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ON NONCOMMUTATIVE VNL-RINGS AND GVNL-RINGS

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Abstract. It is proved that every abelian VNL-ring is an SVNL-ring, which gives a positive answer to a question of Osba et al. [**7**]. Some characterizations of duo VNL-rings are given and some main results of Osba et al. [**7**] on commutative VNL-rings are extended to right duo VNL-rings and even abelian GVNL-rings.

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1. Introduction. Rings considered are associative with identity unless the contrary is stated explicitly. An element a of a ring R is π*-regular* if there exist a positive integer n and $x \in R$ such that $a^n = a^n x a^n$. In the case of $n = 1$, a is regular. A ring R is π*-regular* (*regular*) if every element in R is π-regular (regular). A ring R is an *exchange ring* if for every $a \in R$ there exists an idempotent $e \in R$ such that $e \in aR$ and $(1 - e) \in (1 - a)R$. A ring R is *right* (*left*) *duo* if every right (left) ideal is two-sided, and R is *duo* if R is right and left duo. A ring R is *abelian* if all idempotents are contained in the center and a ring R is reduced if it does not contain nonzero nilpotent elements.

Following Contessa [**4**], a commutative ring R is a *VNL-ring* if for every $a \in R$, at least one of a or $1 - a$ is regular. According to Osba et al. [**7**], a commutative ring R is an *SVNL-ring* if whenever $(S) = R$ for some nonempty subset S of R, at least one of the elements in S is regular, where (S) is an ideal generated by S. Some properties of VNL-rings and SVNL-rings are investigated in Osba et al. [**7**]. But they are unable to solve the question whether every VNL-ring is an SVNL-ring. Because of this they are unable to characterize VNL-rings abstractly in the sense of relating them to more familiar classes of rings. In the present paper we define a noncommutative ring R to be a *VNL-ring* (*GVNL-ring*) if for every $a \in R$ at least one of a or $1 - a$ is regular (π-regular) and a ring R is an SVNL-ring, if whenever $(S)_r = R$ for some nonempty subset S of R, at least one element in S is regular, where $(S)_r$ is a right ideal generated by S. The main purpose of this paper is to prove that every abelian VNL-ring is an SVNL-ring, giving an answer to the question of Osba et al. [**7**] in the affirmative. Furthermore we give some characterizations of duo VNL-rings, and extend and improve some main results of Osba et al. [**7**] on commutative VNL-rings.

Throughout this paper we use the symbol $J(R)$ to denote the Jacobson radical of a ring R, $Id(R)$ its set of idempotents, and $Max(R)$ its maximal spectrum. The left annihilator of an element a in R is denoted by $A_l(a)$. For an integer $n \geq 2$, the symbol

$\mathbb{Z}_n$ stands for the ring of integers modulo n. Also for two sets A and B with $B \subseteq A$, the symbol $A \backslash B$ or $A - B$ denotes the supplementary set of B in A.

2. Abelian VNL-rings. We start this section with the following definition.

DEFINITION 2.1. An element a of a ring R is called an *exchange element* if there exists an idempotent $e \in R$ such that $e \in aR$ and $1 - e \in (1 - a)R$.

Obviously, a ring R is an exchange ring if and only if every element of R is an exchange element. And an element a of a ring R is an exchange element if and only if so is $1 - a$. It is known from Nicholson ([**6**], Proposition 1.6) or Stock ([**8**], Example 2.3) that a regular element (π-regular element) of a ring R is an exchange element. So we have the following.

THEOREM 2.2. *Every VNL-ring (GVNL-ring) is an exchange ring.*

Recall that an ideal I in a ring R is regular if for each $a \in I$, there exists $x \in I$ such that $a = axa$ (Goodearl, [**5**], P_2).

The following facts are essential to obtaining our main results.

