Proceedings of the Edinburgh Mathematical Society (2011) **54**, 599–612
DOI:10.1017/S001309151000060X

ON SIMULTANEOUS RATIONAL APPROXIMATION TO A p-ADIC NUMBER AND ITS INTEGRAL POWERS

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(Received 26 April 2010)

Abstract Let p be a prime number. For a positive integer n and a p-adic number ξ, let $\lambda_n(\xi)$ denote the supremum of the real numbers λ such that there are arbitrarily large positive integers q such that $\|q\xi\|_p, \|q\xi^2\|_p, \dots, \|q\xi^n\|_p$ are all less than $q^{-\lambda-1}$. Here, $\|x\|_p$ denotes the infimum of $|x-n|_p$ as n runs through the integers. We study the set of values taken by the function λ_n.

Keywords: Diophantine approximation; Hausdorff dimension; p-adic number

2010 *Mathematics subject classification:* Primary 11J13; 11J61

1. Introduction

Throughout the paper, p denotes a prime number and $|\cdot|_p$ denotes the usual p-adic absolute value, normalized by $|p|_p = p^{-1}$.

In 1935, in order to define his classification of p-adic numbers, Mahler [**11**] introduced the exponents of Diophantine approximation w_n.

Definition 1.1. Let $n \geqslant 1$ be an integer and let ξ be a p-adic number. We denote by $w_n(\xi)$ the supremum of the real numbers w such that, for arbitrarily large real numbers X, the inequalities

$$0 < |x_n\xi^n + \dots + x_1\xi + x_0|_p \leqslant X^{-w-1}, \qquad \max_{0\leqslant m\leqslant n} |x_m| \leqslant X$$

have a solution in integers $x_0, \dots, x_n$.

The p-adic version of the Dirichlet Theorem implies that $w_n(\xi) \geqslant n$ for every p-adic number ξ which is not algebraic of degree at most n. Furthermore, it follows from the p-adic version of the Schmidt Subspace Theorem that $w_n(\xi) = \min\{n, d-1\}$ for every positive integer n and every p-adic algebraic number ξ of degree d. Moreover, Sprindžuk [**15**] proved that $w_n(\xi) = n$ for every $n \geqslant 1$ and almost every p-adic number ξ, with respect to the Haar measure; see [**5**, § 9.3] for an overview of the known results on the exponents w_n.

Another exponent of Diophantine approximation, which measures the quality of the simultaneous rational approximation to a number and its n first integral powers, has been introduced recently [**7**] in the real case.

Definition 1.2. Let $n \geqslant 1$ be an integer and let ξ be a p-adic number. We denote by $\lambda_n(\xi)$ the supremum of the real numbers λ such that, for arbitrarily large real numbers X, the inequalities

$$0 < |x_0| \leqslant X, \quad \max_{1 \leqslant m \leqslant n} |x_0 \xi^m - x_m|_p \leqslant X^{-\lambda-1}$$

have a solution in integers $x_0, \dots, x_n$.

The p-adic version of the Dirichlet Theorem implies that $\lambda_n(\xi) \geqslant 1/n$ for every irrational p-adic number ξ. Furthermore, it follows from the p-adic form of the Schmidt Subspace Theorem that $\lambda_n(\xi) = \max\{1/n, 1/(d-1)\}$ for every positive integer n and every p-adic algebraic number ξ of degree d. Moreover, $\lambda_n(\xi) = 1/n$ for every $n \geqslant 1$ and almost every p-adic number ξ.

In the present paper, by the spectrum of a function we mean the set of values taken by this function on the set of transcendental p-adic numbers. For $n \geqslant 1$, the spectrum of w_n is equal to the whole interval $[n, \infty]$, but nothing seems to be known regarding the spectrum of λ_n when $n \geqslant 2$. We address the following question.

Problem 1.3. *Let $n \geqslant 1$ be an integer. Is the spectrum of the function λ_n equal to $[1/n, \infty]$?*

The real analogue of Problem 1.3 was recently investigated in [**6**]. The goal of the present paper is twofold. Firstly, we show that, for any $n \geqslant 1$, the spectrum of the function λ_n contains the interval $[1, \infty]$, proving thereby the exact analogue of [**6**, Theorem 3.4]. Secondly, we establish the p-adic analogue of the metrical result from [**4**].

The notation $a \gg_d b$ means that there exists a constant $c > 0$ such that $a \geqslant b$ and c depends only on d. When $\gg$ is written without any subscript, it means that the constant is absolute. We write $a \asymp b$ if both $a \gg b$ and $a \ll b$ hold.

2. Main results

Our first result is a p-adic analogue of [**6**, Corollary 2.3], which slightly improved an old theorem of Güting [**9**]. This seems to be the first result of this type for p-adic numbers.

Theorem 2.1. *Let $n \geqslant 1$ be an integer. For any real number $w \geqslant 2n-1$, there exist uncountably many p-adic integers ξ such that*

$$w_1(\xi) = \cdots = w_n(\xi) = w.$$

The key tool for the proof is a construction inspired by the theory of continued fractions.

