r^*(\mathbf{Z})$-equivariant map $\alpha_r^* : J_r \twoheadrightarrow B_r^*$, they are themselves $\mathfrak{H}_r^*$-equivariant. Similarly, since α_r^* is defined over $\mathbf{Q}$ and compatible with change in r as in Lemma 2.2.6, the isomorphism (3.2.11) is compatible with the given actions of Γ (arising via the Néron mapping property from the semilinear action of Γ over K_r giving the descent data of $J_{r K_r}$ and $B_{r K_r}$ to $\mathbf{Q}_p$) and change in r. Reducing (3.2.11) modulo p^{n+1} and using the canonical isomorphism (3.2.9) yields the identifications

$$e^{*\prime}H(\mathcal{J}_r)/p^{n+1} \simeq e^{*\prime}H(\mathcal{B}_r^*)/p^{n+1} \simeq e^{*\prime}H(\mathcal{B}_{r,n}^*[p^\infty]) \simeq H(e^{*\prime}\mathcal{B}_{r,n}^*[p^\infty]) =: H(\mathcal{G}_{r,n}), \tag{3.2.12}$$

which are clearly compatible with change in n, and which are easily checked (using the naturality of (3.2.9) and our remarks above) to be $\mathfrak{H}_r^*$- and Γ-equivariant, and compatible with change in r. Passing to inverse limits (with respect to n) on (3.2.12) then yields the claimed isomorphism (3.2.8).

To finish the proof of the proposition, it remains to prove that the bottom rows of (3.2.6) and (3.2.8) are naturally isomorphic. But since $\mathcal{X}_r$ is regular and proper flat over R_r with reduced special fiber (see [Cai17, Propositions B.2, B.3, and B.14]), this follows from [Cai10, Theorem 1.2 and (the proof of) Corollary 5.6]. □

Corollary 3.2.4. *Let r be a positive integer. Then the short exact sequence of free R_r-modules*

$$0 \longrightarrow e^{*\prime}H^0(\omega_r) \longrightarrow e^{*\prime}H^1_{\mathrm{dR},r} \longrightarrow e^{*\prime}H^1(\mathscr{O}_r) \longrightarrow 0 \tag{3.2.13}$$

is functorially split; in particular, it is split compatibly with the actions of Γ *and* $\mathfrak{H}_r^*$*. Moreover,* (3.2.13) *admits a functorial descent to* $\mathbf{Z}_p$*: there is a natural isomorphism of split short exact sequences*

$$\begin{tikzcd} 0 \arrow[r] & e^{*\prime}H^0(\omega_r) \arrow[r] \arrow[d, "\simeq"'] & e^{*\prime}H^1_{\mathrm{dR},r} \arrow[r] \arrow[d, "\simeq"] & e^{*\prime}H^1(\mathscr{O}_r) \arrow[r] \arrow[d, "\simeq"] & 0 \\ 0 \arrow[r] & \mathbf{D}(\overline{\mathcal{G}}_r^{\mathrm{m}}) \underset{\mathbf{Z}_p}{\otimes} R_r \arrow[r] & \mathbf{D}(\overline{\mathcal{G}}_r) \underset{\mathbf{Z}_p}{\otimes} R_r \arrow[r] & \mathbf{D}(\overline{\mathcal{G}}_r^{et}) \underset{\mathbf{Z}_p}{\otimes} R_r \arrow[r] & 0 \end{tikzcd} \tag{3.2.14}$$

that is $\mathfrak{H}^*$*- and* Γ*-equivariant, with* Γ *acting trivially on* $\overline{\mathcal{G}}_r^{\text{ét}}$ *and through* $\langle\chi\rangle^{-1}$ *on* $\overline{\mathcal{G}}_r^{\mathrm{m}}$*; here the mappings on the lower row are inclusion and projection, coming from the canonical splitting* $\overline{\mathcal{G}}_r \simeq \overline{\mathcal{G}}_r^{\mathrm{m}} \times_k \overline{\mathcal{G}}_r^{\text{ét}}$ *of the connected-étale sequence of* $\overline{\mathcal{G}}_r$ *over* k*. The identification* (3.2.14) *is compatible with change in* r *using the maps* ρ_* *on the top row and the maps induced by*

$$\overline{\mathcal{G}}_r = \overline{\mathcal{G}}_r^{\mathrm{m}} \times \overline{\mathcal{G}}_r^{\text{ét}} \xrightarrow{V^{-1}\times F} \overline{\mathcal{G}}_r^{\mathrm{m}} \times \overline{\mathcal{G}}_r^{\text{ét}} = \overline{\mathcal{G}}_r \xrightarrow{\overline{\rho}} \overline{\mathcal{G}}_{r+1}$$

on the bottom row.

Proof. Consider the isomorphism (3.2.6) of Proposition 3.2.3. As $\mathcal{G}_r$ is an ordinary p-divisible group by Corollary 2.3.5, the top row of (3.2.6) is functorially split by Lemma 3.2.1, and this gives our first assertion. The rest follows easily from Proposition 3.2.3 and Lemmas 3.2.1–3.2.2, bearing in mind the construction of the isomorphisms in Lemma 3.2.2 via (3.2.5a)–(3.2.5b). $\square$

Proof of Theorem 1.2.4. Applying $\otimes_{R_r} R_\infty$ to (3.2.14) and passing to projective limits yields an isomorphism of split exact sequences

$$\begin{tikzcd} 0 \arrow[r] & e^{*\prime}H^0(\omega) \arrow[r] \arrow[d, "\simeq"'] & e^{*\prime}H^1_{\mathrm{dR}} \arrow[r] \arrow[d, "\simeq"] & e^{*\prime}H^1(\mathscr{O}) \arrow[r] \arrow[d, "\simeq"] & 0 \\ 0 \arrow[r] & \varprojlim_{\overline{\rho}\circ V^{-1}} \Big(\mathbf{D}(\overline{\mathcal{G}}_r^{\mathrm{m}}) \underset{\mathbf{Z}_p}{\otimes} R_\infty\Big) \arrow[r] & \varprojlim_{\overline{\rho}\circ(V^{-1}\times F)} \Big(\mathbf{D}(\overline{\mathcal{G}}_r) \underset{\mathbf{Z}_p}{\otimes} R_\infty\Big) \arrow[r] & \varprojlim_{\overline{\rho}\circ F}\Big(\mathbf{D}(\overline{\mathcal{G}}_r^{et}) \underset{\mathbf{Z}_p}{\otimes} R_\infty\Big) \arrow[r] & 0 \end{tikzcd}$$

On the other hand, the isomorphisms $\overline{\mathcal{G}}_r = \overline{\mathcal{G}}_r^{\mathrm{m}} \times \overline{\mathcal{G}}_r^{\text{ét}} \xrightarrow{V^{-r}\times F^r} \overline{\mathcal{G}}_r^{\mathrm{m}} \times \overline{\mathcal{G}}_r^{\text{ét}} = \overline{\mathcal{G}}_r$ induce an isomorphism of projective limits

$$\varprojlim_{\overline{\rho}}\Big(\mathbf{D}(\overline{\mathcal{G}}_r) \underset{\mathbf{Z}_p}{\otimes} R_\infty\Big) \xrightarrow{\simeq} \varprojlim_{\overline{\rho}\circ(V^{-1}\times F)} \Big(\mathbf{D}(\overline{\mathcal{G}}_r) \underset{\mathbf{Z}_p}{\otimes} R_\infty\Big),$$

which is visibly compatible with the canonical splittings of source and target. The result now follows from [Cai17, Lemma 3.2(5)] and the proof of Theorem 1.2.1, which guarantee that the canonical map $\mathbf{D}_\infty \otimes_\Lambda \Lambda_{R_\infty} \to \varprojlim_{\overline{\rho}}(\mathbf{D}(\overline{\mathcal{G}}_r) \otimes_{\mathbf{Z}_p} R_\infty)$ is an isomorphism respecting the splittings. $\square$

Proof of Corollary 1.2.5. We claim that there are natural isomorphisms of finite free Λ_{R_∞}-modules

$$\mathbf{D}_\infty^{\mathrm{m}} \otimes_\Lambda \Lambda_{R_\infty} \simeq e^{*\prime}H^0(\omega) \simeq e'S(N, \Lambda_{R_\infty}) \simeq e'S(N,\Lambda) \otimes_\Lambda \Lambda_{R_\infty} \tag{3.2.15}$$

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and that the resulting composite isomorphism intertwines $T^* \in \mathfrak{H}^*$ on $\mathbf{D}_\infty^{\mathrm{m}}$ with $T \in \mathfrak{H}$ on $e'S(N,\Lambda)$ and is Γ-equivariant, with $\gamma \in \Gamma$ acting as $\langle \chi(\gamma)\rangle^{-1} \otimes \gamma$ on each tensor product. Indeed, the first and second isomorphisms are due to Theorem 1.2.4 and [Cai17, Corollary 3.14], respectively, while the final isomorphism is a consequence of the definition of $e'S(N;\Lambda_R)$ and the facts that this Λ_R-module is free of finite rank [Oht95, Corollary 2.5.4] and specializes as in [Oht95, 2.6.1]. Twisting the Γ-action on the source and target of the composite (3.2.15) by $\langle\chi\rangle$ therefore gives a Γ-equivariant isomorphism

$$\mathbf{D}_\infty^{\mathrm{m}} \otimes_\Lambda \Lambda_{R_\infty} \simeq S(N,\Lambda) \otimes_\Lambda \Lambda_{R_\infty} \tag{3.2.16}$$

with $\gamma \in \Gamma$ acting as $1 \otimes \gamma$ on source and target. Passing to Γ-invariants on (3.2.16) yields the first isomorphism in Corollary 1.2.5. Via Theorem 1.2.2 and the natural Λ-adic duality between $e\mathfrak{H}$ and $eS(N;\Lambda)$ [Oht95, Theorem 2.5.3], we then obtain a canonical $\mathrm{Gal}(K_0'/K_0)$-equivariant isomorphism

$$e'\mathfrak{H} \mathop{\otimes}_{\Lambda} \Lambda_{R_0'} \simeq \mathbf{D}_\infty^{\text{ét}}(\langle a\rangle_N) \mathop{\otimes}_{\Lambda} \Lambda_{R_0'}$$

of $\Lambda_{R_0'}$-modules that intertwines $T \otimes 1$ for $T \in \mathfrak{H}$ with $T^* \otimes 1$, where U_p^* acts as F on $\mathbf{D}_\infty^{\text{ét}}$. $\square$

In order to relate the slope filtration (3.1.2) of $\mathbf{D}_\infty$ to the ordinary filtration of $e^{*\prime}H^1_{\text{ét}}$, we require the following result.

