RECENT PROGRESS IN THE THEORY OF TROJAN ASTEROIDS

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§1 SUMMARY OF PART I

In order to provide the necessary background for this report, this section summarizes the previously published[5] "Theory of the Trojan Asteroids, Part I". Treating the system as the case of 1:1 resonance in the restricted problem of three bodies, the author constructs a formal long-periodic solution of O(m), where m is the mass-parameter of the system, assumed to be sufficiently small.

The variables of the problem are the angular momentum G, the conjugate mean synodic longitude λ measured from the line of syzygies in a rotating coordinate system, and the complex Poincaré eccentric variable,

$$z = \xi + i\eta = \sqrt{2\Gamma} \ \exp i\ell,$$

where ℓ is the mean anomaly and Γ is defined in terms of the Delaunay variables by

$$\Gamma \equiv L - G.$$

The Hamiltonian of the system is written as

$$F = \frac{1}{2}(1-m)^2 L^{-2} + G + mR , \tag{1}$$

and the heliocentric disturbing function,

$$R = (1 + r^2 - 2r \cos \Theta)^{-1/2} - r \cos \Theta,$$

is expanded into a Taylor series about the unit circle r=1. This mode of expansion assures the solution a wider range than the expansion about L_4 adopted by Deprit et al[2]. With the aid of the formulas

$$r = G^2 - \xi + m + \ldots, \qquad \Theta = \lambda + 2\eta + \ldots \ .$$

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of elliptic motion, the expression (1) is put into the form

$$F = \frac{1}{2}G^{-2} + G - \frac{1}{2}(\xi^2 + \eta^2) + \frac{3}{2}\rho(\xi^2 + \eta^2) - \frac{1}{2}m + mf_o$$
$$+ m[-2f_1\rho + (1 + f_1)\xi + f_2\eta] - m^2(f_1 + \frac{1}{2} + \psi) + \ldots \tag{2}$$

where the various symbols are defined below:

$$\rho = G - 1,$$
$$f_o = \frac{1}{2s} + 2s^2 - \frac{3}{2} + m\psi,$$
$$f_1 = \frac{1}{4s} - 2s^2, \qquad s \equiv \sin(\lambda/2), \tag{3}$$
$$f_2 = 4c(s - \frac{1}{8s^2}). \qquad c \equiv \cos(\lambda/2).$$

Here the function $\Psi(\lambda)$ is the so-called "regularizing function" to be determined later so as to remove the Poincaré singularity in the solution.

The intermediate Hamiltonian F_o is chosen as

$$F_o = \frac{1}{2}G^{-2} + G + mf_o(\lambda) - \frac{1}{2}\omega_2(\xi^2 + \eta^2) - \frac{1}{2}m, \tag{4}$$

where ω_2 is the frequency associated with the short period. Clearly, the system splits into two autonomous subsystems of one degree of freedom each. The first subsystem, constituted by the first three terms of F_o, is identified with the previously formulated[4](1976) Ideal Resonance Problem; the second subsystem, constituted by the last two terms of F_o, is the Simple Harmonic Oscillator.

As shown in 5), the equations of the intermediate orbit can be written down immediately as

$$G^o = 1 - \frac{1}{3}\sqrt{6m(\alpha^2 - f_o)} + \frac{4}{9}m(\alpha^2 - f_o) + \ldots,$$
$$z^o = \sqrt{2\Gamma}\ \exp i(\omega_2 t + \phi), \tag{5}$$
$$\int_{\lambda_1}^{\lambda}(\alpha^2 - f_o)^{-1/2}\,d\lambda = \sqrt{6m}\,[t - t_1 + \frac{4}{9}(\lambda - \lambda_1)] + \ldots$$

The time-dependence $t(\lambda)$ is furnished by the hyperelliptic integral in the last equation of (5), where λ_1 is the lower bound of the libration

$$\lambda_1 \leq \lambda \leq \lambda_2.$$

The function $\lambda(t)$ is obtained by inversion. Clearly, $\lambda(t)$ and $G(\lambda)$ are periodic in t, of long period T_1 given by

$$T_1 = 2\pi/\omega_1 = (6m)^{-1/2}\oint(\alpha^2 - f_o(\lambda))^{-1/2}\,d\lambda. \tag{6}$$

In contrast, $z(t)$ is short-periodic, inasmuch as $\omega_2 = 0(1)$.