Proceeding as in [**6**,**8**], we combine Theorem 2.1 and a transference principle of Mahler [**12**] to get our main result on the spectra of the functions λ_n.

Theorem 2.2. *Let $n \geqslant 1$ be an integer and $\lambda \geqslant 1$ be a real number. There are uncountably many p-adic integers ξ, which can be constructed explicitly, such that $\lambda_n(\xi) = \lambda$. In particular, the spectrum of λ_n contains the interval $[1, \infty]$.*

It is with the help of metric Diophantine approximation that we are able to show that the spectrum of w_n is equal to $[n, \infty]$. Thus, it is meaningful to try to compute the Hausdorff dimension (for background, the reader is directed to [**3**]) of the set of p-adic numbers ξ with a prescribed value for $\lambda_n(\xi)$. For $n = 1$, this was done by Melničuk [**13**], who proved that

$$\dim\{\xi \in \mathbb{Q}_p \colon \lambda_1(\xi) \geqslant \lambda\} = \frac{2}{1+\lambda}.$$

Actually, there is a slightly more precise result [**3**], namely

$$\dim\{\xi \in \mathbb{Q}_p \colon \lambda_1(\xi) = \lambda\} = \frac{2}{1+\lambda}.$$

In this respect, we are able to establish the p-adic analogue of [**4**, Theorem 2].

Theorem 2.3. *Let $n \geqslant 2$ be an integer. Let $\lambda > n - 1$ be a real number. Then*

$$\dim\{\xi \in \mathbb{Q}_p \colon \lambda_n(\xi) = \lambda\} = \frac{2}{n(1+\lambda)}.$$

For $n = 2$ and $\frac{1}{2} \leqslant \lambda \leqslant 1$, it is expected that

$$\dim\{\xi \in \mathbb{Q}_p \colon \lambda_2(\xi) = \lambda\} = \frac{2-\lambda}{1+\lambda},$$

in analogy with the real case [**2**, **16**]. We plan to investigate this problem in a subsequent work.

3. *p*-adic continued fractions

This section was inspired by [**10**].

Set

$$p_{-1} = 1, \qquad q_{-1} = 0, \qquad p_0 = 1, \qquad q_0 = 1.$$

Let $\boldsymbol{v} = (v_n)_{n \geqslant 1}$ be a sequence of positive integers and set

$$p_n = p^{v_n} p_{n-2} + p_{n-1}, \quad q_n = p^{v_n} q_{n-2} + q_{n-1}, \quad n \geqslant 1.$$

A rapid calculation shows that

$$q_1 = 1, \qquad q_2 = p^{v_2} + 1, \qquad q_3 = p^{v_3} + p^{v_2} + 1, \qquad q_4 = p^{v_2+v_4} + p^{v_4} + p^{v_3} + p^{v_2} + 1,$$

and

$$\frac{p_n}{q_n} = \cfrac{p^{v_1}}{1 + \cfrac{p^{v_2}}{1 + \cfrac{p^{v_3}}{\cdots + p^{v_n}}}}.$$

The reader may note the differences between these continued fractions and the classical continued fraction algorithm for real numbers. In the latter case, the convergents p_n/q_n are given by the recurrences $p_n = a_n p_{n-1} + p_{n-2}$ and $q_n = a_n q_{n-1} + q_{n-2}$, where the partial quotients a_n are positive integers.

Observe that

$$\left|\frac{p_1}{q_1} - \frac{p_0}{q_0}\right|_p = p^{-v_1},$$

and that, for $n \geqslant 2$, we have

$$\begin{aligned}\left|\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}}\right|_p &= \left|\frac{(p^{v_n} p_{n-2} + p_{n-1})q_{n-1} - (p^{v_n} q_{n-2} + q_{n-1})p_{n-1}}{q_n q_{n-1}}\right|_p \\ &= p^{-v_n}\left|\frac{p_{n-1}}{q_{n-1}} - \frac{p_{n-2}}{q_{n-2}}\right|_p,\end{aligned}$$

since p does not divide $q_n q_{n-1} q_{n-2}$.

Consequently, for $n \geqslant 0$ and $k \geqslant 1$, we have

$$\left|\frac{p_{n+k}}{q_{n+k}} - \frac{p_n}{q_n}\right|_p = \left|\frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n}\right|_p = p^{-v_{n+1}-v_n-\cdots-v_1}, \tag{3.1}$$

since $v_1, v_2, \ldots$ are positive. Here, we have used that

$$|a+b|_p = \max\{|a|_p, |b|_p\}$$

holds for all p-adic numbers a and b such that $|a|_p \neq |b|_p$. This fact will be used repeatedly in the course of the proof of Theorem 2.1.

