GROUPTHEORETICAL CHARACTERIZATION OF PROJECTIVE SPACE AND CONFORMAL SPACE

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In this paper we characterize a projective space and a conformal space, namely a space of inversive geometry of point and sphere, from the standpoint of a homogeneous space. In such spaces a covariant differential of a vector-field is not a vector, contrary to the case stated in my previous paper "On the vector in homogeneous spaces." (This journal vol. 5. This paper will be referred to as [1] below.) But when we restrict a rotation about a point to a certain subgroup of the full rotation group, we get a covariant differential which is also a vector, and this situation holds good in a general homogeneous space. If the fundamental group $\mathfrak{G}$ of a homogeneous space is generated by the full group of rotation about a point and a commutative subgroup of $\mathfrak{G}$ which operates simply transitively on our space, a translation of a vector can be defined smoothly on the whole space. We call such a space a space of projective type, a projective space and a conformal space being such. We characterize among the homogeneous spaces of projective type a projective space and conformal space by certain properties. In particular a conformal space of dimension greater than 2 can be characterized as a homogeneous space of projective type whose linear group of isotropy is conformal.

1. Projective space and conformal space

1.1 Let $\mathfrak{G}$ be a group of projective transformations of n-dimensional projective space, and $A_0, A_1, \ldots, A_n$ be a set of analytic points in the space, which we can take as a frame of our space. Then putting $dA_i = \sum_{j=0}^{n} \omega_{ij} A_j$ ($\sum_{i=0}^{n} \omega_{ii} = 0$, $i = 0, 1, \ldots, n$) we have as structure equations

$$
\begin{aligned}
&d\omega_{0i} = [\omega_{00}\omega_{0i}] + \sum_k [\omega_{0k}\omega_{ki}], \quad d\omega_{00} = \sum_i [\omega_{0i}\omega_{i0}] \\
&d\omega_{i0} = [\omega_{i0}\omega_{00}] + \sum_k [\omega_{ik}\omega_{k0}], \quad d\omega_{ij} = [\omega_{i0}\omega_{0j}] + \sum_k [\omega_{ik}\omega_{kj}].
\end{aligned} \tag{1.1}
$$

Denoting by A an $n \times 1$ matrix with $A_0, A_1, \ldots, A_n$ as its coefficients we can write $dA_i = \sum_j \omega_{ij} A_j$ in the form $dA = \Omega A$, $\Omega = (\omega_{ij})$. We take any projective transformation P which fixes the point A_0 and whose determinant is ± 1 and

Received June 16, 1952.

put

$$\bar{A}=\begin{pmatrix}\bar{A}_0\\ \bar{A}_1\\ \vdots\\ \bar{A}_n\end{pmatrix}=PA,\quad P=\begin{pmatrix}\rho & 0\\ p & P_0\end{pmatrix},\quad P^{-1}=\begin{pmatrix}\frac{1}{\rho} & 0\\ q & P_0^{-1}\end{pmatrix}$$

where p and q are $n\times 1$ matrices, P_0 is an $n\times n$ matrix and moreover $p+\rho P_0 q=0$, $\rho|P_0|=\pm 1$. We denote the set of all such transformations by $\mathfrak{H}$. Putting $d\bar{A}=\bar{\Omega}\bar{A}$ we get

$$\bar{\Omega}=P\Omega P^{-1}+dPP^{-1}.$$

When we put

$$\Omega=\begin{pmatrix}\omega_{00} & \omega_1\\ \omega_2 & \Omega_0\end{pmatrix},\quad \bar{\Omega}=\begin{pmatrix}\bar{\omega}_{00} & \bar{\omega}_1\\ \bar{\omega}_2 & \bar{\Omega}_0\end{pmatrix}$$

where ω_1 is a $1\times n$-matrix and ω_2 is an $n\times 1$ matrix, we get

(1.2) $$\bar{\omega}_1=\rho\omega_1 P_0^{-1}$$

(1.3)
$$\bar{\omega}_{00}=\rho\omega_1 q+\omega_{00}+\frac{d\rho}{\rho},\quad \bar{\Omega}_0=p\omega_1 P_0^{-1}+P_0\Omega_0 P_0^{-1}+dP_0P_0^{-1}$$
$$\bar{\omega}_2=\frac{1}{\rho}(p\omega_{00}+P_0\omega_2)+p\omega_1 q+P_0\Omega_0 q+\frac{dp}{\rho}+dP_1 q.$$

(1.2) is the transformation of principal relative components by a linear group of isotropy in the projective space with points as its elements. We define a vector in such a homogeneous space by a set of numbers $v=(v_1,\ldots,v_n)$ which is transformed in such a way that

(1.4) $$\bar{v}=\rho v P_0^{-1}$$

by a projective transfomation which fixes A_0. We take a vectorfield which is differentiable. Then we get by (1.2) (1.3) (1.4)

$$d\bar{v}-\bar{\omega}_{00}\bar{v}+\bar{v}\bar{\Omega}_0+\rho\omega_1 q\bar{v}=\rho(dv-\omega_{00}v+v\Omega_0-\rho vq\omega_1)P_0^{-1}.$$

Hence the term in the bracket of the second side does not give a vector. But if we restrict the rotation about a point A_0 to the element of its subgroup satisfying the relation $q=0$ we get

(1.5) $$d\bar{v}-\bar{\omega}_{00}\bar{v}+\bar{v}\bar{\Omega}_0=\rho(dv-\omega_{00}v+v\Omega_0)P_0^{-1}.$$

For such elements we have on account of the relation $q=0$

(1.6) $$\omega_{0i}=0,\quad \omega_{i0}=0.$$

If we add the relation $\rho=1$ we have

(1.7) $$\omega_{0i}=0,\quad \omega_{00}=0,\quad \omega_{i0}=0.$$

https://doi.org/10.1017/S0027763000015440 Published online by Cambridge University Press