Lemma 3.2.5. *Let r be a positive integer and, for $\star \in \{\text{ét}, \mathrm{m}, \mathrm{null}\}$, let $G_r^\star$ be the unique $\mathbf{Q}_p$-descent of the generic fiber of $\mathcal{G}_r^\star$, as in Definition 2.2.10 and Remark 2.2.11. There are canonical isomorphisms*

$$\mathbf{D}(\overline{\mathcal{G}}_r^{\text{ét}}) \mathop{\otimes}_{\mathbf{Z}_p} W(\overline{\mathbf{F}}_p) \simeq \mathrm{Hom}_{\mathbf{Z}_p}(T_pG_r^{\text{ét}}, \mathbf{Z}_p) \mathop{\otimes}_{\mathbf{Z}_p} W(\overline{\mathbf{F}}_p), \tag{3.2.17a}$$

$$\mathbf{D}(\overline{\mathcal{G}}_r^{\mathrm{m}})(-1) \mathop{\otimes}_{\mathbf{Z}_p} W(\overline{\mathbf{F}}_p) \simeq \mathrm{Hom}_{\mathbf{Z}_p}(T_pG_r^{\mathrm{m}}, \mathbf{Z}_p) \mathop{\otimes}_{\mathbf{Z}_p} W(\overline{\mathbf{F}}_p) \tag{3.2.17b}$$

that are $\mathfrak{H}_r^*$-equivariant and $\mathscr{G}_{\mathbf{Q}_p}$-compatible for the diagonal action on source and target, with $\mathscr{G}_{\mathbf{Q}_p}$ acting trivially on $\mathbf{D}(\overline{\mathcal{G}}_r^{\text{ét}})$ and via $\chi^{-1}\cdot\langle\chi^{-1}\rangle$ on $\mathbf{D}(\overline{\mathcal{G}}_r^{\mathrm{m}})(-1) := \mathbf{D}(\overline{\mathcal{G}}_r^{\mathrm{m}}) \otimes_{\mathbf{Z}_p} \mathbf{Z}_p(-1)$. The isomorphism (3.2.17a) intertwines $F \otimes \varphi$ with $1 \otimes \varphi$, while (3.2.17b) intertwines $V \otimes \varphi^{-1}$ with $1 \otimes \varphi^{-1}$.

Proof. The semilinear Γ-action on $\mathcal{G}_r^\star$ gives the $\mathbf{Z}_p[\mathscr{G}_{K_r}]$-module $T_p\mathcal{G}_r^\star := \mathrm{Hom}_{\mathscr{O}_{\mathbf{C}_p}}(\mathbf{Q}_p/\mathbf{Z}_p, \mathcal{G}^\star_{r\,\mathscr{O}_{\mathbf{C}_p}})$ the natural structure of a $\mathbf{Z}_p[\mathscr{G}_{\mathbf{Q}_p}]$-module via $g \cdot f := g^{-1} \circ g^* f \circ g$. It is straightforward to check that the natural map $T_p\mathcal{G}_r^\star \to T_pG_r^\star$, which is an isomorphism of $\mathbf{Z}_p[\mathscr{G}_{K_r}]$-modules by Tate's theorem, is an isomorphism of $\mathbf{Z}_p[\mathscr{G}_{\mathbf{Q}_p}]$-modules as well.

For *any* étale p-divisible group H over a perfect field k, one has a canonical isomorphism of $W(\overline{k})$-modules with semilinear $\mathscr{G}_k$-action

$$\mathbf{D}(H) \mathop{\otimes}_{W(k)} W(\overline{k}) \simeq \mathrm{Hom}_{\mathbf{Z}_p}(T_pH, \mathbf{Z}_p) \mathop{\otimes}_{\mathbf{Z}_p} W(\overline{k})$$

that intertwines $F \otimes \varphi$ with $1 \otimes \varphi$ and $1 \otimes g$ with $g \otimes g$ for $g \in \mathscr{G}_k$; for example, this can be deduced by applying [BM79, § 4.1a)] to $H_{\overline{k}}$ and using the fact that the Dieudonné crystal is compatible with base change. In our case, the étale p-divisible group $\mathcal{G}_r^{\text{ét}}$ lifts $\overline{\mathcal{G}}_r^{\text{ét}}$ over R_r, and we obtain a natural isomorphism of $W(\overline{\mathbf{F}}_p)$-modules with semilinear $\mathscr{G}_{K_r}$-action

$$\mathbf{D}(\overline{\mathcal{G}}_r^{\text{ét}}) \mathop{\otimes}_{\mathbf{Z}_p} W(\overline{\mathbf{F}}_p) \simeq \mathrm{Hom}_{\mathbf{Z}_p}(T_p\mathcal{G}_r^{\text{ét}}, \mathbf{Z}_p) \mathop{\otimes}_{\mathbf{Z}_p} W(\overline{\mathbf{F}}_p).$$

By naturality in $\mathcal{G}_r$, this identification respects the semilinear Γ-actions on both sides (which are trivial, as Γ acts trivially on $\mathcal{G}_r^{\text{ét}}$); as explained in our initial remarks, it is precisely this action which allows us to view $T_p\mathcal{G}_r^{\text{ét}}$ as a $\mathbf{Z}_p[\mathscr{G}_{\mathbf{Q}_p}]$-module, and we deduce (3.2.17a). The proof of (3.2.17b) is similar, using the natural isomorphism (proved as above) for any multiplicative p-divisible group H/k

$$\mathbf{D}(H) \underset{W(k)}{\otimes} W(\bar{k}) \simeq T_pH^\vee \underset{\mathbf{Z}_p}{\otimes} W(\bar{k}),$$

which intertwines $V \otimes \varphi^{-1}$ with $1 \otimes \varphi^{-1}$ and $1 \otimes g$ with $g \otimes g$, for $g \in \mathscr{G}_k$. $\square$

Proof of Theorem 1.2.6 and Corollary 1.2.8. For a p-divisible group H over a field K, we will write $H^1_{\text{ét}}(H) := \operatorname{Hom}_{\mathbf{Z}_p}(T_pH, \mathbf{Z}_p)$; our notation is justified by the standard fact that, for J_X the Jacobian of a curve X over K, there is a natural isomorphism of $\mathbf{Z}_p[\mathscr{G}_K]$-modules

$$H^1_{\text{ét}}(J_X[p^\infty]) \simeq H^1_{\text{ét}}(X_{\overline{K}}, \mathbf{Z}_p). \tag{3.2.18}$$

It follows from (3.2.17a)–(3.2.17b) and Theorem 1.2.1 that for $\star \in \{\text{ét}, \text{m}\}$, the scalar extension $H^1_{\text{ét}}(G_r^\star) \otimes_{\mathbf{Z}_p} W(\overline{\mathbf{F}}_p)$ is a free $W(\overline{\mathbf{F}}_p)[\Delta/\Delta_r]$-module of rank d'. Using [Cai17, Lemma 3.3], we conclude that $H^1_{\text{ét}}(G_r^\star)$ itself is a free $\mathbf{Z}_p[\Delta/\Delta_r]$-module of rank d'. In a similar manner, using the faithful flatness of $W(\overline{\mathbf{F}}_p)[\Delta/\Delta_r]$ over $\mathbf{Z}_p[\Delta/\Delta_r]$, we deduce that the canonical trace mappings

$$H^1_{\text{ét}}(G_r^\star) \longrightarrow H^1_{\text{ét}}(G_{r'}^\star) \tag{3.2.19}$$

are surjective for all $r \geqslant r'$. From the commutative algebra formalism of [Cai17, Lemma 3.2], we conclude that $H^1_{\text{ét}}(G_\infty^\star) := \varprojlim_r H^1_{\text{ét}}(G_r^\star)$ is a free Λ-module of rank d', and that there are isomorphisms

$$H^1_{\text{ét}}(G_\infty^\star) \underset{\Lambda}{\otimes} \Lambda_{W(\overline{\mathbf{F}}_p)} \simeq \varprojlim_r \Big(H^1_{\text{ét}}(G_r^\star) \underset{\mathbf{Z}_p}{\otimes} W(\overline{\mathbf{F}}_p) \Big)$$

of $\Lambda_{W(\overline{\mathbf{F}}_p)}$-modules for $\star \in \{\text{ét}, \text{m}\}$. Since we likewise have canonical identifications

$$\mathbf{D}_\infty^\star \underset{\Lambda}{\otimes} \Lambda_{W(\overline{\mathbf{F}}_p)} \simeq \varprojlim_r \Big(\mathbf{D}(G_r^\star) \underset{\mathbf{Z}_p}{\otimes} W(\overline{\mathbf{F}}_p) \Big)$$

thanks again to [Cai17, Lemma 3.2] and (the proof of) Theorem 1.2.1, passing to inverse limits on (3.2.17a)–(3.2.17b) gives, for $\star \in \{\text{ét}, \text{m}\}$, a canonical isomorphism of $\Lambda_{W(\overline{\mathbf{F}}_p)}$-modules

$$\mathbf{D}_\infty^\star \underset{\Lambda}{\otimes} \Lambda_{W(\overline{\mathbf{F}}_p)} \simeq H^1_{\text{ét}}(G_\infty^\star) \underset{\Lambda}{\otimes} \Lambda_{W(\overline{\mathbf{F}}_p)}. \tag{3.2.20}$$

Applying the functor $H^1_{\text{ét}}(\cdot)$ to the $\mathbf{Q}_p$-descent of the generic fiber of the connected-étale sequence of $\mathcal{G}_r$ yields a short exact sequence of $\mathbf{Z}_p[\mathscr{G}_{\mathbf{Q}_p}]$-modules

$$0 \longrightarrow H^1_{\text{ét}}(G_r^{\text{ét}}) \longrightarrow H^1_{\text{ét}}(G_r) \longrightarrow H^1_{\text{ét}}(G_r^{\text{m}}) \longrightarrow 0$$

which naturally identifies $H^1_{\text{ét}}(G_r^\star)$ with the invariants (respectively covariants) of $H^1_{\text{ét}}(G_r)$ under the inertia subgroup $\mathscr{I} \subseteq \mathscr{G}_{\mathbf{Q}_p}$ for $\star = \text{ét}$ (respectively $\star = \text{m}$). As $G_r = e^{*\prime}J_r[p^\infty]$ by definition, we deduce from this and (3.2.18) a natural isomorphism of short exact sequences of $\mathbf{Z}_p[\mathscr{G}_{\mathbf{Q}_p}]$-modules

$$\begin{tikzcd}
0 \arrow[r] & H^1_{\text{ét}}(G_r^{\text{ét}}) \arrow[r] \arrow[d, "\simeq"] & H^1_{\text{ét}}(G_r) \arrow[r] \arrow[d, "\simeq"] & H^1_{\text{ét}}(G_r^{\text{m}}) \arrow[r] \arrow[d, "\simeq"] & 0 \\
0 \arrow[r] & (e^{*\prime}H^1_{\text{ét},r})^{\mathscr{I}} \arrow[r] & e^{*\prime}H^1_{\text{ét},r} \arrow[r] & (e^{*\prime}H^1_{\text{ét},r})_{\mathscr{I}} \arrow[r] & 0
\end{tikzcd} \tag{3.2.21}$$

where for notational ease we abbreviate $H^1_{\text{ét},r} := H^1_{\text{ét}}(X_{r\overline{\mathbf{Q}}_p}, \mathbf{Z}_p)$. As the trace maps (3.2.19) are surjective, passing to inverse limits on (3.2.21) yields an isomorphism of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & H^1_{\text{ét}}(G^{\text{ét}}_{\infty}) \arrow[r] \arrow[d, "\simeq"] & H^1_{\text{ét}}(G_{\infty}) \arrow[r] \arrow[d, "\simeq"] & H^1_{\text{ét}}(G^{\text{m}}_{\infty}) \arrow[r] \arrow[d, "\simeq"] & 0 \\ 0 \arrow[r] & \varprojlim_r (e^{*\prime} H^1_{\text{ét},r})^{\mathscr{I}} \arrow[r] & \varprojlim_r e^{*\prime} H^1_{\text{ét},r} \arrow[r] & \varprojlim_r (e^{*\prime} H^1_{\text{ét},r})_{\mathscr{I}} \arrow[r] & 0 \end{tikzcd} \tag{3.2.22}$$

Since inverse limits commute with group invariants, the bottom row of (3.2.22) is canonically isomorphic to the ordinary filtration of Hida's $e^{*\prime}H^1_{\text{ét}}$, and Theorem 1.2.6 follows immediately from (3.2.20). Corollary 1.2.8 is then an easy consequence of Theorem 1.2.6 and [Cai17, Lemma 3.3]; alternately, one can prove Corollary 1.2.8 directly from [Cai17, Lemma 3.1.2], using what we have seen above. □

4. Λ-adic crystals and (φ, Γ)-modules

Using the family of Dieudonné crystals attached to the tower of p-divisible groups $\{\mathcal{G}_r/R_r\}_{r\geqslant 1}$ given by Definition 2.2.10, we now construct a crystalline analogue of Hida's ordinary Λ-adic étale cohomology. To do this, we will make critical use of the theory of [CL17], which we first briefly recall.

