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MULTIALGEBRAS, UNIVERSAL ALGEBRAS AND IDENTITIES

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Abstract

In this paper we determine the smallest equivalence relation on a multialgebra for which the factor multialgebra is a universal algebra satisfying a given identity. We also establish an important property for the factor multialgebra (of a multialgebra) modulo this relation.

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1. Introduction

The starting point of this paper can be found in [7] where Freni presents the smallest equivalence on a (semi)hypergroup for which the factor (semi)hypergroup is a commutative (semi)group.

Multialgebras (also called hyperstructures) are particular cases of relational systems which are generalizations of universal algebras. They have been studied for more than 60 years and have been used in different areas of mathematics (algebra, geometry, graph theory) as well as in applied sciences (see [5]).

It follows from [7] (and also from [6] and [13]) that, among the equivalence relations of a multialgebra, of great importance are those equivalence relations for which the factor multialgebra is a universal algebra. For a multialgebra $\mathfrak{A}$, the class of these relations is an algebraic closure system. It follows that we can always obtain a smaller such equivalence, which contains a relation R on A. Using this, some of the results of [7] can be established in the general case of multialgebras. More precisely, we determine the smallest equivalence relation $\alpha^*_{\mathbf{qr}}$ on a multialgebra that has the property that the factor multialgebra is a universal algebra satisfying a given identity $\mathbf{q} = \mathbf{r}$.

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In [8], Grätzer proved that any multialgebra $\mathfrak{A}$ is obtained as a factor of a universal algebra $\mathfrak{B}$ by an appropriate equivalence relation $\rho \subseteq B \times B$. For a multialgebra $\mathfrak{B}/\rho$, we consider the universal algebra $(\mathfrak{B}/\rho)/\alpha^*_{\mathbf{qr}}$ and we prove, in Theorem 5.3, that this algebra is isomorphic to the factor algebra of $\mathfrak{B}$ modulo the smallest congruence relation θ of $\mathfrak{B}$ which has the property that $\rho \subseteq \theta$ and $\mathbf{q} = \mathbf{r}$ is satisfied on $\mathfrak{B}/\theta$. In the last section we give an application to hypergroups, which are factor of a group modulo an equivalence relation determined by a subgroup.

While studying some properties of the factor multialgebra of a multialgebra we have found an answer to the first part of Problem 4 from [8]: *What are the factor multialgebras of a group, abelian group, lattice, ring and so on? Characterize these with a suitable axiom system.* In the third section of this paper, we prove that an n-ary identity $\mathbf{q} = \mathbf{r}$ on an algebra $\mathfrak{B}$ gives the weak identity $\mathbf{q} \cap \mathbf{r} \neq \emptyset$ on the multialgebra $\mathfrak{B}/\rho$. Yet, as mentioned at the end of [8], there exist multialgebras with one binary associative multioperation, which are not factor multialgebras of a semigroup. So, a multialgebra that satisfies a set of given weak (or strong) identities does not have to be a factor multialgebra of a universal algebra satisfying the corresponding identities. This means that our answer does not cover the second part of this problem.

2. Preliminaries

Let $\mathbb{N}$ be the set of the nonnegative integers, let $\tau = (n_\gamma)_{\gamma < o(\tau)}$ be a sequence over $\mathbb{N}$, where $o(\tau)$ is an ordinal, let $\mathbf{f}_\gamma$ be a symbol of an n_γ-ary (multi)operation for any $\gamma < o(\tau)$, and let $\mathfrak{P}^{(n)}(\tau) = (\mathbf{P}^{(n)}(\tau), (f_\gamma)_{\gamma < o(\tau)})$ be the algebra of n-ary terms (of type τ).

Let A be a set and $P^*(A)$ the family of nonempty subsets of A. Let $\mathfrak{A} = (A, (f_\gamma)_{\gamma < o(\tau)})$ be a multialgebra, where for any $\gamma < o(\tau)$, $f_\gamma : A^{n_\gamma} \to P^*(A)$ is the multioperation of arity n_γ that corresponds to the symbol $\mathbf{f}_\gamma$. If the multialgebra $\mathfrak{A}$ has no nullary multioperations, then we allow the support set A to be empty. Of course, any universal algebra is a multialgebra (we identify a one element set with its element).

If for any $\gamma < o(\tau)$ and for any $A_0, \ldots, A_{n_\gamma - 1} \in P^*(A)$, we define

$$f_\gamma(A_0, \ldots, A_{n_\gamma-1}) = \bigcup \left\{ f_\gamma(a_0, \ldots, a_{n_\gamma-1}) \mid a_i \in A_i,\ i \in \{0, \ldots, n_\gamma - 1\} \right\},$$

then we obtain a universal algebra on $P^*(A)$ (see [12]). We denote this algebra by $\mathfrak{P}^*(\mathfrak{A})$ and consider the algebra $\mathfrak{P}^{(n)}(\mathfrak{P}^*(\mathfrak{A}))$ of the n-ary term functions on $\mathfrak{P}^*(\mathfrak{A})$ ($n \in \mathbb{N}$). Clearly, any term from $\mathbf{p} \in \mathbf{P}^{(n)}(\tau)$ induces a term function from $\mathfrak{P}^{(n)}(\mathfrak{P}^*(\mathfrak{A}))$ ([9, Corollary 8.1]). We denote this term function by p (or by $(\mathbf{p})_{\mathfrak{P}^*(\mathfrak{A})}$ when necessary).

Denote by $\mathfrak{P}^{(n)}_{P^*(A)}(\mathfrak{P}^*(\mathfrak{A}))$ the algebra of the n-ary polynomial functions of the universal algebra $\mathfrak{P}^*(\mathfrak{A})$ (see [3]) and by $\mathfrak{P}^{(n)}_A(\mathfrak{P}^*(\mathfrak{A}))$ its subalgebra generated by

$$\{c^n_a \mid a \in A\} \cup \{e^n_i \mid i \in \{0, \dots, n-1\}\},$$

where $c^n_a, e^n_i : P^*(A)^n \to P^*(A)$ are defined by

$$c^n_a(A_0, \dots, A_{n-1}) = \{a\} \quad \text{and} \quad e^n_i(A_0, \dots, A_{n-1}) = A_i.$$

REMARK 1. For a multialgebra $\mathfrak{A}$, $P^{(n)}(\mathfrak{P}^*(\mathfrak{A}))$ is a subalgebra of $\mathfrak{P}^{(n)}_A(\mathfrak{P}^*(\mathfrak{A}))$.

REMARK 2. Let $\mathfrak{A}$ be a universal algebra and let $P^{(n)}_A(\mathfrak{A})$ be the set of the n-ary polynomial functions of $\mathfrak{A}$. For any polynomial function $p \in P^{(n)}_A(\mathfrak{P}^*(\mathfrak{A}))$, the map

$$A^n \to A, \ (a_0, \dots, a_{n-1}) \mapsto p(a_0, \dots, a_{n-1})$$

defines a polynomial function from $P^{(n)}_A(\mathfrak{A})$. Moreover, any polynomial function from $P^{(n)}_A(\mathfrak{A})$ can be obtained in this way from a polynomial function from $P^{(n)}_A(\mathfrak{P}^*(\mathfrak{A}))$.

REMARK 3. It is known that if $\gamma < o(\tau)$ and $A_0, \dots, A_{n_\gamma - 1}$, $B_0, \dots, B_{n_\gamma - 1}$ are nonempty subsets of A such that $A_0 \subseteq B_0, \dots, A_{n_\gamma-1} \subseteq B_{n_\gamma-1}$, then

$$f_\gamma(A_0, \dots, A_{n_\gamma-1}) \subseteq f_\gamma(B_0, \dots, B_{n_\gamma-1}).$$

It easily follows that if $n \in \mathbb{N}$, $p \in P^{(n)}_{P^*(A)}(\mathfrak{P}^*(\mathfrak{A}))$, and the nonempty subsets $A_0, \dots, A_{n-1}$, $B_0, \dots, B_{n-1}$ of A are such that $A_0 \subseteq B_0, \dots, A_{n-1} \subseteq B_{n-1}$, then

$$p(A_0, \dots, A_{n-1}) \subseteq p(B_0, \dots, B_{n-1}).$$

A map $h : A \to B$ between the multialgebras $\mathfrak{A}$ and $\mathfrak{B}$ of the same type τ is called a homomorphism if for any $\gamma < o(\tau)$ and for all $a_0, \dots, a_{n_\gamma-1} \in A$, we have

$$h(f_\gamma(a_0, \dots, a_{n_\gamma-1})) \subseteq f_\gamma(h(a_0), \dots, h(a_{n_\gamma-1})). \tag{1}$$

A bijective map h is a *multialgebra isomorphism* if both h and h^{-1} are multialgebra homomorphisms. It follows from [12] that the multialgebra isomorphisms can be characterized as being the bijective homomorphisms h for which the inclusion (1) is an equality.

REMARK 4. By the construction of a polynomial (symbol) it follows that for a homomorphism $h : A \to B$, if $n \in \mathbb{N}$, $\mathbf{p} \in \mathbf{P}^{(n)}(\tau)$, and $a_0, \dots, a_{n-1} \in A$, then

$$h(p(a_0, \dots, a_{n-1})) \subseteq p(h(a_0), \dots, h(a_{n-1})). \tag{2}$$

Let ρ be an equivalence relation on A and $A/\rho = \{\rho\langle x\rangle \mid x \in A\}$ (where $\rho\langle x\rangle$ denotes the class of x modulo ρ). For a $\gamma < o(\tau)$, the equalities

$$\begin{aligned} &f_\gamma(\rho\langle a_0\rangle, \dots, \rho\langle a_{n_\gamma-1}\rangle) \\ &\quad = \left\{\rho\langle b\rangle \mid b \in f_\gamma(b_0, \dots, b_{n_\gamma-1}),\ a_i \rho b_i,\ i \in \{0, \dots, n_\gamma - 1\}\right\} \end{aligned}$$

define a multioperation f_γ on A/ρ (see [8]). One obtains a multialgebra $\mathfrak{A}/\rho$ on A/ρ called the *factor multialgebra of* $\mathfrak{A}$ *modulo* ρ.