Now the integral manifold of the differential equation

$$\omega_{i0} = 0 \tag{1.8}$$

containing an identity of $\mathfrak{G}$ forms a subgroup and if we restrict the rotation to the elements of such a subgroup we get an affine space of dimension n. By such a restriction of the rotation a covariant differential of a vector v which is defined by $Dv = dv - \omega_{00}v + v\Omega_0$ is also a vector and the geodesic can be defined by the curve get by solving

$$\frac{d}{d\sigma}\left(\frac{\omega_1}{d\sigma}\right) - \frac{\omega_{00}}{d\sigma}\frac{\omega_1}{d\sigma} + \frac{\omega_1}{d\sigma}\frac{\Omega_0}{d\sigma} = 0 \quad (\omega_{i0} = 0)$$

which also can be obtained by solving $\omega_1 = c_1 d\sigma$, $\omega_{00} = 0$, $\omega_{i0} = 0$, $\Omega_0 = 0$ and is nothing but a straight line. Next for the elements of $\mathfrak{H}$ satisfying $\rho = 1$ and $P_0 =$ unit matrix we get by (1.4) $\bar{v} = v$. For such elements of $\mathfrak{H}$ we have

$$\omega_{0i} = 0, \quad \omega_{00} = 0, \quad \omega_{ij} = 0. \tag{1.9}$$

These are the guiding ideas for our investigation. When we put $\omega_i = \omega_{0i}$, $\pi_{ij} = \omega_{ij} - \delta_{ij}\omega_{00}$ we get instead of (1.1) the following

$$\begin{cases} d\omega_i = \sum_j [\omega_j \pi_{ji}], \quad d\omega_{00} = \sum_i [\omega_i \omega_{i0}], \quad d\omega_{i0} = \sum_j [\pi_{ij}\omega_{j0}] \\ d\pi_{ij} = \sum_k [\pi_{ik}\pi_{kj}] + [\omega_{i0}\omega_j] \quad (i \neq j) \\ d\pi_{ii} = \sum_k [\pi_{ik}\pi_{ki}] + [\omega_{i0}\omega_i] - \sum_j [\omega_j \omega_{j0}]. \end{cases} \tag{1.10}$$

1.2 An analogous situation holds in the conformal space with points as its elements. We take two points A_0, A_∞ and n spheres $A_1, A_2, \ldots, A_n$ such that

$$(A_0A_0) = 0, \quad (A_0A_i) = 0, \quad (A_0A_\infty) = -1,$$
$$(A_iA_j) = \delta_{ij}, \quad (A_iA_\infty) = 0, \quad (A_\infty A_\infty) = 0$$

and take these as a frame of our space. Then putting $dA_i = \sum_{j=0}^{\infty} \omega_{ij}A_j$ $(i, j = 0, 1, \ldots, n, \infty)$ and $\omega_{0i} = \omega_i$, $\pi_{ij} = \omega_{ij} - \delta_{ij}\omega_{00}$ $(i, j = 1, \ldots, n)$ we get

$$\begin{cases} d\omega_i = \sum_j [\omega_j \pi_{ji}], \quad d\omega_{00} = \sum_i [\omega_i \omega_{i0}], \quad d\omega_{i0} = \sum_j [\pi_{ij}\omega_{j0}] \\ d\pi_{ij} = \sum_k [\pi_{ik}\pi_{kj}] + [\omega_{i0}\omega_j] + [\omega_i \omega_{j0}] \quad (i \neq j) \\ \pi_{ij} = -\pi_{ji} \ (i \neq j), \quad \pi_{ii} = -\omega_{00}. \end{cases} \tag{1.11}$$

As to (1.6) (1.7) (1.8) the assertion concerning the projective space holds good in the conformal space too.

Thus we get the following schema of subgroups for these two spaces. In this schema $\omega_i = 0$, $\pi_{ij} = 0$, for example, means a subgroup satisfying these conditions.

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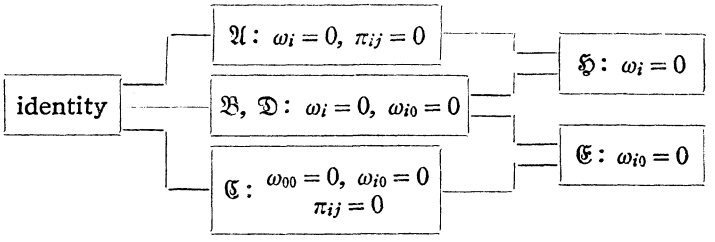

2. Homogeneous space of projective type

2.1 Let $\mathfrak{G}$ be r-parametric Lie group which operates transitively on the space of dimension n and let an element of $\mathfrak{G}$ be given by $S_a : x_i' = f_i(x_1, \ldots, x_n; a_1, \ldots, a_r)$ $(i = 1, \ldots, n)$. Let the subgroup $\mathfrak{H}$ of $\mathfrak{G}$ which fixes a certain fixed point of the n-dimensional space be closed and connected. We denote by $\omega_1, \ldots, \omega_n$ principal relative components of our homogeneous space $\mathfrak{G}/\mathfrak{H}$ and by $\omega_{n+1}, \ldots, \omega_r$ secondary relative components. We use indices in the following manner throughout the discussion

$$i, j, k, l, h = 1, 2, \ldots, n \qquad p, q, s, t, u = 1, 2, \ldots, r$$
$$\alpha, \beta, \gamma, \delta = n+1, \ldots, r.$$

We have

$$d\omega_i = \sum_{(pj)} c_{pji}[\omega_p \omega_j], \quad d\omega_\alpha = \sum_{(pq)} c_{pq\alpha}[\omega_p \omega_q]. \tag{2.1}$$

Let a frame at a point x be $S_a R$ for which relative components are ω_i, ω_α. If we rotate the frame about the point x and get $S_a S_t R$ $(S_t \in \mathfrak{H})$ we obtain for the relative components $\bar{\omega}_i$, $\bar{\omega}_\alpha$ of $S_{a'} = S_a S_t$ the relations

$$\bar{\omega}_i = \sum_j \tau_{ij}\omega_j, \quad \bar{\omega}_\alpha = \sum_p \tau_{\alpha p}\omega_p + \omega_\alpha^{(0)} \tag{2.2}$$

where $(\tau_{pq}) = (\tau_{pq}(t))$ is an element of a linear adjoint group corresponding to S_t and $\omega_\alpha^{(0)}$'s are the relative components of S_t. As $\mathfrak{H}$ is connected we have

$$c_{\alpha\beta i} = 0, \quad \tau_{i\alpha} = 0 \tag{2.3}$$

$$d\tau_{pq} = \sum_{\alpha u} c_{\alpha u p}\omega_\alpha^{(0)}\tau_{uq}, \quad d\tau_{ij} = \sum_{\alpha k} c_{\alpha k i}\omega_\alpha^{(0)}\tau_{kj}. \tag{2.4}$$

$$\sum_q c_{stq}\tau_{pq} = \sum_{qr} c_{qrp}\tau_{qs}\tau_{rt}, \quad \sum_j c_{\alpha kj}\tau_{ij} = \sum_{\beta j} c_{\beta ji}\tau_{\beta\alpha}\tau_{jk}. \tag{2.5}$$

Moreover we assume $\mathfrak{G}$ operates effectively on $\mathfrak{G}/\mathfrak{H}$ namely the transformation which fixes all the points of $\mathfrak{G}/\mathfrak{H}$ is nothing but the identity.

