SOME REGULAR $[F, d_n]$ MATRICES WITH COMPLEX ELEMENTS

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1. Introduction. Let $A = (a_{nk})$ and $\{s_n\}$ $(n, k = 0, 1, 2, \ldots)$ be a matrix and a sequence of complex numbers, respectively. Let the members of the sequence $\{\sigma_n\}$ be defined by

$$\sigma_n = \sum_{k=0}^{\infty} a_{nk}\, s_k,$$

then we say $\{\sigma_n\}$ is the A-transform of $\{s_n\}$. The matrix $A = (a_{nk})$ is called regular if

$$\lim_{n\to\infty} \sigma_n = \lim_{n\to\infty} s_n$$

whenever the second limit exists. Necessary and sufficient conditions for a matrix $A = (a_{nk})$ to be regular are the well-known Silverman-Toeplitz conditions:

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} = 1, \tag{1.1}$$

$$\sum_{k=0}^{\infty} |a_{nk}| \leqslant K \qquad (n = 0, 1, 2, \ldots), \tag{1.2}$$

$$\lim_{n\to\infty} a_{nk} = 0 \qquad (k = 0, 1, 2, \ldots), \tag{1.3}$$

where K is a constant independent of n.

In this paper we shall be concerned with the $[F, d_n]$-matrix that was first introduced by Jakimovski **(4)**. Given a sequence $\{d_n\}$ $(n = 1, 2, 3, \ldots)$, we shall throughout this paper define the $[F, d_n]$-matrix corresponding to this sequence to be the matrix with elements P_{nk} $(n, k = 0, 1, 2, \ldots)$ defined by

$$P_{00} = 1, \qquad P_{0k} = 0 \qquad (k \neq 0),$$

$$\prod_{j=1}^{n} \left(\frac{\theta + d_j}{1 + d_j}\right) = \sum_{k=0}^{\infty} P_{nk}\, \theta^k. \tag{1.4}$$

Note that $P_{nk} = 0$ for $k > n$. Jakimovski **(4)** proves that the $[F, d_n]$-matrix is regular if $d_n \geqslant 0$ for n sufficiently large and if

$$\sum_{n=0}^{\infty} d_n^{-1} = +\infty.$$

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He then asks the question: Is the condition $d_n \geqslant 0$ necessary? This question was answered in the negative by Meir **(6)** using a sequence of real numbers. In **(3)** it is shown that d_n need not be real in order for the $[F, d_n]$-matrix to be regular, but it was required that the sequence $\{\arg d_n\}$ approach zero with a certain rapidity in order to give regularity. The next logical problem, then, is to try and construct regular matrices for which the sequence $\{\arg d_n\}$ has no limit-points that are integral multiples of π. In Section 2 of this paper we shall give three regular $[F, d_n]$-matrices that satisfy this condition.

It has long been a problem in summability to find matrices which sum the geometric series to its analytic continuation for all z except z real and $z \geqslant 1$. The Taylor matrix found in **(2)** sums the geometric series to its analytic continuation for z in a small finite portion of the half-plane $\mathrm{Re}(z) \geqslant 1$. However, in Section 3 we show that each of the $[F, d_n]$-matrices given in Section 2 sums the geometric series to its analytic continuation for all z in an infinite portion of the half-plane $\mathrm{Re}(z) \geqslant 1$, and as much in one case as two-thirds of this half-plane.

We shall use the following notation throughout this paper:

$$d_n = \rho_n e^{i\theta_n}, \tag{1.5}$$

$$\psi(k, n, \lambda_i) = \sum_{s_1+s_2+\ldots+s_n+k=n} \lambda_1^{s_1} \lambda_2^{s_2} \ldots \lambda_n^{s_n}, \tag{1.6}$$

where s_i is either 0 or 1 and the sum is taken over all possible products of the λ's such that $s_1 + s_2 + \ldots + s_n = n - k$. The above summation symbolism will be used throughout the paper and will always denote a sum of the same type as the sum in (1.6). Also when the subscripts of P are complicated, we shall use $P_{n,k}$ to mean P_{nk}.

2. Three regular $[F, d_n]$-matrices.

THEOREM 2.1. *Suppose that* $\{\lambda_n\}$ $(n = 1, 2, \ldots)$ *is a sequence of positive numbers such that*

$$\sum_{n=1}^{\infty} \lambda_n^{-1} = +\infty.$$

Let the terms of the sequence $\{d_n\}$ *be defined by*

$$d_{2n-1} = i\sqrt{\lambda_n} \quad \textit{and} \quad d_{2n} = -i\sqrt{\lambda_n} \qquad (n = 1, 2, \ldots). \tag{2.1}$$

Then the $[F, d_n]$*-matrix corresponding to this sequence* $\{d_n\}$ *is regular.*

Proof. Setting $\theta = 1$ in (1.4), we get

$$\sum_{k=0}^{\infty} P_{nk} = \sum_{k=0}^{n} P_{nk} = 1 \tag{2.2}$$

for any sequence $\{d_n\}$. Hence any $[F, d_n]$-matrix satisfies (1.1).

The proof that (1.2) is satisfied is divided into two cases.

Case 1. Suppose that $n = 2m$ $(m = 1, 2, \ldots)$. Then using the definition of d_n from (2.1) in the Cauchy integral formula for the coefficients P_{nk} in (1.4), we have

$$P_{2m,k} = \frac{1}{2\pi i} \int_c \prod_{j=1}^{m} \left(\frac{t^2 + \lambda_j}{1 + \lambda_j}\right) \frac{dt}{t^{k+1}}, \tag{2.3a}$$

where c is any circle about the origin. Expanding the numerator of the product under the integral sign as a sum, we obtain

$$P_{2m,k} = \frac{1}{2\pi i \prod_{j=1}^{m} (1 + \lambda_j)} \int_c \sum_{s_1+s_2+\ldots+s_m+\beta=m} \lambda_1^{s_1} \lambda_2^{s_2} \ldots \lambda_m^{s_m} t^{2\beta-k-1}\, dt. \tag{2.3b}$$

Upon integrating in (2.3b) we see that $P_{nk} = 0$ if k is odd. If k is even, let $k = 2\alpha$ $(\alpha = 0, 1, \ldots, m)$. Then integrating in (2.3b), we get

$$P_{2m,2\alpha} = \psi(\alpha, m, \lambda_i) \Big/ \prod_{j=1}^{m} (1 + \lambda_j). \tag{2.4}$$

Since the λ_n's are by hypothesis positive, it follows that $\psi(\alpha, m, \lambda_i) > 0$ from (1.6). Thus, from (2.4), we have that $P_{2m,2\alpha} = |P_{2m,2\alpha}|$. Using this fact and (2.2), we can easily show that

$$\sum_{k=0}^{\infty} |P_{2m,k}| = 1 \qquad (m = 0, 1, \ldots). \tag{2.5}$$

Case 2. Suppose that $n = 2m + 1$ $(m = 1, 2, \ldots)$. Using the definition of d_n from (2.1) in the Cauchy integral formula for the coefficients P_{nk} in (1.4), we get

$$P_{2m+1,k} = \frac{1}{2\pi i} \int_c \prod_{j=1}^{m} \left(\frac{t^2 + \lambda_j}{1 + \lambda_j}\right)\left(\frac{t + d_{2m+1}}{1 + d_{2m+1}}\right) \frac{dt}{t^{k+1}}. \tag{2.6a}$$

Expanding the numerator product as a sum, we have

$$P_{2m+1,k} = \frac{1}{2\pi i} \frac{1}{\prod_{j=1}^{n} (1 + d_j)} \times \int_c \sum_{s_1+s_2+\ldots+s_m+\beta=m} \lambda_1^{s_1} \lambda_2^{s_2} \ldots \lambda_m^{s_m} t^{2\beta}(t + d_{2m+1}) \frac{dt}{t^{k+1}}. \tag{2.6b}$$

Suppose $k = 2\alpha + 1$, where $\alpha = 0, 1, \ldots, m$; then, integrating in (2.6b), we obtain

$$P_{2m+1,2\alpha+1} = \psi(\alpha, m, \lambda_i) \Big/ \prod_{j=1}^{n} (1 + d_j). \tag{2.7}$$