The definition of the multioperations from $\mathfrak{A}/\rho$ allows us to see the canonical map π_ρ from A onto A/ρ as an multialgebra homomorphism for any equivalence relation ρ on A (see [12]). Applying (2) for π_ρ we have

$$\left\{\rho\langle a\rangle \mid a \in (\mathbf{p})_{\mathfrak{P}^*(\mathfrak{A})}(a_0, \dots, a_{n-1})\right\} \subseteq (\mathbf{p})_{\mathfrak{P}^*(\mathfrak{A}/\rho)}(\rho\langle a_0\rangle, \dots, \rho\langle a_{n-1}\rangle) \tag{3}$$

for any $n \in \mathbb{N}$, $\mathbf{p} \in \mathbf{P}^{(n)}(\tau)$, and $a_0, \dots, a_{n-1} \in A$.

REMARK 5. The inclusion (3) holds if we replace the n-ary term functions $(\mathbf{p})_{\mathfrak{P}^*(\mathfrak{A})}$ and $(\mathbf{p})_{\mathfrak{P}^*(\mathfrak{A}/\rho)}$ by the n-ary polynomial functions

$$p \in P_A^{(n)}(\mathfrak{P}^*(\mathfrak{A})) \quad \text{and} \quad p' \in P_{A/\rho}^{(n)}(\mathfrak{P}^*(\mathfrak{A}/\rho)),$$

respectively, where the polynomial function p' (which corresponds to p) is given as follows

(i) if $p = c_a^n$, then $p' = c_{\rho\langle a\rangle}^n$;
(ii) if $p = e_i^n = (\mathbf{x}_i)_{\mathfrak{P}^*(\mathfrak{A})}$, then $p' = e_i^n = (\mathbf{x}_i)_{\mathfrak{P}^*(\mathfrak{A}/\rho)}$;
(iii) if $p = f_\gamma(p_0, \dots, p_{n_\gamma-1})$ and the polynomial functions which correspond to $p_0, \dots, p_{n_\gamma-1} \in P_A^{(n)}(\mathfrak{P}^*(\mathfrak{A}))$ are $p'_0, \dots, p'_{n_\gamma-1} \in P_{A/\rho}^{(n)}(\mathfrak{P}^*(\mathfrak{A}/\rho))$ respectively, then $p' = f_\gamma(p'_0, \dots, p'_{n_\gamma-1})$.

Since the polynomial function p' is obtained by using the same steps as in the construction of p, if we take into account (i) from above, and write p instead of p' then

$$\left\{\rho\langle a\rangle \mid a \in p(a_0, \dots, a_{n-1})\right\} \subseteq p(\rho\langle a_0\rangle, \dots, \rho\langle a_{n-1}\rangle). \tag{3'}$$

Let $\mathbf{q}, \mathbf{r} \in \mathbf{P}^{(n)}(\tau)$ and let $\mathfrak{A}$ be a multialgebra of type τ. We say that the n-ary (strong) identity $\mathbf{q} = \mathbf{r}$ is satisfied in the multialgebra $\mathfrak{A}$ if

$$q(a_0, \dots, a_{n-1}) = r(a_0, \dots, a_{n-1})$$

for all $a_0, \dots, a_{n-1} \in A$, where $q = (\mathbf{q})_{\mathfrak{P}^*(\mathfrak{A})}$ and $r = (\mathbf{r})_{\mathfrak{P}^*(\mathfrak{A})}$. We also say that a weak identity $\mathbf{q} \cap \mathbf{r} \neq \emptyset$ is satisfied in the multialgebra $\mathfrak{A}$ if

$$q(a_0, \dots, a_{n-1}) \cap r(a_0, \dots, a_{n-1}) \neq \emptyset$$

for all $a_0, \dots, a_{n-1} \in A$.

REMARK 6. Many important particular multialgebras can be defined by using identities.

3. On the factor multialgebra of an algebra

THEOREM 3.1 ([8]). *For any multialgebra $\mathfrak{A}$ of type τ, there exists a universal algebra $\mathfrak{B}$ of type τ and an equivalence relation ρ on B such that $\mathfrak{A} \cong \mathfrak{B}/\rho$.*

Since $\mathfrak{B}$ is a universal algebra we can rewrite the multioperations from $\mathfrak{B}/\rho$ as follows, for any $\gamma < o(\tau)$ and any $b_0, \dots, b_{n_\gamma - 1} \in B$,

$$\begin{aligned} &f_\gamma(\rho\langle b_0\rangle, \dots, \rho\langle b_{n_\gamma-1}\rangle) \\ &\quad = \left\{\rho\langle c\rangle \mid c = f_\gamma(c_0, \dots, c_{n_\gamma-1}),\ b_i \rho c_i,\ i \in \{0, \dots, n_\gamma - 1\}\right\}. \end{aligned}$$

REMARK 7. If we consider p as in Remark 5 and if $b_0, \dots, b_{n-1} \in B$, then we have

$$\begin{aligned} &p(\rho\langle b_0\rangle, \dots, \rho\langle b_{n-1}\rangle) \\ &\quad \supseteq \left\{\rho\langle c\rangle \mid c = p(c_0, \dots, c_{n-1}),\ b_i \rho c_i,\ i \in \{0, \dots, n - 1\}\right\}. \end{aligned} \tag{4}$$

REMARK 8. It follows immediately that if $n \in \mathbb{N}$, $\mathbf{q}, \mathbf{r} \in \mathbf{P}^{(n)}(\tau)$, and $\mathbf{q} = \mathbf{r}$ is satisfied on the universal algebra $\mathfrak{B}$, then for any $b_0, \dots, b_{n-1} \in B$ the class of

$$q(b_0, \dots, b_{n-1}) = r(b_0, \dots, b_{n-1})$$

modulo ρ is in $q(\rho\langle b_0\rangle, \dots, \rho\langle b_{n-1}\rangle) \cap r(\rho\langle b_0\rangle, \dots, \rho\langle b_{n-1}\rangle)$, thus the weak identity $\mathbf{q} \cap \mathbf{r} \neq \emptyset$ is satisfied on the multialgebra $\mathfrak{B}/\rho$.

Next, we give an example which shows that, in general, the inclusion (4) is not an equality. It also follows that the above weak identity on the multialgebra $\mathfrak{B}/\rho$ does not need to be strong.

EXAMPLE 1. Let $(\mathbb{Z}_5, +)$ be the cyclic group of order 5 and let us consider on $\mathbb{Z}_5$ the equivalence relation $\rho = \{0, 1\} \times \{0, 1\} \cup \{2\} \times \{2\} \cup \{3, 4\} \times \{3, 4\}$. Then $\rho\langle 0\rangle = \rho\langle 1\rangle = \{0, 1\}$, $\rho\langle 2\rangle = \{2\}$, and $\rho\langle 3\rangle = \rho\langle 4\rangle = \{3, 4\}$. Using the above considerations we obtain a multialgebra with one binary multioperation (that is, a hypergroupoid) on $\mathbb{Z}_5/\rho = \{\{0, 1\}, \{2\}, \{3, 4\}\}$ with the table

$+$	$\{0, 1\}$	$\{2\}$	$\{3, 4\}$
$\{0, 1\}$	$\{0, 1\}$, $\{2\}$	$\{2\}$, $\{3, 4\}$	$\{0, 1\}$, $\{3, 4\}$
$\{2\}$	$\{2\}$, $\{3, 4\}$	$\{3, 4\}$	$\{0, 1\}$
$\{3, 4\}$	$\{0, 1\}$, $\{3, 4\}$	$\{0, 1\}$	$\{0, 1\}$, $\{2\}$, $\{3, 4\}$

The inclusion (4) is not always an equality since

$$\begin{aligned}\{\rho\langle c\rangle \mid c=(b_0+b_1)+b_2, b_0=b_1=2, b_2\in\{3,4\}\} &= \{\rho\langle c\rangle \mid c\in\{2,3\}\}\\ &= \{\rho\langle 2\rangle, \rho\langle 3\rangle\}\end{aligned}$$

and

$$(\rho\langle 2\rangle+\rho\langle 2\rangle)+\rho\langle 3\rangle=\rho\langle 3\rangle+\rho\langle 3\rangle=\{\rho\langle 0\rangle, \rho\langle 2\rangle, \rho\langle 3\rangle\}.$$

We also have $\rho\langle 2\rangle + (\rho\langle 2\rangle + \rho\langle 3\rangle) = \rho\langle 2\rangle + \rho\langle 0\rangle = \{\rho\langle 2\rangle, \rho\langle 3\rangle\}$. Thus the associativity holds only in a weak manner for the hypergroupoid $(\mathbb{Z}_5/\rho, +)$.

REMARK 9. Some identities, such as those which characterize the commutativity of an operation of a universal algebra, hold strongly on the factor multialgebra.

EXAMPLE 2. Let $\mathfrak{B}$ be a universal algebra of type τ. Let $\gamma < o(\tau)$ and assume that for a permutation σ of the set $\{0, \dots, n_\gamma - 1\}$,

$$\mathbf{f}_\gamma(\mathbf{x}_0, \dots, \mathbf{x}_{n_\gamma-1}) = \mathbf{f}_\gamma(\mathbf{x}_{\sigma(0)}, \dots, \mathbf{x}_{\sigma(n_\gamma-1)})$$

is satisfied on $\mathfrak{B}$.