2.2 As defined in [1] 2.2 a vector in a homogeneous space is a set of real numbers $v = (v_1, \ldots, v_n)$ corresponding to $S_a R$ which is transformed by a frame transformation from $S_a R$ to $S_a S_t R$ in such a way that

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$$\bar{v}_i = \sum_j \tau_{ij} v_j. \tag{2.6}$$

Then using the notations as in [1] **2.2** we get by [1] (2.10)

$$d\bar{v}_i - \sum_{\alpha k} c_{\alpha k i} \bar{\omega}_\alpha \bar{v}_k = \sum_j \tau_{ij}(dv_j - \sum_{\alpha k} c_{\alpha k j} \omega_\alpha v_k) - \sum_{\alpha k h j} c_{\alpha k i} \tau_{\alpha h} \tau_{kj} \omega_h v_j. \tag{2.7}$$

Putting as in [1] **2.2**

$$B_{ijh} = \sum_\alpha c_{\alpha ij} \tau_{\alpha h}, \tag{2.8}$$

$$dB_{ijh} = \sum_\Upsilon (-\sum_k c_{k\Upsilon j} B_{ikh} + \sum_k c_{i\Upsilon k} B_{kjh} + \sum_{\alpha k} c_{\alpha ij} c_{\Upsilon k \alpha} \tau_{kh}) \omega_\Upsilon^{(0)}. \tag{2.9}$$

If we restrict the rotations about a point x to elements of $\mathfrak{H}$ satisfying

$$\omega_i = 0, \quad \sum_{\alpha\Upsilon} c_{\alpha ij} c_{\Upsilon k \alpha} \omega_\Upsilon = 0 \tag{2.10}$$

we get $B_{ijh} = 0$ on account of (2.9) and $B_{ijh} = 0$ for $S_t =$ identity. Then we have by (2.7) that $Dv_i = dv_i - \sum_{\alpha k} c_{\alpha k i} \omega_\alpha v_k$ is a component of a vector. That the integral manifold of (2.10) containing the identity transformation gives a subgroup can be verified by Frobenius's theorem in the following way.

$$d(\sum_\alpha c_{\alpha k \delta} \omega_\alpha) = \sum_{\alpha(ij)} c_{ij\alpha} c_{\alpha k \delta} [\omega_i \omega_j] + \sum_{\alpha\beta i} c_{\alpha k \delta} c_{\beta i \alpha} [\omega_\beta \omega_i] + \sum_{\alpha(\beta\Upsilon)} c_{\alpha k \delta} c_{\beta\Upsilon\alpha} [\omega_\beta \omega_\Upsilon] \tag{2.11}$$

$$\sum_\alpha c_{\alpha k \delta} c_{\beta\Upsilon\alpha} = -\sum_i c_{\Upsilon k i} c_{i\beta\delta} - \sum_\alpha c_{\Upsilon k \alpha} c_{\alpha\beta\delta} - \sum_i c_{k\beta i} c_{i\Upsilon\delta} - \sum_\alpha c_{k\beta\alpha} c_{\alpha\Upsilon\delta}$$

$$\sum_\delta c_{\alpha\beta\delta} c_{\delta jh} = -\sum_l c_{\beta jl} c_{l\alpha h} - \sum_l c_{j\alpha l} c_{l\beta h}. \tag{2.12}$$

Hence $d\omega_i$, $d(\sum_{\alpha\delta} c_{\delta jh} c_{\alpha k \delta} \omega_\alpha)$ vanish under the relation (2.10) and the integral manifold of (2.10) containing an identity forms a subgroup. We denote this by $\mathfrak{D}$ in correspondence with that in the schema (1. 12). It can also be verified by (2.11) and (2.12) that

$$\omega_i = 0, \quad \sum_\Upsilon c_{\Upsilon k \alpha} \omega_\Upsilon = 0 \tag{2.13}$$

determines a subgroup. But the equation

$$\sum_\Upsilon c_{\Upsilon k \alpha} \omega_\Upsilon = 0 \tag{2.14}$$

is, in general, not completely integrable. It is so when and only when

$$\sum_\alpha c_{ij\alpha} c_{\alpha k \delta} = 0 \tag{2.15}$$

as is evident from the consideration of (2.11).

Thus we get the following theorem

THEOREM 2.1 *Let $(v_1, \ldots, v_n)$ be components of a vector of a differentiable vectorfield on the homogeneous space corresponding to the frame $S_a S_t R$ (S_t*

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$\in \mathfrak{H}$). *Then* $Dv_i = dv_i - \sum_{\alpha k} c_{\alpha k i}\omega_\alpha v_k$ *is not in general a component of a vector. But if we restrict* S_t *to the elements of a subgroup satisfying the relations* (2.10) Dv_i *forms a component of a vector.*

THEOREM 2.2 *The equation* (2.13) *determines a subgoup. Let it be* $\mathfrak{B}$. *Even if we choose* S_a *suitably for each point of our homogeneous space corresponding to* $S_a\mathfrak{H}$, *a set of all* $S_a\mathfrak{B}$ *does not constitute a homogeneous space. It is so when and only when* (2.15) *holds.*

2.3 Now we consider the space in which (2.15) holds. In such a space we can define a parallel translation of a vector by restricting the frames at each point to $S_a\mathfrak{B}R$, mentioned to in theorem 2.2. Then the translation of a vector can be defined in our space by the equation

$$Dv_i = dv_i - \sum_{\alpha j} c_{\alpha j i}\omega_\alpha v_j = 0 \tag{2.16}$$

and this definition is intrinsic in the sense that the equation does not depend on the choice of the frame S_aS_tR ($S_t \in \mathfrak{B}$). When we translate a vector from one point to another the resulting vector depends in general on the path joining these two points. It is independent when and only when the equation (2.16) is completely integrable. By Frobenius's theorem this condition can be paraphrazed into

$$\sum_{\alpha} c_{ij\alpha}c_{\alpha kh} = 0 \tag{2.17}$$

as was shown in [1] **3.2.** Here we state a lemma.

LEMMA. *A set of all equations* $\sum_\alpha e_\alpha c_{\alpha k\tau} = 0$, $\sum_\alpha e_\alpha c_{\alpha kh} = 0$ *leads to* $e_\alpha = 0$ ($\alpha = n+1, \ldots, r$).