Equalities (3.1) show that the sequence $(p_n/q_n)_{n\geqslant 1}$ converges p-adically. Let $\xi_{\boldsymbol{v}}$ denote its limit. It follows from (3.1) that

$$\left|\xi_{\boldsymbol{v}} - \frac{p_n}{q_n}\right|_p \leqslant p^{-v_{n+1}-v_n-\cdots-v_1}. \tag{3.2}$$

If

$$\left|\xi_{\boldsymbol{v}} - \frac{p_n}{q_n}\right|_p < p^{-v_{n+1}-v_n-\cdots-v_1},$$

then, by (3.1), we get

$$\left|\xi_{\boldsymbol{v}} - \frac{p_{n+1}}{q_{n+1}}\right|_p = \max\left\{\left|\xi_{\boldsymbol{v}} - \frac{p_n}{q_n}\right|_p, \left|\frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n}\right|_p\right\} = p^{-v_{n+1}-v_n-\cdots-v_1},$$

which contradicts (3.2) since $v_{n+2} \geqslant 1$. Consequently, we have proved that

$$\left|\xi_{\boldsymbol{v}} - \frac{p_n}{q_n}\right|_p = p^{-v_{n+1}-v_n-\cdots-v_1}, \quad n \geqslant 1. \tag{3.3}$$

4. Proof of Theorem 2.1

Let $w > 1$ be a real number. Set $v_1 = \lceil w \rceil$ and $v_2 = \lceil w^2 \rceil$, where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x. For $n \geqslant 3$, let v_n be the integer such that

$$v_n + v_{n-2} + \cdots + v_{\varepsilon(n)} = \lceil w^n + w^{n-2} + \cdots + w^{\varepsilon(n)} \rceil,$$

where $\varepsilon(n) = 2$ if n is even and $\varepsilon(n) = 1$ otherwise. Let $\xi = \xi_{\boldsymbol{v}}$ be the p-adic number constructed by the algorithm described in § 3 applied with $\boldsymbol{v} = (v_n)_{n \geqslant 1}$.

To shorten the notation, for $n \geqslant 1$, we set

$$u_n = v_n + v_{n-2} + \cdots + v_{\varepsilon(n)}.$$

Note that

$$u_n \geqslant u_{n-1}, \quad n \geqslant 2. \tag{4.1}$$

Observe that

$$\begin{aligned} u_n &\leqslant w^n + w^{n-2} + \cdots + w^{\varepsilon(n)} + 1 \\ &\leqslant w(w^{n-1} + w^{n-3} + \cdots + w^{\varepsilon(n-1)}) + w + 1 \\ &\leqslant w u_{n-1} + w + 1 \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} u_n &\geqslant w^n + w^{n-2} + \cdots + w^{\varepsilon(n)} \\ &\geqslant w(w^{n-1} + w^{n-3} + \cdots + w^{\varepsilon(n-1)}) \\ &\geqslant w(u_{n-1} - 1) = w u_{n-1} - w. \end{aligned} \tag{4.3}$$

We begin with an easy lemma.

Lemma 4.1. *Using the above notation, we have*

$$q_j \geqslant p^{u_j}, \quad j \geqslant 2,$$

and there exists C_1, depending only on p and w, such that

$$q_j \leqslant C_1 p^{u_j}, \quad j \geqslant 1.$$

Proof. The first statement of the lemma is straightforward, since $q_j \geqslant p^{v_j} q_{j-2}$ for $j \geqslant 2$. For the second, we first check inductively that

$$q_j \leqslant 2^j p^{u_j}, \quad j \geqslant 1. \tag{4.4}$$

Indeed, $q_1 = 1$, $q_2 = p^{u_2} + 1$ and, assuming that (4.4) holds for $j = n-1$ and $j = n-2$ for an integer $n \geqslant 3$, we have

$$q_n \leqslant 2^{n-2} p^{u_n} + 2^{n-1} p^{u_{n-1}} \leqslant 2^n p^{u_n},$$

by (4.1), showing that (4.4) holds for $j = n$. We conclude that (4.4) holds for $j \geqslant 1$.

Let n_0 be such that
$$p^{w^n - w^{n-1}} \geqslant 2^{2n} p, \quad n \geqslant n_0,$$
and set $C_1 = 2^{n_0} + 1$. Since $u_n \geqslant w^n$, we have
$$p^{u_n(1-1/w)} \geqslant 2^n C_1 p, \quad n \geqslant n_0 + 1. \tag{4.5}$$
Furthermore, by (4.4), we have
$$q_n \leqslant (C_1 - 1) p^{u_n}, \quad 1 \leqslant n \leqslant n_0. \tag{4.6}$$

We prove by induction on n that
$$q_n \leqslant (C_1 - 1/n) p^{u_n}, \quad n \geqslant 1. \tag{4.7}$$
By (4.6), inequality (4.7) holds for $n \leqslant n_0$. Let $n \geqslant n_0 + 1$ be an integer such that (4.7) holds for $n - 1$ and for $n - 2$. Observe that, by (4.3) and (4.5),
$$2^n C_1 p^{u_{n-1}} \leqslant 2^n C_1 p p^{u_n/w} \leqslant p^{u_n};$$
thus,
$$\begin{aligned} q_n = p^{v_n} q_{n-2} + q_{n-1} &\leqslant (C_1 - 1/(n-2)) p^{u_n} + C_1 p^{u_{n-1}} \\ &\leqslant (C_1 - 1/(n-2) + 2^{-n}) p^{u_n} \\ &\leqslant (C_1 - 1/n) p^{u_n}. \end{aligned}$$
This proves the lemma. □