Consider $b_0, \dots, b_{n_\gamma-1} \in B$. If $\rho\langle c\rangle \in f_\gamma(\rho\langle b_0\rangle, \dots, \rho\langle b_{n_\gamma-1}\rangle)$, then there exist $c_0, \dots, c_{n_\gamma-1} \in B$ with $b_0\rho c_0, \dots, b_{n_\gamma-1}\rho c_{n_\gamma-1}$ such that $c = f_\gamma(c_0, \dots, c_{n_\gamma-1})$. However,

$$b_{\sigma(0)}\rho c_{\sigma(0)}, \dots, b_{\sigma(n_\gamma-1)}\rho c_{\sigma(n_\gamma-1)} \quad\text{and}\quad f_\gamma(c_0, \dots, c_{n_\gamma-1}) = f_\gamma(c_{\sigma(0)}, \dots, c_{\sigma(n_\gamma-1)}),$$

thus $\rho\langle c\rangle \in f_\gamma(\rho\langle b_{\sigma(0)}\rangle, \dots, \rho\langle b_{\sigma(n_\gamma-1)}\rangle)$. Clearly, the identity

$$\mathbf{f}_\gamma(\mathbf{x}_{\sigma^{-1}(0)}, \dots, \mathbf{x}_{\sigma^{-1}(n_\gamma-1)}) = \mathbf{f}_\gamma(\mathbf{x}_0, \dots, \mathbf{x}_{n_\gamma-1})$$

also holds on the universal algebra $\mathfrak{B}$, hence

$$f_\gamma(\rho\langle b_{\sigma(0)}\rangle, \dots, \rho\langle b_{\sigma(n_\gamma-1)}\rangle) \subseteq f_\gamma(\rho\langle b_0\rangle, \dots, \rho\langle b_{n_\gamma-1}\rangle).$$

It means that for any $b_0, \dots, b_{n_\gamma-1} \in B$, we have

$$f_\gamma(\rho\langle b_0\rangle, \dots, \rho\langle b_{n_\gamma-1}\rangle) = f_\gamma(\rho\langle b_{\sigma(0)}\rangle, \dots, \rho\langle b_{\sigma(n_\gamma-1)}\rangle),$$

hence the identity $\mathbf{f}_\gamma(\mathbf{x}_0, \dots, \mathbf{x}_{n_\gamma-1}) = \mathbf{f}_\gamma(\mathbf{x}_{\sigma(0)}, \dots, \mathbf{x}_{\sigma(n_\gamma-1)})$ is satisfied on the multialgebra $\mathfrak{B}/\rho$.

Now we describe the factor multialgebras of semigroups, groups, abelian groups, rings and lattices.

3.1. The case of semigroups Consider a semigroup $(S, \cdot)$ and an equivalence relation ρ on S. According to Remark 8, the hypergroupoid $(S/\rho, \cdot)$ satisfies the associativity in a weak manner. Such a hyperstructure is called H_v-semigroup (see [14]).

3.2. The case of groups Consider a group $(G, \cdot)$ and let ρ be an equivalence relation on G. The existence and uniqueness of solutions of the equations $ya = b$ and $ax = b$ allow us to define on G the operations / and \ by

$$b/a = \{y \in G \mid b = ya\} \quad \text{and} \quad a\backslash b = \{x \in G \mid b = ax\}.$$

So, the group G can be seen as a universal algebra $(G, \cdot, /, \backslash)$ satisfying the following identities

$$(\mathbf{x}_0 \cdot \mathbf{x}_1) \cdot \mathbf{x}_2 = \mathbf{x}_0 \cdot (\mathbf{x}_1 \cdot \mathbf{x}_2), \quad \mathbf{x}_1 = \mathbf{x}_0 \cdot (\mathbf{x}_0 \backslash \mathbf{x}_1), \quad \mathbf{x}_1 = (\mathbf{x}_1/\mathbf{x}_0) \cdot \mathbf{x}_0,$$
$$\mathbf{x}_1 = \mathbf{x}_0 \backslash (\mathbf{x}_0 \cdot \mathbf{x}_1), \quad \mathbf{x}_1 = (\mathbf{x}_1 \cdot \mathbf{x}_0)/\mathbf{x}_0.$$

We obtain the multialgebra $(G/\rho, \cdot, /, \backslash)$, which satisfies the above identities in a weak manner. It follows that $(G/\rho, \cdot)$ is an H_v-semigroup satisfying

$$\rho\langle a\rangle \cdot G/\rho = G/\rho = G/\rho \cdot \rho\langle a\rangle, \quad \text{for any } a \in G,$$

that is, is an H_v-group (see [14]). In general, an H_v-group does not have an identity element. In our case the class $\rho\langle 1\rangle$ of the identity element of G satisfies the condition

$$\rho\langle a\rangle \in \rho\langle a\rangle \cdot \rho\langle 1\rangle \cap \rho\langle 1\rangle \cdot \rho\langle a\rangle, \text{ for any } a \in G,$$

hence the weak identities $\mathbf{x}_0 \cdot \mathbf{1} \cap \mathbf{x}_0 \neq \emptyset$ and $\mathbf{1} \cdot \mathbf{x}_0 \cap \mathbf{x}_0 \neq \emptyset$ are satisfied on the H_v-group G/ρ. Moreover, any class $\rho\langle a\rangle \in G/\rho$ has an inverse since

$$\rho\langle 1\rangle \in \rho\langle a^{-1}\rangle \cdot \rho\langle a\rangle \cap \rho\langle a\rangle \cdot \rho\langle a^{-1}\rangle$$

(a^{-1} denotes the inverse of a in G).

If the group G is abelian, then the H_v-group G/ρ is commutative (see Example 2).

3.3. The case of rings In [14], a hyperstructure $(R, +, \cdot)$ is called *H_v-ring* if $(R, +)$ is an H_v-group, $(R, \cdot)$ is an H_v-semigroup and for any $a, b, c \in R$ we have

$$a(b + c) \cap (ab + ac) \neq \emptyset \quad \text{and} \quad (b + c)a \cap (ba + ca) \neq \emptyset.$$

It is easy to see that the factor multialgebra of a ring is an H_v-ring (for which the first multioperation is commutative).

REMARK 10. Since the absorption (which is required in the definition of a hyperlattice, see [1]) is not satisfied in a strong manner in the factor multialgebra, the factor multialgebra of a lattice is not necessarily a hyperlattice.

EXAMPLE 3. Consider the lattice $(\mathbb{N}, \wedge, \vee)$, where $\mathbb{N} = \{0, 1, 2, \ldots\}$ is the set of the nonnegative integers, $a \wedge b = \gcd(a, b)$, and $a \vee b = \operatorname{lcm}(a, b)$. Let us denote by $\mathbb{P}$ the set $\{2, 3, 5, \ldots\}$ of prime numbers and consider

$$\rho = \mathbb{P} \times \mathbb{P} \cup \{(a, a) \mid a \in \mathbb{N} \setminus \mathbb{P}\}.$$

Clearly, ρ is an equivalence relation on $\mathbb{N}$ and we have

$$\begin{aligned}\rho\langle 2\rangle \in \rho\langle 2\rangle \vee (\rho\langle 2\rangle \wedge \rho\langle 6\rangle) &= \rho\langle 2\rangle \vee \{\rho\langle 1\rangle, \rho\langle 2\rangle\} \\ &= \{\rho\langle 2\rangle\} \cup \{\rho\langle pq\rangle \mid p, q \in \mathbb{P}, p \neq q\},\end{aligned}$$

so the absorption holds only in a weak manner.

4. A class of equivalence relations on a multialgebra

Let ρ be an equivalence relation on the set A. We denote by $\overline{\overline{\rho}}$ the relation defined on $P^*(A)$ as follows. If $X, Y \in P^*(A)$, then

$$X \overline{\overline{\rho}} Y \iff x\rho y,\ \forall x \in X,\ \forall y \in Y\ (\iff X \times Y \subseteq \rho).$$

It follows immediately that $\overline{\overline{\rho}}$ is symmetric and transitive. In general, $\overline{\overline{\rho}}$ is not reflexive. Indeed, let us take, for example, the equality relation on A, denoted here by δ_A. The relation $\overline{\overline{\delta_A}}$ is reflexive if and only if $|A| = 1$.

PROPOSITION 4.1. *Let $\mathfrak{A} = (A, (f_\gamma)_{\gamma<o(\tau)})$ be a multialgebra of type τ and let ρ be an equivalence relation on A. The following conditions are equivalent:*

(a) *$\mathfrak{A}/\rho$ is a universal algebra;*

(b) *If $\gamma < o(\tau)$, $a, b, x_i \in A$, $i \in \{0, \ldots, n_\gamma - 1\}$, with $a\rho b$, then*

$$f_\gamma(x_0, \ldots, x_{i-1}, a, x_{i+1}, \ldots, x_{n_\gamma-1}) \overline{\overline{\rho}} f_\gamma(x_0, \ldots, x_{i-1}, b, x_{i+1}, \ldots, x_{n_\gamma-1}),$$

for all $i \in \{0, \ldots, n_\gamma - 1\}$;

(c) *If $\gamma < o(\tau)$ and $x_i, y_i \in A$ with $x_i \rho y_i$ for any $i \in \{0, \ldots, n_\gamma - 1\}$, then*

$$f_\gamma(x_0, \ldots, x_{n_\gamma-1}) \overline{\overline{\rho}} f_\gamma(y_0, \ldots, y_{n_\gamma-1});$$

(d) *If $n \in \mathbb{N}$, $\mathbf{p} \in P_A^{(n)}(\mathfrak{P}^*(\mathfrak{A}))$, and $x_i, y_i \in A$ with $x_i \rho y_i$ for any $i \in \{0, \ldots, n-1\}$, then $p(x_0, \ldots, x_{n-1}) \overline{\overline{\rho}} p(y_0, \ldots, y_{n-1})$.*

PROOF. (a) if and only if (b) was proved in [2, Remark 12.a]. From the proof of (b) implies (a) it follows that (b) implies (c) and (c) implies (a). Since (d) implies (c) is obvious, it is enough to show that (c) implies (d) to complete the proof. This was proved in [2, Theorem 13] for the case when $p \in P^{(n)}(\mathfrak{P}^*(\mathfrak{A}))$. The reasoning is almost the same, that is why we skip it here. □

EXAMPLE 4. A hypergroupoid $(H, \cdot)$ is called semihypergroup if

$$(a \cdot b) \cdot c = a \cdot (b \cdot c), \quad \text{for any } a, b, c \in H.$$

An equivalence relation ρ on a semihypergroup $(H, \cdot)$ is called *strongly regular* if for any $a, b, x \in H$ with $a \rho b$ we have $ax\overline{\overline{\rho}}bx$ and $xa\overline{\overline{\rho}}xb$. It is clear that the strongly regular equivalences of a semihypergroup are those relations ρ for which $(H/\rho, \cdot)$ is a groupoid. Note that $(H/\rho, \cdot)$ is a semigroup (see [4, Theorem 31]).