4.1 Dieudonné crystals and (φ, Γ)-modules for p-divisible groups

Throughout this section, we fix a perfect field k of characteristic p, write $W = W(k)$ for its ring of Witt vectors, and set $K := \text{Frac}(W)$. We fix an algebraic closure $\overline{K}$ of K, as well as a compatible sequence $\{\varepsilon^{(r)}\}_{r\geqslant 0}$ of primitive p^rth roots of unity in $\overline{K}$, and set $\mathscr{G}_K := \text{Gal}(\overline{K}/K)$. For $r \geqslant 0$, we put $K_r := K(\mu_{p^r})$ and $R_r := W[\mu_{p^r}]$, and we set $\Gamma_r := \text{Gal}(K_\infty/K_r)$ and $\Gamma := \Gamma_0$. Let $\mathfrak{S}_r := W[\![u_r]\!]$ be the power series ring in one variable u_r over W, endowed with the (p, u_r)-adic topology and the unique continuous action of Γ and semilinear extension of φ determined by $\gamma u_r := (1+u_r)^{\chi(\gamma)} - 1$ for $\gamma \in \Gamma$ and $\varphi(u_r) := (1+u_r)^p - 1$. Let $\theta : \mathfrak{S}_r \twoheadrightarrow R_r$ be the continuous W-algebra surjection sending u_r to $\varepsilon^{(r)}-1$, and denote by $\tau : \mathfrak{S}_r \twoheadrightarrow W$ the continuous surjection of W-algebras determined by $\tau(u_r) = 0$. We lift the inclusion $R_r \hookrightarrow R_{r+1}$ to a Γ- and φ-equivariant W-algebra injection $\mathfrak{S}_r \hookrightarrow \mathfrak{S}_{r+1}$ determined by $u_r \mapsto \varphi(u_{r+1})$, and identify $\mathfrak{S}_r$ with its image in $\mathfrak{S}_{r+1}$, which coincides with $\varphi(\mathfrak{S}_{r+1})$; under this convention, for $r > 0$ the kernel of $\theta : \mathfrak{S}_r \twoheadrightarrow R_r$ is principally generated by $E_r(u_r) := \varphi^r(u_r)/\varphi^{r-1}(u_r) = u_0/u_1$, so we simply write $\omega := E_r(u_r)$ for this common element of $\mathfrak{S}_r$ for $r > 0$.

Definition 4.1.1. We write $\text{BT}^{\varphi,\Gamma}_{\mathfrak{S}_r}$ for the category whose objects are pairs $(\mathfrak{M}, \varphi_{\mathfrak{M}})$, where:

- $\mathfrak{M}$ is a free $\mathfrak{S}_r$-module of finite rank equipped with a continuous semilinear action of Γ;
- $\varphi_{\mathfrak{M}} : \mathfrak{M} \to \mathfrak{M}$ is a φ-semilinear map that commutes with the action of Γ;
- the cokernel of the linearization $1 \otimes \varphi_{\mathfrak{M}} : \varphi^*\mathfrak{M} \to \mathfrak{M}$ is annihilated by ω;
- the induced action of Γ_r on $\mathfrak{M}/u_r\mathfrak{M}$ is trivial.

Morphisms in $\text{BT}^{\varphi,\Gamma}_{\mathfrak{S}_r}$ are φ- and Γ-equivariant $\mathfrak{S}_r$-module homomorphisms. We often abuse notation by writing $\mathfrak{M}$ for the pair $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ and φ for $\varphi_{\mathfrak{M}}$.

If $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ is any object of $\text{BT}^{\varphi,\Gamma}_{\mathfrak{S}_r}$, then $1 \otimes \varphi_{\mathfrak{M}} : \varphi^*\mathfrak{M} \to \mathfrak{M}$ is injective with cokernel killed by ω, so there is a unique $\mathfrak{S}_r$-linear homomorphism $\psi_{\mathfrak{M}} : \mathfrak{M} \to \varphi^*\mathfrak{M}$ with the property that

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the composition of $1 \otimes \varphi_{\mathfrak{M}}$ and $\psi_{\mathfrak{M}}$ (in either order) is multiplication by ω. Clearly, $\varphi_{\mathfrak{M}}$ and $\psi_{\mathfrak{M}}$ determine each other. We warn the reader that the action of Γ does *not* commute with $\psi_{\mathfrak{M}}$: instead, for any $\gamma \in \Gamma$, one has

$$(\gamma \otimes \gamma) \circ \psi_{\mathfrak{M}} = (\gamma\omega/\omega) \cdot \psi_{\mathfrak{M}} \circ \gamma. \tag{4.1.1}$$

DEFINITION 4.1.2. Let $\mathfrak{M}$ be an object of $\mathrm{BT}^{\varphi,\Gamma}_{\mathfrak{S}_r}$. The *dual of* $\mathfrak{M}$ is the object $(\mathfrak{M}^\vee, \varphi_{\mathfrak{M}^\vee})$ of $\mathrm{BT}^{\varphi,\Gamma}_{\mathfrak{S}_r}$ whose underlying $\mathfrak{S}_r$-module is $\mathfrak{M}^\vee := \mathrm{Hom}_{\mathfrak{S}_r}(\mathfrak{M}, \mathfrak{S}_r)$, equipped with the φ-semilinear endomorphism

$$\varphi_{\mathfrak{M}^\vee} : \mathfrak{M}^\vee \xrightarrow{1\otimes \mathrm{id}_{\mathfrak{M}^\vee}} \varphi^*\mathfrak{M}^\vee \simeq (\varphi^*\mathfrak{M})^\vee \xrightarrow{\psi_{\mathfrak{M}}^\vee} \mathfrak{M}^\vee$$

and the commuting[10] action of Γ given by $(\gamma f)(m) := \chi(\gamma)^{-1}\varphi^{r-1}(\gamma u_r/u_r) \cdot \gamma(f(\gamma^{-1}m))$.

For an algebraic extension k'/k, we write $W' := W(k')$, $R'_r := W'[\mu_{p^r}]$, $\mathfrak{S}'_r := W'[\![u_r]\!]$, and so on. The inclusion $W \hookrightarrow W'$ extends to a φ- and Γ-compatible injection $\iota_r : \mathfrak{S}_r \hookrightarrow \mathfrak{S}'_{r+1}$ of W-algebras extension of scalars along which yields a base-change functor $\iota_{r*} : \mathrm{BT}^{\varphi,\Gamma}_{\mathfrak{S}_r} \to \mathrm{BT}^{\varphi,\Gamma}_{\mathfrak{S}'_{r+1}}$, which one checks is compatible with duality.

Let us write $\mathrm{pdiv}^{\Gamma}_{R_r}$ for the category of p-divisible groups $\mathcal{G}$ over R_r that are equipped with a descent G of $\mathcal{G}_{K_r}$ to $K = K_0$. As in Remark 2.2.11, this is equivalent to the category of p-divisible groups $\mathcal{G}$ over R_r that are equipped with isomorphisms $\mathcal{G} \simeq \overline{\gamma}^*(\mathcal{G})$ for each $\overline{\gamma}$ in Γ/Γ_r satisfying the obvious cocycle condition. For any algebraic extension k'/k, base change along the inclusion $\iota_r : R_r \hookrightarrow R'_{r+1}$ gives a covariant functor $\iota_{r*} : \mathrm{pdiv}^{\Gamma}_{R_r} \to \mathrm{pdiv}^{\Gamma}_{R'_{r+1}}$. The main result of [CL17] is the following theorem.

THEOREM 4.1.3. *For $r > 0$, there is a contravariant functor $\mathfrak{M}_r : \mathrm{pdiv}^{\Gamma}_{R_r} \to \mathrm{BT}^{\varphi,\Gamma}_{\mathfrak{S}_r}$ such that:*

(i) *the functor $\mathfrak{M}_r$ is an exact anti-equivalence of categories, compatible with duality;*

(ii) *the functor $\mathfrak{M}_r$ is of formation compatible with base change: for any algebraic extension k'/k, there is a natural isomorphism of composite functors $\iota_{r*} \circ \mathfrak{M}_r \simeq \mathfrak{M}_{r+1} \circ \iota_{r*}$ on $\mathrm{pdiv}^{\Gamma}_{R_r}$;*

(iii) *for $\mathcal{G} \in \mathrm{pdiv}^{\Gamma}_{R_r}$, put $\overline{\mathcal{G}} := \mathcal{G} \times_{R_r} k$ and $\mathcal{G}_0 := \mathcal{G} \times_{R_r} R_r/pR_r$.*

  (a) *There is a functorial and Γ-equivariant isomorphism of W-modules*

$$\mathfrak{M}_r(\mathcal{G}) \underset{\mathfrak{S}_r, \tau\circ\varphi}{\otimes} W \simeq \mathbf{D}(\overline{\mathcal{G}})_W$$

  carrying $\varphi_{\mathfrak{M}} \otimes \varphi$ to $F : \mathbf{D}(\overline{\mathcal{G}})_W \to \mathbf{D}(\overline{\mathcal{G}})_W$ and $\psi_{\mathfrak{M}} \otimes 1$ to $V \otimes 1 : \mathbf{D}(\overline{\mathcal{G}})_W \to \varphi^*\mathbf{D}(\overline{\mathcal{G}})_W$.

  (b) *There is a functorial and Γ-equivariant isomorphism of R_r-modules*

$$\mathfrak{M}_r(\mathcal{G}) \underset{\mathfrak{S}_r, \theta\circ\varphi}{\otimes} R_r \simeq \mathbf{D}(\mathcal{G}_0)_{R_r}.$$

Here the action of Γ on the right-hand side of each isomorphism above is through the descent data isomorphisms $\mathcal{G} \simeq \gamma^*(\mathcal{G})$, via the functoriality of the Dieudonné crystal.