Lemma 4.2. *With the above notation there are positive real numbers C_2 and C_3, depending only on p and w, such that*
$$C_2 q_j^w \leqslant q_{j+1} \leqslant C_3 q_j^w, \quad j \geqslant 1.$$

Proof. Let j be a positive integer. By Lemma 4.1 and (4.2), we have
$$q_{j+1} \leqslant C_1 p^{u_{j+1}} \leqslant C_1 p^{w u_j + w + 1} \leqslant (C_1 p^{w+1}) q_j^w,$$
while, by Lemma 4.1 and (4.3),
$$q_{j+1} \geqslant p^{u_{j+1}} \geqslant p^{w u_j - w} \geqslant (pC)^{-w} q_j^w.$$
This proves the lemma. □

We end these preliminaries with a lemma, which follows from an immediate induction.

Lemma 4.3. *For $j \geqslant 0$, we have*
$$p_j \leqslant (p^{v_1} + 1) q_j.$$

For $j \geqslant 2$, it follows from (3.3) that

$$\left|\xi - \frac{p_j}{q_j}\right|_p = p^{-v_{j+1}-v_j-\cdots-v_1} = p^{-u_{j+1}-u_j};$$

thus, by Lemma 4.1, we get

$$q_j^{-1}q_{j+1}^{-1} \leqslant \left|\xi - \frac{p_j}{q_j}\right|_p \leqslant C_1^2 q_j^{-1}q_{j+1}^{-1},$$

and, by Lemma 4.2,

$$\frac{C_3^{-1}}{q_j^{w+1}} \leqslant \left|\xi - \frac{p_j}{q_j}\right|_p \leqslant \frac{C_1^2 C_2^{-1}}{q_j^{w+1}}. \tag{4.8}$$

Consequently, we get

$$w \leqslant w_1(\xi) \leqslant \cdots \leqslant w_d(\xi) \tag{4.9}$$

for every positive integer d (note that the unknown x_n occurring in the definition of w_n can be equal to 0).

Let d be a positive integer with $d < w$. Let $P(X)$ be an integer polynomial of degree at most d and of large height $H(P)$ (recall that the height of an integer polynomial is the maximum of the absolute values of its coefficients). Assume first that $P(X)$ does not vanish at any element of the sequence $(p_j/q_j)_{j\geqslant 1}$. Let j be defined by $q_j \leqslant H(P) < q_{j+1}$. Observe that, by Lemma 4.3, the numerator of the rational number $P(p_j/q_j)$ is at most equal to $(d+1)(p^{v_1}+1)^d H(P) q_j^d$; thus,

$$|P(p_j/q_j)|_p \geqslant (d+1)^{-1}(p^{v_1}+1)^{-d} H(P)^{-1} q_j^{-d}.$$

To shorten the formulae, set

$$C_4 = (d+1)^{-1}(p^{v_1}+1)^{-d}.$$

Since ξ and p_j/q_j are p-adic integers, the Mean Value Theorem (see, for example, [**14**, § 5.3]) gives

$$|P(p_j/q_j) - P(\xi)|_p \leqslant |\xi - p_j/q_j|_p \leqslant p^{-u_{j+1}-u_j}$$

by (3.3). Consequently, since

$$|P(p_j/q_j)|_p \geqslant C_4 H(P)^{-1} q_j^{-d},$$

we get

$$|P(\xi)|_p = |P(p_j/q_j)|_p \geqslant C_4 H(P)^{-1-d}$$

as soon as $p^{-u_{j+1}-u_j} < C_4 H(P)^{-1} q_j^{-d}$, that is, whenever

$$H(P) < C_4 q_j^{-d} p^{u_{j+1}+u_j}. \tag{4.10}$$

Similarly, we observe that

$$|P(p_{j+1}/q_{j+1})|_p \geqslant C_4 H(P)^{-1} q_{j+1}^{-d}$$

and

$$|P(p_{j+1}/q_{j+1}) - P(\xi)|_p \leqslant p^{-u_{j+2}-u_{j+1}} \leqslant C_1^2 C_2^{-1} q_{j+1}^{-1-w}.$$

Since $w > d$ and $H(P) < q_{j+1}$, this implies that, if j (that is, if $H(P)$) is large enough, we have $|P(\xi)|_p \geqslant C_4 H(P)^{-1} q_{j+1}^{-d}$. In other words, for any positive real number $C_5 < C_4$, we have $|P(\xi)|_p \geqslant C_5 H(P)^{-1-w}$ if $H(P)^{-w} \leqslant C_5^{-1} C_4 q_{j+1}^{-d}$, that is, if

$$H(P) \geqslant C_4^{-1/w} C_5^{1/w} q_{j+1}^{d/w}. \tag{4.11}$$

By Lemma 4.1, inequality (4.10) holds if

$$H(P) < C_4 q_j^{-d} C_1^{-2} q_j q_{j+1} = C_1^{-2} C_4 q_{j+1} q_j^{1-d}. \tag{4.12}$$