REMARK 11. If the equivalence relation ρ satisfies one of the equivalent conditions in Proposition 4.1, then the operations in the factor multialgebra (which is a universal algebra) are defined as follows: if $\gamma < o(\tau)$, $a_0, \ldots, a_{n_\gamma - 1} \in A$, and $b \in f_\gamma(a_0, \ldots, a_{n_\gamma - 1})$, then

$$f_\gamma(\rho\langle a_0\rangle, \ldots, \rho\langle a_{n_\gamma - 1}\rangle) = \rho\langle b\rangle. \tag{5}$$

LEMMA 4.2. *Let $\mathfrak{A} = (A, (f_\gamma)_{\gamma < o(\tau)})$ be a multialgebra of type τ. The set $E_{ua}(\mathfrak{A})$ of the equivalence relations ρ on A for which $\mathfrak{A}/\rho$ is a universal algebra is an algebraic closure system on $A \times A$.*

PROOF. Consider a family $(\rho_i \mid i \in I)$ of relations from $E_{ua}(\mathfrak{A})$, let $\gamma < o(\tau)$ and $x_j, y_j \in A$ $(j \in \{0, \ldots, n_\gamma - 1\})$.

If we assume that $(x_j, y_j) \in \bigcap_{i \in I} \rho_i$ for all $j \in \{0, \ldots, n_\gamma - 1\}$, it follows that $x_j \rho_i y_j$ for any $i \in I$ and any $j \in \{0, \ldots, n_\gamma - 1\}$. Thus, for any $i \in I$ we have

$$f_\gamma(x_0, \ldots, x_{n_\gamma - 1})\overline{\overline{\rho_i}} f_\gamma(y_0, \ldots, y_{n_\gamma - 1}).$$

Therefore, $x \rho_i y$ for any $i \in I$, $x \in f_\gamma(x_0, \ldots, x_{n_\gamma - 1})$, and $y \in f_\gamma(y_0, \ldots, y_{n_\gamma - 1})$. It means that $(x, y) \in \bigcap_{i \in I} \rho_i$ for any $x \in f_\gamma(x_0, \ldots, x_{n_\gamma - 1})$ and $y \in f_\gamma(y_0, \ldots, y_{n_\gamma - 1})$ or, equivalently,

$$(f_\gamma(x_0, \ldots, x_{n_\gamma - 1}), f_\gamma(y_0, \ldots, y_{n_\gamma - 1})) \in \overline{\overline{\bigcap_{i \in I} \rho_i}}.$$

Thus, we have proved that $\bigcap_{i \in I} \rho_i \in E_{ua}(\mathfrak{A})$.

If $(x_j, y_j) \in \bigcup_{i \in I} \rho_i$ for all $j \in \{0, \ldots, n_\gamma - 1\}$, then for each $j \in \{0, \ldots, n_\gamma - 1\}$ there exists $i_j \in I$ such that $(x_j, y_j) \in \rho_{i_j}$. Assuming the ordered set $(\{\rho_i \mid i \in I\}, \subseteq)$ is directed and $I \neq \emptyset$, we obtain an element $m \in I$ such that $\rho_{i_j} \subseteq \rho_m$ for all $j \in \{0, \ldots, n_\gamma - 1\}$. Consequently, we have $f_\gamma(x_0, \ldots, x_{n_\gamma - 1})\overline{\overline{\rho_m}} f_\gamma(y_0, \ldots, y_{n_\gamma - 1})$. Hence $(x, y) \in \rho_m \subseteq \bigcup_{i \in I} \rho_i$ for any $x \in f_\gamma(x_0, \ldots, x_{n_\gamma - 1})$ and $y \in f_\gamma(y_0, \ldots, y_{n_\gamma - 1})$, which means that

$$(f_\gamma(x_0, \ldots, x_{n_\gamma - 1}), f_\gamma(y_0, \ldots, y_{n_\gamma - 1})) \in \overline{\overline{\bigcup_{i \in I} \rho_i}}.$$

Thus $\bigcup_{i \in I} \rho_i \in E_{ua}(\mathfrak{A})$. $\square$

COROLLARY 4.3. *Let* $\mathfrak{A} = (A, (f_\gamma)_{\gamma<o(\tau)})$ *be a multialgebra of type* τ. *If* $R \subseteq A \times A$, *then the relation* $\alpha(R) = \bigcap\{\rho \in E_{ua}(\mathfrak{A}) \mid R \subseteq \rho\}$ *is the smallest equivalence relation on A containing R for which the factor multialgebra is a universal algebra.*

REMARK 12. If the multialgebra $\mathfrak{A}$ is not a universal algebra, then the smallest element of $E_{ua}(\mathfrak{A})$ is not δ_A.

For a multialgebra $\mathfrak{A}$ the smallest equivalence from $E_{ua}(\mathfrak{A})$ will be denoted by α^* (or $\alpha^*_{\mathfrak{A}}$ when necessary) and it will be called *the fundamental relation of* $\mathfrak{A}$. We recall that the fundamental relation α^* of the multialgebra $\mathfrak{A}$ is the transitive closure of the relation α defined by $x\alpha y$ if and only if $x, y \in p(a_0, \dots, a_{n-1})$ for some $n \in \mathbb{N}$, $p \in P_A^{(n)}(\mathfrak{P}^*(\mathfrak{A}))$, and $a_0, \dots, a_{n-1} \in A$ (see [10]).

Let $n \in \mathbb{N}$ and $\mathbf{q}, \mathbf{r} \in \mathbf{P}^{(n)}(\tau)$. In [11, Proposition 3] we proved that if the weak identity $\mathbf{q} \cap \mathbf{r} \neq \emptyset$ is satisfied on the multialgebra $\mathfrak{A}$, then the identity $\mathbf{q} = \mathbf{r}$ is satisfied in the factor multialgebra $\mathfrak{A}/\alpha^*$ (which is a universal algebra). This happens because α^* contains the relation

$$R_{\mathbf{qr}} = \left\{(x, y) \in A \times A \;\middle|\; \begin{array}{l} x \in q(a_0, \dots, a_{n-1}),\ y \in r(a_0, \dots, a_{n-1}), \\ a_0, \dots, a_{n-1} \in A \end{array}\right\}.$$

REMARK 13. Even if the weak identity $\mathbf{q} \cap \mathbf{r} \neq \emptyset$ is not satisfied on the multialgebra $\mathfrak{A}$, we can obtain a factor multialgebra of $\mathfrak{A}$ that is a universal algebra satisfying the identity $\mathbf{q} = \mathbf{r}$ by taking the factor multialgebra determined by a relation from $E_{ua}(\mathfrak{A})$ which contains $R_{\mathbf{qr}}$. Also, each relation from $E_{ua}(\mathfrak{A})$ that gives a factor multialgebra satisfying the identity $\mathbf{q} = \mathbf{r}$ must contain the relation $R_{\mathbf{qr}}$. It means that the smallest relation from $E_{ua}(\mathfrak{A})$ for which the factor multialgebra is a universal algebra satisfying the identity $\mathbf{q} = \mathbf{r}$ is the relation $\alpha(R_{\mathbf{qr}})$. From now on, we will denote the relation $\alpha(R_{\mathbf{qr}})$ by $\alpha^*_{\mathbf{qr}}$.

REMARK 14. It follows immediately that $\alpha^* = \alpha(\emptyset) = \alpha(\delta_A) = \alpha^*_{\mathbf{x}_0\mathbf{x}_0}$, and it is obvious that $\alpha^* \subseteq \alpha^*_{\mathbf{qr}}$ for any $\mathbf{q}, \mathbf{r} \in \mathbf{P}^{(n)}(\tau)$.

We give a characterization of the relation $\alpha^*_{\mathbf{qr}}$ in terms of unary polynomial functions from $P_A^{(1)}(\mathfrak{P}^*(\mathfrak{A}))$.

REMARK 15. By the construction of the polynomial functions from $P_A^{(1)}(\mathfrak{P}^*(\mathfrak{A}))$ it follows that the usual mapping composition of two elements from $P_A^{(1)}(\mathfrak{P}^*(\mathfrak{A}))$ is an element of $P_A^{(1)}(\mathfrak{P}^*(\mathfrak{A}))$.

