S. Kawai
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SCALAR CURVATURES OF CONFORMAL METRICS ON S^n

SHIGEO KAWAI

§0.

In this paper we consider the following problem: Given a smooth function K on the n-dimensional unit sphere $S^n (n \geq 3)$ with its canonical metric g_0, is it possible to find a pointwise conformal metric $g = fg_0$ $(f > 0)$ which has K as its scalar curvature? This problem was presented by J. L. Kazdan and F. W. Warner. The associated problem for Gaussian curvature in dimension 2 had been presented by L. Nirenberg several years before.

In both cases, the problems can be reduced to solving nonlinear partial differential equations: For $n = 2$,

$$-\Delta u + 1 = K \exp(2u) \tag{1}$$

where $f = \exp(2u)$, and for $n \geq 3$,

$$-\frac{4(n-1)}{n-2}\Delta u + n(n-1)u = Ku^{\frac{n+2}{n-2}}, \quad u > 0 \tag{2}$$

where $f = u^{\frac{4}{n-2}}$.

It is known that there exist functions K with no solutions. This is shown by the obstruction of J. L. Kazdan and F. Warner which we now recall.

PROPOSITION 1 ([3], [13], [14]). *If u is a solution of the equation* (1) (*resp.* (2)), *then we have*

$$\int_{S^2} \exp(2u) \langle \nabla K, \nabla F \rangle dV_{g_0} = 0$$

$$\left(resp. \int_{S^2} u^{\frac{2n}{n-2}} \langle \nabla K, \nabla F \rangle dV_{g_0} = 0 \right)$$

for all spherical harmonics F of degree 1, *where dV_{g_0} denotes the canonical volume form on S^n.*

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Taking $K=c_1+c_2F_0$ with c_1, c_2 constants and F_0 a spherical harmonic of degree 1, we conclude there is no solution of the equation (1) and (2). Note that the spherical harmonics of degree 1 are the restrictions of linear functions on $\mathbf{R}^{n+1}$ to the unit sphere and their critical sets have simple structures. Thus the nice sufficient conditions may include assumptions on the complexity of the critical set of K. Some existence results are known under symmetry assumptions on the function K or in low dimensions ([2], [5], [6], [8], [10], [11], [12]). Recently Chang and Yang [7] presented a result in general dimensions.

The purpose of this paper is to present a consequence of min-max method, following Chen and Ding [8], applied to the n-dimensional case with $n \geq 3$. We also use the argument of Bahri and Coron [2] on the deformation of functions along the gradient line of a functional. For simplicity, we consider the following equation instead of the equation (2):

$$\mathscr{L}u := -\Delta u + \frac{n(n-2)}{4}u = Ku^{\frac{n+2}{n-2}}, \quad u > 0 \tag{3}$$

and assume that $\max K = 1$, $n \geq 3$ throughout the paper.

THEOREM. *Assume that a function K on S^n with $\max K = 1$ satisfies the following conditions:*

(i) *There exist nondegenerate local maximum points a and b such that*

$$\left(\frac{1}{2}\right)^{\frac{2}{n-2}} < \nu < K(b) \leq K(a)$$

where

$$\nu = \sup_{h\in\Gamma} \min_{x\in h([0,1])} K(x),$$
$$\Gamma = \{h \in C^0([0,1], S^n) \mid h(0) = a, h(1) = b\}.$$

(ii) *There exists an element h_0 in Γ such that*

$$\min_{x\in h_0([0,1])} K(x) = \nu$$

and $\Delta K(x) > 0$ for any x with $K(x) = \nu$.

(iii) *There is no critical value of K in the interval $(\nu, K(b))$.*

Then we have a positive solution of the equation (3).

Though the restrictions on K are rather stringent, we can apply the theorem, for example, to a function K which has a saddle point c with $K(c) > \left(\frac{1}{2}\right)^{\frac{2}{n-2}}$.

To prove this theorem, we use the max-min method. Let us define functionals F, G and J on $W^{1,2}(S^n)$ as follows:

$$F[u] = \fint |\nabla u|^2 + \frac{n(n-2)}{4} \fint u^2,$$

$$G[u] = \fint u_+^{\frac{2n}{n-2}},$$

$$J[u] = \fint K u_+^{\frac{2n}{n-2}}$$

where $\fint f = (\mathrm{vol}\, S^n)^{-1} \left(\int_{S^n} f dV_{g_0} \right)$ and $u_+(x) = \max\{u(x), 0\}$.

Taking families of functions $\phi_{a,\varepsilon}$ and $\phi_{b,\varepsilon}$ which will be specified later, we consider for some $\varepsilon > 0$, the following max-min problem:

$$\mu(\varepsilon) = \sup_{l \in L(\varepsilon)} \min_{u \in l([0,1])} J[u]$$

where

$$H = \{u \in W^{1,2}(S^n) \mid F[u] = 1\},$$
$$L(\varepsilon) = \{l \in C^0([0,1], H) \mid l(0) = \Phi_{a,\varepsilon},\ l(1) = \phi_{b,\varepsilon}\}.$$

Under the assumptions of the theorem, this number $\mu(\varepsilon)$ turns out to be a critical value of $J_{|H}$ for sufficiently small ε, and we get a desired solution of the equation (3).