Using Lemma 4.2, we see that (4.11) certainly holds for

$$H(P) \geqslant C_4^{-1/w} C_5^{1/w} q_{j+1} (C_3 q_j^w)^{-1+d/w}. \tag{4.13}$$

Selecting C_5 such that

$$C_4^{-1/w} C_5^{1/w} C_3^{-1+d/w} < C_4 C_1^{-2},$$

we get that, if $1-d \geqslant -w+d$, then for every polynomial $P(X)$ whose height is in the interval $[q_j, q_{j+1})$ at least one of the inequalities (4.12) and (4.13) is satisfied. This means that the whole range of values $q_j \leqslant H(P) < q_{j+1}$ is covered as soon as

$$w \geqslant 2d - 1. \tag{4.14}$$

To summarize, we have proved that if j is sufficiently large, then, for $w \geqslant 2d-1$ and for any polynomial $P(X)$ of degree at most d that does not vanish at p_j/q_j and whose height satisfies $q_j \leqslant H(P) < q_{j+1}$, we have

$$|P(\xi)|_p \geqslant C_5 H(P)^{-w-1}.$$

In particular, if the polynomial $P(X)$ of degree at most d does not vanish at any element of the sequence $(p_j/q_j)_{j\geqslant 1}$ and has sufficiently large height, then it satisfies

$$|P(\xi)|_p \geqslant C_5 H(P)^{-w-1}. \tag{4.15}$$

Assume now that there are positive integers $a_1, \ldots, a_h$, distinct positive integers $n_1, \ldots, n_h$ and an integer polynomial $R(X)$ such that the polynomial $P(X)$ of degree at most d can be written as

$$P(X) = (q_{n_1} X - p_{n_1})^{a_1} \cdots (q_{n_h} X - p_{n_h})^{a_h} R(X),$$

where $R(X)$ does not vanish at any element of the sequence $(p_j/q_j)_{j\geqslant 1}$. It follows from (4.8), (4.15), Lemma 4.3 and the so-called Gelfond inequality (see, for example, [**5**, Lemma A.3])

$$H(P) \asymp_{d,w} q_{n_1}^{a_1} \cdots q_{n_h}^{a_h} H(R)$$

that

$$\begin{aligned}|P(\xi)|_p &\gg_{d,w} q_{n_1}^{-a_1(w+1)}\cdots q_{n_h}^{-a_h(w+1)}|R(\xi)|_p\\ &\gg_{d,w} q_{n_1}^{-a_1(w+1)}\cdots q_{n_h}^{-a_h(w+1)}H(R)^{-w-1}\\ &\gg_{d,w} (q_{n_1}^{a_1}\cdots q_{n_h}^{a_h}H(R))^{-w-1}\\ &\gg_{d,w} H(P)^{-w-1}.\end{aligned}$$

We conclude that, if (4.14) is satisfied, then

$$|P(\xi)|_p \gg_{d,w} H(P)^{-w-1}$$

holds for every polynomial $P(X)$ of degree at most d and sufficiently large height; hence, $w_d(\xi) \leqslant w$. Combined with (4.9), this completes the proof of Theorem 2.1, since our construction is flexible enough to yield uncountably many p-adic integers with the required property.

5. Proof of Theorem 2.2

Let ξ be an irrational p-adic number. Clearly, we have

$$\lambda_1(\xi) = w_1(\xi) \geqslant 1$$

and

$$\lambda_1(\xi) \geqslant \lambda_2(\xi) \geqslant \cdots .$$

Our first lemma establishes a relation between the exponents λ_n and λ_m when m divides n.

Lemma 5.1. *For any positive integers k and n and any transcendental p-adic number ξ we have*

$$\lambda_{kn}(\xi) \geqslant \frac{\lambda_k(\xi) - n + 1}{n}.$$

Proof. Let v be a positive real number and let q be a positive integer such that

$$\max_{1\leqslant j\leqslant k} |q\xi^j - p_j|_p \leqslant q^{-v-1}$$

for suitable integers $p_1, \ldots, p_k$. Let h be an integer with $1 \leqslant h \leqslant kn$. Write $h = j_1 + \cdots + j_m$ with $m \leqslant n$ and $1 \leqslant j_1, \ldots, j_m \leqslant k$. Then there are p-adic numbers $\varepsilon_1, \ldots, \varepsilon_m$ such that

$$|\varepsilon_i|_p \leqslant q^{-v-1}, \quad q\xi^{j_i} = p_{j_i} + \varepsilon_i, \quad i = 1, \ldots, m.$$

Consequently, we have

$$q^m\xi^h = \prod_{i=1}^m q\xi^{j_i} = \prod_{i=1}^m (p_{j_i} + \varepsilon_i) = \varepsilon' + \prod_{i=1}^m p_{j_i}$$

for a p-adic number ε' satisfying $|\varepsilon'|_p \leqslant q^{-v-1}$. This shows that

$$|q^m \xi^h - p_{j_1} \cdots p_{j_m}|_p \leqslant q^{-v-1}$$

and

$$|q^n \xi^h - p_{j_1} \cdots p_{j_m} q^{n-m}|_p \leqslant q^{-v-1} = (q^n)^{-1-(v-n+1)/n},$$

independently of h. This proves the lemma. □

We display an immediate consequence of Lemma 5.1.