Remark 4.1.4. We warn the reader that, while the action of Γ on $\mathfrak{M}_r(\mathcal{G}) \otimes_{\theta\circ\varphi} R_r \simeq \mathbf{D}(\mathcal{G}_0)_{R_0}$ is through the finite quotient Γ/Γ_r, the natural Γ-action on $\mathfrak{M}_r(\mathcal{G}) \otimes_\theta R_r$, which is a p-integral incarnation of $\mathbf{D}_{\mathrm{Sen}}((V_pG)^\vee)$, is in general not through any finite quotient; cf. [CL17, §§ 4.3–4.4] and [BB10, 3.2.3].

[10] As one checks using the intertwining relation (4.1.1).

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The $\mathfrak{S}_r$-module $\mathfrak{M}_r(\mathcal{G})$ is a functorial descent of the evaluation of the Dieudonné crystal $\mathbf{D}(\mathcal{G}_0)_\star$ on a certain 'universal' PD thickening of R_r/pR_r. This descent is determined by the Hodge filtration of $\mathbf{D}(\mathcal{G}_0)_{R_r}$ coming from the lift of $\mathcal{G}_0$ to R_r provided by $\mathcal{G}$, and one should think of $\mathfrak{M}_r(\mathcal{G})$ as an incarnation of the first crystalline cohomology of $\mathcal{G}$, enhanced by the data of the Hodge filtration of $\mathbf{D}(\mathcal{G}_0)_{R_r}$. As we will need to make use of the relationship between $\mathfrak{M}_r(\mathcal{G})$ and the Dieudonné crystal of $\mathcal{G}_0$ in what follows, we briefly recall the construction of $\mathfrak{M}_r(\mathcal{G})$, and refer to [CL17] for details, including the proofs of Theorems 4.1.3–4.1.6 and Corollary 4.1.7.

Let S_r be the p-adic completion of the PD envelope of $\mathfrak{S}_r$ with respect to the ideal $\ker\theta$, viewed as a topological ring via the p-adic topology, so that the unique continuous extension $\theta : S_r \twoheadrightarrow R_r$ realizes S_r as a PD thickening of R_r with kernel $\mathrm{Fil}^1 S_r := \ker\theta$ that is equipped with topologically PD-nilpotent[11] divided powers. The map $\tau : \mathfrak{S}_r \twoheadrightarrow W$ likewise uniquely extends to a continuous PD thickening $\tau : S_r \twoheadrightarrow W$, where W is given the PD structure coming from the ideal (p). One shows that there is a unique continuous extension $\varphi : S_r \to S_r$ of φ on $\mathfrak{S}_r$, and that $\varphi(\mathrm{Fil}^1 S_r) \subseteq pS_r$; in particular, we may define $\varphi_1 : \mathrm{Fil}^1 S_r \to S_r$ by $\varphi_1 := \varphi/p$. Similarly, the action of Γ on $\mathfrak{S}_r$ uniquely extends to a continuous action by PD endomorphisms that commute with φ on the PD extension $S_r \twoheadrightarrow R_r$, with Γ_r acting trivially on R_r. Set $t := \log(1+u_0)$, where $u_0 = \varphi^r(u_r) \in S_r$ and $\log(1+X) : \mathrm{Fil}^1 S_r \to S_r$ is the usual (convergent for the p-adic topology) power series, and let $v_r := \varphi(E_r)/p \in S_r^\times$. Evaluating the Dieudonné crystal of $\mathcal{G}_0$ on S_r yields a finite free S_r-module $\mathscr{M}(\mathcal{G}) := \varprojlim_n \mathbf{D}(\mathcal{G}_0)_{S_r/p^nS_r}$ that is equipped with an integrable and topologically nilpotent connection ∇ coming from the canonical HPD stratification on the crystal $\mathbf{D}(\mathcal{G}_0)_\star$, as well as a natural horizontal action of Γ_r. The relative Frobenius of $\mathcal{G}_0$ equips $\mathscr{M}(\mathcal{G})$ with a semilinear Frobenius endomorphism $\varphi_{\mathscr{M}}$ that commutes with the action of Γ_r, while the Hodge filtration $\omega_{\mathcal{G}} \subseteq \mathbf{D}(\mathcal{G}_0)_{R_r}$ provides $\mathscr{M}(\mathcal{G})$ with a canonical S_r-submodule $\mathrm{Fil}^1\mathscr{M}(\mathcal{G})$ that is by definition the preimage of $\omega_{\mathcal{G}}$ under $\mathscr{M}(\mathcal{G}) \twoheadrightarrow \mathscr{M}(\mathcal{G})/(\mathrm{Fil}^1 S)\mathscr{M}(\mathcal{G}) \simeq \mathbf{D}(\mathcal{G}_0)_{R_r}$. As pullback by the p-power map kills $\omega_{\mathcal{G}_0}$, it follows that the restriction of $\varphi_{\mathscr{M}}$ to $\mathrm{Fil}^1\mathscr{M}(\mathcal{G})$ is divisible by p, so it makes sense to define $\varphi_{\mathscr{M},1} := p^{-1}\varphi_{\mathscr{M}} : \mathrm{Fil}^1\mathscr{M}(\mathcal{G}) \to \mathscr{M}(\mathcal{G})$, and one shows that the image of $\varphi_{\mathscr{M},1}$ generates $\mathscr{M}(\mathcal{G})$ as an S_r-module. The descent data isomorphisms $\mathcal{G} \simeq \overline{\gamma}^*(\mathcal{G})$ for $\overline{\gamma} \in \Gamma/\Gamma_r$ provide an extension of the action of Γ_r on $\mathscr{M}(\mathcal{G})$ to all of Γ, and in this way one functorially obtains from $\mathcal{G}$ an object of the category $\mathrm{MF}_{S_r}^{\varphi,\nabla,\Gamma}$ of quadruples $(\mathscr{M}, \mathrm{Fil}^1\mathscr{M}, \varphi_{\mathscr{M},1}, \nabla)$, where:

(i) $\mathscr{M}$ is a finite free S_r-module with a continuous and semilinear action of Γ;
(ii) $\mathrm{Fil}^1\mathscr{M} \subseteq \mathscr{M}$ is a Γ-stable S_r-submodule containing $(\mathrm{Fil}^1 S_r)\mathscr{M}$;
(iii) $\mathscr{M}/\mathrm{Fil}^1\mathscr{M}$ is a free $S_r/\mathrm{Fil}^1 S_r = R_r$-module, and Γ_r acts trivially on $\mathscr{M}\otimes_{S_r,\tau} W$;
(iv) $\varphi_{\mathscr{M},1} : \mathrm{Fil}^1\mathscr{M}_r \to \mathscr{M}$ is a φ-semilinear, Γ-equivariant map with surjective linearization;
(v) ∇ is an integrable and topologically nilpotent connection on $\mathscr{M}$ for which the action of Γ and $\varphi_{\mathscr{M}}$ are horizontal, with $\varphi_{\mathscr{M}} : \mathscr{M} \to \mathscr{M}$ determined by $\varphi_{\mathscr{M}}(\alpha) := v_r^{-1}\varphi_{\mathscr{M},1}(E_r\alpha)$.

Writing $\mathrm{MF}_{S_r}^{\varphi,\Gamma}$ for the category of triples $(\mathscr{M}, \mathrm{Fil}^1\mathscr{M}, \varphi_{\mathscr{M},1})$ satisfying (i)–(iv), one has functors

$$\mathrm{MF}_{S_r}^{\varphi,\nabla,\Gamma} \longrightarrow \mathrm{MF}_{S_r}^{\varphi,\Gamma} \longleftarrow \mathrm{BT}_{\mathfrak{S}_r}^{\varphi,\Gamma} \tag{4.1.2}$$

with the first arrow given by forgetting the connection, and the second arrow induced by 'twisted' base change $\mathfrak{M} \rightsquigarrow \mathscr{M} := \varphi^*\mathfrak{M}\otimes_{\mathfrak{S}_r} S_r$. One of the main results of [CL17] is that both of these functors are *exact equivalences*, compatible with the natural notions of duality. The point here is that the action of Γ on an object $\mathscr{M}$ of $\mathrm{MF}_{S_r}^{\varphi,\Gamma}$ determines an integrable and topologically nilpotent connection ∇ on $\mathscr{M}$ by imposing the condition that the evaluation of ∇ on the

[11] Here we use our assumption that $p > 2$.

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derivation $(1+u_0)td/du_0$ is the differential operator $N_{\mathscr{M}} := p^r \lim_{\gamma\to 1}(\gamma-1)/(\chi(\gamma)-1)$ on $\mathscr{M}(\mathcal{G})$. That the action of Γ determines a connection and *vice versa* is well known in the theory of (φ,Γ)-modules and p-adic differential equations (e.g. [Ber02, § 4.1]); the point here is that this correspondence works *p-integrally* at the level of S_r-modules, which one shows via a fairly delicate calculation. Since the functors (4.1.2) are equivalences, we obtain from crystalline Dieudonné theory in this way the desired functorial association $\mathcal{G} \rightsquigarrow \mathfrak{M}_r(\mathcal{G})$, and by construction one has a natural φ- and Γ-equivariant isomorphism of S_r-modules $\varphi^*\mathfrak{M}_r(\mathcal{G})\otimes_{\mathfrak{S}_r} S_r \simeq \mathbf{D}(\mathcal{G}_0)_{S_r}$.

Example 4.1.5. For $\mathcal{G}=\mathbf{G}_m$ and $r>0$, one has $\mathfrak{M}_r(\mathcal{G})=\mathfrak{S}_r$, with $\varphi_{\mathfrak{M}}=E_r\varphi$ and $\gamma\in\Gamma$ acting as $\chi(\gamma)^{-1}(\gamma u_1/u_1)\gamma$. In particular, $\gamma\in\Gamma$ acts in the standard manner on $\mathfrak{M}_r(\mathcal{G})\otimes_{\theta\circ\varphi} R_r \simeq R_r$ and as $\chi(\gamma)^{-1}\eta(\gamma)\cdot\gamma$ on $\mathfrak{M}_r(\mathcal{G})\otimes_\theta R_r\simeq R_r$, for $\eta(\gamma):=(\gamma\varepsilon^{(1)}-1)/(\varepsilon^{(1)}-1)\in R_r^\times$.

We now explain how to functorially recover the $\mathscr{G}_K$-representation afforded by the p-adic Tate module T_pG from $\mathfrak{M}_r(\mathcal{G})$. To do so, we first recall the necessary period rings; for a more detailed synopsis of these rings and their properties, we refer the reader to [Col08, §§ 6–8].