§1.

In this section we make some preparation for the later parts. First we define several functions. For every point p on S^n, $\sigma(p)$ denotes the streographic projection from $S^n \setminus \{p\}$ to $\mathbf{R}^n$, and $\rho(p)$ its inverse. Denoting the canonical metric of $\mathbf{R}^n$ and S^n by g and g' respectively, we have

$$\rho(p)^* g' = \left(\frac{2}{1 + |x|^2} \right)^2 g.$$

Let us define $\delta(\alpha) : \mathbf{R}^n \to \mathbf{R}^n$ by $\delta(\alpha)(x) = \dfrac{x}{\alpha}$. Then

$$\delta(\alpha)^* (\rho(p)^* g') = \left(\frac{\alpha(1 + |x|^2)}{\alpha^2 + |x|^2} \right)^2 \rho(p)^* g',$$

and consequently we obtain

$$\det \delta(\alpha) = \left(\frac{\alpha(1+|x|^2)}{\alpha^2+|x|^2}\right)^2$$

with respect to the spherical volume form $dV_{\rho(p)^*g'} = \left(\frac{2}{1+|x|^2}\right)^n dV_g$ on $\mathbf{R}^n$. Let us define $\psi_\alpha = (\det \delta(\alpha))^{\frac{n-2}{2n}}$ and $\phi'_{p,\alpha} = \sigma(-p)^*\psi_\alpha$ where $-p$ denotes the antipodal point of p. Since

$$\delta(\alpha)^*(\rho(-p)^*g') = \psi_\alpha^{\frac{4}{n-2}}\rho(-p)^*g',$$

the function $\phi'_{p,\alpha}$ satisfies the equation

$$-\frac{4(n-1)}{n-2}\Delta\phi'_{p,\alpha} + n(n-1)\phi'_{p,\alpha} = n(n-1)\phi'^{\frac{n+2}{n-2}}_{p,\alpha}$$

on S^n. In fact, every positive solution of this equation coincides with $\phi'_{p,\alpha}$ for some p and α. Denoting the volume of S^n by ω_n, we have

$$\begin{aligned}
\omega_n G[\phi'_{p,\alpha}] &= \int_{\mathbf{R}^n} \psi_\alpha^{\frac{2n}{n-2}} dV_{\rho(-p)^*g'} = \int_{\mathbf{R}^n} u'^{\frac{2n}{n-2}}_\alpha dV_g,\\
\omega_n J[\phi'_{p,\alpha}] &= \int_{\mathbf{R}^n} (\rho(-p)^*K)\psi_\alpha^{\frac{2n}{n-2}} dV_{\rho(-p)^*g'}\\
&= \int_{\mathbf{R}^n} (\rho(-p)^*K)u'^{\frac{2n}{n-2}}_\alpha dV_g,\\
\omega_n F[\phi'_{p,\alpha}] &= \frac{n(n-2)}{4}\int_{S^n} \phi'^{\frac{2n}{n-2}}_{p,\alpha} dV_{g'}\\
&= \int_{\mathbf{R}^n} |\nabla u'_\alpha|^2 dV_g
\end{aligned}$$

where $u'_\alpha = \left(\frac{2\alpha}{\alpha^2+|x|^2}\right)^{\frac{n-2}{2}}$

Since $G[\phi'_{p,\alpha}] = 1$ and $F[\phi'_{p,\alpha}] = \frac{n(n-2)}{4}$, we define $\phi_{p,\alpha} = \sqrt{\frac{4}{n(n-2)}}\,\phi'_{p,\alpha}$ and $u_\alpha = \sqrt{\frac{4}{n(n-2)}}\,u'_\alpha$. Then we have $\phi_{p,\alpha} \in H$ and

$$\mu_0 := \left(\frac{4}{n(n-2)}\right)^{\frac{n}{n-2}} = G[\phi_{p,\alpha}] = \max_{u\in H} G[u],$$

$$\omega_n J[\phi_{p,\alpha}] = \int_{\mathbf{R}^n} (\rho(-p)^*K)u_\alpha^{\frac{2n}{n-2}} dV_g.$$

The following property of functions u_α will be used: If we fix a ball $B(R)$ of radius R centered at the origin,

$$\int_{B(R)^c} u_\alpha^{\frac{2n}{n-2}} dV_g = O(\alpha^n), \int_{B(R)} u_\alpha^{\frac{2n}{n-2}} |x|^2 dV_g \sim \alpha^2,$$

$$\int_{B(R)} u_\alpha^{\frac{2n}{n-2}} |x|^3 dV_g = \begin{cases} O(\alpha^3) & (n \geq 4) \\ O\left(\alpha^3 \log\left(\frac{1}{\alpha}\right)\right) & (n = 3) \end{cases}$$

as $\alpha \to 0$.