Now let $k = 2\alpha$, where $\alpha = 0, 1, \ldots, m$; then, integrating (2.6b), we get

$$P_{2m+1,2\alpha} = d_{2m+1}\, \psi(\alpha, m, \lambda_i) \Big/ \prod_{j=1}^{n} (1 + d_j). \tag{2.8}$$

Since all the λ_n's are real and since (2.1) implies that

$$\prod_{j=1}^{2m+1} (1 + d_j) = (1 + d_{2m+1}) \prod_{j=1}^{m} (1 + \lambda_j),$$

it follows from (2.7) that

$$P_{2m+1,2\alpha+1} = \frac{|1 + d_{2m+1}|}{(1 + d_{2m+1})} |P_{2m+1,2\alpha+1}|,$$

and from (2.8) that

$$P_{2m+1,2\alpha} = \frac{d_{2m+1}\,|1 + d_{2m+1}|}{|d_{2m+1}|\,(1 + d_{2m+1})} |P_{2m+1,2\alpha}|.$$

Summing both sides of these equations over α from 0 to m and adding the resulting sums together, we obtain

$$\sum_{k=0}^{n} P_{nk} = \frac{|1 + d_n|}{(1 + d_n)} \sum_{\alpha=0}^{m} |P_{2m+1,2\alpha+1}| + \frac{d_n\,|1 + d_n|}{|d_n|\,(1 + d_n)} \sum_{\alpha=0}^{m} |P_{2m+1,2\alpha}|,$$

where $n = 2m + 1$. Using (2.2) and the definitions of d_n from (2.1), this equation reduces to

$$\sum_{\alpha=0}^{m} |P_{2m+1,2\alpha+1}| + i \sum_{\alpha=0}^{m} |P_{2m+1,2\alpha}| = e^{i\phi_m},$$

where $\phi_m = \arg(1 + d_{2m+1})$. This equation implies that

$$\sum_{\alpha=0}^{m} |P_{2m+1,2\alpha+1}| \leqslant 1 \qquad \text{and} \sum_{\alpha=0}^{m} |P_{2m+1,2\alpha}| \leqslant 1.$$

After adding these two inequalities together we see that

$$\sum_{k=0}^{n} |P_{nk}| \leqslant 2 \quad (n = 1, 3, 5, \ldots) \qquad \text{since } |P_{10}| + |P_{11}| < 2.$$

From this inequality and (2.5) we conclude that

$$\sum_{k=0}^{\infty} |P_{nk}| \leqslant 2$$

independent of n. Hence (1.2) is satisfied.

Since $1 + w \leqslant e^w$ for w real, we have

$$\left|\frac{t^2 + \lambda_j}{1 + \lambda_j}\right|^2 \leqslant \exp\left\{-1 + \left|\frac{t^2 + \lambda_j}{1 + \lambda_j}\right|^2\right\}. \tag{2.9}$$

Now if $t = x + iy$ and $|t| = 2^{-1}$, we get

$$-1 + \left|\frac{t^2 + \lambda_j}{1 + \lambda_j}\right|^2 = \frac{(x^2 + y^2)^2 - 1 + 2\lambda_j(x^2 - y^2 - 1)}{(1 + \lambda_j)^2}.$$

Since $(x^2 + y^2)^2 < \frac{1}{2}$ and $x^2 - y^2 - 1 \leqslant -\frac{3}{4}$ when $|t| = 2^{-1}$, this equation implies that

$$-1 + \left|\frac{t^2 + \lambda_j}{1 + \lambda_j}\right|^2 < -\frac{3\lambda_j + 1}{2(1 + \lambda_j)^2} \qquad (j = 1, 2, \ldots).$$

Using this inequality in (2.9) and taking positive square roots of both sides, we get

$$\left|\frac{t^2+\lambda_j}{1+\lambda_j}\right| < \exp\left\{\frac{-3\lambda_j-1}{4(1+\lambda_j)^2}\right\}. \tag{2.10}$$

Inserting absolute values on both sides of (2.3a) and making use of the inequality (2.10), it follows that

$$|P_{2m,k}| < 2^k \exp\left\{-\sum_{j=1}^{m}\frac{3\lambda_j+1}{4(1+\lambda_j)^2}\right\}. \tag{2.11}$$

Using the definition of d_{2m+1} given by (2.1), we have

$$-1+\left|\frac{t+d_{2m+1}}{1+d_{2m+1}}\right|^2 = \frac{x^2+y^2-1+2y(\lambda_{m+1})^{\frac{1}{2}}}{1+\lambda_{m+1}} < \frac{(\lambda_{m+1})^{\frac{1}{2}}}{1+\lambda_{m+1}} \leqslant \frac{1}{2},$$

if $t = x+iy$ and $|t| = 2^{-1}$. Since $1+w \leqslant e^w$ for w real it follows, after taking square roots, that

$$\left|\frac{t+d_{2m+1}}{1+d_{2m+1}}\right| < e^{1/4} < 2.$$

Inserting absolute values on both sides of (2.6a), and making use of the above inequality and the inequality (2.10), it follows that

$$|P_{2m+1,k}| < 2^{k+1} \exp\left\{-\sum_{j=1}^{m}\frac{3\lambda_j+1}{4(1+\lambda_j)^2}\right\}. \tag{2.12}$$

But by hypothesis

$$\sum_{n=1}^{\infty}\lambda_n^{-1} = +\infty.$$

This implies that if

$$\sum_{n=1}^{\infty}\lambda_n(1+\lambda_n)^{-2} < +\infty,$$

then zero is a limit-point of the set of numbers λ_n. Hence

$$\sum_{n=1}^{\infty}\frac{-3\lambda_n-1}{4(1+\lambda_n)^2} = -\infty.$$

Using this fact, (2.11) implies that

$$\lim_{m\to\infty}|P_{2m,k}| = 0,$$

and (2.12) implies that

$$\lim_{m\to\infty}|P_{2m+1,k}| = 0,$$

independent of k. Therefore

$$\lim_{n\to\infty}P_{nk} = 0,$$

and (1.3) is satisfied.

There are several ways that one might try to get a simple extension of this theorem. One is to assume $\arg \lambda_n = \gamma$, but $\gamma \neq 0$. However, if $|\lambda_n| \geqslant \eta > 0$, (**3**, Theorem 2.4) implies that this does not yield a regular $[F, d_n]$-matrix.

THEOREM 2.2. *Suppose that* $\{\lambda_n\}$ $(n = 1, 2, \ldots)$ *is a sequence of positive numbers such that*

$$\sum_{n=1}^{\infty} \lambda_n^{-1} = +\infty.$$

Define the terms of the sequence $\{d_n\}$ *by*

$$(2.13)\qquad \begin{aligned} d_{3n-2} &= 2^{-1}\lambda_n^{\frac{1}{3}}(-1 + i\sqrt{3}),\\ d_{3n-1} &= \lambda_n^{\frac{1}{3}}, \end{aligned}$$

and

$$d_{3n} = 2^{-1}\lambda_n^{\frac{1}{3}}(-1 - i\sqrt{3}) \qquad (n = 1, 2, \ldots).$$

Then the $[F, d_n]$*-matrix formed using this sequence is regular.*

Proof. From (2.2), (1.1) is satisfied.

We divide the proof that (1.2) is satisfied into three cases.