Corollary 5.2. *Let ξ be a p-adic irrational number. Then $\lambda_n(\xi) = \infty$ holds for every positive n if, and only if, $\lambda_1(\xi) = \infty$.*

We recall two relations between the exponents w_n and λ_n deduced from the p-adic analogue of Khintchine's transference principle due to Mahler [**12**].

Proposition 5.3. *For any positive integer n and any p-adic number ξ which is not algebraic of degree at most n we have*

$$\frac{w_n(\xi)}{(n-1)w_n(\xi) + n} \leqslant \lambda_n(\xi) \leqslant \frac{w_n(\xi) - n + 1}{n}.$$

Proof. See [**12**]. Note that the value of $w_n(\xi)$ does not change if, in Definition 1.1, we only consider tuples $(x_0, x_1, \ldots, x_n)$ such that there exists at least one index i for which p does not divide x_i. Similarly, the value of $\lambda_n(\xi)$ does not change if, in Definition 1.2, we only consider tuples $(x_0, x_1, \ldots, x_n)$ such that p does not divide x_0. □

We are now able to complete the proof of Theorem 2.2.

Proof of Theorem 2.2. Let $n \geqslant 2$ be an integer and let ξ be a transcendental p-adic number. Lemma 5.1 with $k = 1$ implies the lower bound

$$\lambda_n(\xi) \geqslant \frac{w_1(\xi) - n + 1}{n}.$$

On the other hand, Proposition 5.3 gives the upper bound

$$\lambda_n(\xi) \leqslant \frac{w_n(\xi) - n + 1}{n}.$$

Now, Theorem 2.1 asserts that for any given real number $w \geqslant 2n-1$ there exist uncountably many p-adic integers ξ_w such that

$$w_1(\xi_w) = \cdots = w_n(\xi_w) = w.$$

Then,

$$\lambda_k(\xi_w) = \frac{w}{k} - 1 + \frac{1}{k}, \quad k = 1, \ldots, n.$$

In particular,

$$\lambda_n(\xi_w) = \frac{w}{n} - 1 + \frac{1}{n},$$

and this gives the required result. □

6. Proof of Theorem 2.3

As $\mathbb{Q}_p$ can be covered by a countable collection of balls of radius 1, we will only prove the theorem for one such ball, namely $\mathbb{Z}_p$. The arguments are the same for any other ball but some of the constants will change. The proof follows that of [**4**]. Fix an integer $n \geqslant 2$. Define the curve $\Gamma \subset \mathbb{Z}_p^n$ as $\Gamma = \{(\xi, \xi^2, \dots, \xi^n) \colon \xi \in \mathbb{Z}_p\}$. We will use the notation $|a, b, c|$ to denote the maximum of $|a|$, $|b|$ and $|c|$. If $\boldsymbol{a}$ is a vector, then $|\boldsymbol{a}|$ is the maximum of the vector entries. The set of points $(\xi, \xi^2, \dots, \xi^n) \in \Gamma$ which satisfy the inequalities $|q\xi - r|_p \leqslant |q, r, \boldsymbol{t}|^{-\tau}$ and $|q\xi^i - t_i|_p \leqslant |q, r, \boldsymbol{t}|^{-\tau}$ for infinitely many $q, r \in \mathbb{Z}$ and $\boldsymbol{t} \in \mathbb{Z}^{n-1}$ will be denoted by $W_\tau(\Gamma)$. The set $W_\tau(\Gamma)$ is closely related to the set of exact order in the statement of Theorem 2.3 and in order to prove the theorem we will first obtain the Hausdorff dimension and measure of $W_\tau(\Gamma)$ for sufficiently large τ. The proof relies on the following lemma, which shows that if $(\xi, \xi^2, \dots, \xi^n) \in W_\tau(\Gamma)$, then the rational points $(r/q, \boldsymbol{t}/q)$ also lie on Γ for τ sufficiently large.

Lemma 6.1. *Let $(\xi, \xi^2, \dots, \xi^n) \in W_\tau(\Gamma)$ be such that there exist infinitely many $D, r \in \mathbb{Z}$, $\boldsymbol{t} \in \mathbb{Z}^{n-1}$ such that $|D\xi - r|_p < |D, r, \boldsymbol{t}|^{-\tau}$ and $|D\xi^i - t_i|_p < |D, r, \boldsymbol{t}|^{-\tau}$. If $\tau > n$, then $(r/D, \boldsymbol{t}/D) \in \Gamma$.*