Next Proposition is a form of maximum principle by Stampacchia [17] (see also Kazdan and Warner [13]).

Proposition 2. *If a function $u \in W^{1,2}(S^n)$ satisfies the equation $\mathcal{L}u = fu_+^{\frac{n+2}{n-2}}$ for a function f, then $u \geq 0$.*

Proof. Let us denote $u_-(x) = \min\{0, u(x)\}$. Then we have $u_- \in W^{1,2}(S^n)$ and

$$⨍ (\mathcal{L}u)u_- = ⨍ fu_+^{\frac{n+2}{n-2}} u_- = 0.$$

Hence we obtain

$$⨍ \left(|\nabla u_-|^2 + \frac{n(n-1)}{4} u_-^2\right) = 0$$

which implies $u_- \equiv 0$, i.e., $u \geq 0$. □

We need the concept of *renormalization* or *rescaling*. Consider for $\alpha > 0$ and $p \in S^n$, a diffeomorphism $\gamma_{\alpha,p} : S^n \to S^n$ defined by $\gamma_{\alpha,p} = \rho(-p) \circ \delta(\alpha) \circ \sigma(-p)$. For $u \in W^{1,2}(S^n)$, $\alpha > 0$ and $p \in S^n$, we define the rescaled function $\tilde{u}(\alpha, p)$ by

$$\tilde{u}(\alpha, p) = (u \circ \gamma_{\alpha,p})(\det \gamma_{\alpha,p})^{\frac{n-2}{2n}} = (u \circ \gamma_{\alpha,p})\phi'_{p,\alpha}.$$

In this notation, $\tilde{u}(\alpha, p)$ concentrates at p as $\alpha \to 0$. The functionals G and F are invariant by rescaling, i.e.,

$$G[\tilde{u}(\alpha, p)] = G[u], \quad F[\tilde{u}(\alpha, p)] = F[u].$$

Let us consider an inner product $\langle , \rangle_1$ on $W^{1,2}(S^n)$ defined by

$$\langle u, v\rangle_1 = ⨍ \langle \nabla u, \nabla v\rangle + \frac{n(n-2)}{4} ⨍ uv.$$

This is apparently equivalent to the usual inner product in $W^{1,2}(S^n)$. The gradient of $J_{|H}$ with respect to $\langle\ ,\ \rangle_1$, which we denote by $\operatorname{grad} J$, is given by

$$(\operatorname{grad} J)[u] = \frac{2n}{n-2}\{-J[u]u + \mathscr{L}^{-1}(Ku_+^{\frac{n+2}{n-2}})\}.$$

PROPOSITION 3. *If* $\{u_i\}$ *is a sequence in* H *with* $(\operatorname{grad} J)[u_i] \to 0$ *and* $\lim J[u_i] > \left(\frac{1}{2}\right)^{\frac{2}{n-2}}\mu_0$, *then the equation* (3) *has a positive solution or a subsequence concentrates at exactly one point in* S^n.

Proof. We follow the proof of Theorem 3.1 in Struwe [18] which proves the case $K \equiv 1$. For a function f, let us define a functional E_f on $W^{1,2}(S^n)$ by

$$E_f[u] = \frac{1}{2}F[u] - \frac{n-2}{2n}\int f u_+^{\frac{2n}{n-2}}.$$

Then $(\operatorname{grad} J)[u_i] \to 0$ $(i \to \infty)$ if and only if $\| E_K'[v_i] \| \to 0$ $(i \to \infty)$ where $v_i = J[u_i]^{\frac{2-n}{4}} u_i$. By the definition of E_K, we have

$$-\Delta v_i + \frac{n(n-2)}{4} v_i - K v_{i_+}^{\frac{n+2}{n-2}} \to 0$$

weakly as $i \to \infty$.

We note that the sequence $\{v_i\}$ is bounded in $W^{1,2}(S^n)$ because $F[u_i] = 1$ and $\lim J[u_i] > \left(\frac{1}{2}\right)^{\frac{2}{n-2}}\mu_0$. Consequently a subsequence converges to $v \in W^{1,2}(S^n)$ weakly. Then from the fact stated above, the function v is a weak solution of the equation $\mathscr{L}v = Kv_+^{\frac{n+2}{n-2}}$, and we have $v \geq 0$ by Proposition 2. This weak solution v is smooth from Theorem 3 in Trudinger [19]. Using maximum principle for smooth solution of elliptic equations, we obtain either $v \equiv 0$ or $v > 0$ everywhere. In the latter case, we get a desired positive solution of the equation $\mathscr{L}v = Kv^{\frac{n+2}{n-2}}$. Hence we may only consider the case that a subsequence of $\{v_i\}$, which is still denoted by $\{v_i\}$, converges to zero weakly.

Next we show $\| E_K'[v_i] \| \to 0$ and $\liminf E_K[v_i] < \frac{1}{n}\left(\frac{n(n-2)}{4}\right)^{\frac{n}{2}}$ implies that $\{v_i\}$ is relatively compact. Though this is true even if the weak limit does not equal to zero, we treat only the case $v_i \to 0$ weakly.