Case 1. Suppose that $n = 3m$ $(m = 1, 2, \ldots)$. Then, using the Cauchy integral formula in (1.4) to evaluate P_{nk} and the definition of d_n from (2.13), we have

$$(2.14)\qquad P_{3m,k} = \frac{1}{2\pi i}\int_c \prod_{j=1}^{m} \left(\frac{t^3 + \lambda_j}{1 + \lambda_j}\right) \frac{dt}{t^{k+1}},$$

where c is any circle with centre at the origin. Suppose $k = 3\alpha$ $(\alpha = 0, 1, \ldots, m)$. Then expanding the numerator of the product in (2.14) in a sum of the same type as the sum in (2.3b) and integrating, we get

$$P_{3m,3\alpha} = \psi(\alpha, m, \lambda_i) \Big/ \prod_{j=1}^{m} (1 + \lambda_j).$$

Since $\lambda_n > 0$ and since from (1.6) $\psi(\alpha, m, \lambda_i) > 0$, it follows from this equation that $P_{3m,3\alpha} = |P_{3m,3\alpha}|$. If k is not of the form $k = 3\alpha$ $(\alpha = 0, 1, \ldots, m)$, then integrating in (2.14), we get $P_{3m,k} = 0$. Therefore, $P_{3m,k} = |P_{3m,k}|$ $(k = 0, 1, \ldots)$. It now follows, using (2.2), that

$$(2.15)\qquad \sum_{k=0}^{\infty} |P_{nk}| = 1 \qquad (n = 0, 3, 6, \ldots).$$

Case 2. Suppose $n = 3m + 1$ $(m = 1, 2, \ldots)$. Then, using the Cauchy integral formula to evaluate P_{nk} in (1.4) and the definition of d_n from (2.13), we have

$$P_{3m+1,k} = \frac{1}{2\pi i}\int_c \prod_{j=1}^{m} \left(\frac{t^3 + \lambda_j}{1 + \lambda_j}\right)\left(\frac{t + d_{3m+1}}{1 + d_{3m+1}}\right) \frac{dt}{t^{k+1}},$$

where c is any circle about the origin. Expanding the numerator of the product inside the integral, we obtain

$$(2.16)\qquad P_{3m+1,k} = \int_c \frac{\sum_{s_1+s_2+\ldots+s_m+\beta=m} \lambda_1^{s_1}\lambda_2^{s_2}\ldots\lambda_m^{s_m} t^{3\beta}(t+d_{3m+1})\,dt}{2\pi i(1+d_{3m+1})\prod_{j=1}^{m}(1+\lambda_j)t^{k+1}}.$$

Suppose k takes on successively the values $k = 3\alpha$, $k = 3\alpha + 1$, and $k = 3\alpha+2$, where $\alpha = 0, 1, \ldots, m$. Then, integrating in (2.16) for each successive value of k, we have

$$P_{3m+1,3\alpha} = \frac{d_{3m+1}}{(1+d_{3m+1})}\frac{\psi(\alpha, m, \lambda_i)}{\prod_{j=1}^{m}(1+\lambda_j)},$$

$$P_{3m+1,3\alpha+1} = \frac{\psi(\alpha, m, \lambda_i)}{(1+d_{3m+1})\prod_{j=1}^{m}(1+\lambda_j)},$$

and

$$P_{3m+1,3\alpha+2} = 0.$$

Since $\lambda_n > 0$ and $\psi(\alpha, m, \lambda_i) > 0$, it easily follows from these three equations that

$$(2.17\text{a})\qquad P_{3m+1,3\alpha} = \frac{d_{3m+1}|1+d_{3m+1}|}{|d_{3m+1}|(1+d_{3m+1})}|P_{3m+1,3\alpha}|,$$

$$(2.17\text{b})\qquad P_{3m+1,3\alpha+1} = \frac{|1+d_{3m+1}|}{(1+d_{3m+1})}|P_{3m+1,3\alpha+1}|,$$

$$(2.17\text{c})\qquad P_{3m+1,3\alpha+2} = 0.$$

Summing both sides of (2.17a), (2.17b), and (2.17c) over α from 0 to m and adding the resulting sums together, we get

$$(2.18)\qquad \sum_{k=0}^{n} P_{nk} = \frac{|1+d_{3m+1}|}{(1+d_{3m+1})}\sum_{\alpha=0}^{m}\left\{\frac{d_{3m+1}}{|d_{3m+1}|}|P_{3m+1,3\alpha}| + |P_{3m+1,3\alpha+1}|\right\}.$$

Using the value of d_{3m+1} from (2.13), we have

$$d_{3m+1}\,|d_{3m+1}|^{-1} = 2^{-1}(-1+i\sqrt{3}).$$

Let $1 + d_{3m+1} = r_m e^{i\phi_m}$; then, substituting the value of $d_{3m+1}|d_{3m+1}|^{-1}$ into (2.18) and using (2.2), we obtain

$$e^{i\phi_m} = \sum_{\alpha=0}^{m}\{2^{-1}(-1+i\sqrt{3})|P_{3m+1,3\alpha}| + |P_{3m+1,3\alpha+1}|\}.$$

Equating real and imaginary parts in this equation, we get

$$\sum_{\alpha=0}^{m}|P_{3m+1,3\alpha}| = \frac{2}{\sqrt{3}}\sin\phi_m$$

and

$$-\sum_{\alpha=0}^{m}|P_{3m+1,3\alpha}| + 2\sum_{\alpha=0}^{m}|P_{3m+1,3\alpha+1}| = 2\cos\phi_m.$$

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Multiplying the first equation through by 3 and then adding the two equations together, we have

$$\sum_{k=0}^{n} |P_{3m+1,k}| = 3^{1/2} \sin \phi_m + \cos \phi_m \leqslant 2 \qquad (m = 1, 2, \ldots).$$

Since $|P_{10}| + |P_{11}| \leqslant 2$, it follows that

$$(2.19) \qquad \sum_{k=0}^{\infty} |P_{3m+1,k}| \leqslant 2 \qquad (m = 0, 1, 2, \ldots).$$

Case 3. Suppose $n = 3m + 2$ $(m = 1, 2, \ldots)$. Let

$$g_m = d_{3m+1} + d_{3m+2} = 2^{-1}(\lambda_{m+1})^{1/3}(1 + i\sqrt{3})$$

and

$$h_m = d_{3m+1}d_{3m+2} = 2^{-1}(\lambda_{m+1})^{2/3}(-1 + i\sqrt{3}).$$

Using the Cauchy integral formula for P_{nk} in (1.4) and then expanding the numerator product under the integral sign, we get

$$(2.20) \qquad P_{3m+2,k} = \int_c \frac{\sum_{s_1+s_2+\ldots+s_m+\beta=m} \lambda_1^{s_1} \lambda_2^{s_2} \ldots \lambda_m^{s_m} t^{3\beta}(t^2 + g_m t + h_m)\, dt}{2\pi i(1 + d_{3m+1})(1 + d_{3m+2}) \prod_{j=1}^{m} (1 + \lambda_j)t^{k+1}}.$$

Let k take on successively the values $k = 3\alpha$, $k = 3\alpha + 1$, and $k = 3\alpha + 2$, where $\alpha = 0, 1, \ldots, m$. Integrating in succession in (2.20) using these values of k, we get

$$P_{3m+2,3\alpha} = \frac{h_m \psi(\alpha, m, \lambda_i)}{(1 + d_{3m+1})(1 + d_{3m+2}) \prod_{j=1}^{m} (1 + \lambda_j)},$$

$$P_{3m+2,3\alpha+1} = \frac{g_m \psi(\alpha, m, \lambda_i)}{(1 + d_{3m+1})(1 + d_{3m+2}) \prod_{j=1}^{m} (1 + \lambda_j)},$$

$$P_{3m+2,3\alpha+2} = \frac{\psi(\alpha, m, \lambda_i)}{(1 + d_{3m+1})(1 + d_{3m+2}) \prod_{j=1}^{m} (1 + \lambda_j)}.$$

Since $\lambda_n > 0$ and $\psi(\alpha, m, \lambda_i) > 0$, these equations imply, respectively, that

$$(2.21) \qquad P_{3m+2,3\alpha} = e^{-i\phi_m} h_m |h_m|^{-1} |P_{3m+2,3\alpha}|,$$

$$(2.22) \qquad P_{3m+2,3\alpha+1} = e^{-i\phi_m} g_m |g_m|^{-1} |P_{3m+2,3\alpha+1}|,$$

$$(2.23) \qquad P_{3m+2,3\alpha+2} = e^{-i\phi_m} |P_{3m+2,3\alpha+2}|,$$

where $(1 + d_{3m+1})(1 + d_{3m+2}) = r_m e^{i\phi_m}$. Summing both sides of (2.21), (2.22), and (2.23) from $\alpha = 0$ to $\alpha = m$, adding the resulting equations together, and making use of (2.2), we get

$$\sum_{\alpha=0}^{m} \left\{ \frac{h_m}{|h_m|} |P_{3m+2,3\alpha}| + \frac{g_m}{|g_m|} |P_{3m+2,3\alpha+1}| + |P_{3m+2,3\alpha+2}| \right\} = e^{i\phi_m}.$$