LEMMA 2.3 (Goodearl, [**5**], Proposition 1.5). *Let R be a ring, and set $M(R) = \{x \in R | RxR$ is a regular ideal$\}$. Then the following hold:*

(a) *$M(R)$ is a regular ideal of R.*
(b) *$M(R)$ contains all regular ideals of R.*
(c) *$R/M(R)$ has no nonzero regular ideal.*

LEMMA 2.4 (Goodearl, [**5**], P_2). *Let R be a ring. If $x, y \in R$ and $x' = x - xyx$, and if $x' = x'ax'$ for some $a \in R$, then $x = xbx$ for some $b \in R$.*

LEMMA 2.5 (Warfield, [**9**], Theorem 1). *An exchange ring with only two idempotents is a local ring.*

LEMMA 2.6. *Let $R = \prod_{\alpha \in \Lambda} R_\alpha$ be a ring. Then R is a VNL-ring (abelian VNL-ring) if and only if there exists $\alpha_0 \in \Lambda$ such that R_{α_0} is a VNL-ring (abelian VNL-ring) and for each $\alpha \in \Lambda - \{\alpha_0\}$, R_α is a regular (abelian regular) ring.*

Proof. It is very similar to that of Osba et al. ([**4**], Theorem 3.1).

LEMMA 2.7. *Let R be an abelian VNL-ring and not regular. Then $R/M(R)$ is a local ring, where the meaning of $M(R)$ is the same as that in Lemma 2.3.*

Proof. It is known and easy to prove that the homomorphic image of a VNL-ring is a VNL-ring. Since R is a VNL-ring, $R/M(R)$ is a VNL-ring and so is an exchange ring by Theorem 2.2. Let $\overline{R} = R/M(R)$. If $R/M(R)$ is not a local ring, then by Lemma 2.5 there exists a nontrivial idempotent $\overline{e} \in \overline{R}$. And hence $\overline{R} = \overline{e}\overline{R} \bigoplus (\overline{1} - \overline{e})\overline{R}$. Since R is an abelian exchange ring, the idempotents of $R/M(R)$ can be lifted to R, so $R/M(R)$ is an abelian exchange ring and hence an abelian VNL-ring. Now Lemma 2.6 implies that at least one of $\overline{e}\overline{R}$ or $(\overline{1} - \overline{e})\overline{R}$ is an abelian regular ring, so is a nonzero regular ideal of $\overline{R}$, which contradicts Lemma 2.3. And the proof is completed.

THEOREM 2.8. *Let R be an abelian ring. Then R is a VNL-ring if and only if it is an SVNL-ring.*

Proof. ($\Rightarrow$) If R is regular, then we are done. Otherwise R is a VNL-ring which is not regular. Now for any nonempty subset S of R with $(S)_r = R$, there exist

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$s_1, s_2, \ldots, s_n \in S$ such that $s_1R + s_2R + \cdots + s_nR = R$, hence there exist $r_1, r_2, \ldots, r_n \in R$ satisfying $s_1r_1 + s_2r_2 + \cdots + s_nr_n = 1$ and so $\bar{s_1}\bar{r_1} + \bar{s_2}\bar{r_2} + \cdots + \bar{s_n}\bar{r_n} = \bar{1}$ in $\overline{R} = R/M(R)$. According to Lemma 2.7, $\overline{R}$ is a local ring. It follows that not all $\bar{s_i}$ are in $J(\overline{R})$. So there exists an $\bar{s_k}$ such that $\bar{s_k}$ is a unit in $\overline{R}$, hence $\bar{s_k}$ is a regular element in $\overline{R}$. Assume that $\bar{s_k} = \bar{s_k}\bar{x_k}\bar{s_k}$ for some $\bar{x_k} \in \overline{R}$. Then we have $s_k - s_kx_ks_k \in M(R)$ and $s_k - s_kx_ks_k = (s_k - s_kx_ks_k)y_k(s_k - s_kx_ks_k)$ for some $y_k \in R$, thus s_k is a regular element by Lemma 2.4 and so R is an SVNL-ring.

($\Leftarrow$) Assume that R is an SVNL-ring. For any $a \in R$, let $S = \{a, 1 - a\}$. Then $(S)_r = R$ since $1 = a + 1 - a \in (S)_r$. Hence either a or $1 - a$ is regular and R is a VNL-ring.

Corollary 2.9. *Let R be a commutative ring. Then R is a VNL-ring if and only if it is an SVNL-ring.*

The above corollary gives a positive answer to the question whether every commutative VNL-ring is an SVNL-ring.

Although the following corollary is observed by Osba et al. [**5**], its proof may not be trivial until we obtain the above theorem.

Corollary 2.10. *The homomorphic image of a commutative (abelian) SVNL-ring is a commutative (abelian) SVNL-ring.*

Proof. Since the homomorphic image of a commutative (abelian) VNL-ring is a commutative (abelian) VNL-ring, we get the desired conclusion by Theorem 2.8.