Since

$$o(1) = \langle v_i, E_K'[v_i]\rangle = \fint |\nabla v_i|^2 - \fint K v_{i+}^{\frac{2n}{n-2}} + o(1),$$

we obtain

$$\fint |\nabla v_i|^2 = nE_K[v_i] + o(1) < \left(\frac{n(n-2)}{4}\right)^{\frac{n}{2}} + o(1).$$

From the assumption $K \leq 1$ and Sobolev inequality, we get

$$\begin{aligned} o(1) &= \fint |\nabla v_i|^2 - \fint K v_{i+}^{\frac{2n}{n-2}} + o(1) \\ &\geq \fint |\nabla v_i|^2 - \fint |v_i|^{\frac{2n}{n-2}} + o(1) \\ &\geq \fint |\nabla v_i|^2 - \left(\frac{4}{n(n-2)}\right)^{\frac{n}{n-2}} \left(\fint |\nabla v_i|^2\right)^{\frac{n}{n-2}} + o(1) \\ &\geq \left(\fint |\nabla v_i|^2\right)\left\{1 - \left(\frac{4}{n(n-2)}\right)^{\frac{n}{n-2}} \left(\fint |\nabla v_i|^2\right)^{\frac{n}{n-2}}\right\} + o(1). \end{aligned}$$

Thus it follows that $\| v_i \|_{1,2} \to 0$ as $i \to \infty$.

We can derive, as in the proof of Lemma 3.3 in [18], the following fact: There exist a sequence $\{x_i\}$ of points in S^n with $x_i \to x_0 \in S^n$, a sequence $\{\alpha_i\}$ of positive numbers with $\alpha_i \to 0$ and a nontrivial solution v_0 of the equation

$$\mathscr{L} v_0 = K(x_0) v_{0+}^{\frac{n+2}{n-2}} \tag{4}$$

such that the sequence $\{w_i\}$ of functions defined by $w_i = v_i - \tilde{v}_0(\alpha_i, x_i)$ satisfies

$$E_K[w_i] = E_K[v_i] - E_K[\tilde{v}_0(\alpha_i, x_i)] + o(1) = E_K[v_i] - E_{K(x_0)}[v_0] + o(1)$$

and

$$\| E_K'[w_i] \| \to 0.$$

The argument required to prove this is almost the same as in [18], and we only point out the differences.

First we use the following identity which can be proved by the method of Theorem 2 in [4]:

$$\fint K\cdot(\bar{v}_i)_+^{\frac{2n}{n-2}} = \fint K\cdot(\bar{v}_i - v_0)_+^{\frac{2n}{n-2}} + \fint K v_{0+}^{\frac{2n}{n-2}} + o(1),$$

where $\{\bar{v}_i\}$ is a rescaled sequence of $\{v_i\}$ and v_0 is the weak limit of $\{\bar{v}_i\}$. The corresponding relation for $K \equiv 1$ is used for example in p.173 (3.3) of [18] ($\bar{v}_i$ and v_0 are written there as $\tilde{v}_m$ and v^0 respectively.). Secondly v^0 in [18] is a solution of

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the "limiting problem" (3.1) (p.169), while our "limiting problem" for v_0 becomes the equation (4). This is because rescalings are done from the points x_i, and we take the subsequence of $\{x_i\}$ so that it converges to a point $x_0 \in S^n$.

We show in the following that $w_i \to 0$ in $W^{1,2}(S^n)$ which means that the sequence $\{v_i\}$ approaches $\tilde{v}_0(\alpha_i, x_i)$. From the equation (4), we get $v_0 > 0$ and $K(x_0) > 0$. Let us set $v_0' = \left(\frac{4K(x_0)}{n(n-2)}\right)^{\frac{n-2}{4}} v_0$. Then this positive function v_0' satisfies the equation

$$-\frac{4(n-1)}{n-2}\Delta v_0' + n(n-1)v_0' = n(n-1)v_0'^{\frac{n+2}{n-2}},$$

and consequently $v_0' = \phi'_{p,\alpha}$ for some p and α. Hence we have

$$F[v_0] = \left(\frac{n(n-2)}{4}\right)^{\frac{n}{2}} K(x_0)^{\frac{2-n}{2}}, \quad G[v_0] = \left(\frac{n(n-2)}{4K(x_0)}\right)^{\frac{n}{2}}$$

which implies

$$E_K[w_i] = \frac{1}{n} J[u_i]^{\frac{2-n}{2}} - \frac{1}{n}\left(\frac{n(n-2)}{4}\right)^{\frac{n}{2}} K(x_0)^{\frac{2-n}{2}} + o(1).$$

Suppose that $\{u_i\}$ satisfies the condition

$$J[u_i] > (1 + K(x_0)^{\frac{2-n}{2}})^{-\frac{2}{n-2}}\mu_0$$

where $\mu_0 = \left(\frac{4}{n(n-2)}\right)^{\frac{n}{n-2}}$, then the inequality

$$E_K[w_i] < \frac{1}{n}\left(\frac{n(n-2)}{4}\right)^{\frac{n}{2}}$$

holds and we get $w_i \to 0$ in $W^{1,2}(S^n)$ because $\| E_K'[w_i] \| \to 0$. Since the function $(1 + t^{\frac{2-n}{2}})^{-\frac{2}{n-2}}$ of t on the interval $(0,1]$ takes the maximum value $\left(\frac{1}{2}\right)^{\frac{2}{n-2}}$ at $t = 1$, the proof is completed. □

The following is a variant of Mountain Pass Lemma.