Proof. Let X_i and X_α be infinitesimal operators of our group $\mathfrak{G}$ corresponding to relative components ω_i and ω_α. Then the set of all operators $\sum_\alpha e_\alpha X_\alpha$ such that $\sum_\alpha e_\alpha c_{\alpha k\tau} = 0$, $\sum_\alpha e_\alpha c_{\alpha kh} = 0$ generates a subgroup $\mathfrak{H}_0$ of $\mathfrak{H}$ which commutes with any one-parametric subgroup generated by $\sum_i a_i X_i$ and any transformation of $\mathfrak{H}_0$ fixes each point of $\mathfrak{G}/\mathfrak{H}$. Owing to the effectivity of $\mathfrak{G}$ stated at the end of **2.1** $\mathfrak{H}_0$ reduces to an identity and hence $e_\alpha = 0$. That the set of all $\sum_\alpha e_\alpha X_\alpha$ above stated forms an infinitesimal group and commutes with $\sum_i a_i X_i$ can be verified as follows.

$$\left(\sum_\alpha e_\alpha X_\alpha, \sum_\beta e_\beta X_\beta\right) = \sum_{\alpha\beta\tau} e_\alpha e_\beta c_{\alpha\beta\tau} X_\tau$$

$$\sum_\tau \left(\sum_{\alpha\beta} e_\alpha e_\beta c_{\alpha\beta\tau}\right) c_{\tau k\delta} = \sum_{\alpha\beta} e_\alpha e_\beta \left(\sum_\tau c_{\alpha\beta\tau}c_{\tau k\delta}\right)$$

$$= \sum_{\alpha\beta} e_\alpha e_\beta \left(-\sum_i c_{\beta ki}c_{i\alpha\delta} - \sum_\tau c_{\beta k\tau}c_{\tau\alpha\delta} - \sum_i c_{k\alpha i}c_{i\beta\delta} - \sum_\tau c_{k\alpha\tau}c_{\tau\beta\delta}\right) = 0$$

$$\sum_{\tau}(\sum_{\alpha\beta} e_\alpha e_\beta c_{\alpha\beta\tau})c_{\tau ki} = \sum_{\alpha\beta} e_\alpha e_\beta(-\sum_j c_{\beta kj}c_{j\alpha i} - \sum_j c_{k\alpha j}c_{j\beta i}) = 0$$

$$(\sum_\alpha e_\alpha X_\alpha,\ X_i) = \sum_\alpha e_\alpha(\sum_j c_{\alpha ij}X_j + \sum_\beta c_{\alpha i\beta}X_\beta) = 0.$$

By virtue of this lemma we get from (2.15) and (2.17)

$$c_{ij\alpha} = 0. \tag{2.18}$$

This relation is equivalent to the condition that $X_1, \ldots, X_n$ generate a subgroup of $\mathfrak{G}$. Thus we get the following

THEOREM 2.3 *In the homogeneous space where* $\sum_\alpha c_{ij\alpha}c_{\alpha k\delta} = 0$ *holds the translation of a vector is absolute when and only when* $c_{ij\alpha} = 0$ *hold. This condition is also equivalent to the following: in* $\mathfrak{G}/\mathfrak{H}$ $\mathfrak{G}$ *is generated locally by* $\mathfrak{H}$ *and an n-parametric subgroup of* $\mathfrak{G}$.

2.4 Now we consider a space satisfying $c_{ij\alpha} = 0$. Then we have

$$d\omega_\alpha = \sum_{\beta k} c_{\beta k\alpha}[\omega_\beta\omega_k] + \sum_{(\beta\tau)} c_{\beta\tau\alpha}[\omega_\beta\omega_\tau]$$

and so $\omega_\alpha = 0$ is completely integrable. We can restrict a frame at each point of the space $\mathfrak{G}/\mathfrak{H}$ to that satisfying this relation. We get then a homogeneous space which is transformed simply transitively, and the structure equation of such a space is given by $d\omega_i = \sum_{(jk)} c_{jki}[\omega_j\omega_k]$. We call a transformation of our space $\mathfrak{G}/\mathfrak{H}$ satistying $\omega_\alpha = 0$ a translation and denote the group of all translations by $\mathfrak{T}$. The commutativity of the group $\mathfrak{T}$ can be expressed by $c_{ijk} = 0$. We call the space satisfying the relations $c_{ij\alpha} = 0$ and $c_{ijk} = 0$ a homogeneous space of *projective type*. A projective space and a conformal space with points as their elements are the spaces of this type. In the space of projective type we have the relations

$$\sum_\alpha c_{k\beta\alpha}c_{\alpha ji} = \sum_\alpha c_{j\beta\alpha}c_{\alpha ki}, \quad \sum_\alpha c_{k\beta\alpha}c_{\alpha j\tau} = \sum_\alpha c_{j\beta\alpha}c_{\alpha k\tau} \tag{2.19}$$

which can easily be verified by Jacobi's identity.

2.5 In a homogeneous space the differential equation

$$\omega_i = 0, \quad \sum_\alpha c_{\alpha ij}\omega_\alpha = 0 \tag{2.20}$$

is completely integrable on account of the relations (2.1) and

$$d(\sum_\alpha c_{\alpha ij}\omega_\alpha) = \sum_{(kh)\alpha} c_{kha}c_{\alpha ij}[\omega_k\omega_h] + \sum_{\beta k\alpha} c_{\beta k\alpha}c_{\alpha ij}[\omega_\beta\omega_k] + \sum_{(\beta\tau)\alpha} c_{\beta\tau\alpha}c_{\alpha ij}[\omega_\beta\omega_\tau] \tag{2.21}$$

$$\sum_\alpha c_{\beta\tau\alpha}c_{\alpha ij} = -\sum_k c_{\tau ik}c_{k\beta j} - \sum_k c_{i\beta k}c_{k\tau j}. \tag{2.22}$$

In fact $d\omega_i = 0$, $d(\sum_\alpha c_{\alpha ij}\omega_\alpha) = 0$ under the condition (2.20).

Thus we have a schema of subgroups in a general homogeneous space.