Substituting the values of g_m and h_m into this equation and equating real and imaginary parts, we get

$$\text{(2.24a)} \qquad \sum_{\alpha=0}^{m} |P_{3m+2,3\alpha}| + \sum_{\alpha=0}^{m} |P_{3m+2,3\alpha+1}| = \frac{2}{\sqrt{3}} \sin \phi_m,$$

$$\text{(2.24b)} \quad -\sum_{\alpha=0}^{m} |P_{3m+2,3\alpha}| + \sum_{\alpha=0}^{m} |P_{3m+2,3\alpha+1}| + 2\sum_{\alpha=0}^{m} |P_{3m+2,3\alpha+2}| = 2\cos\phi_m.$$

Equation (2.24a) implies that

$$\sum_{\alpha=0}^{m} |P_{3m+2,3\alpha}| \leqslant 2(3)^{-1/2} \qquad \text{and} \sum_{\alpha=0}^{m} |P_{3m+2,3\alpha+1}| \leqslant 2(3)^{-1/2}.$$

Adding (2.24a) and (2.24b), we obtain an equation which implies that

$$\sum_{\alpha=0}^{m} |P_{3m+2,3\alpha+2}| \leqslant 2(3)^{-1/2}.$$

Adding these last three inequalities together, we get

$$\text{(2.25)} \qquad \sum_{k=0}^{\infty} |P_{3m+2,k}| \leqslant 6(3)^{-1/2} < 4 \qquad (m = 1, 2, \ldots).$$

Since $|P_{20}| + |P_{21}| + |P_{22}| < 4$, the inequalities (2.15), (2.19), and (2.25) imply that

$$\sum_{k=0}^{\infty} |P_{nk}| < 4 \qquad (n = 0, 1, 2, \ldots).$$

Hence (1.2) is satisfied.

If $t = x + iy$ and $|t| = 2^{-1}$, then

$$-1 + \left|\frac{t^3 + \lambda_j}{1 + \lambda_j}\right|^2 = \frac{-(63/64) + 2\lambda_j(x^3 - 3xy^2 - 1)}{1 + 2\lambda_j + \lambda_j^2}.$$

If $y^2 = 4^{-1} - x^2$, then $x^3 - 3xy^2 - 1 \leqslant -(7/8)$. Using this inequality in the above equation, we get

$$-1 + \left|\frac{t^3 + \lambda_j}{1 + \lambda_j}\right|^2 < -\frac{7\lambda_j + 3}{4(1 + \lambda_j)^2}.$$

Using the fact that $1 + w \leqslant e^w$ for real w, we obtain, after taking positive square roots of both sides,

$$\text{(2.26)} \qquad \left|\frac{t^3 + \lambda_j}{1 + \lambda_j}\right| < \exp\left\{\frac{-7\lambda_j - 3}{8(1 + \lambda_j)^2}\right\}.$$

If $w \geqslant 0$, $\cos\theta_n \geqslant -(1/2)$, and $x^2 + y^2 = 4^{-1}$, then

$$(x - 1)\cos\theta_n + y\sin\theta_n < 1$$

and $w(1 + 2w\cos\theta_n + w^2)^{-1} \leqslant 1$. Hence, if $|t| = 2^{-1}$, then

$$-1 + \left|\frac{t + d_n}{1 + d_n}\right|^2 < \frac{2\rho_n}{1 + 2\rho_n\cos\theta_n + \rho_n^2} \leqslant 2.$$

Since $1 + w \leqslant e^w$ for real w, we obtain, after taking square roots of both sides,

$$\left|\frac{t + d_n}{1 + d_n}\right| < e \qquad (n = 1, 2, \ldots). \tag{2.27}$$

Let m denote the greatest integer in $n/3$. Using the values of d_n from (2.13) and the inequality (2.27) it then follows that

$$\prod_{j=1}^{n} \left|\frac{t + d_j}{1 + d_j}\right| < 2^3 \prod_{j=1}^{m} \left|\frac{t^3 + \lambda_j}{1 + \lambda_j}\right| .$$

If we use (2.26) in this inequality, it follows that

$$\prod_{j=1}^{n} \left|\frac{t + d_j}{1 + d_j}\right| < 2^3 \exp\left\{ \sum_{j=1}^{m} \frac{-7\lambda_j - 3}{8(1 + \lambda_j)^2} \right\} . \tag{2.28}$$

Inserting absolute value bars on both sides of the Cauchy integral formula for the P_{nk} in (1.4) and applying the inequality (2.28), we obtain

$$|P_{nk}| < 2^{k+3} \exp\left\{ \sum_{j=1}^{m} \frac{-7\lambda_j - 3}{8(1 + \lambda_j)^2} \right\} . \tag{2.29}$$

But

$$\sum_{n=1}^{\infty} \lambda_n^{-1} = +\infty$$

implies that if

$$\sum_{n=1}^{\infty} \lambda_n (1 + \lambda_n)^{-2} < +\infty, \quad \text{then} \quad (7\lambda_n + 3)(1 + \lambda_n)^{-2} > 1$$

must hold for infinitely many integers. Hence (2.29) implies that

$$\lim_{n\to\infty} P_{nk} = 0 \qquad (k = 0, 1, 2, \ldots),$$

and so Condition (1.3) is satisfied.

THEOREM 2.3. *Suppose that* $\{\lambda_n\}$ $(n = 1, 2, \ldots)$ *is a sequence of positive numbers such that*

$$\sum_{n=1}^{\infty} \lambda_n^{-1} = +\infty .$$

Define the sequence $\{d_n\}$ *by*

$$\begin{aligned} d_{4n-3} &= 2^{-\frac{1}{2}}(1 + i)(\lambda_n)^{\frac{1}{4}}. \\ d_{4n-2} &= 2^{-\frac{1}{2}}(-1 + i)(\lambda_n)^{\frac{1}{4}}, \\ d_{4n-1} &= 2^{-\frac{1}{2}}(-1 - i)(\lambda_n)^{\frac{1}{4}}, \\ d_{4n} &= 2^{-\frac{1}{2}}(1 - i)(\lambda_n)^{\frac{1}{4}} \qquad (n = 1, 2, \ldots). \end{aligned} \tag{2.30}$$

Then the $[F, d_n]$*-matrix is regular.*

The proof of this theorem is similar to the proofs of the two previous theorems.

3. Analytic continuation of geometric series. In considering a new regular sequence-to-sequence matrix method of summability, it is of interest to determine the region in which it sums the geometric series to its analytic continuation. The terms of the sequence of partial sums $\{s_n(z)\}$ of the geometric series are given by

$$s_n(z) = \sum_{k=0}^{n} z^k = (1 - z)^{-1} - z^{n+1}(1 - z)^{-1}. \tag{3.1}$$

The analytic continuation of the geometric series is given by $(1 - z)^{-1}$. If $\{\sigma_n(z)\}$ denotes the $[F, d_n]$-transform of $\{s_n(z)\}$, then using (2.1) we have

$$\sigma_n(z) = (1 - z)^{-1} - z(1 - z)^{-1} \sum_{k=0}^{n} P_{nk}\, z^k.$$

Using (1.4) with $\theta = z$, this becomes

$$\sigma_n(z) = (1 - z)^{-1} - z(1 - z)^{-1} \prod_{j=1}^{n} \left(\frac{z + d_j}{1 + d_j}\right) \qquad (n > 0). \tag{3.2}$$

The $[F, d_n]$-matrix sums the geometric series to its analytic continuation if and only if

$$\lim_{n\to\infty} \sigma_n(z) = (1 - z)^{-1}.$$

Let us use the notation

$$Q_n(z) = \prod_{j=1}^{n} \left(\frac{z + d_j}{1 + d_j}\right) \qquad (n > 0). \tag{3.3}$$

Then the [F, d_n]-matrix sums the geometric series to its analytic continuation $(1 - z)^{-1}$ if and only if z is such that

$$\lim_{n\to\infty} Q_n(z) = 0.$$

First we shall discuss the regular $[F, d_n]$-transform given in Theorem 2.1 to determine the region in which it sums the geometric series to its analytic continuation. We shall determine this region for two special cases of this particular $[F, d_n]$-transform. First, we consider the case in which the sequence $\{\lambda_n\}$ satisfies

$$\lim_{n\to\infty} \lambda_n = +\infty.$$

In this case the $[F, d_n]$-matrix is, roughly speaking, a generalization of the Lototsky matrix (see **5**). Second, we consider the case in which

$$\lim_{n\to\infty} \lambda_n = \lambda$$

is finite. In this case the $[F, d_n]$-matrix is, roughly speaking, a generalization of the Euler matrix (see **1**). The case in which $\{\lambda_n\}$ is bounded but not convergent cannot be handled neatly. However, certain of these cases can be handled fairly easily by using sub-sequences if one is interested. For instance, the case in which the set of points λ_n has two limit-points is only slightly more difficult than the case in which $\{\lambda_n\}$ is convergent to a finite limit. In general, the greater the number of limit-points in the set of numbers λ_n $(n = 1, 2, \ldots)$ the more difficult the theorem becomes in statement and proof.