PROPOSITION 4. *Let f be a C^1-function defined on a closed smooth submanifold X of a Hilbert space. Assume that for two points p and q in X,*

$$\mu = \sup_{c \in \Gamma} \min_{t \in [0,1]} f(c(t)) < \min\{f(p), f(q)\}$$

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where

$$\Gamma = \{c \in C^0([0,1], X) \mid c(0) = p,\ c(1) = q\}.$$

Then for every sequence $\{c_n\}$ *in* Γ *with* $\min_{t\in[0,1]} f(c_n(t)) \to \mu\ (n \to \infty)$, *there exist sequences* $\{c_n'\}$ *in* Γ, *which can be taken as close to* $\{c_n\}$ *as possible, and* $\{t_n\}$ *in* $[0,1]$ *such that*

$$f(c_n'(t_n)) = \min_{t\in[0,1]} f(c_n'(t))$$

and

$$\|(\operatorname{grad} f)(c_n'(t_n))\| \to 0\ (n \to \infty).$$

If X itself is a linear space, this proposition is proved in Aubin & Ekeland [1] and Shuzhong [16]. The key ingredient in both proofs is Ekeland's variational principle ([9]) which is valid for functions on complete metric spaces. In our case, we consider the function I defined by $I(c) = \max_{t\in[0,1]} f(c(t))$ on the space Γ, and apply Ekeland's variational principle to this function I. Since the space Γ is a complete metric space, we can easily modify the proof to fit in with our case.

§2.

The purpose of this section is to show that if $J_{|H}$ has no critical point, then there exists some constant $c > 0$ such that

$$\mu_0 \nu < \mu(\varepsilon) < (K(b) - c)\mu_0$$

for sufficiently small ε. For every $u \in W^{1,2}(S^n)$ with $u_+ \not\equiv 0$, we define $P(u) = (P(u)_1, P(u)_2, \cdots, P(u)_{n+1})$ by the equality

$$P(u)_i = \left(\fint u_+^{\frac{2n}{n-2}} x_i\right)\left(\fint u_+^{\frac{2n}{n-2}}\right)^{-1}$$

where $(x_1, x_2, \cdots, x_{n+1})$ is an orthogonal coordinate system of $\mathbf{R}^{n+1}$. When $|P(u)| \neq 0$, we write $Q(u) = P(u)/|P(u)|$ and $d(u) = |Q(u) - P(u)|$. If we are considering a sequence $\{u_i\}$ of functions which concentrates around at most one point, then $\{u_i\}$ actually concentrates if and only if $d(u_i) \to 0\ (i \to \infty)$. The following lemma can be proved by the same way as the proof of Lemma 1.1 in [8]. We present the proof because we need later some of the estimates in it.

Lemma 1. *There exists a constant* C_0 *which depends only on the* C^1*-norm of* K *such that for* $u \in W^{1,2}(S^n)$ *with* $P(u) \neq 0$,

$$\left| \fint \{K - K(Q(u))\} u_+^{\frac{2n}{n-2}} \right| \leq C_0 (d(u))^{\frac{1}{3}} \fint u_+^{\frac{2n}{n-2}}.$$

Proof. Let us define $B(r, Q) = \{x \in S^n \mid \operatorname{dist}(x, Q) < r\}$ where dist denotes the geodesic distance. Then we have

$$\int_{S^n \setminus B(r,Q)} \left\{1 - \sum_i x_i Q_i\right\} u_+^{\frac{2n}{n-2}} dV' \leq d(u) \int_{S^n} u_+^{\frac{2n}{n-2}} dV'$$

and

$$1 - \sum_i x_i Q_i \geq \frac{r^2}{4}$$

for every $x \in S^n \setminus B(r, Q)$. Taking $r = (d(u))^{\frac{1}{3}}$, we get

$$\int_{S^n \setminus B(r,Q)} u_+^{\frac{2n}{n-2}} dV' \leq 4 (d(u))^{\frac{1}{3}} \int_{S^n} u_+^{\frac{2n}{n-2}} dV'.$$