Indeed, if $f, p \in P_A^{(1)}(\mathfrak{P}^*(\mathfrak{A}))$, then $f \circ p : P^*(A) \to P^*(A)$, and we have:

(i) if $a \in A$, $f = c_a^1$ and $X \in P^*(A)$ is arbitrary, then $(f \circ p)(X) = c_a^1(p(X)) = a = c_a^1(X)$. Thus $f \circ p = c_a^1 \in P_A^{(1)}(\mathfrak{P}^*(\mathfrak{A}))$;

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(ii) if $f = e_0^1$ and $X \in P^*(A)$ is arbitrary, then $(f \circ p)(X) = e_0^1(p(X)) = p(X)$. Thus $f \circ p = p$;

(iii) if $\gamma < o(\tau)$ and for $f^0, \dots, f^{n_\gamma - 1} \in P_A^{(1)}(\mathfrak{P}^*(\mathfrak{A}))$ was proved that

$$f^0 \circ p = p_0 \in P_A^{(1)}(\mathfrak{P}^*(\mathfrak{A})), \dots, f^{n_\gamma - 1} \circ p = p_{n_\gamma - 1} \in P_A^{(1)}(\mathfrak{P}^*(\mathfrak{A})),$$

then for $f = f_\gamma(f^0, \dots, f^{n_\gamma - 1})$ we have

$$\begin{aligned}(f \circ p)(X) = f(p(X)) &= f_\gamma(f^0, \dots, f^{n_\gamma - 1})(p(X)) \\ &= f_\gamma(f^0(p(X)), \dots, f^{n_\gamma - 1}(p(X))) \\ &= f_\gamma(p_0(X), \dots, p_{n_\gamma - 1}(X)) \\ &= f_\gamma(p_0, \dots, p_{n_\gamma - 1})(X),\end{aligned}$$

and thus $f \circ p = f_\gamma(p_0, \dots, p_{n_\gamma - 1}) \in P_A^{(1)}(\mathfrak{P}^*(\mathfrak{A}))$.

Now we can prove the main result of this section.

THEOREM 4.4. *Let $n \in \mathbb{N}$, $\mathbf{q}, \mathbf{r} \in \mathbf{P}^{(n)}(\tau)$, let $\mathfrak{A} = (A, (f_\gamma)_{\gamma < o(\tau)})$ be a multialgebra of type τ, and let $\alpha_{\mathbf{qr}} \subseteq A \times A$ be the relation defined by*

$$\begin{aligned} x\alpha_{\mathbf{qr}}y \;\Leftrightarrow\; &\exists\, p \in P_A^{(1)}(\mathfrak{P}^*(\mathfrak{A})),\ \exists\, a_0, \dots, a_{n-1} \in A \text{ such that} \\ &x \in p(q(a_0, \dots, a_{n-1})),\ y \in p(r(a_0, \dots, a_{n-1})) \text{ or} \\ &y \in p(q(a_0, \dots, a_{n-1})),\ x \in p(r(a_0, \dots, a_{n-1})).\end{aligned}$$

The relation $\alpha^*_{\mathbf{qr}}$ is the transitive closure of the relation $\alpha_{\mathbf{qr}}$.

PROOF. Let $\alpha'_{\mathbf{qr}}$ be the transitive closure of the relation $\alpha_{\mathbf{qr}}$. We will show that $\alpha^*_{\mathbf{qr}} = \alpha'_{\mathbf{qr}}$ by proving the following statements:

(A) The relation $\alpha'_{\mathbf{qr}}$ is an equivalence relation on A containing $R_{\mathbf{qr}}$.

(B) If $f \in P_A^{(1)}(\mathfrak{P}^*(\mathfrak{A}))$, and for the elements $a, b \in A$ we have $a\alpha'_{\mathbf{qr}}b$, then $f(a)\overline{\overline{\alpha'_{\mathbf{qr}}}}f(b)$.

(C) The factor multialgebra $\mathfrak{A}/\alpha'_{\mathbf{qr}}$ is a universal algebra satisfying the identity $\mathbf{q} = \mathbf{r}$.

(D) If $\rho \in E_{ua}(\mathfrak{A})$ such that the identity $\mathbf{q} = \mathbf{r}$ holds on the universal algebra $\mathfrak{A}/\rho$, then $\alpha'_{\mathbf{qr}} \subseteq \rho$.

PROOF OF (A). It is obvious that $\alpha_{\mathbf{qr}}$ is symmetric. Taking $p = c_a^1$ for any $a \in A$, we obtain the reflexivity of $\alpha_{\mathbf{qr}}$. Thus $\alpha'_{\mathbf{qr}}$ is the smallest equivalence relation on A containing $\alpha_{\mathbf{qr}}$. The inclusion $R_{\mathbf{qr}} \subseteq \alpha'_{\mathbf{qr}}$ follows by considering $p = e_0^1$.

PROOF OF (B). From $a\alpha'_{\mathbf{qr}}b$ it follows that there exist $m \in \mathbb{N}^*$ and $z_0, \dots, z_{m-1} \in A$ such that $a = z_0\alpha_{\mathbf{qr}}z_1\alpha_{\mathbf{qr}} \dots \alpha_{\mathbf{qr}}z_{m-1} = b$. Let us consider $j \in \{0, \dots, m-2\}$. Since

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$z_j\alpha_{\mathbf{qr}}z_{j+1}$, there exist $p \in P_A^{(1)}(\mathfrak{P}^*(\mathfrak{A}))$ and $a_0, \dots, a_{n-1} \in A$ such that

$$z_j \in p(q(a_0, \dots, a_{n-1})), \quad z_{j+1} \in p(r(a_0, \dots, a_{n-1})) \quad \text{or}$$
$$z_{j+1} \in p(q(a_0, \dots, a_{n-1})), \quad z_j \in p(r(a_0, \dots, a_{n-1})).$$

However, $p' = f \circ p \in P_A^{(1)}(\mathfrak{P}^*(\mathfrak{A}))$ and we have

$$f(z_j) \subseteq p'(q(a_0, \dots, a_{n-1})), \quad f(z_{j+1}) \subseteq p'(r(a_0, \dots, a_{n-1})) \quad \text{or}$$
$$f(z_{j+1}) \subseteq p'(q(a_0, \dots, a_{n-1})), \quad f(z_j) \subseteq p'(r(a_0, \dots, a_{n-1})).$$

However, for any $u_j \in f(z_j)$ and any $u_{j+1} \in f(z_{j+1})$ we have $u_j\alpha_{\mathbf{qr}}u_{j+1}$ and, consequently, $u_0\alpha'_{\mathbf{qr}}u_{m-1}$. Since $u_0 \in f(a)$ and $u_{m-1} \in f(b)$ are arbitrary, we obtain

$$f(a)\overline{\overline{\alpha'_{\mathbf{qr}}}}f(b).$$

PROOF OF (C). Consider $\gamma < o(\tau)$ and the arbitrary elements $a, b, x_0, \dots, x_{n_\gamma-1} \in A$ such that $a\alpha'_{\mathbf{qr}}b$. Applying (B) to the unary polynomial functions (from $P_A^{(1)}(\mathfrak{P}^*(\mathfrak{A}))$)

$$f_\gamma(e_0^1, c_{x_1}^1, \dots, c_{x_{n_\gamma-1}}^1),\ f_\gamma(c_{x_0}^1, e_0^1, c_{x_2}^1, \dots, c_{x_{n_\gamma-1}}^1), \dots,\ f_\gamma(c_{x_0}^1, \dots, c_{x_{n_\gamma-2}}^1, e_0^1),$$

it follows that $\alpha'_{\mathbf{qr}}$ verifies (b) from Proposition 4.1, hence $\mathfrak{A}/\alpha'_{\mathbf{qr}}$ is a universal algebra. Remark 13 and (A) complete the proof of (C).

PROOF OF (D). Let ρ be a relation from $E_{ua}(\mathfrak{A})$ such that the identity $\mathbf{q} = \mathbf{r}$ is satisfied on the universal algebra $\mathfrak{A}/\rho$, let $p \in P^{(1)}(\mathfrak{P}^*(\mathfrak{A}))$, $a_0, \dots, a_{n-1} \in A$, and $x \in p(q(a_0, \dots, a_{n-1}))$, $y \in p(r(a_0, \dots, a_{n-1}))$. We will show by induction over the steps of the construction of a polynomial function from $P_A^{(1)}(\mathfrak{P}^*(\mathfrak{A}))$ that $x\rho y$.

If $a \in A$ and $p = c_a^1$, then $x = y = a$ and $x\rho y$.

If $p = e_0^1$, then $x \in q(a_0, \dots, a_{n-1})$, $y \in r(a_0, \dots, a_{n-1})$ and using Remark 13 we have $(x, y) \in R_{\mathbf{qr}} \subseteq \rho$.

Assume that the statement is true for $p_0, \dots, p_{n_\gamma-1}$ $(\gamma < o(\tau))$ and consider $p = f_\gamma(p_0, \dots, p_{n_\gamma-1})$. If

$$\begin{aligned} x \in p(q(a_0, \dots, a_{n-1})) &= f_\gamma(p_0, \dots, p_{n_\gamma-1})(q(a_0, \dots, a_{n-1})) \\ &= f_\gamma(p_0(q(a_0, \dots, a_{n-1})), \dots, p_{n_\gamma-1}(q(a_0, \dots, a_{n-1}))) \end{aligned}$$

and

$$\begin{aligned} y \in p(r(a_0, \dots, a_{n-1})) &= f_\gamma(p_0, \dots, p_{n_\gamma-1})(r(a_0, \dots, a_{n-1})) \\ &= f_\gamma(p_0(r(a_0, \dots, a_{n-1})), \dots, p_{n_\gamma-1}(r(a_0, \dots, a_{n-1}))), \end{aligned}$$

then there exist $x_i \in p_i(q(a_0, \dots, a_{n-1}))$, $y_i \in p_i(r(a_0, \dots, a_{n-1}))$, $i \in \{0, \dots, n_\gamma - 1\}$, such that $x \in f_\gamma(x_0, \dots, x_{n_\gamma-1})$ and $y \in f_\gamma(y_0, \dots, y_{n_\gamma-1})$. Since $x_i\rho y_i$ for all $i \in \{0, \dots n_\gamma - 1\}$, using Proposition 4.1, it results $x\rho y$. Analogously, if we take $x \in p(r(a_0, \dots, a_{n-1}))$ and $y \in p(q(a_0, \dots, a_{n-1}))$, then $x\rho y$.