THEOREM 3.1. *Suppose* $\{\lambda_n\}$ $(n = 1, 2, \ldots)$ *satisfies the conditions*

$$\lambda_n > 0, \qquad \lim_{n\to\infty} \lambda_n = +\infty, \qquad \text{and} \qquad \sum_{n=1}^{\infty} \lambda_n^{-1} = +\infty.$$

Let d_n *be given by* (2.1). *Then the corresponding* $[F, d_n]$*-matrix sums the geometric series to its analytic continuation for all* z *such that* $\operatorname{Re}(z^2) < 1$.

Proof. This theorem is proved if we show that

$$\lim_{n\to\infty} Q_n(z) = 0$$

for $\operatorname{Re}(z^2) < 1$, where $Q_n(z)$ is given by (3.3). To accomplish this we shall first show that

$$\lim_{n\to\infty} Q_{2n}(z) = 0,$$

and then that

$$\lim_{n\to\infty} Q_{2n+1}(z) = 0.$$

Let $z = x + iy$; then

$$-1 + \left|\frac{z^2 + \lambda_j}{1 + \lambda_j}\right|^2 = \frac{(x^2 + y^2)^2 - 1 + 2\lambda_j(x^2 - y^2 - 1)}{(1 + \lambda_j)^2}. \tag{3.4}$$

Suppose z is such that $\operatorname{Re}(z^2) < 1$ is given; then there exists $\delta > 0$ such that $x^2 - y^2 \leqslant 1 - \delta$. Since $1 + w \leqslant e^w$ for w real, it follows from (3.4), after taking the positive square root of both sides, that

$$\left|\frac{z^2 + \lambda_j}{1 + \lambda_j}\right| \leqslant \exp\left\{\frac{(x^2 + y^2)^2 - 1 - 2\delta\lambda_j}{2(1 + \lambda_j)^2}\right\}.$$

This inequality implies that

$$\prod_{j=1}^{n} \left|\frac{z^2 + \lambda_j}{1 + \lambda_j}\right| \leqslant \exp\left\{\sum_{j=1}^{n} \frac{(x^2 + y^2)^2 - 1 - 2\delta\lambda_j}{2(1 + \lambda_j)^2}\right\},$$

where $\delta > 0$. Now since

$$\lim_{n\to\infty} \lambda_n = +\infty \qquad \text{and} \sum_{n=1}^{\infty} \lambda_n^{-1} = +\infty,$$

it follows from this inequality that

$$\prod_{j=1}^{\infty} \left(\frac{z^2 + \lambda_j}{1 + \lambda_j}\right) = 0 \tag{3.5a}$$

for z such that $\operatorname{Re}(z^2) < 1$. Using the defining relation (3.3) for $Q_n(z)$, we see, after a short computation using the values of d_j given in (2.1), that (3.5a) is equivalent to

$$\lim_{n\to\infty} Q_{2n}(z) = 0. \tag{3.5b}$$

Let us now consider

$$-1 + \left|\frac{z + d_{2n+1}}{1 + d_{2n+1}}\right|^2 = \frac{x^2 + y^2 - 1 + 2y\rho_{2n+1}}{1 + \rho_{2n+1}^2}.$$

Since $1 + w \leqslant e^w$ for real w, we have, after taking square roots,

$$\left|\frac{z + d_{2n+1}}{1 + d_{2n+1}}\right| \leqslant \exp\left\{\frac{x^2 + y^2 - 1 + 2y\rho_{2n+1}}{2(1 + \rho_{2n+1}^2)}\right\}.$$

Since

$$\lim_{n\to\infty} \lambda_n = +\infty \qquad \text{or} \qquad \lim_{n\to\infty} \rho_n = +\infty,$$

this inequality implies that there exists a constant M, independent of n, such that

$$\left|\frac{z + d_{2n+1}}{1 + d_{2n+1}}\right| < M.$$

It follows from this inequality and (3.3) that

$$|Q_{2n+1}(z)| < M\,|Q_{2n}(z)|.$$

This inequality, with (3.5b), implies that

$$\lim_{n\to\infty} Q_{2n+1}(z) = 0.$$

THEOREM 3.2. *Suppose that* $\{\lambda_n\}$ $(n = 1, 2, \ldots)$ *is a sequence of positive numbers such that*

$$\lim_{n\to\infty} \lambda_n = \lambda.$$

Suppose the numbers d_n *are given by* (2.1). *Then the* $[F, d_n]$*-matrix sums the geometric series to its analytic continuation for all* z *such that* $|z^2 + \lambda| < 1 + \lambda$.

Proof. This theorem is proved using the same approach that was used to prove Theorem 3.1. Let $z = x + iy$; then

$$-1 + \left|\frac{z^2 + \lambda_j}{1 + \lambda_j}\right|^2 = \frac{-1 + (x^2 + y^2)^2 - 2\lambda(1 - x^2 + y^2)}{(1 + \lambda)^2} + \epsilon_j(x, y, \lambda). \tag{3.6}$$

Notice that

$$\lim_{j\to\infty} \epsilon_j = 0 \qquad \text{since} \qquad \lim_{j\to\infty} \lambda_j = \lambda.$$

Let $z = x + iy$ be given such that $(x^2 + y^2)^2 - 1 < 2\lambda(1 - x^2 + y^2)$; then there exists $\delta > 0$ such that

$$(x^2 + y^2)^2 - 1 - 2\lambda(1 - x^2 + y^2) \leqslant -2\delta(1 + \lambda)^2.$$

Since

$$\lim_{j\to\infty} \epsilon_j(x, y, \lambda) = 0,$$

there exists $N(x, y, \lambda)$ such that if $j \geqslant N$, then $|\epsilon_j(x, y, \lambda)| \leqslant \delta$. Using these inequalities in (3.6), it follows that

$$-1 + \left|\frac{z^2 + \lambda_j}{1 + \lambda_j}\right|^2 \leqslant -\delta \quad \text{for } j \geqslant N.$$

Since $1 + w \leqslant e^w$ for real w, it follows, after taking the positive square root of both sides, that

$$\left|\frac{z^2 + \lambda_j}{1 + \lambda_j}\right| \leqslant \exp(-\delta/2) \quad \text{for } j \geqslant N.$$

If we use the definition of d_j from (2.1) in (3.3), we see that this inequality implies that

$$\lim_{n\to\infty} Q_{2n}(z) = \upsilon, \tag{3.7}$$

for all z such that $|z^2 + \lambda| < 1 + \lambda$.

Since $\{\lambda_n\}$ is a convergent sequence, there exists a constant $M(z)$, independent of n, such that

$$\left|\frac{z + d_n}{1 + d_n}\right| < M.$$

Using this inequality, it follows from (3.3) that $|Q_{2n+1}(z)| < M\,|Q_{2n}(z)|$. This inequality and (3.7) imply that

$$\lim_{n\to\infty} Q_{2n+1}(z) = 0$$

whenever $|z^2 + \lambda| < 1 + \lambda$, and the theorem is proved.