Next extend some main results of Osba et al. [**4**] on commutative VNL-rings to right duo VNL-rings and we give some characterizations of duo VNL-rings.

Recall that an ideal I of a ring R is (left) pure if $I = mI$, where $mI = \{a \in R | a = ia$ *for some* $i \in I\}$. It is easy to check that for a right duo ring R and any $a \in R$, $A_l(a)$ is a two-sided ideal and $mI = \{a \in R | I + A_l(a) = R\}$.

Lemma 2.11. *Let R be a right duo ring and $a \in R$. Then aR is pure if and only if $aR + A_l(a) = R$.*

Proof. Suppose that aR is pure. Then there exists $x = ar \in aR$ such that $a = xa = ara$. So $(1 - ar)a = 0$ which implies that $1 - ar \in A_l(a)$. Hence $1 = ar + (1 - ar) \in aR + A_l(a)$, and $aR + A_l(a) = R$. Conversely, assume that $aR + A_l(a) = R$. Then there exist $r \in R$, $b \in A_l(a)$ such that $ar + b = 1$. So $ara + ba = a$, i.e., $a = ara$ since $ba = 0$. So for any $x \in R$, we have $ax = arax$ with $ar \in aR$, which implies that aR is pure.

Theorem 2.12. *Let R be a right duo ring and $a \in R$. Then a is regular if and only if for every maximal ideal M, $a \in M$ implies $a \in mM$.*

Proof. Suppose that a is regular. If $M \in Max(R)$ is such that $a \in M$, then $a = ara$ for some $r \in R$ with $ar \in M$ and hence $a \in mM$. Conversely, assume that for each maximal ideal M, $a \in M$ implies $a \in mM$. We claim that $aR + A_l(a) = R$. If not, there exists a maximal ideal M such that $aR + A_l(a) \subseteq M$. Note that $a \in M$ implies $a \in mM$. There exists $x \in M$ such that $a = xa$. So $(1 - x)a = 0$, which gives $1 - x \in A_l(a) \subseteq M$. Hence $1 \in M$, a contradiction. Thus $aR + A_l(a) = R$, there exist $r \in R$ and $c \in A_l(a)$ such that $ar + c = 1$, which implies $a = ara$ and a is regular.

Corollary 2.13. *A right duo ring R is regular if and only if all maximal ideals are pure.*

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Proof. If R is regular, then for every $M \in Max(R)$, $a \in M$ implies $a \in mM$ by Theorem 2.12, which gives $mM = M$ and hence M is pure. Conversely, if every $M \in Max(R)$ is pure, then $M = mM$ and so $a \in M$ implies $a \in mM$ for every $a \in R$. Hence for every $a \in R$, a is regular by Theorem 2.12 and so R is regular.

THEOREM 2.14. *The following statements are equivalent for a duo ring R.*
(a) *All maximal ideals of R except may be one are pure.*
(b) *There exists $N \in Max(R)$ such that for each $a \notin N$, a is regular.*
(c) *The ring R is a VNL-ring.*

Proof. (a) $\Rightarrow$ (b) Suppose that there exists $N \in Max(R)$ such that $M = mM$ for each $M \in Max(R)\backslash\{N\}$. If $a \notin N$, then for each $M \in Max(R)$, $a \in M$ implies $M \neq N$, so $M = mM$ and $a \in mM$. Hence a is regular by Theorem 2.12.

(b) $\Rightarrow$ (c) Assume that there exists $N \in Max(R)$ such that for each $a \notin N$, a is regular. For each $a \in R$, if $a \notin N$, then we are done. Otherwise $a \in N$, so $1 - a \notin N$ and hence $1 - a$ is regular.

(c) $\Rightarrow$ (b) First note that a duo ring is abelian. Now suppose that R is a VNL-ring. Then it is an SVNL-ring by Theorem 2.8. Let T be the set of elements which are not regular in R. If T is empty, then we are done. Otherwise consider the right ideal I generated by T, which is an ideal since R is duo. If $1 \in I$, then $1 = \sum_{i=1}^{n} s_i r_i$ with $s_i \in T$ and $r_i \in R$ for each i. Thus $R = s_1R + s_2R + \cdots + s_nR$ and there exists i such that $s_i \notin T$, a contradiction. Hence I is contained in a maximal N and for each $a \notin N$, a is regular.