It follows that $\alpha_{\mathbf{qr}} \subseteq \rho$, thus $\alpha'_{\mathbf{qr}} \subseteq \rho$. $\square$

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EXAMPLE 5. Let $(H, \cdot)$ be a semihypergroup. The smallest strongly regular equivalence on H such that the factor semihypergroup is a commutative semigroup was determined in [7]. This relation, denoted by γ^*, is the transitive closure of the relation $\gamma = \bigcup_{n\in\mathbb{N}^*} \gamma_n$, where $\gamma_1 = \delta_H$ and, for any $n > 1$, γ_n is defined by

$$x\gamma_n y \Leftrightarrow \exists(z_1, \ldots, z_n) \in H^n,\ \exists\sigma \in S_n : x \in \prod_{i=1}^{n} z_i,\ \ y \in \prod_{i=1}^{n} z_{\sigma(i)}$$

(S_n denotes the set of the permutations of the set $\{1, \ldots, n\}$). Since the set of cycles $(1, 2), (2, 3), \ldots, (n-1, n)$ generates the group S_n, it follows that γ^* is the transitive closure of the relation $\gamma' = \bigcup_{n\in\mathbb{N}^*} \gamma'_n$ where $\gamma'_1 = \delta_H$ and for $n > 1$,

> $x\gamma'_n y$ if and only if there exist $(z_1, \ldots, z_n) \in H^n$, and $i \in \{1, \ldots, n-1\}$ such that $x \in z_1 \cdots z_{i-1}(z_i z_{i+1})z_{i+2} \cdots z_n$, and $y \in z_1 \cdots z_{i-1}(z_{i+1}z_i)z_{i+2} \cdots z_n$.

Clearly, $\gamma' = \alpha_{\mathbf{qr}}$ with $\mathbf{q} = \mathbf{x}_0\mathbf{x}_1$ and $\mathbf{r} = \mathbf{x}_1\mathbf{x}_0$.

From [7] it follows that if $(H, \cdot)$ is a hypergroup, then the relation γ is transitive and $\gamma^* = \gamma$ is the smallest equivalence relation on H such that H/γ^* is a commutative group.

REMARK 16. If $(H, \cdot)$ is a hypergroup and ρ is a strongly regular equivalence on H, then H/ρ is a group (see [4, Theorem 31]). If $\mathbf{q}, \mathbf{r}$ are two n-ary terms, then the smallest equivalence relation on H such that the factor hypergroupoid is a semigroup satisfying the identity $\mathbf{q} = \mathbf{r}$ is the transitive closure ψ^* of the relation

$$\psi = \bigcup \left\{ \begin{array}{c} p(q(a_1, \ldots, a_n)) \times p(r(a_1, \ldots, a_n)) \\ \cup\, p(r(a_1, \ldots, a_n)) \times p(q(a_1, \ldots, a_n)) \end{array} \middle| \begin{array}{l} p \in P^1_H(\mathfrak{P}^*(H, \cdot)), \\ \text{and } a_1, \ldots, a_n \in H \end{array} \right\}.$$

Since this relation is strongly regular, the factor semihypergroup is a group. It means that ψ^* contains the smallest relation $\alpha^*_{\mathbf{qr}}$ of the hypergroup $(H, \cdot, /, \backslash)$ with the property that the factor hypergroup is a group satisfying the identity $\mathbf{q} = \mathbf{r}$. Since

$$\alpha_{\mathbf{qr}} = \bigcup \left\{ \begin{array}{c} p(q(a_1, \ldots, a_n)) \times p(r(a_1, \ldots, a_n)) \\ \cup p(r(a_1, \ldots, a_n)) \times p(q(a_1, \ldots, a_n)) \end{array} \middle| \begin{array}{l} p \in P^1_H(\mathfrak{P}^*(H, \cdot, /, \backslash)), \\ \text{and } a_1, \ldots, a_n \in H \end{array} \right\}$$

and $P^1_H(\mathfrak{P}^*(H, \cdot)) \subseteq P^1_H(\mathfrak{P}^*(H, \cdot, /, \backslash))$, it follows that $\psi \subseteq \alpha_{\mathbf{qr}}$, thus $\psi^* \subseteq \alpha^*_{\mathbf{qr}}$ and we obtain $\psi^* = \alpha^*_{\mathbf{qr}}$.

So, the smallest strongly regular equivalence on H for which the factor hypergroup satisfies the identity $\mathbf{q} = \mathbf{r}$ can be obtained by considering in Theorem 4.4 only those polynomial functions p that are obtained with the multioperation $\cdot$ (in other words, it is not necessary to use the multioperations / and \ in the construction of p).

It is easy to observe that Theorem 4.4 and Remark 14 lead to the following characterization of the fundamental relation of a multialgebra.

COROLLARY 4.5. *The fundamental relation α^* of a multialgebra $\mathfrak{A}$ is the transitive closure of the relation $\alpha' \subseteq A \times A$ defined by $x\alpha' y$ if and only if there exist $p \in P_A^{(1)}(\mathfrak{P}^*(\mathfrak{A}))$ and $a \in A$ such that $x, y \in p(a)$.*

5. Identities and factor multialgebras

Let $n \in \mathbb{N}$ and $\mathbf{q}, \mathbf{r} \in \mathbf{P}^{(n)}(\tau)$. Let $\mathfrak{B}$ be a universal algebra and ρ an equivalence relation on B. We denote by $\rho_{\mathbf{qr}}$ the smallest equivalence relation on B containing ρ and all the pairs $(q(b_0, \ldots, b_{n-1}), r(b_0, \ldots, b_{n-1}))$ with $b_0, \ldots, b_{n-1} \in B$. We denote by $\theta(\rho_{\mathbf{qr}})$ the smallest congruence relation on $\mathfrak{B}$ containing $\rho_{\mathbf{qr}}$. Clearly $\theta(\rho_{\mathbf{qr}})$ is the smallest congruence relation on $\mathfrak{B}$ containing

$$\rho \cup \{(q(b_0, \ldots, b_{n-1}), r(b_0, \ldots, b_{n-1})) \mid b_0, \ldots, b_{n-1} \in B\}.$$

Theorem 10.4 from [9] presents a characterization for the smallest congruence relation of a universal algebra, which contains a given relation. According to this, $x\theta(\rho_{\mathbf{qr}})y$ if and only if there exist $m \in \mathbb{N}^*$, a sequence $x = t_0, t_1, \ldots, t_m = y$, and pairs of elements

$$(x_i, y_i) \in \rho \cup \{(q(b_0, \ldots, b_{n-1}), r(b_0, \ldots, b_{n-1})) \mid b_0, \ldots, b_{n-1} \in B\}$$

and unary algebraic functions p_i, $i \in \{1, \ldots, m\}$, such that

$$\{p_i(x_i), p_i(y_i)\} = \{t_{i-1}, t_i\}, \ i \in \{1, \ldots, m\}.$$

Clearly, if we take $\mathbf{q} = \mathbf{r} = \mathbf{x}_0$, then $\theta(\rho_{\mathbf{qr}})$ is the smallest congruence relation on $\mathfrak{B}$ which contains ρ. We denote it by $\theta(\rho)$.

LEMMA 5.1. *Let ρ be an equivalence relation on a universal algebra $\mathfrak{B}$. If $n \in \mathbb{N}$, $p \in P_{B/\rho}^{(n)}(\mathfrak{P}^*(\mathfrak{B}/\rho))$, and $x, y, z_0, \ldots, z_{n-1} \in B$ are such that*

$$\rho\langle x\rangle, \rho\langle y\rangle \in p(\rho\langle z_0\rangle, \ldots, \rho\langle z_{n-1}\rangle),$$

then $x\theta(\rho)y$.

PROOF. We prove this lemma by induction over the steps of construction of the polynomial functions from $P_{B/\rho}^{(n)}(\mathfrak{P}^*(\mathfrak{B}/\rho))$.
Step 1. If $p = c^n_{\rho\langle b\rangle}$ for some $b \in B$, then $\rho\langle x\rangle = \rho\langle y\rangle = \rho\langle b\rangle$ and hence $x\theta(\rho)y$.