Consequently we obtain

$$\begin{aligned} \left| \int_{S^n} \{K - K(Q(u))\} u_+^{\frac{2n}{n-2}} dV' \right| &\leq \left| \int_{B(r,Q)} \cdots \right| + \left| \int_{S^n \setminus B(r,Q)} \cdots \right| \\ &\leq (\max |\nabla K|) r \int_{S^n} u_+^{\frac{2n}{n-2}} dV' \\ &\quad + 8 (\max |K|) (d(u))^{\frac{1}{3}} \int_{S^n} u_+^{\frac{2n}{n-2}} dV' \\ &\leq C_0 (d(u))^{\frac{1}{3}} \int_{S^n} u_+^{\frac{2n}{n-2}} dV' \end{aligned}$$

which is the desired result. □

LEMMA 2. *Under the assumption on K in the theorem, we have $\mu(\varepsilon) > \mu_0 \nu$ for sufficiently small ε.*

Proof. Using the path h_0 in the assumption (ii) of the theorem. We set $l_\varepsilon(t) = \phi_{h_0(t),\varepsilon}$ for $t \in [0,1]$. To prove $\mu(\varepsilon) > \mu_0 \nu$, we have only to show

(5) $$\fint K (\phi_{h_0(t),\varepsilon})^{\frac{2n}{n-2}} > \mu_0 \nu$$

for $0 \leq t \leq 1$. Let us set $N_\delta = \{x \in h_0([0,1]) \mid \operatorname{dist}(x, A) < \delta\}$ where $A = \{y \in h_0([0,1]) \mid K(y) = \nu\}$. Then we can choose a small δ so that $\Delta K_{|N_\delta} > 0$.

We first show the inequality (5) for t with $h_0(t) \in S^n \setminus N_\delta$. Take a sufficiently small $\beta > 0$ such that

$$K(\zeta) \geq \nu + \beta$$

for every $\zeta \in h_0([0,1]) \setminus N_\delta$, and let ε be sufficiently small so that

$$d(\phi_{\zeta,\varepsilon}) \leq \left(\frac{\beta}{2C_0}\right)^3.$$

Here note that $d(\phi_{\zeta,\varepsilon})$ does not depend on ζ. Then by Lemma 1, we have

$$\begin{aligned}
\fint K(\phi_{h_0(t),\varepsilon})^{\frac{2n}{n-2}} &\geq \{K(h_0(t)) - C_0(d(\phi_{h_0(t),\varepsilon}))^{\frac{1}{3}}\} \fint (\phi_{h_0(t),\varepsilon})^{\frac{2n}{n-2}} \\
&\geq \left(\nu + \beta - \frac{\beta}{2}\right) \fint (\phi_{h_0(t),\varepsilon})^{\frac{2n}{n-2}} \\
&\geq \left(\nu + \frac{\beta}{2}\right)\mu_0 \\
&> \mu_0 \nu.
\end{aligned}$$

Next we consider the case $h_0(t) \in N_\delta$. Due to the continuity of J and the relative compactness of N_δ, it suffices to show that for each $\zeta \in N_\delta$, there is a number $\varepsilon(\zeta)$ such that the inequality (5) holds for ε with $\varepsilon \leq \varepsilon(\zeta)$.

Since $\rho(-\zeta)$ is a conformal diffeomorphism,

$$\Delta(\rho(-\zeta)^* K) = \Delta(K \circ \rho(-\zeta)) = (dK)(\tau(\rho(-\zeta))) + f(\Delta K) \circ (\rho(-\zeta))$$

where f is a positive C^∞ function on $\mathbf{R}^n$ and $\tau(\rho(-\zeta))$ denotes the tension field of the map $\rho(-\zeta)$. By straight forward computation, we see $\tau(\rho(-\zeta))(O) = 0$ which implies

$$\Delta(\rho(-\zeta)^* K)(O) = f(O)(\Delta K)(\zeta).$$

Denoting by $B(R)$ the ball of radius R centered at the origin O and considering Taylor expansion of $\rho(-\zeta)^* K$ at the origin, we get

$$\begin{aligned}
\omega_n J[\phi_{\zeta,\varepsilon}] &= \int_{B(R)} (\rho(-\zeta)^* K) u_\varepsilon^{\frac{2n}{n-2}} dV_g + \int_{B(R)^C} (\rho(-\zeta)^* K) u_\varepsilon^{\frac{2n}{n-2}} dV_g \\
&= \int_{B(R)} \Big\{ (\rho(-\zeta)^* K)(O) + \sum_i \left(\frac{\partial}{\partial x_i} (\rho(-\zeta)^* K)(O)\right) x_i \\
&\quad + \sum_{i,j} \left\{ \frac{\partial^2}{\partial x_i \partial x_j} (\rho(-\zeta)^* K)(O)\right) x_i x_j + O(|x|^3) \Big\} u_\varepsilon^{\frac{2n}{n-2}} dV_g \\
&\quad + \int_{B(R)^C} (\rho(-\zeta)^* K) u_\varepsilon^{\frac{2n}{n-2}} dV_g
\end{aligned}$$

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$$= K(\zeta)\mu_0\omega_n + \frac{f(O)}{n}(\Delta K)(\zeta)\int_{B(R)} u_\varepsilon^{\frac{2n}{n-2}}|x|^2 dV_g + \int_{B(R)} O(|x|^3) u_\varepsilon^{\frac{2n}{n-2}} dV_g + O(\varepsilon^n).$$

In this calculation, we used the property that the functions $u_\varepsilon^{\frac{2n}{n-2}}$ are radial and hence the integrals over $B(R)$ involving $x_i u_\varepsilon^{\frac{2n}{n-2}}$ and $x_i x_j u_\varepsilon^{\frac{2n}{n-2}}$ $(i \neq j)$ must vanish.