As usual, we put $\widetilde{\mathbf{E}}^+ := \varprojlim_{x\mapsto x^p} \mathscr{O}_{\overline{K}}/(p)$, equipped with its canonical $\mathscr{G}_K$-action via 'coordinates' and p-power Frobenius map φ. This is a perfect valuation ring of characteristic p with residue field $\overline{k}$ and fraction field $\widetilde{\mathbf{E}} := \mathrm{Frac}(\widetilde{\mathbf{E}}^+)$ that is algebraically closed. We view $\widetilde{\mathbf{E}}$ as a topological field via its valuation topology, with respect to which it is complete. Our fixed choice of p-power compatible sequence $\{\varepsilon^{(r)}\}_{r\geqslant 0}$ induces an element $\underline{\varepsilon} := (\varepsilon^{(r)} \bmod p)_{r\geqslant 0}$ of $\widetilde{\mathbf{E}}^+$ and we set $\mathbf{E}_K := k(\!(\underline{\varepsilon}-1)\!)$, viewed as a topological subring of $\widetilde{\mathbf{E}}$; note that this is a φ- and $\mathscr{G}_K$-stable subfield of $\widetilde{\mathbf{E}}$ that is independent of our choice of $\underline{\varepsilon}$. We write $\mathbf{E}:=\mathbf{E}_K^{\mathrm{sep}}$ for the separable closure of $\mathbf{E}_K$ in $\widetilde{\mathbf{E}}$. The natural $\mathscr{G}_K$-action on $\widetilde{\mathbf{E}}$ induces a canonical identification $\mathrm{Gal}(\mathbf{E}/\mathbf{E}_K)=\mathscr{H}:=\ker(\chi)\subseteq\mathscr{G}_K$, so $\mathbf{E}^{\mathscr{H}}=\mathbf{E}_K$. If E is any subring of $\widetilde{\mathbf{E}}$, we write $E^+:=E\cap\widetilde{\mathbf{E}}^+$ for the intersection (taken inside $\widetilde{\mathbf{E}}$).

We define $\widetilde{\mathbf{A}}^+ := W(\widetilde{\mathbf{E}}^+)$ and $\widetilde{\mathbf{A}} := W(\widetilde{\mathbf{E}})$, equipped with their canonical Frobenius automorphisms φ and $\mathscr{G}_K$-actions via Witt functoriality. Set-theoretically identifying $W(\widetilde{\mathbf{E}})$ with $\prod_{m=0}^\infty \widetilde{\mathbf{E}}$ in the usual way, we endow each factor with its valuation topology and give $\widetilde{\mathbf{A}}$ the product topology.[12] For each $r\geqslant 0$, there is a unique continuous W-algebra map $\mathfrak{S}_r \hookrightarrow \widetilde{\mathbf{A}}^+$ determined by $u_r\mapsto\varphi^{-r}([\underline{\varepsilon}]-1)$, and we write $\mathbf{A}_{K,r}^+$ for its image. We denote by $\mathbf{A}_{K,r}$ the p-adic completion of the localization of $\mathbf{A}_{K,r}^+$ at the ideal (p), which is a complete discrete valuation ring with uniformizer p and residue field $\varphi^{-r}(\mathbf{E}_K)$. We henceforth identify $\mathfrak{S}_r$ with its image $\mathbf{A}_{K,r}^+$ in $\widetilde{\mathbf{A}}^+$. Let $\mathbf{A}_K^{\mathrm{sh}}$ be the strict Henselization of $\mathbf{A}_K := \mathbf{A}_{K,0}$ with respect to the separable closure of its residue field inside $\widetilde{\mathbf{E}}$. Since $\widetilde{\mathbf{A}}$ is strictly Henselian, there is a unique local morphism $\mathbf{A}_K^{\mathrm{sh}}\to\widetilde{\mathbf{A}}$ recovering the given inclusion on residue fields, and we henceforth view $\mathbf{A}_K^{\mathrm{sh}}$ as a subring of $\widetilde{\mathbf{A}}$. We denote by $\mathbf{A}$ the topological closure of $\mathbf{A}_K^{\mathrm{sh}}$ inside $\widetilde{\mathbf{A}}$ with respect to the strong topology, which is a φ- and $\mathscr{G}_K$-stable subring of $\widetilde{\mathbf{A}}$. If A is any subring of $\widetilde{\mathbf{A}}$, we define $A^+:=A\cap\widetilde{\mathbf{A}}^+$, with the intersection taken inside $\widetilde{\mathbf{A}}$, and put $A_r:=\varphi^{-r}(A)$; observe that this notation is consistent as $\mathbf{A}_{K,r}^+=\varphi^{-r}(\mathbf{A}_K^+)$ by construction.

THEOREM 4.1.6. *Let $\mathcal{G}\in\mathrm{pdiv}_{R_r}^\Gamma$, and write $H^1_{\text{ét}}(G):=(T_pG)^\vee$ for the $\mathbf{Z}_p$-linear dual of T_pG. There is a canonical mapping of finite free $\mathbf{A}_r^+$-modules with semilinear Frobenius and $\mathscr{G}_K$-actions*

$$\mathfrak{M}_r(\mathcal{G}) \underset{\mathfrak{S}_r,\varphi}{\otimes} \mathbf{A}_r^+ \longrightarrow H^1_{\text{ét}}(G)\otimes_{\mathbf{Z}_p}\mathbf{A}_r^+ \tag{4.1.3}$$

[12] This is what is called the *weak topology* on $\widetilde{\mathbf{A}}$. If each factor of $\widetilde{\mathbf{E}}$ is instead given the discrete topology, then the product topology on $\widetilde{\mathbf{A}}=W(\widetilde{\mathbf{E}})$ is the familiar p-adic topology, called the *strong* topology.

that is injective with cokernel killed by $u_1 = \varphi^{-1}([\underline{\varepsilon}] - 1)$. *Here* φ *acts as* $\varphi_{\mathfrak{M}_r(\mathcal{G})} \otimes \varphi$ *on source and as* $1 \otimes \varphi$ *on target, while* $\mathscr{G}_K$ *acts diagonally on source and target through the quotient* $\mathscr{G}_K \twoheadrightarrow \Gamma$ *on* $\mathfrak{M}_r(\mathcal{G})$. *In particular, there is a natural* φ*- and* $\mathscr{G}_K$*-equivariant isomorphism*

$$\mathfrak{M}_r(\mathcal{G}) \underset{\mathfrak{S}_r,\varphi}{\otimes} \mathbf{A}_r \simeq H^1_{\text{ét}}(G) \otimes_{\mathbf{Z}_p} \mathbf{A}_r. \tag{4.1.4}$$

These mappings are compatible with duality and with change in r *in the obvious manner.*

Corollary 4.1.7. *For* $\mathcal{G} \in \mathrm{pdiv}^{\Gamma}_{R_r}$, *there are functorial isomorphisms of* $\mathbf{Z}_p[\mathscr{G}_K]$*-modules*

$$T_pG \simeq \mathrm{Hom}_{\mathfrak{S}_r,\varphi}(\mathfrak{M}_r(\mathcal{G}), \mathbf{A}_r^+), \tag{4.1.5a}$$

$$H^1_{\text{ét}}(G) \simeq \Big(\mathfrak{M}_r(\mathcal{G}) \underset{\mathfrak{S}_r,\varphi}{\otimes} \mathbf{A}_r\Big)^{\varphi_{\mathfrak{M}_r(\mathcal{G})}\otimes\varphi=1} \tag{4.1.5b}$$

which are compatible with duality and change in r. *In the first isomorphism, we view* $\mathbf{A}_r^+$ *as a* $\mathfrak{S}_r$*-algebra via the composite of the usual structure map with* φ.

Remark 4.1.8. By definition, the map φ^r on $\mathbf{A}_{K,r}$ is injective with image $\mathbf{A}_K := \mathbf{A}_{K,0}$, and so induces a semilinear isomorphism of W-algebras $\varphi^r : \mathbf{A}_{K,r} \simeq \mathbf{A}_K$. It follows from (4.1.5b) and Fontaine's theory of (φ,Γ)-modules over $\mathbf{A}_K$ that $\mathfrak{M}_r(\mathcal{G}) \otimes_{\mathfrak{S}_r,\varphi^r} \mathbf{A}_K$ *is* the (φ,Γ)-module $M(L)$ associated to $L := H^1_{\text{ét}}(G)$. In this way, $\mathfrak{M}_r(\mathcal{G}) \otimes_{\mathfrak{S}_r,\varphi^r} \mathbf{A}_K^+$ is naturally a φ- and Γ-stable $\mathbf{A}_K^+$-lattice in the (φ,Γ)-module $M(L)$. The literature provides *two* other distinguished such lattices: the *Wach module* $\mathfrak{N}(L)$ constructed in [BB10] and the module $\mathfrak{M}_{\mathrm{KR}}(L)$ associated to L by the theory of Kisin and Ren [KR09] applied to the special case of the Lubin–Tate group $\widehat{\mathbf{G}}_m$. By [CL17, §§ 4.4–4.5], one has

$$\mathfrak{M}_r(\mathcal{G}) \otimes_{\mathfrak{S}_r,\varphi^r} \mathbf{A}_K^+ = \mathfrak{M}_{\mathrm{KR}}(L) \subseteq \mathfrak{N}(L) \tag{4.1.6}$$

inside $M(L)$, with the inclusion $\mathfrak{M}_{\mathrm{KR}}(L) \subseteq \mathfrak{N}(L)$ an equality when $r=1$. For $r>1$, this inclusion is in general not an equality, and the precise relationship between $\mathfrak{M}_{\mathrm{KR}}(L)$ and $\mathfrak{N}(L)$ is more complicated; see [CL17, Proposition 4.5.3]. It thus follows from [CL17] that the theory of Kisin and Ren [KR09] does *a posteriori* provide the 'right' $\mathbf{A}_K^+$-lattice inside $M(L)$ (cf. § 1), but we stress that their work makes no connections with *geometry*, which are essential for this paper.

As in Remark 2.2.11, for any $\mathcal{G} \in \mathrm{pdiv}^{\Gamma}_{R_r}$ and $\star \in \{\text{ét}, \mathrm{m}, \mathrm{null}\}$, the p-divisible group $\mathcal{G}^\star$ is again naturally an object of $\mathrm{pdiv}^{\Gamma}_{R_r}$. We thus (functorially) obtain objects $\mathfrak{M}_r(\mathcal{G}^\star)$ of $\mathrm{BT}^{\varphi,\Gamma}_{\mathfrak{S}_r}$ which admit particularly simple descriptions when $\star = \text{ét}$ or m, as we now explain.

As usual, we write $\overline{\mathcal{G}}^\star$ for the special fiber of $\mathcal{G}^\star$ and $\mathbf{D}(\overline{\mathcal{G}}^\star)_W$ for its Dieudonné module. Twisting the W-algebra structure on $\mathfrak{S}_r$ by the automorphism φ^{r-1} of W, we define objects

$$\mathfrak{M}_r^{\text{ét}}(\mathcal{G}) := \mathbf{D}(\overline{\mathcal{G}}^{\text{ét}})_W \underset{W,\varphi^{r-1}}{\otimes} \mathfrak{S}_r, \quad \varphi_{\mathfrak{M}_r^{\text{ét}}} := F \otimes \varphi, \quad \gamma := \gamma \otimes \gamma, \tag{4.1.7a}$$

$$\mathfrak{M}_r^{\mathrm{m}}(\mathcal{G}) := \mathbf{D}(\overline{\mathcal{G}}^{\mathrm{m}})_W \underset{W,\varphi^{r-1}}{\otimes} \mathfrak{S}_r, \quad \varphi_{\mathfrak{M}_r^{\mathrm{m}}} := V^{-1} \otimes E_r \cdot \varphi, \quad \gamma := \gamma \otimes \chi(\gamma)^{-1}\varphi^{r-1}(\gamma u_r/u_r) \cdot \gamma \tag{4.1.7b}$$

of $\mathrm{BT}^{\varphi,\Gamma}_{\mathfrak{S}_r}$, with $\gamma \in \Gamma$ acting as indicated. Note that these formulae make sense and do indeed give objects of $\mathrm{BT}^{\varphi,\Gamma}_{\mathfrak{S}_r}$ as V is invertible[13] on $\mathbf{D}(\overline{\mathcal{G}}^{\mathrm{m}})_W$ and $\gamma u_r/u_r \in \mathfrak{S}_r^\times$.