Step 2. If $p = e_i^n$ for some $i \in \{0, \dots, n-1\}$, then $\rho\langle x\rangle = \rho\langle y\rangle = \rho\langle z_i\rangle$ and hence $x\theta(\rho)y$.
Step 3. We consider the statement proved for $p_0, \dots, p_{n_\gamma-1}$ ($\gamma < o(\tau)$) and we take $p = f_\gamma(p_0, \dots, p_{n_\gamma-1})$. Since

$$p(\rho\langle z_0\rangle, \dots, \rho\langle z_{n-1}\rangle) = f_\gamma(p_0, \dots, p_{n_\gamma-1})(\rho\langle z_0\rangle, \dots, \rho\langle z_{n-1}\rangle)$$
$$= f_\gamma(p_0(\rho\langle z_0\rangle, \dots, \rho\langle z_{n-1}\rangle), \dots, p_{n_\gamma-1}(\rho\langle z_0\rangle, \dots, \rho\langle z_{n-1}\rangle)),$$

from $\rho\langle x\rangle, \rho\langle y\rangle \in p(\rho\langle z_0\rangle, \dots, \rho\langle z_{n-1}\rangle)$, we deduce that there exist some elements $x_i, y_i \in B$ with $\rho\langle x_i\rangle, \rho\langle y_i\rangle \in p_i(\rho\langle z_0\rangle, \dots, \rho\langle z_{n-1}\rangle)$ such that

$$\rho\langle x\rangle \in f_\gamma(\rho\langle x_0\rangle, \dots, \rho\langle x_{n_\gamma-1}\rangle), \quad \rho\langle y\rangle \in f_\gamma(\rho\langle y_0\rangle, \dots, \rho\langle y_{n_\gamma-1}\rangle).$$

From the definition of the multioperation f_γ in $\mathfrak{B}/\rho$ it follows that there exist $x_i', y_i' \in B$ with $x_i\rho x_i'$ and $y_i\rho y_i'$ ($i \in \{0, \dots, n_\gamma-1\}$) such that $x = f_\gamma(x_0', \dots, x_{n_\gamma-1}')$, and $y = f_\gamma(y_0', \dots, y_{n_\gamma-1}')$. Since the statement holds for $p_0, \dots, p_{n_\gamma-1}$, it follows that $x_i\theta(\rho)y_i$ for all $i \in \{0, \dots, n_\gamma-1\}$. Hence $x_i'\rho x_i$, $x_i\theta(\rho)y_i$, $y_i\rho y_i'$ which implies $x_i'\theta(\rho)y_i'$ for all $i \in \{0, \dots, n_\gamma-1\}$. However, $\theta(\rho)$ is a congruence on $\mathfrak{B}$, thus

$$x = f_\gamma(x_0', \dots, x_{n_\gamma-1}')\theta(\rho)f_\gamma(y_0', \dots, y_{n_\gamma-1}') = y,$$

which ends the proof of the lemma. □

LEMMA 5.2. *Let $n \in \mathbb{N}$, $\mathbf{q}, \mathbf{r} \in \mathbf{P}^{(n)}(\tau)$ and let ρ be an equivalence relation on the universal algebra $\mathfrak{B}$. If $p \in P_{B/\rho}^{(1)}(\mathfrak{P}^*(\mathfrak{B}/\rho))$ and $x, y, z_0, \dots, z_{n-1} \in B$ are such that*

$$\rho\langle x\rangle \in p(q(\rho\langle z_0\rangle, \dots, \rho\langle z_{n-1}\rangle)) \quad \text{and} \quad \rho\langle y\rangle \in p(r(\rho\langle z_0\rangle, \dots, \rho\langle z_{n-1}\rangle)),$$

then $x\theta(\rho_{\mathbf{qr}})y$.

PROOF. We prove this lemma by using the steps of construction of the polynomial functions from $P_{B/\rho}^{(1)}(\mathfrak{P}^*(\mathfrak{B}/\rho))$.
Step 1. If $p = c_{\rho\langle b\rangle}^1$ for some $b \in B$, then $\rho\langle x\rangle = \rho\langle y\rangle = \rho\langle b\rangle$ and hence $x\theta(\rho_{\mathbf{qr}})y$.
Step 2. If $p = e_0^1$, then $\rho\langle x\rangle \in q(\rho\langle z_0\rangle, \dots, \rho\langle z_{n-1}\rangle)$ and $\rho\langle y\rangle \in r(\rho\langle z_0\rangle, \dots, \rho\langle z_{n-1}\rangle)$. According to Remark 7 we also have $\rho\langle q(z_0, \dots, z_{n-1})\rangle \in q(\rho\langle z_0\rangle, \dots, \rho\langle z_{n-1}\rangle)$ and $\rho\langle r(z_0, \dots, z_{n-1})\rangle \in r(\rho\langle z_0\rangle, \dots, \rho\langle z_{n-1}\rangle)$, so, using the previous lemma, it follows that $x\theta(\rho)q(z_0, \dots, z_{n-1})$ and $y\theta(\rho)r(z_0, \dots, z_{n-1})$. However, $\theta(\rho) \subseteq \theta(\rho_{\mathbf{qr}})$ and $q(z_0, \dots, z_{n-1})\theta(\rho_{\mathbf{qr}})r(z_0, \dots, z_{n-1})$, thus $x\theta(\rho_{\mathbf{qr}})y$.
Step 3. We consider the statement proved for $p_0, \dots, p_{n_\gamma-1}$ ($\gamma < o(\tau)$) and we take $p = f_\gamma(p_0, \dots, p_{n_\gamma-1})$. Since

$$p(q(\rho\langle z_0\rangle, \dots, \rho\langle z_{n-1}\rangle))$$
$$= f_\gamma(p_0, \dots, p_{n_\gamma-1})(q(\rho\langle z_0\rangle, \dots, \rho\langle z_{n-1}\rangle))$$
$$= f_\gamma(p_0(q(\rho\langle z_0\rangle, \dots, \rho\langle z_{n-1}\rangle)), \dots, p_{n_\gamma-1}(q(\rho\langle z_0\rangle, \dots, \rho\langle z_{n-1}\rangle)))$$

and

$$\begin{aligned}
&p(r(\rho\langle z_0\rangle,\dots,\rho\langle z_{n-1}\rangle))\\
&\quad= f_\gamma(p_0,\dots,p_{n_\gamma-1})(r(\rho\langle z_0\rangle,\dots,\rho\langle z_{n-1}\rangle))\\
&\quad= f_\gamma(p_0(r(\rho\langle z_0\rangle,\dots,\rho\langle z_{n-1}\rangle)),\dots,p_{n_\gamma-1}(r(\rho\langle z_0\rangle,\dots,\rho\langle z_{n-1}\rangle))),
\end{aligned}$$

from $\rho\langle x\rangle \in p(q(\rho\langle z_0\rangle,\dots,\rho\langle z_{n-1}\rangle))$ and $\rho\langle y\rangle \in p(r(\rho\langle z_0\rangle,\dots,\rho\langle z_{n-1}\rangle))$, we deduce that there exist $x_i, y_i \in B$ with $\rho\langle x_i\rangle \in p_i(q(\rho\langle z_0\rangle,\dots,\rho\langle z_{n-1}\rangle))$ and $\rho\langle y_i\rangle \in p_i(r(\rho\langle z_0\rangle,\dots,\rho\langle z_{n-1}\rangle))$ such that

$$\rho\langle x\rangle \in f_\gamma(\rho\langle x_0\rangle,\dots,\rho\langle x_{n_\gamma-1}\rangle),\quad \rho\langle y\rangle \in f_\gamma(\rho\langle y_0\rangle,\dots,\rho\langle y_{n_\gamma-1}\rangle).$$

From the definition of the multioperation f_γ in $\mathfrak{B}/\rho$ it results that there exist $x_i', y_i' \in B$ with $x_i\rho x_i'$ and $y_i\rho y_i'$ ($i \in \{0,\dots,n_\gamma-1\}$) such that $x = f_\gamma(x_0',\dots,x_{n_\gamma-1}')$, and $y = f_\gamma(y_0',\dots,y_{n_\gamma-1}')$. Since the statement holds for $p_0,\dots,p_{n_\gamma-1}$ it follows that $x_i\theta(\rho_{\mathbf{qr}})y_i$ for all $i \in \{0,\dots,n_\gamma-1\}$. Hence $x_i'\rho x_i$, $x_i\theta(\rho_{\mathbf{qr}})y_i$, $y_i\rho y_i'$ which implies $x_i'\theta(\rho_{\mathbf{qr}})y_i'$ for all $i \in \{0,\dots,n_\gamma-1\}$. However, $\theta(\rho_{\mathbf{qr}})$ is a congruence on $\mathfrak{B}$, thus

$$x = f_\gamma(x_0',\dots,x_{n_\gamma-1}')\theta(\rho_{\mathbf{qr}})f_\gamma(y_0',\dots,y_{n_\gamma-1}') = y,$$

which ends the proof of the lemma. □

Now we can prove the main result of this paper.

THEOREM 5.3. *Let $n \in \mathbb{N}$ and $\mathbf{q},\mathbf{r} \in \mathbf{P}^{(n)}(\tau)$. If ρ is an equivalence relation on a universal algebra $\mathfrak{B}$, then* $(\mathfrak{B}/\rho)/\alpha^*_{\mathbf{qr}} \cong \mathfrak{B}/\theta(\rho_{\mathbf{qr}})$.

PROOF. First we will prove that the correspondence $\alpha^*_{\mathbf{qr}}\langle\rho\langle a\rangle\rangle \mapsto \theta(\rho_{\mathbf{qr}})\langle a\rangle$ defines a bijective map $h : (B/\rho)/\alpha^*_{\mathbf{qr}} \to B/\theta(\rho_{\mathbf{qr}})$.

For this, we will show that if $a, b \in B$, we have $\rho\langle a\rangle\alpha^*_{\mathbf{qr}}\rho\langle b\rangle$ if and only if $a\theta(\rho_{\mathbf{qr}})b$.

If $\rho\langle a\rangle\alpha^*_{\mathbf{qr}}\rho\langle b\rangle$, then there exist $m \in \mathbb{N}$ and $z_0,\dots,z_m \in B$ such that

$$\rho\langle a\rangle = \rho\langle z_0\rangle\alpha_{\mathbf{qr}}\rho\langle z_1\rangle\alpha_{\mathbf{qr}}\dots\alpha_{\mathbf{qr}}\rho\langle z_m\rangle = \rho\langle b\rangle.$$

Thus for each $i \in \{1,\dots,m\}$ there exist $p_i \in P^{(1)}_{B/\rho}(\mathfrak{P}^*(\mathfrak{B}/\rho))$ and $z_0^i,\dots,z_{n-1}^i \in B$ such that

$$\begin{aligned}
&\rho\langle z_{i-1}\rangle \in p_i(q(\rho\langle z_0^i\rangle,\dots,\rho\langle z_{n-1}^i\rangle)),\quad \rho\langle z_i\rangle \in p_i(r(\rho\langle z_0^i\rangle,\dots,\rho\langle z_{n-1}^i\rangle))\quad\text{or}\\
&\rho\langle z_{i-1}\rangle \in p_i(r(\rho\langle z_0^i\rangle,\dots,\rho\langle z_{n-1}^i\rangle)),\quad \rho\langle z_i\rangle \in p_i(q(\rho\langle z_0^i\rangle,\dots,\rho\langle z_{n-1}^i\rangle)).
\end{aligned}$$

According to Lemma 5.2 it follows that for any $i \in \{1,\dots,m\}$ we have $z_{i-1}\theta(\rho_{\mathbf{qr}})z_i$. We deduce that $z_0\theta(\rho_{\mathbf{qr}})z_m$. However, $a\rho z_0$, $z_m\rho b$ and $\rho \subseteq \theta(\rho_{\mathbf{qr}})$, thus $a\theta(\rho_{\mathbf{qr}})b$.

