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TRIVIALITY OF THE GENERALISED LAU PRODUCT ASSOCIATED TO A BANACH ALGEBRA HOMOMORPHISM

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Abstract

Several papers have, as their raison d'être, the exploration of the *generalised Lau product* associated to a homomorphism $T : B \to A$ of Banach algebras. In this short note, we demonstrate that the generalised Lau product is isomorphic as a Banach algebra to the usual direct product $A \oplus B$. We also correct some misleading claims made about the relationship between this generalised Lau product and an older construction of Monfared ['On certain products of Banach algebras with applications to harmonic analysis', *Studia Math.* **178**(3) (2007), 277–294].

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1. The generalised Lau product

Motivated by a construction of Monfared [9], several authors have in recent years written papers on a certain construction, which manufactures a Banach algebra $A \times_T B$ given a pair of Banach algebras (A, B) and a continuous algebra homomorphism $T : B \to A$.

Definition 1.1. The underlying Banach space of $A \times_T B$ is the usual product/sum $A \oplus B$; multiplication is defined by the following rule:

$$(a_1, b_1) \bullet_T (a_2, b_2) = (a_1a_2 + T(b_1)a_2 + a_1T(b_2), b_1b_2).$$

It is clear that $\bullet_T$ is continuous and easily verified by hand that it is associative (although this will also follow from the proof of Proposition 1.3). By renorming if necessary to obtain a submultiplicative norm, one obtains a Banach algebra $A \times_T B$.

The construction in Definition 1.1 has gone by various names: 'generalised Lau product', 'the T-Lau product' or 'the Lau product defined by a Banach algebra morphism'. This last phrase is the one used in [3], which appears to be the earliest occurrence of this construction. Since [3] there have been several papers on the theme of the 'generalised Lau product', by various authors: see [1, 2, 4–8, 10–12].

REMARK 1.2. In [3], it is claimed that this construction extends that of [9] and that it might have some bearing on extensions of C^*-algebras in the sense of the Brown–Douglas–Fillmore (BDF) theory. The second claim is not given clear justification in the article; and the first claim is misleading, to say the least, as will be explained in Section 2.

Unfortunately, the following elementary observation casts doubt on the whole enterprise. We leave the proof to the reader.

PROPOSITION 1.3. *Let $A \oplus_{\text{alg}} B$ denote the usual sum of Banach algebras, that is, we equip the Banach space $A \oplus B$ with coordinatewise product*

$$(a_1, b_1) \cdot (a_2, b_2) = (a_1 a_2, b_1 b_2).$$

Define $\phi : A \times_T B \to A \oplus_{\text{alg}} B$ by $\phi(a, b) = (a + T(b), b)$. Then ϕ is continuous, linear and bijective; and $\phi((a, b) \bullet_T (c, d)) = \phi(a, b) \cdot \phi(c, d)$. In particular, $A \times_T B$ is isomorphic as a Banach algebra to $A \oplus_{\text{alg}} B$.

REMARK 1.4. The proof of Proposition 1.3 was found when the present author was reading the preprint [11], in particular the proof of Proposition 4.1 in that paper. It seems that the authors of [11] noticed something very similar to the isomorphism of Proposition 1.3, but only in a restricted setting. Similarly, it is observed just after [6, Theorem 4.2] that there is an isomorphism $A \times_T B \to A \oplus_{\text{alg}} B$, but the authors of [6] only state this in the case where A and B are commutative and semisimple, and do not mention that Proposition 1.3 applies in full generality.

Hopefully, any future attempts to study the generalised Lau product, as defined in Definition 1.1, will bear in mind that $A \times_T B \cong A \oplus_{\text{alg}} B$ regardless of the choice of T. It is a basic theme, when defining a property of Banach algebras, to see how it behaves under forming binary sums/products of algebras. For many of the properties considered in the items of the bibliography, stability of such properties under forming binary sums/products is either known or refuted by old work.

2. Comparison with an older construction

It is claimed in [3] that the construction presented there extends the one studied by Monfared in [9]. If this were true, then Monfared's construction would be a special case of Definition 1.1 and hence would be trivial for the same reason.

In fact, it is *not* true that Monfared's construction is a special case of Definition 1.1. In this section we shall briefly explain why.

DEFINITION 2.1 (Monfared [9]). Let A and B be Banach algebras and let $\varphi : B \to \mathbb{C}$ be a character, that is, a nonzero homomorphism. The *Lau product* of A and B with respect to φ is defined to be the Banach algebra whose underlying Banach space is $A \oplus B$, equipped with the multiplication operation

$$(a_1, b_1) \bullet_\varphi (a_2, b_2) = (a_1 a_2 + \varphi(b_1) a_2 + a_1 \varphi(b_2), b_1 b_2).$$

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As observed by Bhatt and Dabhi [3], it is immediate that when A has an identity element e_A and φ is a character on B, we may define a continuous homomorphism $T : B \to A$ by $T(a) = \varphi(b)e_A$. If we do this, then the Lau product of A and B with respect to φ does indeed coincide with $A \times_T B$ and hence by Proposition 1.3 it is isomorphic to the usual direct product of algebras $A \oplus_{\text{alg}} B$.

What seems to have gone unremarked in [3] is that when A does not have an identity element, Definition 2.1 is no longer a special case of Definition 1.1. Indeed, in general the Lau product of A and B is not isomorphic as an algebra to $A \oplus_{\text{alg}} B$. This is not a new observation—it is implicit in [9]—but for the sake of completeness we shall give an illustrative example.

Example 2.2. Let A be an arbitrary Banach algebra. Take $B = \mathbb{C}$ and let id denote the identity map on $\mathbb{C}$. It was observed in [9] (and it is also clear from Definition 2.1) that the corresponding Lau product of A and B is isomorphic to $A^\sharp$, the usual unitisation of A. Observe that $A \oplus_{\text{alg}} \mathbb{C}$ has an identity element if and only if A does. Hence, if $A = c_0(\mathbb{N})$ (for example), then $A \oplus_{\text{alg}} \mathbb{C}$ cannot be isomorphic to $A^\sharp$.

A minor variation on the proof of Proposition 1.3 shows that given A, B, φ as in Definition 2.1, there is a natural way to identify $A \times_\varphi B$ with a closed subalgebra of codimension one (not an ideal, in general) of $A^\sharp \oplus_{\text{alg}} B$. Given that there continue to be papers exploring the Lau product (as in Definition 2.1), it therefore seems worthwhile to record the following consequence, whose proof we omit since it is straightforward.

Proposition 2.3. *Let Q be a property of Banach algebras which is preserved by all isomorphisms of Banach algebras (not just the isometric ones). Suppose also that:*

- *if A has the property Q, then so does $A^\sharp$;*
- *if A_1 and A_2 have the property Q, then so does $A_1 \oplus_{\text{alg}} A_2$;*
- *if A has the property Q and B is a closed subalgebra of A with finite codimension, then B has the property Q.*

Then the property Q is preserved by taking Lau products in the sense of Definition 2.1.

Note that two examples of such properties are Arens regularity and the property of being isomorphic to a closed subalgebra of $\mathcal{B}(H)$ for some Hilbert space H.

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