The disturbing Hamiltonian is given by

$$\delta F = F - F_o = F_1 + F_2 + \dots , \tag{7}$$

and the perturbations of $0(m)$, arising from F_1 are calculated by the method of Lie-series in the version of Hori[8]. The result is of the form

$$\begin{aligned} \delta G &= \frac{2}{3} m f_1 + 0(\sqrt{\Gamma}) + \dots \\ \delta z &= m\ (1 + f + g + 0(\sqrt{\Gamma})) + \dots \\ \delta \lambda &= 0 \end{aligned} \tag{8}$$

Here the complex variable $f(\lambda)$ is defined by

$$f \equiv f_1 + i f_2,$$

while g is the resonant term of the form

$$\begin{aligned} g &= c_\kappa \exp(ik\omega_1 t)/D, \\ D &\equiv \omega_2 - k\omega_1 \end{aligned} \tag{9}$$

Here k is the integer nearest the ratio ω_2/ω_1, and D is the critical divisor.

From F_2 of (7), the regularizing function ψ is calculated[5] as

$$\psi = \frac{1}{6} f_1^2 + \frac{1}{2} f_2^2 \ . \tag{10}$$

Since the short-periodic terms in (5) and (8) carry $\sqrt{\Gamma}$ as a factor, such terms can be removed from the solution by a choice of initial conditions corresponding to Γ=0. The result is a one-parameter family of long-periodic orbits,

$$\begin{aligned} G &= G^\circ(\lambda) + \frac{2}{3} m f_1(\lambda), \\ z &= m[1 + f(\lambda) + g(t)], \\ t &= t(\lambda) \end{aligned} \tag{11}$$

The family-parameter α^2 is related to the Jacobi constant C by[5]

$$C = 2m\alpha^2 + 3.$$

Instead of α^2, it is convenient to use the normalized Jacobi constant α_o^2 defined by

$$\alpha_o^2 = \alpha^2 - m\psi(\lambda_2).$$

Then the range $0 \leq \alpha_o^2 \leq 1$ corresponds to the family of the tadpole-shaped orbits librating about L_4, while $\alpha_o^2 > 1$ includes the "horseshoes" encircling both L_4 and L_5.

The problem illustrates double resonance. For, in addition to the 1:1 resonance between the asteroid and Jupiter, there is also a k:1 resonance between the long and the short periods of the asteroid. Because of the apparently irremovable critical divisor D, the solution is local, rather than global. For its domain is restricted by the inequality of the form $|D| > \varepsilon$, or

$$|m - m_k| > \delta,$$

interpreted as avoidance of the set $\{m_k(\alpha)\}$ of the critical mass ratios corresponding to the exact commensurability of ω_1 and ω_2.

The mean value of the rapidly oscillating resonant term g(t), averaged over the "short" period T_1/k, is

$$\bar{g} = 0.$$

Accordingly, we define the mean orbit by writing

$$z = \bar{z} = m(1 + f)$$

in the second equation of (11). Clearly, the resonant term imparts to the mean orbit an epicyclic character. As m varies, D varies accordingly, and the epicycle develops cusps and loops, in qualitative accord with the results of numerical integration by Deprit and Henrard[3].

The presence of the resonant term g also serves to refute the Brown conjecture (1911) that the family of tadpoles terminates at L_3, thus confirming the earlier finding of Deprit and Henrard[3]. However, the conjecture is valid for the mean orbits and for the (G,λ)-projection.