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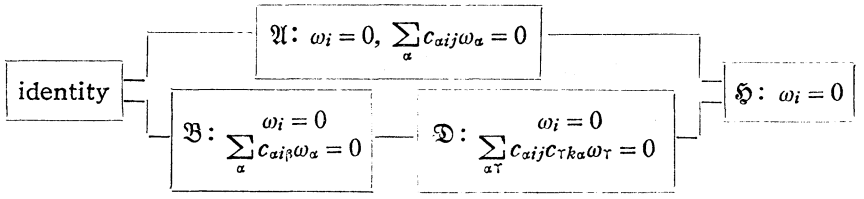
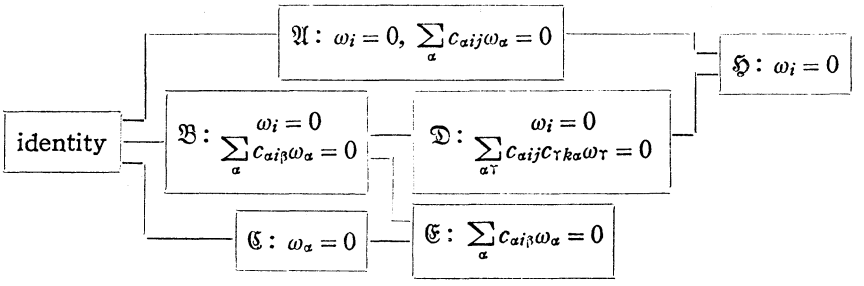


In the homogeneous space of projective type we have the following. We give attention to the correspondence with (1.12).

(2.24)



3. Characterization of projective space

3.1 In a homogeneous space of projective type we have

$$d\omega_i = \sum_{\alpha j} c_{\alpha ji}[\omega_\alpha \omega_j] \tag{3.1}$$

$$d\omega_\alpha = \sum_{\beta k} c_{\beta k\alpha}[\omega_\beta \omega_k] + \sum_{(\beta\tau)} c_{\beta\tau\alpha}[\omega_\beta \omega_\tau]. \tag{3.2}$$

In contrast with the equation in the case of projective space (1.10) we put

$$\pi_{ij} = -\sum_\alpha c_{\alpha ij}\omega_\alpha. \tag{3.3}$$

Then we have

$$d\omega_i = \sum_j [\omega_j \pi_{ji}]. \tag{3.4}$$

By virtue of (2.21) (2.19) (2.22) and the relation $c_{kh\alpha} = 0$ we get

$$d\pi_{ij} = \sum_\alpha c_{i\beta\alpha}c_{\alpha kj}[\omega_\beta \omega_k] + \sum_k [\pi_{ik}\pi_{kj}]. \tag{3.5}$$

In a projective space we have by (1.10)

$$d\pi_{ij} = [\omega_{i0}\omega_j] + \sum_k [\pi_{ik}\pi_{kj}] \quad (i \neq j). \tag{3.6}$$

So if there exist such $A_{i\beta}$ that

$$\sum_\alpha c_{i\beta\alpha}c_{\alpha kj} = \sum_\alpha c_{k\beta\alpha}c_{\alpha ij} = A_{i\beta}\delta_{kj} \quad (i \neq j) \tag{3.7}$$

we obtain (3.6) by putting

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$$\omega_{i0} = \sum_{\beta} A_{i\beta}\omega_{\beta}. \tag{3.8}$$

(3.7) being assumed we get from (3.8)

$$d\omega_{i0} = \sum_{\beta} A_{i\beta}\Big(\sum_{\gamma k} c_{\gamma k\beta}[\omega_{\gamma}\omega_{k}] + \sum_{(\gamma\delta)} c_{\gamma\delta\beta}[\omega_{\gamma}\omega_{\delta}]\Big). \tag{3.9}$$

In order that this reduces to

$$d\omega_{i0} = \sum_{k} [\pi_{ik}\omega_{k0}] \tag{3.10}$$

with ω_{i0}, π_{ij} defined by (3.8) and (3.3) it is sufficient that we have the relations

$$\sum_{\beta} A_{i\beta}c_{\gamma k\beta} = 0 \tag{3.11}$$

$$\sum_{\beta} A_{i\beta}c_{\gamma\delta\beta} = -\sum_{k} c_{\gamma ik}A_{k\delta} + \sum_{k} c_{\delta ik}A_{k\gamma}. \tag{3.12}$$

Next we consider the condition in order that

$$d\pi_{ii} = \sum_{k} [\pi_{ik}\pi_{ki}] + [\omega_{i0}\omega_{i}] - \sum_{j} [\omega_{j}\omega_{j0}] \tag{3.13}$$

holds. By (3.3) and (3.7) we have

$$\begin{aligned} d\pi_{ii} &= -\sum_{\alpha} c_{\alpha ii}\Big(\sum_{\beta k} c_{\beta k\alpha}[\omega_{\beta}\omega_{k}] + \sum_{(\beta\gamma)} c_{\beta\gamma\alpha}[\omega_{\beta}\omega_{\gamma}]\Big) \\ &= \sum_{k\beta} A_{k\beta}[\omega_{\beta}\omega_{k}] - \sum_{\beta} A_{i\beta}[\omega_{\beta}\omega_{i}] - \sum_{\alpha\beta} c_{\alpha ii}c_{\beta i\alpha}[\omega_{\beta}\omega_{i}] \\ &\qquad - \sum_{\alpha(\beta\gamma)} c_{\alpha ii}c_{\beta\gamma\alpha}[\omega_{\beta}\omega_{\gamma}]. \end{aligned} \tag{3.14}$$

and by (3.3) and (3.8) we have

$$\begin{aligned} &\sum_{k} [\pi_{ik}\pi_{ki}] + [\omega_{i0}\omega_{i}] - \sum_{j} [\omega_{j}\omega_{j0}] \\ &\qquad = \sum_{k} \Big[\sum_{\alpha} c_{\alpha ik}\omega_{\alpha},\ \sum_{\beta} c_{\beta ki}\omega_{\beta}\Big] + \sum_{\beta} A_{i\beta}[\omega_{\beta}\omega_{i}] - \sum_{\beta i} A_{i\beta}[\omega_{i}\omega_{\beta}]. \end{aligned} \tag{3.15}$$

That the coefficients of $[\omega_{\beta}\omega_{\gamma}]$ in (3.14) and (3.15) are the same can be verified by the Jacobi's identity (2.22). As for the coefficients of $[\omega_{\beta}\omega_{i}]$ it is sufficient that

$$\sum_{\alpha} c_{i\beta\alpha}c_{\alpha ii} = 2A_{i\beta} \tag{3.16}$$

holds. Thus we get (3.7) (3.11) (3.12) (3.16) as sufficient conditions in order that (1.10) except $d\omega_{00} = \sum_{i} [\omega_{i}\omega_{i0}]$ holds for π_{ij}, π_{i0} given by (3.3) (3.8). Now we remark that (3.12) is a consequence of other relations. In fact if we take k which is not equal to i, the dimension of the space being assumed to be greater than one, then we have $A_{i\beta} = \sum_{\alpha} c_{i\beta\alpha}c_{\alpha kk}$ and hence