It is now of some interest to know what sort of region in the z-plane is defined by $|z^2 + \lambda| < 1 + \lambda$. It is the region inside the closed curve

$$(x^2 + y^2)^2 - 1 = 2\lambda(1 - x^2 + y^2).$$

This curve has y-intercepts equal to $\pm(2\lambda + 1)^{\frac{1}{2}}$ and x-intercepts equal to ± 1. It has horizontal tangents at $x = 0$, and vertical tangents at $y = 0$ and $y^2 = (3\lambda/4) - (1/2) - (1/4\lambda)$. Notice also that as $\lambda \to \infty$ the region $(x^2 + y^2)^2 - 1 < 2\lambda(1 - x^2 + y^2)$ approaches the region $x^2 - y^2 < 1$ of Theorem 3.1. This is in a certain sense an extension of a well-known analogous relationship concerning the Euler matrix and the Lototsky matrix. The Euler matrix E_r sums the geometric series to its analytic continuation inside a circle of radius r^{-1} (see **1**). The Lototsky matrix sums the geometric series to its analytic continuation for all z such that $\text{Re}(z) < 1$ (see **4**). As $r \to 0$ the circle of the Euler matrix approaches the half-plane of the Lototsky matrix.

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In Theorem 3.2 we showed that a certain $[F, d_n]$-matrix sums the geometric series to its analytic continuation for all z such that $\mathrm{Re}(z^2) < 1$. One might then ask the following question. Do there exist any values of z for which $\mathrm{Re}(z^2) \geqslant 1$ and such that the $[F, d_n]$-matrix of Theorem 3.1 sums the geometric series to its analytic continuation? This question is essentially answered by the following theorem.

THEOREM 3.3. *Suppose that* $\{\lambda_n\}$ $(n = 1, 2, 3, \ldots)$ *is a sequence such that*

$$\lim_{n\to\infty} \lambda_n = \infty, \qquad \lambda_n > 0, \qquad \text{and } \sum_{n=1}^{\infty} \lambda_n^{-1} = +\infty.$$

Suppose that the terms of the sequence $\{d_n\}$ *are given by* (2.1), *and that* $\sigma_n(z)$ *is given by* (3.2). *Then* $\{\sigma_n(z)\}$ *diverges to infinity for all* z *such that* $\mathrm{Re}(z^2) > 1$.

Proof. To prove this theorem it is sufficient to show that the sequence $\{Q_n(z)\}$, where $Q_n(z)$ is given by (3.3), diverges to infinity.

Let $z = x + iy$ be given such that $x^2 - y^2 > 1$; then there exists $\delta > 0$ such that $x^2 - y^2 - 1 \geqslant \delta$. It now follows from (3.4) that

$$-1 + \left|\frac{z^2 + \lambda_n}{1 + \lambda_n}\right|^2 > \frac{2\lambda_n \delta}{(1 + \lambda_n)^2} > 0, \tag{3.8}$$

Since

$$\sum_{n=1}^{\infty} \lambda_n^{-1} = +\infty$$

by hypothesis, (3.8) implies that the series

$$\sum_{j=1}^{\infty} \left\{-1 + \left|\frac{z^2 + \lambda_j}{1 + \lambda_j}\right|^2\right\} \tag{3.9}$$

diverges to infinity. Since (3.8) is satisfied, we can now make use of a well-known theorem (see **7**, p. 14). It follows from this theorem that as n becomes infinite

$$Q_{2n}(z) = \prod_{j=1}^{n} \left(\frac{z^2 + \lambda_j}{1 + \lambda_j}\right)$$

diverges to infinity if and only if the series (3.9) diverges to infinity. Since the series (3.9) does diverge,

$$\lim_{n\to\infty} Q_{2n}(z) = \infty$$

for all z such that $\mathrm{Re}(z^2) > 1$.

Using the values of d_{2n+1} from (2.1), a short computation shows that

$$\left|\frac{z + d_{2n+1}}{1 + d_{2n+1}}\right| > 2^{-1} \quad \text{if } x^2 - y^2 > 1.$$

From this inequality and (3.3), we have $2\,|Q_{2n+1}(z)| > |Q_{2n}(z)|$. Thus

$$\lim_{n\to\infty} Q_{2n+1}(z) = \infty \quad \text{since } \lim_{n\to\infty} O_{2n}(z) = \infty.$$

Theorem 3.3 answers a certain question regarding the $[F, d_n]$-matrix given in Theorem 3.1. If we ask this same question regarding the $[F, d_n]$-matrix in Theorem 3.2, we get the following theorem.

THEOREM 3.4. *Let the $[F, d_n]$-matrix of Theorem 3.2 be given and let $\sigma_n(z)$ be given by (3.2). Then $\{\sigma_n(z)\}$ diverges to infinity for all z such that*

$$|z^2 + \lambda| > 1 + \lambda.$$

Proof. Let z be given such that $|z^2 + \lambda| > 1 + \lambda$; then there exists $\delta > 0$ such that

$$(x^2 + y^2)^2 - 1 - 2\lambda(1 - x^2 + y^2) > 2\delta(1 + \lambda)^2.$$

Since

$$\lim_{n\to\infty} \epsilon_n = 0$$

in (3.6), there exists $N(x, y, \lambda)$, such that if $n \geqslant N$, then $|\epsilon_n(x, y, \lambda)| < \delta$. Now, using these inequalities in (3.6), we have

$$-1 + \left|\frac{z^2 + \lambda_j}{1 + \lambda_j}\right|^2 > \delta \quad \text{if } j \geqslant N.$$

It now follows that the series (3.9) diverges. The remainder of the proof of this theorem is very similar to the proof of Theorem 3.3.

We shall now look at the regular sequence-to-sequence $[F, d_n]$-transform given in Theorem 2.2 to determine the region in which it sums the geometric series. We shall again consider only two cases. First, the case in which the sequence $\{\lambda_n\}$ satisfies

$$\lim_{n\to\infty} \lambda_n = +\infty.$$

Second, the case in which

$$\lim_{n\to\infty} \lambda_n = \lambda$$

is finite.

THEOREM 3.5. *Suppose that $\{\lambda_n\}$ $(n = 1, 2, \ldots)$ is a sequence of positive numbers such that*

$$\lim_{n\to\infty} \lambda_n = \infty \qquad \text{and} \sum_{n=1}^{\infty} \lambda_n^{-1} = +\infty.$$

Let the terms of the sequence $\{d_n\}$ be given by (2.13). Then the $[F, d_n]$-matrix sums the geometric series to its analytic continuation for all z such that $\operatorname{Re}(z^3) < 1$.

Proof. To prove this theorem it is sufficient to show that

$$\lim_{n\to\infty} Q_n(z) = 0$$

for $\operatorname{Re}(z^3) < 1$, where $Q_n(z)$ is given by (3.3). To accomplish this we show in turn that

$$\lim_{n\to\infty} Q_{3n}(z) = 0, \qquad \lim_{n\to\infty} Q_{3n+1}(z) = 0, \qquad \text{and} \lim_{n\to\infty} Q_{3n+2}(z) = 0.$$

If $z = x + iy$, then

$$(3.10) \qquad -1 + \left|\frac{z^3 + \lambda_j}{1 + \lambda_j}\right|^2 = \frac{-1 + (x^2 + y^2)^3 + 2\lambda_j(x^3 - 3xy^2 - 1)}{(1 + \lambda_j)^2}.$$

Let z be given such that $\mathrm{Re}(z^3) < 1$, i.e., $x^3 - 3xy^2 < 1$. Then there exists $\delta > 0$ such that $x^3 - 3xy^2 - 1 \leqslant \delta$. Using this inequality, (3.10) becomes

$$-1 + \left|\frac{z^3 + \lambda_j}{1 + \lambda_j}\right|^2 \leqslant \frac{-1 + (x^2 + y^2)^3 - 2\delta\lambda_j}{(1 + \lambda_j)^2}.$$

Since $1 + w \leqslant e^w$ for real w, it follows from this inequality, after taking square roots of both sides, that

$$(3.11) \qquad \prod_{j=1}^{n} \left|\frac{z^3 + \lambda_j}{1 + \lambda_j}\right| \leqslant \exp\left\{\sum_{j=1}^{n} \frac{-1 + (x^2 + y^2)^3 - 2\delta\lambda_j}{2(1 + \lambda_j)^2}\right\}.$$

Since

$$\lim_{n\to\infty} \lambda_n = +\infty,$$

we have $-1 + (x^2 + y^2)^3 < \delta\lambda_n$ for n sufficiently large. Since

$$\sum_{n=1}^{\infty} \lambda_n^{-1} = +\infty,$$

it is easy to show that the series in (3.11) approaches negative infinity as n becomes infinite. Hence

$$(3.12) \qquad \prod_{j=1}^{\infty} \left(\frac{z^3 + \lambda_j}{1 + \lambda_j}\right) = 0.$$

Using the values of d_j as given by (2.13), we find that (3.12) is equivalent to

$$(3.13) \qquad \lim_{n\to\infty} Q_{3n}(z) = 0$$

for all z such that $x^3 - 3xy^2 < 1$.