(b) $\Rightarrow$ (a) Assume that there exists $N \in Max(R)$ such that for each $a \notin N$, a is regular. Let $M \in Max(R)\backslash\{N\}$ and $a \in M$. If $a \notin N$, then a is regular and so $a \in mM$ by Theorem 2.12. If $a \in M \bigcap N$, then choose $b \in M\backslash N$, so that $b \in mM$. Clearly $a + b \in M\backslash N$, which implies $a + b \in mM$. We need to prove $a \in mM$. Since $b \in mM$, $M + A_l(b) = R$ and there exist $m_1 \in M, x_1 \in A_l(b)$ such that $m_1 + x_1 = 1$. $(*)$ Similarly, $a + b \in mM$ implies $M + A_l(a + b) = R$ and there exist $m_2 \in M$, $x_2 \in A_l(a + b)$ such that $m_2 + x_2 = 1$. $(**)$ Equations $(*)$ and $(**)$ imply $m_3 + x_2x_1 = 1$ with $m_3 = m_2m_1 + m_2x_1 + x_2m_1 \in M$. Since R is duo,we have $x_2x_1 = y_1x_2$ for some $y_1 \in R$. It follows that $x_2x_1(a + b) = 0$ and $x_2x_1b = 0$, which implies $x_2x_1a = 0$. Hence $M + A_l(a) = R$ and so $a \in mM$.

The following corollary characterizes a commutative VNL-ring, and the equivalence of (a) and (b) is known in Osba et al. [**4**].

COROLLARY 2.15. *The following are equivalent for a commutative ring R.*
(a) *All maximal ideals of R except may be one are pure.*
(b) *There exists $N \in Max(R)$ such that for each $a \notin N$, a is regular.*
(c) *The ring R is a VNL-ring.*

3. Abelian GVNL-rings.

In this section we study abelian GVNL-rings, extending some main results of Osba et al. [**4**] on commutative VNL-rings to abelian GVNL-rings.

EXAMPLE 3.1. There is a commutative GVNL-ring R which is not a VNL-ring.

Proof. Let $R = \mathbb{Z}_4 \bigoplus \mathbb{Z}_4$. Then it is easy to check that $(\bar{3}, \bar{2})$ and $(\bar{1}, \bar{1}) - (\bar{3}, \bar{2})$ are not regular, so R is not a VNL-ring. Since R is π-regular, it is a GVNL-ring.

In the above example R satisfies $J(R) \neq 0$. We naturally ask whether there exists an abelian GVNL-ring R such that $J(R) = 0$ but it is not a VNL-ring. The following proposition shows that the answer is negative.

PROPOSITION 3.2. *If R is an abelian GVNL-ring with $J(R) = 0$, then R is a VNL-ring.*

Proof. It is known by Lemma 4.10 in (Stock, [**8**]) that for an exchange ring R with $J(R) = 0$, R is abelian if and only if it is reduced. Now R is a reduced exchange ring by Theorem 2.2. Now for any $a \in R$, if a is π-regular, then $a^n = a^n x a^n$ for some positive integer n and $x \in R$. Clearly, $e = a^n x \in Id(R)$. So $((1-e)a)^n = (1-e)a^n = (1-e)ea^n = 0$, and hence $(1-e)a = 0$. Therefore $a = ea = a(a^{n-1}x)a$ is regular. And the proof is completed.

Since the homomorphic image of a (an abelian) GVNL-ring is also a (an abelian) GVNL-ring, the following corollary is immediate.

COROLLARY 3.3. *If R is an abelian GVNL-ring, then $R/J(R)$ is an abelian VNL-ring.*

LEMMA 3.4. *Let $R = \bigoplus_{i=1}^{k} R_i$ be an abelian ring. Then $(a_1, a_2, \ldots, a_k)$ is π-regular in R if and only if every a_i is π-regular in R_i.*