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(3.17) $$\sum_{\beta} A_{i\beta}c_{\tau\delta\beta} = \sum_{\alpha\beta} c_{\tau\delta\beta}c_{i\beta\alpha}c_{\alpha kk}$$
$$= \sum_{\alpha}(\sum_{j} c_{\delta ij}c_{j\tau\alpha} + \sum_{\beta} c_{\delta i\beta}c_{\beta\tau\alpha} + \sum_{j} c_{i\tau j}c_{j\delta\alpha} + \sum_{\beta} c_{i\tau\beta}c_{\beta\delta\alpha})c_{\alpha kk}.$$

By (3.7) (3.16) (2.19) we have

$$\sum_{\alpha j} c_{\delta ij}c_{j\tau\alpha}c_{\alpha kk} = \sum_{j} c_{\delta ij}A_{j\tau} + c_{\delta ik}A_{k\tau}$$
$$\sum_{\alpha\beta} c_{\delta i\beta}c_{\beta\tau\alpha}c_{\alpha kk} = -\sum_{\beta} c_{\delta i\beta}(\sum_{j} c_{\tau kj}c_{j\beta k} + \sum_{j} c_{k\beta j}c_{j\tau k})$$
$$= \sum_{\beta} c_{\delta i\beta}c_{\beta kk}c_{\tau kk} - \sum_{\beta} c_{\delta i\beta}c_{\beta kk}c_{\tau kk} - \sum_{\beta} c_{k\delta\beta}c_{\beta ii}c_{i\tau k} = c_{\tau ik}A_{k\delta}.$$

Similarly $\sum_{\alpha j} c_{i\tau j}c_{j\delta\alpha}c_{\alpha kk} = -\sum_{j} c_{\tau ij}A_{j\delta} - c_{\tau ik}A_{k\delta}$, $\sum_{\alpha\beta} c_{i\tau\beta}c_{\beta\delta\alpha}c_{\alpha kk} = -c_{\delta ik}A_{k\tau}$. Putting these into the right side of (3.17) we get (3.12). The verification of the case $n=1$ is more easy. Thus we obtain a set of conditions

(I) $$c_{ijk} = 0, \quad c_{ij\alpha} = 0$$

(II) $$\sum_{\alpha} c_{i\beta\alpha}c_{\alpha kj} = 0 \quad (i \neq j,\ k \neq j)$$

$$\frac{1}{2}\sum_{\alpha} c_{i\beta\alpha}c_{\alpha ii} = \sum_{\alpha} c_{i\beta\alpha}c_{\alpha jj} = \sum_{\alpha} c_{j\beta\alpha}c_{\alpha ij} = A_{i\beta} \quad (i \neq j)$$

(III) $$\sum_{\beta} A_{i\beta}c_{\tau k\beta} = 0.$$

Under these conditions we get the structure equations of a projective space (1.10) when we put

$$\pi_{ij} = -\sum_{\alpha} c_{\alpha ij}\omega_{\alpha}, \quad \omega_{i0} = \sum_{\beta} A_{i\beta}\omega_{\beta}, \quad \omega_{00} = \frac{1}{n+1}\sum_{\alpha i} c_{\alpha ii}\omega_{\alpha}.$$

The formula for ω_{00} can be verified as follows. We have by virtue of the relation $\sum_{i=0}^{n} \omega_{ii} = 0$

$$\sum_{i=1}^{n} \pi_{ii} = \sum_{i=1}^{n} (\omega_{ii} - \omega_{00}) = -(n+1)\omega_{00}.$$

Hence $\omega_{00} = -\frac{1}{n+1}\sum_{\alpha i} c_{\alpha ii}\omega_{\alpha}$. That this satisfies the equation $d\omega_{00} = \sum_{i}[\omega_i\omega_{i0}]$ can be verified as follows.

$$d\omega_{00} = \frac{1}{n+1}\sum_{\alpha i} c_{\alpha ii}d\omega_{\alpha} = \frac{1}{n+1}(\sum_{\alpha ij\beta} c_{j\beta\alpha}c_{\alpha ii}[\omega_j\omega_{\beta}] + \sum_{\alpha(\beta\tau)i} c_{\beta\tau\alpha}c_{\alpha ii}[\omega_{\beta}\omega_{\tau}])$$
$$= \frac{1}{n+1}\sum_{j}[\omega_j, \sum_{\alpha i\beta} c_{j\beta\alpha}c_{\alpha ii}\omega_{\beta}] + \frac{1}{n+1}\sum_{\alpha(\beta\tau)i} c_{\beta\tau\alpha}c_{\alpha ii}[\omega_{\beta}\omega_{\tau}]$$
$$= \frac{1}{n+1}\sum_{j}[\omega_j, (n+1)\sum_{\beta} A_{j\beta}\omega_{\beta}] - \frac{1}{n+1}\sum_{(\beta\tau)}(\sum_{k\iota} c_{\tau ik}c_{k\beta i} + \sum_{k\iota} c_{i\beta k}c_{k\tau i})[\omega_{\beta}\omega_{\tau}]$$
$$= \sum_{j}[\omega_j\omega_{j0}].$$

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3.2 The geometrical interpretation of the relation (I) was given in **2.** Now we will interprete (II). In the homogeneous space of projective type we restrict to the frames satisfying $\sum_{\alpha} c_{\alpha k\beta}\omega_\alpha = 0$ as was mentioned to in **2.2**, then a translation of a vector is defined by $Dv_i = dv_i - \sum_{\alpha k} c_{\alpha ki}\omega_\alpha v_k = 0$ and a geodesic can be defined as the curve obtained by solving

$$\frac{d}{d\sigma}\left(\frac{\omega_i}{d\sigma}\right) - \sum_{\alpha k} c_{\alpha ki}\frac{\omega_\alpha}{d\sigma}\frac{\omega_k}{d\sigma} = 0$$

with a suitably chosen parameter σ. The curve can also be obtained by solving

$$\omega_i = c_i d\sigma, \quad \omega_\alpha = 0 \quad (c_i = \text{const.}). \tag{3.18}$$