We have

$$(3.14) \quad -1 + \left|\frac{z + d_n}{1 + d_n}\right|^2 = \frac{-1 + (x^2 + y^2) + 2\rho_n[(x - 1)\cos\theta_n + y\sin\theta_n]}{1 + 2\rho_n\cos\theta_n + \rho_n^2}.$$

Since

$$\lim_{n\to\infty} \rho_n = +\infty,$$

it follows that the right member of (3.14) approaches zero as n becomes infinite. Therefore, the left member of (3.14) is a bounded function of n. Hence, there exists a constant M, independent of n, such that

$$(3.15) \qquad \left|\frac{z + d_n}{1 + d_n}\right| < M.$$

Using the definition of $Q_n(z)$ given by (3.3) and (3.15), it easily follows that

$$|Q_{3n+\nu}(z)| < M^{\nu}|Q_{3n}(z)| \qquad (\nu = 1, 2).$$

This inequality taken together with (3.13) implies that

$$\lim_{n\to\infty} Q_{3n+\nu}(z) = 0 \qquad (\nu = 1, 2).$$

This completes the proof of the theorem.

The region defined by $x^3 - 3xy^2 < 1$ is the region inside the curve $x^3 - 3xy^2 = 1$, i.e., the simply connected region containing the origin and bounded by $x^3 - 3xy^2 = 1$. The curve $x^3 - 3xy^2 = 1$ has three branches. One branch goes through the point $(1, 0)$ and is asymptotic to the two lines $y = \pm[\tan(\pi/6)]x$ and symmetric about the x-axis. The other two branches are exact replicas of this branch except they pass through the points ($\cos 2\pi/3$, $\sin 2\pi/3$) and ($\cos 4\pi/3$, $\sin 4\pi/3$) and are symmetric, respectively, to the lines $y = (\tan 2\pi/3)x$ and $y = (\tan 4\pi/3)x$.

If we follow the pattern used in the case of Theorem 2.1 in studying the region in which the $[F, d_n]$-transform sums the geometric series to its analytic continuation, then the next theorem to be proved would be the following.

THEOREM A. *Suppose* $\lambda_n > 0$,

$$\lim_{n\to\infty} \lambda_n = \lambda,$$

and the numbers d_n *are defined by* (2.13). *Then the* $[F, d_n]$*-matrix sums the geometric series to its analytic continuation for all* z *such that* $|z^3 + \lambda| < 1 + \lambda$.

Theorem A is indeed true. The proof follows using equations and ideas found in the proofs of Theorems 2.2 and 3.5 in a manner similar to the manner in which Theorem 3.2 follows from equations and ideas found on the proofs of Theorems 2.1 and 3.1. However, instead of actually proving Theorem A here, we shall prove the following theorem.

THEOREM 3.6. *Let* $\lambda > 0$ *be given. Define the numbers* d_n *by*

$$\begin{aligned} d_{3n-2} &= 2^{-1}\lambda^{\frac{1}{3}}(-1 + i3^{\frac{1}{2}}), \\ d_{3n-2} &= \lambda^{\frac{1}{3}}, \\ d_{3n} &= 2^{-1}\lambda^{\frac{1}{3}}(-1 - i3^{\frac{1}{2}}) \qquad (n = 1, 2, \ldots). \end{aligned}$$

This $[F, d_n]$*-matrix sums the geometric series to its analytic continuation if and only if* z *is such that* $|z^3 + \lambda| < 1 + \lambda$.

We are considering this special case for two reasons. First, even though Theorem 3.6 is much shorter and easier to prove, it is an "if and only if" theorem. When we used the general approach similar to Theorem A for the matrix of Theorem 2.1, we had to prove two theorems, i.e., Theorems 3.2 and 3.4, to determine just exactly those values of z for which we had analytic continuation of the geometric series, and still the situation was in doubt for

z such that $|z^3 + \lambda| = 1 + \lambda$. If we use this general approach again we would need to prove Theorem A and also a Theorem B, which says that we do not get analytic continuation for z such that $|z^3 + \lambda| > 1 + \lambda$. Incidently, this Theorem B is true and can be proved in a manner similar to the way Theorem 3.4 was proved. The second reason is that the $[F, d_n]$-matrix given in Theorem 3.6 sums the geometric series to its analytic continuation for z in the same region as does the $[F, d_n]$-matrix given in Theorem A except possibly for some z such that $|z^3 + \lambda| = 1 + \lambda$. Hence, in the problem of summing the geometric series to its analytic continuation, the $[F, d_n]$-matrix of Theorem 3.6 is of almost as much interest as the $[F, d_n]$-matrix of Theorem A. It seems, then, that we should point this fact out and give some attention to the simpler matrix of Theorem 3.6.

We should perhaps remark at this point that if λ is replaced by $\lambda_n = \lambda e^{i\theta_n}$ in Theorem 3.6, then one may arrive at a theorem of the same general type as (**3**; Corollary 4.3). Such a theorem would give an $[F, d_n]$-matrix which sums the geometric series to its analytic continuation for some z such that

$$|z^3 + \lambda| = 1 + \lambda.$$

Proof of Theorem 3.6. A short computation using the values of d_n as given in the statement of the theorem shows that

$$Q_{3n}(z) = \prod_{j=1}^{n} \left(\frac{z^3 + \lambda}{1 + \lambda}\right) = \left(\frac{z^3 + \lambda}{1 + \lambda}\right)^n.$$

Therefore

$$\lim_{n\to\infty} Q_{3n}(z) = 0$$

if and only if z is such that $|z^3 + \lambda| < 1 + \lambda$. Since

$$Q_{3n+\nu}(z) = C_\nu(z, \lambda)Q_{3n}(z) \qquad (\nu = 1, 2),$$

where $C_1(z, \lambda)$ and $C_2(z, \lambda)$ are independent of n, it follows that

$$\lim_{n\to\infty} Q_{3n+\nu}(z) = 0 \qquad (\nu = 1, 2)$$

if and only if z satisfies $|z^3 + \lambda| < 1 + \lambda$.

THEOREM 3.7. *Suppose we are given the* $[F, d_n]$*-matrix of Theorem* 3.5 *and that* $\sigma_n(z)$ *is given by* (3.2). *Then*

$$\lim_{n\to\infty} \sigma_n(z) = \infty$$

for all z *such that* $\operatorname{Re}(z^3) > 1$.

Proof. It follows from (3.2) and (3.3) that this theorem will be proved if we can show that

$$\lim_{n\to\infty} Q_n(z) = \infty$$

for all z such that $\operatorname{Re}(z^3) > 1$.