[13] In the sense that there exists a φ-semilinear map V^{-1} whose composition with V in either order is the identity.

PROPOSITION 4.1.9. *Let $\mathcal{G}$ be an object of $\mathrm{pdiv}_{R_r}^{\Gamma}$ and let $\mathfrak{M}_r^{\text{ét}}(\mathcal{G})$ and $\mathfrak{M}_r^{\mathrm{m}}(\mathcal{G})$ be as in (4.1.7a)–(4.1.7b). The map $F^r : \mathcal{G}_0 \to \mathcal{G}_0^{(p^r)}$ (respectively $V^r : \mathcal{G}_0^{(p^r)} \to \mathcal{G}_0$) induces a natural isomorphism in $\mathrm{BT}_{\mathfrak{S}_r}^{\Gamma}$*

$$\mathfrak{M}_r(\mathcal{G}^{\text{ét}}) \simeq \mathfrak{M}_r^{\text{ét}}(\mathcal{G}) \quad \textit{respectively} \quad \mathfrak{M}_r(\mathcal{G}^{\mathrm{m}}) \simeq \mathfrak{M}_r^{\mathrm{m}}(\mathcal{G}) \tag{4.1.8}$$

which is compatible with change in r in that for $\star = \text{ét}$ (respectively $\star = \mathrm{m}$) it intertwines the isomorphism of Theorem 4.1.3(ii) with the linear map induced by F (respectively V^{-1}) on $\mathbf{D}(\overline{\mathcal{G}}^{\star})_W$.

Proof. For simplicity, we will write $\mathfrak{M}_r^{\star}$ and $\mathbf{D}^{\star}$ for $\mathfrak{M}_r^{\star}(\mathcal{G})$ and $\mathbf{D}(\overline{\mathcal{G}}^{\star})_W$, respectively. Using the explicit description of the equivalences (4.1.2) in [CL17], one finds that the object of $\mathrm{MF}_{S_r}^{\varphi,\Gamma}$ corresponding to $\mathfrak{M}_r^{\text{ét}}$ is given by the triple

$$\mathscr{M}_r^{\text{ét}} := (\mathbf{D}^{\text{ét}} \otimes_{W,\varphi^r} S_r, \mathbf{D}^{\text{ét}} \otimes_{W,\varphi^r} \mathrm{Fil}^1 S_r, F \otimes \varphi_1) \tag{4.1.9a}$$

with Γ acting diagonally on the tensor product. Similarly, the object corresponding to $\mathfrak{M}_r^{\mathrm{m}}$ is

$$\mathscr{M}_r^{\mathrm{m}} := (\mathbf{D}^{\mathrm{m}} \otimes_{W,\varphi^r} S_r, \mathbf{D}^{\mathrm{m}} \otimes_{W,\varphi^r} S_r, V^{-1} \otimes v_r \cdot \varphi), \tag{4.1.9b}$$

where $v_r = \varphi(E_r)/p$ as before and $\gamma \in \Gamma$ acts on $\mathbf{D}^{\mathrm{m}} \otimes_{W,\varphi^r} S_r$ as $\gamma \otimes \chi(\gamma)^{-1}\varphi^r(\gamma u_r/u_r) \cdot \gamma$. Using the relation $\gamma t = \chi(\gamma)t$ for $\gamma \in \Gamma$, one checks that the S_r-module automorphism of $\mathbf{D}^{\mathrm{m}} \otimes_{W,\varphi^r} S_r$ given by multiplication by $\lambda := t/u_0 \in S_r^{\times}$ carries (4.1.9b) isomorphically onto the the triple

$$\mathscr{M}_r^{\mathrm{m}} := (\mathbf{D}^{\mathrm{m}} \otimes_{W,\varphi^r} S_r, \mathbf{D}^{\mathrm{m}} \otimes_{W,\varphi^r} S_r, V^{-1} \otimes \varphi) \tag{4.1.10}$$

of $\mathrm{MF}_{S_r}^{\varphi,\Gamma}$ with Γ acting *diagonally* on the tensor product. As in the proof of Lemma 3.2.2, using the fact that the Dieudonné crystal is compatible with base change and that the inclusion $W \to S_r$ extends to a PD morphism $(W,p) \to (S_r, pS_r + \mathrm{Fil}^1 S_r)$ over $k \to R_r/pR_r$, applying $\mathbf{D}(\cdot)_{S_r}$ to (3.2.5a)–(3.2.5b) with $R := R_r$ yields natural isomorphisms $\mathbf{D}(\mathcal{G}_0^{\star})_{S_r} \simeq \mathbf{D}^{\star} \otimes_{W,\varphi^r} S_r$ for $\star = \text{ét}, \mathrm{m}$ which carry F to $F \otimes \varphi$. Using the construction of $\mathscr{M}_r(\mathcal{G}^{\star})$ outlined above and explained in detail in [CL17], one checks that these isomorphisms extend to give isomorphisms $\mathscr{M}_r(\mathcal{G}^{\text{ét}}) \simeq \mathscr{M}_r^{\text{ét}}$ and $\mathscr{M}_r(\mathcal{G}^{\mathrm{m}}) \simeq \mathscr{M}_r^{\mathrm{m}}$ in $\mathrm{MF}_{S_r}^{\varphi,\Gamma}$. The claimed natural isomorphisms (4.1.8) follow at once from the equivalences (4.1.2), and the asserted compatibility with change in r is a straightforward exercise that we leave to the reader. $\square$

Now suppose that $\mathcal{G} \in \mathrm{pdiv}_{R_r}^{\Gamma}$ is ordinary. As $\mathfrak{M}_r$ is exact by Theorem 4.1.3(i), applying $\mathfrak{M}_r$ to the connected-étale sequence of $\mathcal{G}$ gives a contravariantly functorial short exact sequence

$$0 \longrightarrow \mathfrak{M}_r(\mathcal{G}^{\text{ét}}) \longrightarrow \mathfrak{M}_r(\mathcal{G}) \longrightarrow \mathfrak{M}_r(\mathcal{G}^{\mathrm{m}}) \longrightarrow 0 \tag{4.1.11}$$

in $\mathrm{BT}_{\mathfrak{S}_r}^{\varphi,\Gamma}$. Since $\varphi_{\mathfrak{M}_r}$ linearizes to an isomorphism on $\mathfrak{M}_r(\mathcal{G}^{\text{ét}})$ and is topologically nilpotent on $\mathfrak{M}_r(\mathcal{G}^{\mathrm{m}})$, we think of (4.1.11) as the 'slope filtration' for Frobenius acting on $\mathfrak{M}_r(\mathcal{G})$.

PROPOSITION 4.1.10. *There is a canonical Γ-equivariant isomorphism of short exact sequences*

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{M}_r(\mathcal{G}^{\text{ét}}) \underset{\mathfrak{S}_r,\tau\circ\varphi}{\otimes} W \arrow[r] \arrow[d, "\simeq"] & \mathfrak{M}_r(\mathcal{G}) \underset{\mathfrak{S}_r,\tau\circ\varphi}{\otimes} W \arrow[r] \arrow[d, "\simeq"] & \mathfrak{M}_r(\mathcal{G}^{\mathrm{m}}) \underset{\mathfrak{S}_r,\tau\circ\varphi}{\otimes} W \arrow[r] \arrow[d, "\simeq"] & 0 \\ 0 \arrow[r] & \mathbf{D}(\overline{\mathcal{G}}^{\text{ét}})_W \arrow[r] & \mathbf{D}(\overline{\mathcal{G}})_W \arrow[r] & \mathbf{D}(\overline{\mathcal{G}}^{\mathrm{m}})_W \arrow[r] & 0 \end{tikzcd} \tag{4.1.12a}$$

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intertwining $\varphi_{\mathfrak{M}_r}\otimes\varphi$ *with* F, *and a canonical* Γ*-equivariant isomorphism of short exact sequences*

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{M}_r(\mathcal{G}^{\text{ét}}) \underset{\mathfrak{S}_r,\theta\circ\varphi}{\otimes} R_r \arrow[r] \arrow[d, "\simeq"] & \mathfrak{M}_r(\mathcal{G}) \underset{\mathfrak{S}_r,\theta\circ\varphi}{\otimes} R_r \arrow[r] \arrow[d, "\simeq"] & \mathfrak{M}_r(\mathcal{G}^{\text{m}}) \underset{\mathfrak{S}_r,\theta\circ\varphi}{\otimes} R_r \arrow[r] \arrow[d, "\simeq"] & 0 \\ 0 \arrow[r] & \mathrm{Lie}(\mathcal{G}^t) \arrow[r, "i"'] & \mathbf{D}(\mathcal{G}_0)_{R_r} \arrow[r, "j"'] & \omega_{\mathcal{G}} \arrow[r] & 0 \end{tikzcd} \tag{4.1.12b}$$

where $i:\mathrm{Lie}(\mathcal{G}^t)\hookrightarrow\mathbf{D}(\mathcal{G}_0)_{R_r}$ *and* $j:\mathbf{D}(\mathcal{G}_0)_{R_r}\twoheadrightarrow\omega_{\mathcal{G}}$ *are the canonical splittings of Lemma 3.2.1.*

Proof. This follows immediately from Theorem 4.1.3, using Lemma 3.2.1 and its proof. □

4.2 Ordinary families of (φ,Γ)-modules

For each pair of positive integers $r\geqslant s$, we have a morphism $\rho_{r,s}:\mathcal{G}_s\times_{T_s}T_r\to\mathcal{G}_r$ in $\mathrm{pdiv}^{\Gamma}_{R_r}$; applying the contravariant functor $\mathfrak{M}_r:\mathrm{pdiv}^{\Gamma}_{R_r}\to\mathrm{BT}^{\Gamma}_{\mathfrak{S}_r}$ of Theorem 4.1.3 to the map on connected-étale sequences induced by $\rho_{r,s}$ and using the exactness of $\mathfrak{M}_r$ and its compatibility with base change, we obtain morphisms in $\mathrm{BT}^{\Gamma}_{\mathfrak{S}_r}$

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{M}_r(\mathcal{G}_r^{\text{ét}}) \arrow[r] \arrow[d, "\mathfrak{M}_r(\rho_{r,s})"'] & \mathfrak{M}_r(\mathcal{G}_r) \arrow[r] \arrow[d, "\mathfrak{M}_r(\rho_{r,s})"] & \mathfrak{M}_r(\mathcal{G}_r^{\text{m}}) \arrow[r] \arrow[d, "\mathfrak{M}_r(\rho_{r,s})"] & 0 \\ 0 \arrow[r] & \mathfrak{M}_s(\mathcal{G}_s^{\text{ét}}) \underset{\mathfrak{S}_s}{\otimes} \mathfrak{S}_r \arrow[r] & \mathfrak{M}_r(\mathcal{G}_s) \underset{\mathfrak{S}_s}{\otimes} \mathfrak{S}_r \arrow[r] & \mathfrak{M}_r(\mathcal{G}_s^{\text{m}}) \underset{\mathfrak{S}_s}{\otimes} \mathfrak{S}_r \arrow[r] & 0 \end{tikzcd} \tag{4.2.1}$$

Definition 4.2.1. Let $\star=$ ét or $\star=$ m and define

$$\mathfrak{M}_\infty:=\varprojlim_r\Big(\mathfrak{M}_r(\mathcal{G}_r)\underset{\mathfrak{S}_r}{\otimes}\mathfrak{S}_\infty\Big),\quad \mathfrak{M}_\infty^\star:=\varprojlim_r\Big(\mathfrak{M}_r(\mathcal{G}_r^\star)\underset{\mathfrak{S}_r}{\otimes}\mathfrak{S}_\infty\Big) \tag{4.2.2}$$

with the projective limits taken with respect to the mappings induced by (4.2.1).