The discussion is analogous to that of [1] **4.1.** We call the frames along the geodesic satisfying (3.18) *adapted frames* corresponding to the geodesic. Let these be S_aR and operate on them infinitesimal rotations S_ε. Relative displacement of these frames $S_aS_\varepsilon R$ is $(S_aS_\varepsilon)^{-1}S_{a+da}S_\varepsilon = S_\varepsilon^{-1}(S_a^{-1}S_{a+da})S_\varepsilon$ and its relative components are given by $\sum_k \tau_{pk}c_k d\sigma$ where (τ_{pq}) is an element of a linear adjoint group corresponding to S_ε. Hence by (2.4) we have for an infinitesimal change of relative components corresponding to the frame transformation from S_aR to $S_aS_\varepsilon R$

$$\delta\omega_i = \sum_{\alpha k} c_{\alpha ki}e_\alpha c_k d\sigma, \quad \delta\omega_\alpha = \sum_{\beta h} c_{\beta h\alpha}e_\beta c_h d\sigma$$

where $(0, \ldots, 0, e_{n+1}, \ldots, e_r)$ are parameters for S_ε. Thus if we neglect terms of higher order with respect to e_α we get for secondary relative components of $S_aS_\varepsilon R$ $\omega_\alpha + \delta\omega_\alpha = \delta\omega_\alpha = \sum_{\beta h} c_{\beta h\alpha}e_\beta c_h d\sigma$. When we attach to each point of a curve in the homogeneous space a fixed frame, we can define a translation of a vector v along the curve relative to the frames by the equation

$$Dv_i = dv_i - \sum_{\alpha j} c_{\alpha ji}\omega_\alpha v_j = 0$$

although the translation thus defined is not intrinsic with respect to the curve. In this sense of parallelism $c = (c_1, \ldots, c_n)$ is parallel with respect to the frames S_aR.

Let $c = (c_1, \ldots, c_n)$ be a vector which determines the geodesic (3.18) and we assume for any geodesic and S_ε that

(3.19) *"when we translate a vector c with respect to frames $S_aS_\varepsilon R$ the direction of c is invariant except for a second order with respect to the parameters of S_ε."*

Then we get (II) as follows. The property (3.19) means that

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$$Dc_i = dc_i - \sum_{\alpha k} c_{\alpha ki}(\omega_\alpha + \delta\omega_\alpha)c_k = -\sum_{\alpha k} c_{\alpha ki}(\sum_{\beta h} c_{\beta h\alpha} e_\beta c_h d\sigma)c_k$$

is proportional to c_i. Hence we can put $\sum_{\beta\alpha kh} c_{\alpha ki}c_{\beta h\alpha}e_\beta c_h c_k = c_i P$. As e_β is arbitrary we get

$$\sum_{\alpha kh} c_{\alpha ki}c_{\beta h\alpha}c_h c_k = c_i P_\beta. \tag{3.20}$$

Hence $\sum_{\alpha kh} c_{\alpha ki}c_{\beta h\alpha}c_h c_k/c_i = P_\beta$ does not depend on i. As c_i is arbitrary P_β must be a linear combination of $c_1, \ldots, c_n$. Hence

$$\sum_{\alpha} c_{\beta h\alpha}c_{\alpha ki} = 0 \quad (h \neq i.\ k \neq i) \tag{3.21}$$

and so

$$P_\beta = \sum_\alpha c_{\beta i\alpha}c_{\alpha ii}c_i + \sum_{j(\neq i)}(\sum_\alpha c_{\beta i\alpha}c_{\alpha ji} + \sum_\alpha c_{\beta j\alpha}c_{\alpha ii})c_j.$$

As we have $\sum_\alpha c_{\beta i\alpha}c_{\alpha ji} = \sum_\alpha c_{\beta j\alpha}c_{\alpha ii}$ by (2.19) we can put

$$\sum_\alpha c_{\beta i\alpha}c_{\alpha ii} = -2A_{i\beta}, \quad \sum_\alpha c_{\beta j\alpha}c_{\alpha ii} = -A_{j\beta} \quad (i \neq j). \tag{3.22}$$

Thus we obtain (II).

3.3 Now we will show that if the dimension of our space is greater than one the relation (III) is a consequence of (II). We begin with the case of the dimension greater than 2. Then we can take for any i and k a positive integer j such that $i \neq j$, $k \neq j$. By virtue of the relation (II) and (2.19) we get

$$\sum_\beta A_{i\beta}c_{\gamma k\beta} = -\sum_{\beta\alpha} c_{\gamma k\beta}c_{\beta i\alpha}c_{\alpha jj} = -\sum_{\beta\alpha} c_{\gamma k\beta}c_{\beta j\alpha}c_{\alpha ij} = -\sum_{\beta\alpha} c_{\gamma j\beta}c_{\beta k\alpha}c_{\alpha ij} = 0.$$

In the case of dimension 2 such j as we used above can be taken except for $\sum_\beta A_{1\beta}c_{\gamma 2\beta}$ and $\sum_\beta A_{2\beta}c_{\gamma 1\beta}$. In this case we proceed as follows. We get by (3.22) (2.19)

$$2\sum_\beta A_{1\beta}c_{\gamma 2\beta} = -\sum_{\beta\alpha} c_{\gamma 2\beta}c_{\beta 1\alpha}c_{\alpha 11} = -\sum_{\beta\alpha} c_{\gamma 1\beta}c_{\beta 2\alpha}c_{\alpha 11} = \sum_\beta c_{\gamma 1\beta}A_{2\beta}.$$

Similarly $\sum_\beta A_{2\beta}c_{\gamma 1\beta} = 2\sum_\beta c_{\gamma 2\beta}A_{1\beta}$. Hence $\sum_\beta A_{2\beta}c_{\gamma 1\beta} = \sum_\beta A_{1\beta}c_{\gamma 2\beta} = 0$.

In the case of dimension one I can not give a geometric interpretation to the relation (III) namely

$$\sum_\beta A_{1\beta}c_{\gamma 1\beta} = \sum_{\alpha\beta} c_{\gamma 1\beta}c_{\beta 1\alpha}c_{\alpha 11} = 0.$$

Thus we get a characterization of a projective space which can be stated as follows:

THEOREM 3.1 *A homogeneous space of projective type whose dimension is greater than one can be locally imbedded into the projective space of the same*

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dimension preserving the group-theoretical structure if it satisfies the following condition: when we operate on all the adapted frames along any geodesic any infinitesimal rotation S_ε, the vector which is tangent to this geodesic and has constant components does not change its direction by the translation with respect to the rotated frames, except for an order greater than one.