Let $z = x + iy$ be given such that $\text{Re}(z^3) > 1$; then there exists $\delta > 0$ such that $x^3 - 3xy^2 - 1 > \delta$. Since

$$\lim_{n\to\infty} \lambda_n = +\infty,$$

there exists N such that if $n \geqslant N$, then $-1 + (x^2 + y^2)^3 > -\delta\lambda_n$. Using $x^3 - 3xy^2 - 1 > \delta$ and $-1 + (x^2 + y^2)^3 > -\delta\lambda_j$ in (3.10), we have

$$-1 + \left|\frac{z^3 + \lambda_j}{1 + \lambda_j}\right|^2 > \frac{\delta\lambda_j}{(1 + \lambda_j)^2} > 0 \qquad (j \geqslant N). \tag{3.16}$$

It now follows from a well-known theorem (see **7**, p. 14) that

$$\lim_{n\to\infty} Q_{3n}(z) = \lim_{n\to\infty} \prod_{j=1}^{n} \left(\frac{z^3 + \lambda_j}{1 + \lambda_j}\right) = \infty$$

if the series

$$\sum_{j=1}^{\infty} \left\{-1 + \left|\frac{z^3 + \lambda_j}{1 + \lambda_j}\right|^2\right\} \tag{3.17}$$

diverges to infinity. Since

$$\sum_{n=1}^{\infty} \lambda_n^{-1} = +\infty,$$

(3.16) implies that the series (3.17) does indeed diverge to infinity. Hence

$$\lim_{n\to\infty} Q_{3n}(z) = \infty \tag{3.18}$$

if $\text{Re}(z^3) > 1$.

From (3.14) it follows that

$$\lim_{n\to\infty} \left|\frac{z + d_n}{1 + d_n}\right| = 1.$$

Using this together with (3.3) and (3.18), it is easy to show that

$$\lim_{n\to\infty} Q_{3n+\nu}(z) = \infty \qquad (\nu = 1, 2).$$

When considering the problem of where the $[F, d_n]$-matrix of Theorem 2.3 sums the geometric series to its analytic continuation, we shall again look at the two most important special cases. First, there is the case where

$$\lim_{n\to\infty} \lambda_n = +\infty.$$

Second, the case where

$$\lim_{n\to\infty} \lambda_n = \lambda$$

is finite.

THEOREM 3.8. *Suppose that* $\{\lambda_n\}$ $(n = 1, 2, \ldots)$ *is a sequence of positive numbers such that*

$$\lim_{n\to\infty} \lambda_n = +\infty \qquad \textit{and} \sum_{n=1}^{\infty} \lambda_n^{-1} = +\infty.$$

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Suppose the terms of the sequence $\{d_n\}$ *are given by* (2.30). *Then the* $[F, d_n]$-*matrix sums the geometric series to its analytic continuation for all* z *such that* $\mathrm{Re}(z^4) < 1$.

Proof. To prove this theorem, it is sufficient to show that if z is given such that $\mathrm{Re}(z^4) < 1$, then

$$\lim_{n\to\infty} Q_n(z) = 0,$$

where $Q_n(z)$ is given by (3.3).

If $z = x + iy$, then

$$(3.19)\qquad -1 + \left|\frac{z^4 + \lambda_n}{1 + \lambda_n}\right|^2 = \frac{-1 + (x^2 + y^2)^4 + 2\lambda_n(x^4 - 6x^2y^2 + y^4 - 1)}{(1 + \lambda_n)^2}.$$

Suppose z is given such that $\mathrm{Re}(z^4) < 1$; then there exists $\delta > 0$ such that $x^4 - 6x^2y^2 + y^4 - 1 < -\delta$. Since

$$\lim_{n\to\infty} \lambda_n = +\infty,$$

there exists N such that if $n \geqslant N$, then $-1 + (x^2 + y^2)^4 < \delta\lambda_n$. Using these two inequalities in (3.19), it follows that

$$-1 + \left|\frac{z^4 + \lambda_n}{1 + \lambda_n}\right|^2 < \frac{-\lambda_n\delta}{(1 + \lambda_n)^2}.$$

Since $1 + w \leqslant e^w$ for real w, it follows from this inequality, after taking square roots, that

$$(3.20)\qquad \left|\frac{z^4 + \lambda_n}{1 + \lambda_n}\right| < \exp\left\{\frac{-\delta\lambda_n}{2(1 + \lambda_n)^2}\right\} \qquad (n \geqslant N).$$

Using the values of d_n as given by (2.30) in (3.3) we easily show that

$$Q_{4n}(z) = \prod_{j=1}^{n} \left(\frac{z^4 + \lambda_j}{1 + \lambda_j}\right).$$

It now follows from (3.20) that

$$|Q_{4n}(z)| < \exp\left\{-\sum_{j=N}^{n} \frac{\delta\lambda_j}{2(1 + \lambda_j)^2}\right\} |Q_{4N-4}(z)| \qquad (n \geqslant N).$$

Since

$$\sum_{n=1}^{\infty} \lambda_n^{-1} = +\infty$$

by hypothesis, this inequality implies that

$$(3.21)\qquad \lim_{n\to\infty} Q_{4n}(z) = 0.$$

Since

$$\lim_{n\to\infty} \rho_n = +\infty,$$

it follows from (3.14) that (3.15) is satisfied. It follows from (3.15) and (3.3) that

$$|Q_{4n+\nu}(z)| < M^{\nu}\,|Q_{4n}(z)| \qquad (\nu = 1, 2, 3).$$

This inequality and (3.21) imply that

$$\lim_{n\to\infty} Q_{4n+\nu}(z) = 0 \qquad (\nu = 1, 2, 3).$$

This completes the proof of the theorem.

In view of the conclusion of Theorem 3.8, there is some interest in the region defined by $x^4 - 6x^2y^2 + y^4 < 1$. This is the region inside the curve $x^4 - 6x^2y^2 + y^4 = 1$, i.e., the simply connected region containing the origin and bounded by $x^4 - 6x^2y^2 + y^4 = 1$. The curve $x^4 - 6x^2y^2 + y^4 = 1$ has four branches. One branch goes through the point $(1, 0)$, is symmetric to the x-axis, and is asymptotic to the lines $y = \pm(\tan \pi/8)x$. The other three branches are replicas of this branch except they go through the points $z = i$, $z = -1$, and $z = -i$; and are symmetric, respectively, to the positive imaginary, negative real, and negative imaginary axis.

THEOREM 3.9. *Suppose that* $\{\lambda_n\}$ $(n = 1, 2, \ldots)$ *is a sequence of positive numbers such that*

$$\lim_{n\to\infty} \lambda_n = \lambda$$

and suppose that the terms of the sequence $\{d_n\}$ *are defined by* (2.30). *Then the* $[F, d_n]$*-matrix sums the geometric series to its analytic continuation for all* z *such that* $|z^4 + \lambda| < 1 + \lambda$.

The proof of this theorem is similar to the proof of Theorem 3.2.

THEOREM 3.10. *Suppose the* $[F, d_n]$*-matrix of Theorem* 3.8 *is given and that* $\sigma_n(z)$ *is then given by* (3.2). *Then*

$$\lim_{n\to\infty} \sigma_n(z) = \infty$$

for all z *such that* $\operatorname{Re}(z^4) > 1$.

THEOREM 3.11. *Suppose the* $[F, d_n]$*-matrix of Theorem* 3.9 *is given and that* $\sigma_n(z)$ *is then given by* (3.2). *Then*

$$\lim_{n\to\infty} \sigma_n(z) = \infty$$

for all z *such that* $|z^4 + \lambda| > 1 + \lambda$.

The proofs of Theorems 3.10 and 3.11 are similar to the proofs of Theorems 3.3 and 3.4, respectively.

4. Conclusion. In this paper we have considered three classes of regular $[F, d_n]$-matrices. These $[F, d_n]$-matrices are defined in terms of a sequence $\{d_n\}$ which in turn is defined in terms of a real sequence $\{\lambda_n\}$. The three sequences $\{d_n\}$ were obtained by taking successively the square roots of $-\lambda_n$, the cube roots of λ_n, the fourth roots of $-\lambda_n$. Then why not continue this process and obtain a new $[F, d_n]$-matrix using the fifth roots of λ_n? One can, of course, do this, but there seems little point in continuing this process unless we can prove regularity for the $[F, d_n]$-matrix obtained from the νth roots of $(-1)^{\nu-1}\lambda_n$, where ν is a positive integer. We use the νth roots of $(-1)^{\nu-1}\lambda_n$ and not the roots of λ_n in order that the product of the ν roots thus obtained will be λ_n. We should like to conjecture that a regular $[F, d_n]$-matrix can be obtained by using the successive νth roots of $(-1)^{\nu-1}\lambda_n$ as the values for d_n. The difficulty in proving this conjecture is in managing the manipulation involved in proving that the matrix satisfies (1.2).

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