Note that

$$\int_{B(R)} u_\varepsilon^{\frac{2n}{n-2}} |x|^2 dV_g \sim \varepsilon^2, \int_{B(R)} u_\varepsilon^{\frac{2n}{n-2}} |x|^3 dV_g = O(\varepsilon^3) \text{ or } = O\left(\varepsilon^3 \log\left(\frac{1}{\varepsilon}\right)\right).$$

Since $K(\zeta) > \nu$, $f(O)(\Delta K)(\zeta) > 0$, we obtain a number $\varepsilon(\zeta)$ such that the inequality (4) holds for every ε with $\varepsilon \leq \varepsilon(\zeta)$. This completes the proof of the lemma. □

PROPOSITION 5. *If the function K satisfies the assumption in the theorem, then either we have a critical point of the functional J, or there exists a positive constant c such that*

$$\mu(\varepsilon) < (K(b) - c)\mu_0$$

for every sufficiently small ε.

Proof. By the definition, the functions $\phi_{a,\varepsilon}$ and $\phi_{b,\varepsilon}$ concentrate around the points a and b respectively for small ε. To investigate the behavior of $\Phi_s\phi_{a,\varepsilon}$ and $\Phi_s\phi_{b,\varepsilon}$ by the flow $\{\Phi_s\}$ $(s > 0)$ on H generated by grad J, we use the argument in [2]. In that paper, the set of functions which concentrate at p points is denoted by $W(p, \varepsilon)$ and the definition of "center" a_i is different from that of Q. However if a function sufficiently concentrates at exactly one point, we may think that our Q is nearly equal to a_1 in [2], and $\varepsilon \to 0$ in our notation corresponds to $\lambda_1 \to \infty$ in [2].

The equations (121) and (122) in [2] imply that if a function sufficiently concentrates around a nondegenerate local maximum point, then it more and more concentrates around this point by the flow $\{\Phi_s\}$ as $s \to \infty$ (Incidentally "$-$" in the right hand side of the equation (122) should be replaced by "$+$".). Though [2] treats mainly 3-dimensonal case, this property is valid in all dimensions if we consider functions which concentrate at exactly one point. Thus if ε is sufficiently small, the functions $\Phi_s\phi_{a,\varepsilon}$ and $\Phi_s\phi_{b,\varepsilon}$ concetrate more and more around points a and b respectively, and points $Q(\Phi_s\phi_{a,\varepsilon})$, $Q(\Phi_s\phi_{b,\varepsilon})$ are well defined for all $s \geq 0$.

By the definition of $\mu(\varepsilon)$, there exists $l \in L(\varepsilon)$ such that $\min J_{|l([0,1])}$ is close to $\mu(\varepsilon)$. Then we have for every $v \in l([0,1])$,

$$J[\Phi_s(v)] \geq \left(\frac{1}{2}\right)^{\frac{2}{n-2}}\mu_0, \quad (\operatorname{grad} J)[\Phi_s(v)] \to 0 \ (s \to +\infty).$$

It follows from Section 1, Proposition 3 that there exists a critical point of the functional J or a subsequence $\{\Phi_{s_k}(v)\}$ concentrates at one point in S^n. If the first case does not occur, we have not only $d(\Phi_{s_k}(v)) \to 0$ but also $d(\Phi_s(v)) \to 0$ as $s \to +\infty$. Consequently $Q(\Phi_s(v))$ is well defined for every sufficiently large s.

From the compactness of $l([0,1])$, we get a sufficiently large constant s_0 such that $Q(\Phi_s(v))$ is well defined for all $v \in l([0,1])$ and all $s \geq s_0$. Thus considering the path made by $\{\Phi_s(\phi_{a,\varepsilon})\}_{0 \leq s \leq 2s_0}$, $\{\Phi_{2s_0}(l([0,1]))\}$ and $\{\Phi_s(\phi_{b,\varepsilon})\}_{0 \leq s \leq 2s_0}$, we obtain a path l' between $\phi_{a,\varepsilon}$ and $\phi_{b,\varepsilon}$ such that $Q(v)$ is well defined and $d(v)$ is sufficiently small for every element v in l'.

Since $\{Q(v) \mid v \in l'([0,1])\}$ is a continuous path between a and b, there exists an element $v_0 \in l'([0,1])$ such that $K(Q(v_0)) < K(b)$ by the assumption (i). Because v_0 sufficiently concentrates at $Q(v_0)$, we have

$$J[v_0] < (K(b) - c')\mu_0$$

for some positive constant c'. Thus the inequality $J_{|l([0,1])} \leq \min J_{|l'([0,1])}$ and the closeness of $\min J_{|l([0,1])}$ to $\mu(\varepsilon)$ complete the proof. □

Let us fix a small ε_0 so that Proposition 5 and Lemma 2 hold for every $\varepsilon \leq \varepsilon_0$, and let us write $\mu = \mu(\varepsilon_0)$ and $L = L(\varepsilon_0)$ for simplicity.