Each of (4.2.2) is naturally a module over the completed group ring $\Lambda_{\mathfrak{S}_\infty}$ and is equipped with a semilinear action of Γ and a φ-semilinear Frobenius morphism defined by $F:=\varprojlim(\varphi_{\mathfrak{M}_r}\otimes\varphi)$. Since φ is bijective on $\mathfrak{S}_\infty$, we also have a φ^{-1}-semilinear Verschiebung morphism defined as follows. For notational ease, we provisionally set $M_r:=\mathfrak{M}_r(\mathcal{G}_r)\otimes_{\mathfrak{S}_r}\mathfrak{S}_\infty$ and we define

$$V_r: M_r \xrightarrow{\ \psi\ } \varphi^*M_r \xrightarrow{\alpha\otimes m\mapsto\varphi^{-1}(\alpha)m} M_r \tag{4.2.3}$$

with ψ induced by scalar extension from the map $\psi_{\mathfrak{M}_r}$ defined above 4.1.1. It is easy to see that the V_r are compatible with change in r, and we put $V:=\varprojlim V_r$ on $\mathfrak{M}_\infty$. We define Verschiebung morphisms on $\mathfrak{M}_\infty^\star$ for $\star=$ ét, m similarly. As the composite of $\psi_{\mathfrak{M}_r}$ and $1\otimes\varphi_{\mathfrak{M}_r}$ in either order is multiplication by $E_r(u_r)=:\omega$, one has $FV=\omega$ and $VF=\varphi^{-1}(\omega)$. Due to the functoriality of $\mathfrak{M}_r$, we moreover have a $\Lambda_{\mathfrak{S}_\infty}$-linear action of $\mathfrak{H}^*$ on each of (4.2.2) which commutes with F, V, and Γ.

Proof of Theorem 1.2.9. Since φ is an automorphism of $\mathfrak{S}_\infty$, pullback by φ commutes with projective limits of $\mathfrak{S}_\infty$-modules. As the canonical $\mathfrak{S}_\infty$-linear map $\varphi^*\Lambda_{\mathfrak{S}_\infty}\to\Lambda_{\mathfrak{S}_\infty}$ is an isomorphism of rings (even of $\mathfrak{S}_\infty$-algebras), it therefore suffices to prove the assertions of

Theorem 1.2.9 after pullback by φ, which is more convenient due to the relation between $\varphi^*\mathfrak{M}_r(\mathcal{G}_r)$ and the Dieudonné crystal of $\mathcal{G}_r$.

Pulling back (4.2.1) by φ gives a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \varphi^*\mathfrak{M}_r(\mathcal{G}_r^{\text{ét}}) \arrow[r] \arrow[d] & \varphi^*\mathfrak{M}_r(\mathcal{G}_r) \arrow[r] \arrow[d] & \varphi^*\mathfrak{M}_r(\mathcal{G}_r^{\text{m}}) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \varphi^*\mathfrak{M}_s(\mathcal{G}_s^{\text{ét}}) \underset{\mathfrak{S}_s}{\otimes} \mathfrak{S}_r \arrow[r] & \varphi^*\mathfrak{M}_r(\mathcal{G}_s) \underset{\mathfrak{S}_s}{\otimes} \mathfrak{S}_r \arrow[r] & \varphi^*\mathfrak{M}_r(\mathcal{G}_s^{\text{m}}) \underset{\mathfrak{S}_s}{\otimes} \mathfrak{S}_r \arrow[r] & 0 \end{tikzcd} \tag{4.2.4}$$

and, as in the proof of Theorem 1.2.1, we apply the commutative algebra formalism of [Cai17, § 3.1]. In the notation of [Cai17, Lemma 3.2], we take $A_r := \mathfrak{S}_r$, $I_r := (u_r)$, $B = \mathfrak{S}_\infty$, and M_r each one of the terms in the top row of (4.2.4), and we must check that the hypotheses

(i) $\overline{M}_r := M_r/u_rM_r$ is a free $\mathbf{Z}_p[\Delta/\Delta_r]$-module of rank d';

(ii) for all $s \leqslant r$, the induced transition maps $\overline{\rho}_{r,s} : \overline{M}_r \longrightarrow \overline{M}_s$ are surjective

hold. The isomorphism (4.1.12a) ensures, via Theorem 1.2.1, that hypothesis (i) is satisfied.

Due to the functoriality of (4.1.12a), for any $r \geqslant s$, the mapping obtained from (4.2.4) by reducing modulo I_r is identified with the mapping on (3.1.1) induced (via functoriality of $\mathbf{D}(\cdot)$) by $\overline{\rho}_{r,s}$. As was shown in the proof of Theorem 1.2.1, these mappings are surjective for all $r \geqslant s$, and we conclude that hypothesis (ii) holds as well. Moreover, the vertical mappings of (4.2.4) are then surjective by Nakayama's lemma, so as in the proof of Theorem 1.2.1 (keeping in mind that pullback by φ commutes with projective limits of $\mathfrak{S}_\infty$-modules), we obtain, by applying $\otimes_{\mathfrak{S}_r}\mathfrak{S}_\infty$ to (4.2.4), passing to projective limits, and pulling back by φ^{-1}, the short exact sequence (1.2.9). The final assertion is an immediate consequence of the very construction of $\varphi_{\mathfrak{M}_r}$, the definition (4.2.3) of V, and (4.1.1). $\square$

We can now prove Theorem 1.2.10, which relates the exact sequence (1.2.9) to its dual over $\Lambda_{\mathfrak{S}'_\infty}$ for $\mathfrak{S}'_\infty := \mathfrak{S}_\infty[\mu_N]$.

Proof of Theorem 1.2.10. We first note that there is a natural isomorphism of $\mathfrak{S}'_\infty[\Delta/\Delta_r]$-modules

$$\mathfrak{M}_r(\mathcal{G}_r)(\mu\langle a\rangle_N) \otimes_{\mathfrak{S}_r} \mathfrak{S}'_\infty \simeq \mathrm{Hom}_{\mathfrak{S}'_\infty}(\mathfrak{M}_r(\mathcal{G}_r) \otimes_{\mathfrak{S}_r} \mathfrak{S}'_\infty, \mathfrak{S}'_\infty) \tag{4.2.5}$$

that is $\mathfrak{H}^*$-equivariant and $\mathrm{Gal}(K'_\infty/K_0)$-compatible for the standard action $\gamma\cdot f(m) := \gamma f(\gamma^{-1}m)$ on the right-hand side, and that intertwines F and V with $V^\vee$ and $F^\vee$, respectively. Indeed, this follows immediately from the identifications

$$\mathfrak{M}_r(\mathcal{G}_r)(\langle\chi\rangle\langle a\rangle_N) \underset{\mathfrak{S}_r}{\otimes} \mathfrak{S}'_\infty \simeq \mathfrak{M}_r(\mathcal{G}'_r) \underset{\mathfrak{S}_r}{\otimes} \mathfrak{S}'_\infty =: \mathfrak{M}_r(\mathcal{G}_r^\vee) \underset{\mathfrak{S}_r}{\otimes} \mathfrak{S}'_\infty \simeq \mathfrak{M}_r(\mathcal{G}_r)^\vee \underset{\mathfrak{S}_r}{\otimes} \mathfrak{S}'_\infty \tag{4.2.6}$$

and the definition (Definition 4.1.2) of duality in $\mathrm{BT}^{\varphi,\Gamma}_{\mathfrak{S}_r}$; here the first isomorphism in (4.2.6) results from Proposition 2.2.12 and Theorem 4.1.3(ii), while the final identification is due to Theorem 4.1.3(i). The identification (4.2.5) carries F (respectively V) on its source to $V^\vee$ (respectively $F^\vee$) on its target due to the compatibility of the functor $\mathfrak{M}_r(\cdot)$ with duality.

From (4.2.5), we obtain a natural $\mathrm{Gal}(K'_r/K_0)$-compatible evaluation pairing of $\mathfrak{S}'_\infty$-modules

$$\langle\cdot,\cdot\rangle_r : \mathfrak{M}_r(\mathcal{G}_r)(\mu\langle a\rangle_N) \underset{\mathfrak{S}_r}{\otimes} \mathfrak{S}'_\infty \times \mathfrak{M}_r(\mathcal{G}_r) \underset{\mathfrak{S}_r}{\otimes} \mathfrak{S}'_\infty \longrightarrow \mathfrak{S}'_\infty \tag{4.2.7}$$

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with respect to which the natural action of $\mathfrak{H}^*$ is self-adjoint, due to the fact that (4.2.6) is $\mathfrak{H}^*$-equivariant by Proposition 2.2.12. Due to the compatibility with change in r of the identification (2.2.9) of Proposition 2.2.12 together with the definitions (2.2.8) of $\rho_{r,s}$ and $\rho'_{r,s}$, the identification (4.2.6) intertwines the map induced by $\operatorname{Pic}^0(\rho)$ on its source with the map induced by $U_p^{*-1}\operatorname{Alb}(\sigma)$ on its target. For $r \geqslant s$, we therefore have

$$\langle \mathfrak{M}_r(\rho_{r,s})x, \mathfrak{M}_r(\rho_{r,s})y\rangle_s = \langle x, \mathfrak{M}_r(U_p^{*s-r}\operatorname{Pic}^0(\rho)^{r-s}\operatorname{Alb}(\sigma)^{r-s})y\rangle_r = \sum_{\delta\in\Delta_s/\Delta_r}\langle x, \delta^{-1}y\rangle_r,$$

where the final equality follows from (3.1.7). Thus, the perfect pairings (4.2.7) satisfy the compatibility condition of [Cai17, Lemma 3.4] (as in (3.1.5) of the proof of Theorem 1.2.2), which, together with Theorem 1.2.9, completes the proof. □

The $\Lambda_{\mathfrak{S}_\infty}$-modules $\mathfrak{M}_\infty^{\text{ét}}$ and $\mathfrak{M}_\infty^{\text{m}}$ have a particularly simple structure, as made precise by Theorem 1.2.11, which we now prove.