Proof. It is sufficient to prove the case of $k = 2$. Clearly R is abelian if and only if every R_i is abelian. Now suppose that a_1 and a_2 are π-regular. Then there exist positive integers m, n and $x, y \in R$ such that $a_1^m = a_1^m x a_1^m$ and $a_2^n = a_2^n y a_2^n$. Since xa_1^m, and ya_2^n are idempotents, we have $a_1^{mn} = a_1^{mn}(xa_1^m)^n = a_1^{mn}(xa_1^m)\cdots(xa_1^m) = a_1^{mn}xa_1^m(xa_1^m)(xa_1^m)\cdots(xa_1^m) = a_1^{mn}x^2a_1^m(xa_1^m)(xa_1^m)\cdots(xa_1^m)a_1^m = a_1^{mn}x^3a_1^m(xa_1^m)(xa_1^m)\cdots(xa_1^m)a_1^{2m} = a_1^{mn}x^n a_1^{mn}$. Similarly, $a_2^{mn} = a_2^{mn}y^m a_2^{mn}$ and so $(a_1, a_2)^{mn} = (a_1^{mn}, a_2^{mn}) = (a_1^{mn}x^n a_1^{mn}, a_2^{mn}y^m a_2^{mn}) = (a_1^{mn}, a_2^{mn})(x^n, y^m)(a_1^{mn}, a_2^{mn}) = (a_1, a_2)^{mn} \quad (x^n, y^m) \quad (a_1, a_2)^{mn}$. Hence (a_1, a_2) is π-regular. Using induction on k, we can obtain the desired conclusion. Conversely, if $(a_1, a_2, \ldots, a_k)$ is π-regular in R, then obviously every a_i is π-regular in R_i.

COROLLARY 3.5. *Let $R = \bigoplus_{i=1}^{n} R_i$ be a ring. Then R is abelian π-regular if and only if every R_i is abelian π-regular.*

THEOREM 3.6. *Let $R = \bigoplus_{i=1}^{n} R_i$ be a ring. Then R is an abelian GVNL-ring if and only if there exists an index k such that R_k is a GVNL-ring and for each $i \neq k$, R_i is a π-regular ring.*

Proof. Suppose that R is GVNL-ring. Then R_i is a GVNL-ring as a homomorphic image of R for each i. If every R_i is a π-regular ring, then we are done. Otherwise, there exists an index k such that R_k is not a π-regular ring. Assume that $a_k \in R_k$ is not π-regular. Then for each $i \neq k$ and any $a_i \in R_i$, $(1 - a_1, \ldots, 1 - a_{k-1}, a_k, 1 - a_{k+1}, \ldots, 1 - a_n)$ is not π-regular by Lemma 3.4. Thus $(a_1, \ldots, a_{k-1}, 1 - a_k, a_{k+1}, \ldots, a_n)$ is π-regular, and so is every a_i whenever $i \neq k$. Therefore R_i is π-regular for each $i \neq k$.

Conversely, assume that that there exists an index k such that R_k is a GVNL-ring and, for each $i \neq k$, R_i is π regular. We prove that R is a GVNL-ring. For each $a = (a_1, \ldots, a_{k-1}, a_k, a_{k+1}, \ldots, a_n) \in R$, if a_k is π-regular, then a is π-regular by Lemma 3.4. If a_k is not π-regular, then $1 - a_k$ is π-regular, so is $1 - a$. And the proof is completed.

THEOREM 3.7. *Let R be an abelian GVNL-ring in which 2 is a unit. Then every element in R is a sum of a unit and a square root of* 1 *(i.e., an element a with $a^2 = 1$).*

Proof. Suppose that R is a GVNL-ring. By Theorem 2.2, R is an exchange ring. Since R is abelian, it is a clean ring by Nicholson ([**6**], Proposition 1.8). According to

Camillo and Yu ([**3**], Theorem 11), every element of a clean ring in which 2 is invertible is a sum of a unit and a square root of 1 and so we are done.

COROLLARY 3.8 (Osba et al. 2004, Theorem 3.8). *Every element of a commutative VNL-ring in which 2 is a unit is a sum of no more than three units.*

THEOREM 3.9. *Let R be an abelian ring with only a finite number of idempotents. Then R is a GVNL-ring if and only if it is the direct product of a local ring and finitely many π-regular local rings each of which has the the property that the Jacobson radical coincides its set of nilpotent elements.*

Proof. Suppose that R is an abelian GVNL-ring. Then it is an exchange ring by Theorem 2.2. According to Camillo and Yu ([**3**], Theorem 9), R is a semiperfect ring. There exist $e_1, e_2, \ldots, e_n \in Id(R)$ such that $e_1 + e_2 + \cdots + e_n = 1$ and e_is are mutually orthogonal local idempotents. It follows that $R = e_1R \bigoplus e_2R \bigoplus \cdots \bigoplus e_nR$, where e_iR is a local ring for each i. Now Theorem 3.6 implies that there exists k such that e_kR is a GVNL-ring and e_jR is a π-regular ring for each $j \neq k$. By Badawi ([**2**], Lemma 5), we know that $J(e_jR)$ coincides with the set of nilpotent elements in e_jR. Conversely, if R is the direct product of a local ring and finitely many π-regular local rings, then it is a GVNL-ring by Theorem 3.6.