PROPOSITION 6. *There exist positive constants α_0, δ_0 with the following property: If a sequence $\{v_k\}$ in H satisfies*

$$J[v_k] > \mu - \delta_0, \quad P(v_k) \to \zeta \in S^n$$

as $k \to \infty$, then

$$K(\zeta) \geq \nu + \alpha_0.$$

Proof. From Lemma 2, we get positive constants α_0' and δ_0 such that

$$\mu - \delta_0 \geq \mu_0(\nu + \alpha_0').$$

Then from Lemma 1, the following inequalities hold:

$$(\nu + \alpha_0')\mu_0 \leq \mu - \delta_0 < J[v_k] \leq \{K(Q(v_k)) + C_0(d(v_k))^{\frac{1}{3}}\}\mu_0.$$

Since $d(v_k) \to 0$ and $K(Q(v_k)) \to K(\zeta)$ as $k \to 0$, we obtain the desired result. $\square$

§3.

In this section we complete the proof of the theorem. Since we have only to consider the second case in Proposition 5, let us choose ε_1 so small that $\mu < \mu_0(K(b) - \varepsilon_1)$ and take the set $U_d = \{u \in H \mid P(u) \neq 0,\ K(Q(u)) \leq K(b) - \varepsilon_1,\ d(u) \leq d\}$. From now on, $J^{-1}(a, b)$ (resp. $J^{-1}[a, b]$) denotes the subset $\{u \in H \mid a < J[u] < b\}$ (resp. $\{u \in H \mid a \leq J[u] \leq b\}$).

The proof of the following lemma is almost the same as that of Lemma 5.1 in [8] and we omit it.

LEMMA 3. *There exist positive constants δ, d with $K(b)\mu_0 - \delta > \mu + \delta$, and a continuous map $\mathcal{T} : H \to H$ such that*

(1) $J[\mathcal{T}u] \geq J[u]$ *for every* $u \in H$,

(2) $\mathcal{T}(J^{-1}(\mu - \delta, K(b)\mu_0) \cap U_d) \subset J^{-1}(\mu + \delta, K(b)\mu_0)$,

(3) $\mathcal{T}(u) = u$ *for every* $u \in J^{-1}(K(b)\mu_0 - \delta, \mu_0)$,

(4) $\mathcal{T}(H \setminus U_d) \subset H \setminus U_d$.

Proof of the theorem. Choose a sequence $\{l_k\}_{k=1,2,\dots}$ in L such that

$$\min_{u \in l_k[0,1]} J[u] > \mu - \delta, \quad \min_{u \in l_k[0,1]} J[u] \to \mu$$

as $k \to \infty$. By (1) and (3) in Lemma 3, we have $\mathcal{T}(l_k) \in L$ and

$$\min_{\mathcal{T}(l_k(0,1))} J[u] \to \mu.$$

Also we get from (2) and (4) in Lemma 3,

$$(\mathcal{T}(l_k[0,1])) \cap U_d \subset J^{-1}(\mu + \delta, K(b)\mu_0). \tag{6}$$

By virtue of Proposition 4, we obtain $m_k \in L$ and $v_k \in m_k[0,1]$ such that

$$J[v_k] = \min_{u \in m_k[0,1]} J[u], \quad J[v_k] \to \mu, \quad (\operatorname{grad} J)[v_k] \to 0.$$

Moreover we can take m_k as close to $\mathcal{T}(l_k)$ as possible. Hence the relation (6) implies $v_k \in H \setminus U_d$ for sufficiently large k.

Since $(\operatorname{grad} J)[v_k] \to 0$, and $J[v_k] \to \mu$, $\{v_k\}$ concentrates at most one point from Proposition 3 and Lemma 2. Because $v_k \in H \setminus U_d$, we have only two cases:

(a) A subsequence of $\{v_k\}$ converges in $W^{1,2}$-norm.

(b) $P(v_k) \to \zeta \in S^n$ with $K(\zeta) > K(b) - \varepsilon_1$.

In the case (b), $J[v_k] \to \mu_0 K(\zeta) > \mu_0(K(b) - \varepsilon_1)$ which contradicts the fact $J[v_k] \to \mu < \mu_0(K(b) - \varepsilon_1)$. Thus only the case (a) occurs and we get a critical function v in H for functional J. Namely v weakly satisfies the equation

$$-\Delta v + \frac{n(n-2)}{4} v = \lambda K v_+^{\frac{2n}{n-2}}$$

for some constant λ. Hence from Proposition 2, a constant multiple of v gives the desired solution of the equation (3). Thus the proof of the theorem is completed. □

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Department of Mathematics
Faculty of Education
Saga University
Saga 840, Japan