Proof of Theorem 1.2.11. We twist the identifications (4.1.8) of Proposition 4.1.9 to obtain natural isomorphisms

$$\mathfrak{M}_r(\mathcal{G}_r^{\text{ét}}) \xrightarrow[F^r\circ(4.1.8)]{\simeq} \mathbf{D}(\overline{\mathcal{G}}_r^{\text{ét}})_{\mathbf{Z}_p}\otimes_{\mathbf{Z}_p}\mathfrak{S}_r \quad \text{and} \quad \mathfrak{M}_r(\mathcal{G}_r^{\text{m}}) \xrightarrow[V^{-r}\circ(4.1.8)]{\simeq} \mathbf{D}(\overline{\mathcal{G}}_r^{\text{m}})_{\mathbf{Z}_p}\otimes_{\mathbf{Z}_p}\mathfrak{S}_r$$

that are $\mathfrak{H}_r^*$-equivariant and, thanks to Proposition 4.1.9, compatible with change in r using the maps on source and target induced by $\rho_{r,s}$. Passing to inverse limits and appealing again to [Cai17, Lemma 3.2] and (the proof of) Theorem 1.2.1, we deduce for $\star = \text{ét}, \text{m}$ natural isomorphisms of $\Lambda_{\mathfrak{S}_\infty}$-modules

$$\mathfrak{M}_\infty^\star \simeq \varprojlim_r(\mathbf{D}(\overline{\mathcal{G}}_r^\star)_{\mathbf{Z}_p}\otimes_{\mathbf{Z}_p}\mathfrak{S}_\infty) \simeq \mathbf{D}_\infty^\star\otimes_\Lambda\Lambda_{\mathfrak{S}_\infty}$$

that are $\mathfrak{H}^*$-equivariant and satisfy the asserted compatibility with respect to Frobenius, Verschiebung, and the action of Γ due to Proposition 4.1.9 and the definitions (4.1.7a)–(4.1.7b). □

4.3 Λ-adic crystalline comparison isomorphism

We now prove Theorem 1.2.12, which asserts that the slope filtration (1.2.9) of $\mathfrak{M}_\infty$ specializes, on the one hand, to the slope filtration (3.1.2) of $\mathbf{D}_\infty$, and on the other hand to the Hodge filtration (1.1.2) (in the opposite direction) of $e^{*\prime}H^1_{\text{dR}}$.

Proof of Theorem 1.2.12. To prove the first assertion of Theorem 1.2.12, we apply [Cai17, Lemma 3.2] with $A_r = \mathfrak{S}_r$, $I_r = (u_r)$, $B = \mathfrak{S}_\infty$, $B' = \mathbf{Z}_p$ (viewed as a B-algebra via τ), and $M_r = \varphi^*\mathfrak{M}_r^\star$ for $\star \in \{\text{ét}, \text{m}, \text{null}\}$ and, as in the proofs of Theorems 1.2.1 and 1.2.9, we must verify the hypotheses:

(i) $\overline{M}_r := M_r/u_rM_r$ is a free $\mathbf{Z}_p[\Delta/\Delta_r]$-module of rank d';

(ii) for all $s \leqslant r$, the induced transition maps $\overline{\rho}_{r,s} : \overline{M}_r \longrightarrow \overline{M}_s$ are surjective.

Thanks to (4.1.12a), we have a canonical identification $\overline{M}_r := M_r/I_rM_r \simeq \mathbf{D}(\overline{\mathcal{G}}_r^\star)_{\mathbf{Z}_p}$ that is compatible with change in r in the sense that the induced projective system $\{\overline{M}_r\}_r$ is identified with that of Definition 3.1.1. It follows from this and Theorem 1.2.1 that the hypotheses (i)–(ii) are satisfied, and (1.2.12) is an isomorphism by [Cai17, Lemma 3.2(5)].

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In exactly the same manner, the second assertion of Theorem 1.2.12 follows by appealing to [Cai17, Lemma 3.2] with $A_r = \mathfrak{S}_r$, $I_r = (E_r)$, $B = \mathfrak{S}_\infty$, $B' = R_\infty$ (viewed as a B-algebra via θ), and $M_r = \varphi^* \mathfrak{M}_r^\star$, using (4.1.12b) and Proposition 3.2.3 together with Theorem 1.1.1 (see [Cai17, Theorem 3.7]) to verify the requisite hypotheses in this setting. $\square$

Proof of Theorem 1.2.13 and Corollary 1.2.14. Applying Theorem 4.1.6 to (the connected-étale sequence of) $\mathcal{G}_r$ gives a natural isomorphism of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{M}_r(\mathcal{G}_r^{\text{ét}}) \underset{\mathfrak{S}_r,\varphi}{\otimes} \mathbf{A}_r \arrow[r] \arrow[d, "\simeq"] & \mathfrak{M}_r(\mathcal{G}_r) \underset{\mathfrak{S}_r,\varphi}{\otimes} \mathbf{A}_r \arrow[r] \arrow[d, "\simeq"] & \mathfrak{M}_r(\mathcal{G}_r^{\mathrm{m}}) \underset{\mathfrak{S}_r,\varphi}{\otimes} \mathbf{A}_r \arrow[r] \arrow[d, "\simeq"] & 0 \\ 0 \arrow[r] & H^1_{\text{ét}}(G_r^{\text{ét}}) \underset{\mathbf{Z}_p}{\otimes} \mathbf{A}_r \arrow[r] & H^1_{\text{ét}}(G_r) \underset{\mathbf{Z}_p}{\otimes} \mathbf{A}_r \arrow[r] & H^1_{\text{ét}}(G_r^{\mathrm{m}}) \underset{\mathbf{Z}_p}{\otimes} \mathbf{A}_r \arrow[r] & 0 \end{tikzcd} \tag{4.3.1}$$

Due to Theorem 1.2.9, the terms in the top row of 4.3.1 are free of ranks d', $2d'$, and d' over $\widetilde{\mathbf{A}}_r[\Delta/\Delta_r]$, respectively, so we conclude from [Cai17, Lemma 3.3] (using $A = \mathbf{Z}_p[\Delta/\Delta_r]$ and $B = \mathbf{A}_r[\Delta/\Delta_r]$ in the notation of that result) that $H^1_{\text{ét}}(G_r^\star)$ is a free $\mathbf{Z}_p[\Delta/\Delta_r]$-module of rank d' for $\star = \{\text{ét}, \mathrm{m}\}$ and that $H^1_{\text{ét}}(G_r)$ is free of rank $2d'$ over $\mathbf{Z}_p[\Delta/\Delta_r]$. Using the fact that $\mathbf{Z}_p \to \mathbf{A}_r$ is faithfully flat, it then follows from the surjectivity of the vertical maps in (4.2.4) (which was noted in the proof of Theorem 1.2.9) that the canonical trace mappings $H^1_{\text{ét}}(G_r^\star) \to H^1_{\text{ét}}(G_{r'}^\star)$ for $\star \in \{\text{ét}, \mathrm{m}, \mathrm{null}\}$ are surjective for all $r \geqslant r'$. Applying [Cai17, Lemma 3.2] with $A_r = \mathbf{Z}_p$, $M_r := H^1_{\text{ét}}(G_r^\star)$, $I_r = (0)$, $B = \mathbf{Z}_p$, and $B' = \widetilde{\mathbf{A}}$, we conclude that $H^1_{\text{ét}}(G_\infty^\star)$ is free of rank d' (respectively $2d'$) over Λ for $\star = \text{ét}$, m (respectively $\star = \mathrm{null}$), that the specialization mappings

$$H^1_{\text{ét}}(G_\infty^\star) \underset{\Lambda}{\otimes} \mathbf{Z}_p[\Delta/\Delta_r] \longrightarrow H^1_{\text{ét}}(G_r^\star)$$

are isomorphisms, and that the canonical mappings for $\star \in \{\text{ét}, \mathrm{m}, \mathrm{null}\}$

$$H^1_{\text{ét}}(G_\infty^\star) \underset{\Lambda}{\otimes} \Lambda_{\widetilde{\mathbf{A}}} \longrightarrow \varprojlim_r \Big(H^1_{\text{ét}}(G_r^\star) \underset{\mathbf{Z}_p}{\otimes} \widetilde{\mathbf{A}} \Big) \tag{4.3.2}$$

are isomorphisms. Invoking the isomorphism (3.2.22) gives Corollary 1.2.14. By [Cai17, Lemma 3.2] with $A_r = \mathfrak{S}_r$, $M_r = \mathfrak{M}_r(\mathcal{G}_r^\star)$, $I_r = (0)$, $B = \mathfrak{S}_\infty$, and $B' = \widetilde{\mathbf{A}}$ (viewed as a B-algebra via φ), we similarly conclude from (the proof of) Theorem 1.2.9 that the canonical mappings for $\star \in \{\text{ét}, \mathrm{m}, \mathrm{null}\}$

$$\mathfrak{M}_\infty^\star \underset{\mathfrak{S}_\infty,\varphi}{\otimes} \Lambda_{\widetilde{\mathbf{A}}} \longrightarrow \varprojlim_r \Big(\mathfrak{M}_r(\mathcal{G}_r^\star) \underset{\mathfrak{S}_r,\varphi}{\otimes} \widetilde{\mathbf{A}} \Big) \tag{4.3.3}$$

are isomorphisms. Applying $\otimes_{\mathbf{A}_r} \widetilde{\mathbf{A}}$ to the diagram (4.3.1), passing to inverse limits, and using the isomorphisms (4.3.2) and (4.3.3) gives (again invoking (3.2.22)) the isomorphism (1.2.14). Using the fact that the inclusion $\mathbf{Z}_p \hookrightarrow \widetilde{\mathbf{A}}^{\varphi=1}$ is an equality, the isomorphism (1.2.15) follows immediately from (1.2.14) by taking $F \otimes \varphi$-invariants. $\square$

Using Theorems 1.2.13 and 1.2.10, we can give a new proof of Ohta's duality theorem [Oht95, Theorem 4.3.1] for the Λ-adic ordinary filtration of $e^{*\prime} H^1_{\text{ét}}$.

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Proof of Corollary 1.2.15. As in the proof of Theorem 1.2.2, by using Corollary 1.2.14 and applying [Cai17, Lemma 3.4], one shows that the pairing

$$\langle\cdot,\cdot\rangle_\Lambda : e^{*\prime} H^1_{\text{ét}} \times e^{*\prime} H^1_{\text{ét}} \to \Lambda \quad \text{determined by } \langle x,y\rangle_\Lambda \equiv \sum_{\delta\in\Delta/\Delta_r} (x, w_r U_p^{*r}\langle\delta^{-1}\rangle^* y)_r \delta \bmod I_r \tag{4.3.4}$$

is a Λ-bilinear and perfect duality pairing with respect to which the action of $\mathfrak{H}^*$ is self-adjoint; here $(\cdot,\cdot)_r$ is the usual cup-product pairing on $H^1_{\text{ét},r}$ and $I_r := \ker(\Lambda \twoheadrightarrow \mathbf{Z}_p[\Delta/\Delta_r])$. This pairing is easily seen to induce the claimed isomorphism (1.2.16). □

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