Conversely, if $a\theta(\rho_{\mathbf{qr}})b$, there exist $m \in \mathbb{N}$, a sequence $a = t_0, t_1, \ldots, t_m = b$, pairs of elements $(x_i, y_i) \in \rho \cup \{(q(b_0, \ldots, b_{n-1}), r(b_0, \ldots, b_{n-1})) \mid b_0, \ldots, b_{n-1} \in B\}$, and unary algebraic functions p_i, $i \in \{1, \ldots, m\}$, such that

$$\{t_{i-1}, t_i\} = \{p_i(x_i), p_i(y_i)\}, \quad i \in \{1, \ldots, m\}.$$

We have $\{\rho\langle t_{i-1}\rangle, \rho\langle t_i\rangle\} = \{\rho\langle p_i(x_i)\rangle, \rho\langle p_i(y_i)\rangle\}$.

If $(x_i, y_i) \in \rho$, then $\rho\langle x_i\rangle = \rho\langle y_i\rangle$, and since

$$\rho\langle p_i(x_i)\rangle \in p_i(\rho\langle x_i\rangle) = p_i(\rho\langle y_i\rangle) \ni \rho\langle p_i(y_i)\rangle,$$

we deduce that $\rho\langle t_{i-1}\rangle\alpha^*\rho\langle t_i\rangle$, thus $\rho\langle t_{i-1}\rangle\alpha^*_{\mathbf{qr}}\rho\langle t_i\rangle$.

If $(x_i, y_i) = (q(b_0, \ldots, b_{n-1}), r(b_0, \ldots, b_{n-1}))$ for some $b_0, \ldots, b_{n-1} \in B$, from

$$\begin{aligned}\rho\langle p_i(x_i)\rangle &= \rho\langle p_i(q(b_0, \ldots, b_{n-1}))\rangle \in p_i(\rho\langle q(b_0, \ldots, b_{n-1})\rangle)\\ &\subseteq p_i(q(\rho\langle b_0\rangle, \ldots, \rho\langle b_{n-1}\rangle)),\\ \rho\langle p_i(y_i)\rangle &= \rho\langle p_i(r(b_0, \ldots, b_{n-1}))\rangle \in p_i(\rho\langle r(b_0, \ldots, b_{n-1})\rangle)\\ &\subseteq p_i(r(\rho\langle b_0\rangle, \ldots, \rho\langle b_{n-1}\rangle)),\end{aligned}$$

it follows that $\rho\langle t_{i-1}\rangle\alpha^*_{\mathbf{qr}}\rho\langle t_i\rangle$.

So, we have proved that $\rho\langle a\rangle = \rho\langle t_0\rangle\alpha^*_{\mathbf{qr}}\rho\langle t_m\rangle = \rho\langle b\rangle$.

We deduce that h is well defined and also that h is injective. Its surjectivity is obvious.

Now, we can prove that the map h is an isomorphism between the universal algebras $(\mathfrak{B}/\rho)/\alpha^*_{\mathbf{qr}}$ and $\mathfrak{B}/\theta(\rho_{\mathbf{qr}})$.

Indeed, let us consider $\gamma < o(\tau)$ and $b_0, \ldots, b_{n_\gamma-1} \in B$. Since

$$f_\gamma(\alpha^*_{\mathbf{qr}}\langle\rho\langle b_0\rangle\rangle, \ldots, \alpha^*_{\mathbf{qr}}\langle\rho\langle b_{n_\gamma-1}\rangle\rangle) = \alpha^*_{\mathbf{qr}}\langle\rho\langle b\rangle\rangle$$

for any $\rho\langle b\rangle \in f_\gamma(\rho\langle b_0\rangle, \ldots, \rho\langle b_{n_\gamma-1}\rangle)$ and since

$$\rho\langle f_\gamma(b_0, \ldots, b_{n_\gamma-1})\rangle \in f_\gamma(\rho\langle b_0\rangle, \ldots, \rho\langle b_{n_\gamma-1}\rangle),$$

we have $f_\gamma(\alpha^*_{\mathbf{qr}}\langle\rho\langle b_0\rangle\rangle, \ldots, \alpha^*_{\mathbf{qr}}\langle\rho\langle b_{n_\gamma-1}\rangle\rangle) = \alpha^*_{\mathbf{qr}}\langle\rho\langle f_\gamma(b_0, \ldots, b_{n_\gamma-1})\rangle\rangle$, thus

$$h(f_\gamma(\alpha^*_{\mathbf{qr}}\langle\rho\langle b_0\rangle\rangle, \ldots, \alpha^*_{\mathbf{qr}}\langle\rho\langle b_{n_\gamma-1}\rangle\rangle)) = \theta(\rho_{\mathbf{qr}})\langle f_\gamma(b_0, \ldots, b_{n_\gamma-1})\rangle.$$

We also have

$$\begin{aligned}f_\gamma(h(\alpha^*_{\mathbf{qr}}\langle\rho\langle b_0\rangle\rangle), \ldots, h(\alpha^*_{\mathbf{qr}}\langle\rho\langle b_{n_\gamma-1}\rangle\rangle)) &= f_\gamma(\theta(\rho_{\mathbf{qr}})\langle b_0\rangle, \ldots, \theta(\rho_{\mathbf{qr}})\langle b_{n_\gamma-1}\rangle)\\ &= \theta(\rho_{\mathbf{qr}})\langle f_\gamma(b_0, \ldots, b_{n_\gamma-1})\rangle,\end{aligned}$$

hence h is a homomorphism. □

Let α^* be the fundamental relation of the multialgebra $\mathfrak{A}$. The universal algebra $\mathfrak{A}/\alpha^*$ will be denoted by $\overline{\mathfrak{A}}$ and it will be called *the fundamental algebra of the multialgebra* $\mathfrak{A}$. Since $\alpha^* = \alpha^*_{\mathbf{x}_0\mathbf{x}_0}$ and $\rho_{\mathbf{x}_0\mathbf{x}_0} = \rho$ we have the following result.

COROLLARY 5.4. *Let ρ be an equivalence relation on the universal algebra $\mathfrak{B}$ and let $\theta(\rho)$ be the smallest congruence relation on $\mathfrak{B}$ which contains ρ. Then $\overline{\mathfrak{B}/\rho} \cong \mathfrak{B}/\theta(\rho)$.*

From Corollary 5.4, using the notations from the beginning of this section, we obtain the following.

COROLLARY 5.5. *Let $n \in \mathbb{N}$ and $\mathbf{q}, \mathbf{r} \in \mathbf{P}^{(n)}(\tau)$. If ρ is an equivalence relation on a universal algebra $\mathfrak{B}$, then $\overline{\mathfrak{B}/\rho_{\mathbf{qr}}} \cong \mathfrak{B}/\theta(\rho_{\mathbf{qr}})$.*

From Theorem 5.3 and Corollary 5.5 we obtain the following consequence.

COROLLARY 5.6. *If ρ is an equivalence relation on the universal algebra $\mathfrak{B}$, then $(\mathfrak{B}/\rho)/\alpha^*_{\mathbf{qr}} \cong \overline{\mathfrak{B}/\rho_{\mathbf{qr}}}$.*

6. An application to hypergroups

A hypergroup is an H_v-group satisfying the associativity in a strong manner. A classical example of hypergroup is obtained in [8], by factorizing a group $(G, \cdot)$ through an equivalence relation determined by a subgroup H. The definition of the hyperproduct on $G/H = \{xH \mid x \in G\}$ is

$$(xH)(yH) = \{zH \mid z = x'y',\ x' \in xH,\ y' \in yH\}.$$

Clearly, $(G/H, \cdot)$ is a group if and only if H is a normal subgroup of G.

Let γ be the smallest strongly regular equivalence on G/H such that the factor hypergroup is a commutative group. If G' is the derived subgroup of G, then $G'H$ is the smallest normal subgroup N of G for which $H \subseteq N$ and G/N is abelian. From Theorem 5.3 we obtain the group isomorphism

$$h : (G/H)/\gamma \to G/(G'H), \quad h(\gamma\langle xH\rangle) = x(G'H).$$

The derived subhypergroup $D(K)$ of a hypergroup $(K, \cdot)$ is characterized in [7, Theorem 3.1] as being $\varphi_K^{-1}(1_{K/\gamma})$ where $\varphi_K : K \to K/\gamma$ is the canonical projection and $1_{K/\gamma}$ is the identity of the group $(K/\gamma, \cdot)$.

Let $\pi_H : G \to G/H$ and $\varphi_{G/H} : G/H \to (G/H)/\gamma$ be the canonical projections. Using [7, Theorem 3.1], a connection between the derived subhypergroup of G/H and the derived subgroup of G can be established as follows:

$$D(G/H) = (h \circ \varphi_{G/H})^{-1}(G'H) = \{xH \mid x \in G'H\} = (G'H)/H = \pi_H(G').$$

Of course, if $G' \subseteq H$, then H is an normal subgroup of G and G/H is an abelian group, so $D(G/H) = (G/H)' = H$. If $H \subseteq G'$, then $D(G/H) = G'/H$.

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