COROLLARY 3.10. *Let R be an abelian ring with only a finite number of idempotents. Then R is a VNL-ring if and only if it is the direct product of finitely many division rings and a local ring.*

Proof. By Lemma 2.6 and the fact that a local regular ring is a division ring.

In the case of R is a commutative ring, we have the following corollary.

COROLLARY 3.11 (Osba, [**4**], Theorem 6.1). *If R is a commutative VNL-ring with only a finite number of idempotents. Then it is the direct product of finitely many regular rings and a local ring.*

According to Ara [**1**], a ring I (without unit) is called an *exchange ring* if for each $a \in I$ there exist an idempotent $e \in I$ and $r, s \in I$ such that $e = ar = a + s - as$. Also if I is an ideal of a unital exchange ring, then I satisfies the above condition.

LEMMA 3.12. *Let R be a ring. Then $R[x]$ is not an exchange ring.*

Proof. Suppose that $R[x]$ is an exchange ring. Then for $x \in R[x]$ there exist an idempotent $e(x) \in R[x]$ and $r(x), s(x) \in R[x]$ such that $e(x) = xr(x) = x + s(x) - xs(x)$. Thus $e(x) = a_1x + a_2x^2 + \cdots + a_nx^n$, and hence $e(x) = 0$ by a direct calculation, so that $x + s(x) = xs(x)$, which is impossible by comparing the coefficients.

COROLLARY 3.13. *For any ring R, the ring $R[x]$ is not a GVNL-ring.*

COROLLARY 3.14 (Osba et al., [**4**], Corollary 4.8). *For a commutative ring R, the ring $R[x]$ is not a VNL-ring.*

THEOREM 3.15. *The following statements are equivalent for an abelian ring R.*

(a) *The ring R is a local ring.*
(b) *The ring $R[[x]]$ is a local ring.*
(c) *The ring $R[[x]]$ is a GVNL-ring.*
(d) *The ring R is a GVNL-ring and $Id(R) = \{0, 1\}$.*

Proof. $(a) \Rightarrow (b)$ is known.

$(b) \Rightarrow (c)$ Since a local ring is a VNL-ring, it is a GVNL-ring.

$(c) \Rightarrow (d)$ Since R is a homomorphic image of $R[[x]]$, R is a GVNL-ring. If $Id(R) \neq \{0, 1\}$, then there exists a nontrivial idempotent $e \in R$. Hence $R = eR \bigoplus (1-e)R$ and $R[[x]] = eR[[x]] \bigoplus (1-e)R[[x]]$. By Theorem 3.6, $eR[[x]]$ or $(1-e)R[[x]]$ is π-regular, say for example $eR[[x]]$. Then $(ex)^n = (ex)^n f(x)(ex)^n$ for some $f(x) \in R[[x]]$ and some positive integer n. This implies $e = 0$ by comparing the coefficients, which is a contradiction.

$(d) \Rightarrow (a)$ Since R is abelian, $Id(R) = Id(R[[x]])$. Hence R is an exchange ring with only two idempotents and so R is a local ring.

Combining Theorem 2.8 with Theorem 3.15, we have the following corollary.

COROLLARY 3.16 (Osba, [**4**], Theorem 4.6). *For a commutative ring R, the following statements are equivalent.*

(a) *The ring R is a local ring.*

(b) *The ring $R[[x]]$ is a local ring.*

(c) *The ring $R[[x]]$ is an SVNL-ring.*

(d) *The ring $R[[x]]$ is a VNL-ring.*

(e) *The ring R is an SVNL-ring and $Id(R) = \{0, 1\}$.*

(f) *The ring R is a VNL-ring and $Id(R) = \{0, 1\}$.*

(g) *For each $a \in R$, a is a unit or $1 - a$ is a unit.*

We conclude this paper with the following open question:

QUESTION 3.17. Is every noncommutative VNL-ring an SVNL-ring?

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