Proof. Let $(\xi, \xi^2, \dots, \xi^n) \in W_\tau(\Gamma)$. Hence, there exist integers r, t_i and D such that $|D\xi - r|_p < |D, r, \boldsymbol{t}|^{-\tau}$ and $|D\xi^i - t_i|_p < |D, r, \boldsymbol{t}|^{-\tau}$. Therefore, $|\xi - r/D|_p < |D, r, \boldsymbol{t}|^{-\tau}|D|_p^{-1}$ and $|\xi^i - \boldsymbol{t}/D|_p < |D, r, \boldsymbol{t}|^{-\tau}|D|_p^{-1}$ and there exist $\varepsilon_1, \dots, \varepsilon_n$, such that $\xi - r/D = \varepsilon_1$ and $\xi^i - t_i/D = \varepsilon_i$ for $i = 2, \dots, n$ with $|\varepsilon_i|_p < |D, r, \boldsymbol{t}|^{-\tau}|D|_p^{-1}$. Then

$$\xi^i = \frac{t_i}{D} + \varepsilon_i = \left(\frac{r}{D} + \varepsilon_1\right)^i = \left(\frac{r}{D}\right)^i + R(\varepsilon_1),$$

where $R(X)$ is a rational polynomial divisible by X. Hence, $t_i/D - (r/D)^i = R(\varepsilon_1) - \varepsilon_i$ so that

$$D^{i-1}t_i - r^i = D^i(R(\varepsilon_1) - \varepsilon_i).$$

Clearly, $D^{i-1}R(X) \in \mathbb{Z}[X]$, so that $|D^iR(\varepsilon_1)|_p \leqslant |D|_p|\varepsilon_1|_p < |D, r, \boldsymbol{t}|^{-\tau}$. Thus,

$$|D^{i-1}t_i - r^i|_p \leqslant |D, r, \boldsymbol{t}|^{-\tau}.$$

Since $D^{i-1}t_i - r^i$ is an integer, its p-adic absolute value is either 0 or at least equal to $|D^{i-1}t_i - r^i|^{-1}$. Combined with our assumption that τ exceeds n, the above inequality shows that $D^{i-1}t_i = r^i$ for $i = 2, \dots, n$. This implies that $(r/D, \boldsymbol{t}/D)$ lies on Γ, as asserted. $\square$

Define the point P_{rq} as

$$P_{rq} = \left(\frac{r}{q}, \dots, \frac{r^n}{q^n}\right) = \left(\frac{rq^{n-1}}{q^n}, \dots, \frac{r^n}{q^n}\right).$$

If the highest common factor of r and q is 1, then the lowest common denominator of the coordinates of P_{rq} is q^n. On the other hand, if $(r, q) = h > 1$, then we can write $r = r_1h$

and $q = q_1 h$ so that

$$P_{rq} = \left(\frac{r_1 q_1^{n-1}}{q_1^n}, \dots, \frac{r_1^n}{q_1^n}\right) = P_{r_1 q_1}.$$

We may therefore assume without loss of generality that $(r, q) = 1$. If

$$\Xi = (\xi, \xi^2, \dots, \xi^n) \in W_\tau(\Gamma)$$

and $\tau > n$, then Lemma 6.1 asserts that Ξ must be approximated by infinitely many points P_{rq} with $(r, q) = 1$ and must satisfy the inequalities $|q^n\xi - rq^{n-1}|_p < |q^n, r^n|^{-\tau}$, $|q^n\xi^2 - r^2 q^{n-2}|_p < |q^n, r^n|^{-\tau}$, $\dots$, $|q^n\xi^n - r^n|_p < |q^n, r^n|^{-\tau}$.

The proof of the theorem now follows that in [**4**]. First, we move from the set $W_\tau(\Gamma)$ to the set

$$V_\tau(\Gamma) = \{\xi \in \mathbb{Z}_p \colon (\xi, \xi^2, \dots, \xi^n) \in W_\tau(\Gamma)\}.$$

It is not difficult to show that for all ξ_1, ξ_2 in $\mathbb{Z}_p$ we have

$$|\xi_1 - \xi_2|_p = \max_{i=1,\dots,n} |\xi_1^i - \xi_2^i|_p.$$

Thus, there is a bi-Lipschitz transformation between any ball $B(\xi, r) \subset \mathbb{Z}_p$ and the image of that ball on Γ. To determine the Hausdorff dimension of $W_\tau(\Gamma)$ it is therefore sufficient to find the Hausdorff dimension of $V_\tau(\Gamma)$. It can be readily verified that the following inclusions hold for $V_\tau(\Gamma)$:

$$\bigcap_{N=1}^{\infty} \bigcup_{k>N} \bigcup_{|q,r|=k} B\left(\frac{r}{q}, |r^n, q^n|^{-\tau}\right) \subset V_\tau(\Gamma) \subset \bigcap_{N=1}^{\infty} \bigcup_{k>N} \bigcup_{|q,r|=k} B\left(\frac{r}{q}, |r^n, q^n|^{-\tau} |q^n|_p^{-1}\right). \tag{6.1}$$

To prove the exact order result it is necessary to obtain dimension and measure results for $W_\tau(\Gamma)$. The fact that $\dim W_\tau(\Gamma) = \dim V_\tau(\Gamma) \geqslant 2/n\tau$ and the fact that the Hausdorff $2/n\tau$-measure is infinite follows directly from [**1**, Theorem 16] by using the left-hand side of (6.1) and setting $\psi(r) = r^{-n\tau}$ and $f(r) = r^s$. It is therefore only necessary to prove the upper bound for the Hausdorff dimension.