§2 CONTRIBUTIONS OF PART II

The principal contributions of Part II of "Theory of the Trojan Asteroids", now in press, are itemized below:

1) The solution is carried from $O(m)$ to $O(m^{3/2})$, and the feasibility of a recursive algorithm to generate a solution to any order is shown.

2) The expression (6) for the period T_1 then becomes

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$$T_1(\alpha_o^2,m) = (6m)^{-1/2} \oint (1 + mq)\ (\alpha^2 - f_o)^{-1/2}\, d\lambda,$$
$$q \equiv \frac{1}{36}\,(3s^{-3} - 4s^{-1} + 20\alpha^2 + 78 - 160s^2) \qquad (12)$$

For small oscillations about L_4, we calculate

$$\omega_1(0,m) = \sqrt{\frac{27}{4}\, m}\ (1 + \frac{23}{8}\, m + \ldots),$$

in agreement with the classical theory[9], which provides a further check of the regularizing function $\psi(\lambda)$, entering (12) through f_o of (3).

3) The regularizing function is extended to higher orders in m, and the result is used to prove the periodicity of the solution to any order.

§3 CONTRIBUTIONS OF PART III

Part III of the paper, now in progress, deals with the Hagihara integral:

1) The long period $\tau(\alpha_o^2,m)$, normalized so that $\tau(0,m) = 1$, is a <u>weak</u> function of m. Thus, it can be approximated by the Hagihara integral[7],

$$\tau(\alpha_o^2,0) = \frac{3}{4\pi} \oint\ [z/(1-z^2)(z-z_1)(z_2-z)(z-z_3)]^{1/2} dz$$

which is a relatively simple hyperelliptic integral of <u>class two</u>.

2) The latter is expanded into a convergent series,

$$\tau(\alpha_o^2,0) = \frac{2A}{\pi} \sum_{o}^{\infty} c_i c^i\, C_i,$$

where A, c_i, and c are known functions of α_o^2, and C_i are generated recursively in terms of the standard elliptic integral κ and E.

3) Asymptotic approximations to the Hagihara integral for the cases $\alpha_o \sim 0$ and $\alpha_o \sim 1$ are obtained in the form

$$\tau = 1 + \frac{1}{6}\,\alpha_o^2 + \ldots \qquad (\alpha_o \sim 0)$$
$$= \frac{3}{\pi\sqrt{14}}\, \log[28(3 - \sqrt{2})/(1 - \alpha_o^2) + \ldots \quad (\alpha_o \sim 1)$$

The behavior of τ near $\alpha_o = 0$ had been studied by Deprit and Delie[1], while the logarithmic singularity at $\alpha_o = 1$ has been noted by Hagihara[6]. Incidentally, this singularity in the period serves to confirm the Strömgren Termination Principle when it is applied to the family of "tadpoles".

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4) The small correction ε to the Hagihara integral has been expanded into a Taylor series,

$$\tau(\alpha_o^2, m) = \tau(\alpha_o^2, 0)(1 + m\,\varepsilon_1 + m^2\varepsilon_2 + \ldots),$$

and the functions $\varepsilon_i(\alpha)$ have been calculated.

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DISCUSSION

Message: You call the orbits obtained with $\Gamma=0$ "periodic solutions of the premiere sorte." However, they still contain long-period oscillations, while those solutions of the restricted three-body problem to which Poincaré gave that designation have short-period terms only.

Garfinkel: It is true that Poincaré's examples, as well as Hill's variation orbit, are short-periodic; however, one may define a "premiere sorte" orbit as a periodic orbit that reduces to a circle for m=0. My long-periodic orbit has this property.

Hori: The choice of a Keplerian intermediary, followed by an elimination of short-period terms, leads to simple resonance theories constructed by the previous investigators. It is necessary to include in the intermediary a part of the disturbing function of the same order of magnitude, and this inclusion accounts for the double resonance. Is this true?

Garfinkel: Yes. Indeed, the Ideal Resonance Problem, which I chose as my intermediary, incorporates the dominant term of the external 1:1 resonance.