Lemma 6.2. *For any $n \geqslant 2$ and $\tau > n$ we have*

$$\dim V_\tau(\Gamma) \leqslant \frac{2}{n\tau}.$$

Proof. The proof follows that of [**4**, Lemma 2]. Using the right-hand side of (6.1) gives a cover of $V_\tau(\Gamma)$, so that

$$\begin{aligned}
&\mathcal{H}^s(V_\tau(\Gamma)) \\
&\quad \ll \sum_{k>N} \sum_{r,q\colon \max(r,q)=k} |r^n, q^n|^{-\tau s} |q^n|_p^{-s} \\
&\quad \ll \sum_{k>N} \left(\sum_{r,q\colon \max(r,q)=q=k} |r^n, q^n|^{-\tau s} |q^n|_p^{-s} + \sum_{r,q\colon \max(r,q)=r=k} |r^n, q^n|^{-\tau s} |q^n|_p^{-s} \right) \\
&\quad \ll \sum_{k>N} \left(k k^{-n\tau s} |k|_p^{-ns} + k^{-\tau n s} \sum_{q=1}^{k} |q|_p^{-ns} \right).
\end{aligned}$$

Consider the second sum first and let α be such that $p^\alpha \leqslant k < p^{\alpha+1}$. Then, as $|k|_p = 1$ if p does not divide k, we have

$$\begin{aligned}
\sum_{k>N} k^{-\tau ns} \sum_{q=1}^{k} |q|_p^{-ns} &= \sum_{k>N} k^{-\tau ns}\biggl(\sum_{q\leqslant k, p\nmid q} 1 + \sum_{q\leqslant k:\ p|q \text{ and } p^2\nmid q} p^{ns} + \cdots + \sum_{q\leqslant k:\ p^\alpha|q} p^{\alpha ns}\biggr)\\
&\ll \sum_{k>N} k^{-\tau ns}\biggl(k + \frac{k}{p}p^{ns} + \frac{k}{p^2}p^{2ns} + \cdots + \frac{k}{p^\alpha}p^{\alpha ns}\biggr)\\
&\ll \sum_{k>N} k^{1-\tau ns}\biggl(\sum_{i=0}^{\alpha} p^{i(ns-1)}\biggr)\\
&\ll \sum_{k>N} k^{ns-\tau ns}\\
&< \infty
\end{aligned}$$

for $s > 1/(n\tau - n)$. Clearly, for $\tau > n \geqslant 2$, $2/n\tau > 1/(n\tau - n)$, so for $s > 2/n\tau$ the series converges. Now, using the same arguments, consider the first sum, to obtain

$$\begin{aligned}
&\sum_{k>N} kk^{-n\tau s}|k|_p^{-ns}\\
&\qquad\ll \sum_{k>N:\ p\nmid k} k^{1-n\tau s} + \sum_{r>N:\ p\nmid r} (pr)^{1-n\tau s}p^{ns} + \sum_{r>N:\ p\nmid r} (p^2 r)^{1-n\tau s}p^{2ns} + \cdots\\
&\qquad\ll \sum_{k>N} k^{1-n\tau s} \sum_{i=0}^{\infty} p^{i(1+ns-n\tau s)}
\end{aligned}$$

The last geometric series again converges if $s > 1/(n\tau - n)$. Thus, for $s > 2/n\tau$ both sums converge, which is sufficient to prove $\dim W_\tau(\Gamma) = \dim V_\tau(\Gamma) \leqslant 2/n\tau$ for $\tau > n$. □

It is now possible to obtain the dimension of the set

$$E_\lambda := \{\xi \in \mathbb{Z}_p : \lambda_n(\xi) = \lambda\}$$

when λ exceeds $n-1$. Clearly, $E_\lambda \subset W_{\lambda+1}(\Gamma)$, so that

$$\dim E_\lambda \leqslant \frac{2}{n(1+\lambda)},$$

by Lemma 6.2. Note that

$$E_\lambda = \lim_{m\to\infty} W_{\lambda+1}(\Gamma) \setminus W_{\lambda+1+1/m}(\Gamma).$$

Also, $\mathcal{H}^{2/n(1+\lambda)}(W_{\lambda+1}(\Gamma)) = \infty$ [**1**, Theorem 16] and $\mathcal{H}^{2/n(1+\lambda)}(W_{\lambda+1+1/m}(\Gamma)) = 0$ from the definition of the Hausdorff dimension. Thus,

$$\mathcal{H}^{2/n(1+\lambda)}(W_{\lambda+1}(\Gamma) \setminus W_{\lambda+1+1/m}(\Gamma)) = \infty,$$

which implies that

$$\dim E_\lambda \geqslant \frac{2}{n(1+\lambda)}.$$

This proves Theorem 2.3.

Acknowledgements. This project was undertaken with funding from the ULYSSES project. N.B. is funded under the Science Foundation Ireland Grant RFP08/MTH1512. Thanks are due to the referee for very careful reading.

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