This condition is rather complicated. It is desirable to replace it by one which is simpler. The rotation group about a point when it is imbedded into the projective space is in general a subgroup of that of the projective space.

In one-dimensional case (I) (II) are always satisfied. Hence we get the following theorem.

THEOREM 3.2 *A one-dimensional homogeneous space satisfying the relation* $\sum_{\alpha\beta} c_{\tau 1\beta} c_{\beta 1\alpha} c_{\alpha 11} = 0$ *can be imbedded into the one-dimensional projective space preserving the group-theoretical structure.*

Although a projective space with points as its elements is of projective type and satisfies (3.19) the converse of theorem 3.1 is not true. A simple example shows that there exists a homogeneous space which can be imbedded into the projective space of the same dimension and yet is not of projective type. A homogeneous space which can be imbedded into the projective space of the same dimension and whose fundamental group contains a group of all translations is of projective type. In fact in such a space (1.10) holds, where π_{ij}, ω_{i0} are linear combinations of relative components ω_i, ω_α with constant coefficients. The differential equations $\pi_{ij} = 0$, $\omega_{i0} = 0$ are completely integrable, though π_{ij}, ω_{i0} are not always independent. We take frames satisfying these relations. Then owing to the assumption that the fundamental group contains a group of all translations ω_i's $(i = 1, \ldots, n)$ are independent even when we put $\pi_{ij} = 0$, $\omega_{i0} = 0$. So we can take secondary relative components ω_α anew such that π_{ij}, ω_{i0} are linear combinations of ω_α with constant coefficients. Then we get $c_{ijk} = 0$, $c_{ij\alpha} = 0$ and hence a space of projective type. We have in addition (3.3) (3.8), and (3.19) is satisfied.

4. Characterization of conformal space

4.1 Now we treat the case of a conformal space which is simpler than that of the projective space. In a conformal space we have by (1.11)

$$\pi_{ii} = -\omega_{00}, \quad \pi_{ij} = -\pi_{ji} \quad (i \neq j).$$

Hence the linear group of isotropy is conformal. We take a homogeneous space of projective type and seek for the conditions in order that it can be imbedded into a conformal space. In the first we put

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$$\pi_{ij} = -\sum_{\alpha} c_{\alpha ij}\omega_{\alpha} \tag{4.1}$$

to get $d\omega_i = \sum_j [\omega_j \pi_{ji}]$ from (3.1). We assume that a linear group of isotropy of our space is conformal, namely

$$c_{\alpha ij} = -c_{\alpha ji} \quad (i \neq j), \quad c_{\alpha ii} = c_{\alpha}. \tag{4.2}$$

We have by (2.19) $\sum_{\alpha} c_{k\beta\alpha}c_{\alpha ji} = \sum_{\alpha} c_{j\beta\alpha}c_{\alpha ki}$. So if we put $\sum_{\alpha} c_{k\beta\alpha}c_{\alpha ji} = P_{kji}$ we have $P_{kji} = P_{jki}$ and by virtue of the assumption (4.2) $P_{kji} = -P_{kij}$ for $i \neq j$. Hence for $i \neq j \neq k$ we get

$$P_{kji} = -P_{kij} = -P_{ikj} = P_{ijk} = P_{jik} = -P_{jki} = -P_{kji}.$$

Hence $P_{kji} = 0$ $(i \neq j \neq k)$. For other cases we get

$$P_{jji} = -P_{jij} = -P_{ijj} = -\sum_{\alpha} c_{i\beta\alpha}c_{\alpha jj} = -\sum_{\alpha} c_{i\beta\alpha}c_{\alpha} = -P_{iii} \quad (i \neq j).$$

Hence putting $\sum_{\alpha} c_{i\beta\alpha}c_{\alpha} = A_{i\beta}$ we get

$$\begin{aligned} &\sum_{\alpha} c_{i\beta\alpha}c_{\alpha jk} = 0 \quad (i \neq k \neq j) \\ &\sum_{\alpha} c_{i\beta\alpha}c_{\alpha ji} = -\sum_{\alpha} c_{i\beta\alpha}c_{\alpha ij} = \sum_{\alpha} c_{j\beta\alpha}c_{\alpha ii} = A_{j\beta} \quad (i \neq j). \end{aligned} \tag{4.3}$$

As our space is of projective type we have by (3.5)

$$d\pi_{ij} = \sum_{\alpha\beta k} c_{i\beta\alpha}c_{\alpha kj}[\omega_{\beta}\omega_k] + \sum_k [\pi_{ik}\pi_{kj}].$$

So for $i \neq j$ we obtain by (4.3)

$$\begin{aligned} d\pi_{ij} &= \sum_{\alpha\beta} c_{i\beta\alpha}c_{\alpha jj}[\omega_{\beta}\omega_j] + \sum_{\alpha,\beta} c_{i\beta\alpha}c_{\alpha ij}[\omega_{\beta}\omega_i] + \sum_k [\pi_{ik}\pi_{kj}] \\ &= \sum_{\beta} A_{i\beta}[\omega_{\beta}\omega_j] + \sum_{\beta} A_{j\beta}[\omega_{\beta}\omega_i] + \sum_k [\pi_{ik}\pi_{kj}]. \end{aligned}$$

If we put

$$\omega_{i0} = \sum_{\beta} A_{i\beta}\omega_{\beta} \tag{4.4}$$

we get

$$d\pi_{ij} = [\omega_{i0}\omega_j] - [\omega_{j0}\omega_i] + \sum_k [\pi_{ik}\pi_{kj}]. \tag{4.5}$$

From (4.4) we get

$$d\omega_{i0} = \sum_{\beta\gamma k} A_{i\beta}c_{\gamma k\beta}[\omega_{\gamma}\omega_k] + \sum_{\beta(\gamma\delta)} A_{i\beta}c_{\gamma\delta\beta}[\omega_{\gamma}\omega_{\delta}]$$

and this reduces to

$$d\omega_{i0} = \sum_j [\pi_{ij}\omega_{j0}] \tag{4.5}$$

if the relations

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$$\sum_{\beta} A_{i\beta}c_{\gamma\delta\beta} = -\sum_{j} c_{\gamma ij}A_{j\delta} + \sum_{j} c_{\delta ij}A_{j\gamma} \tag{4.6}$$

$$\sum_{\beta} A_{i\beta}c_{\gamma k\beta} = 0 \tag{4.7}$$

are satisfied. The former is a consequence of other relations as is shown in the following. We have by (4.3)

$$\sum_{\beta} A_{i\beta}c_{\gamma\delta\beta} = -\sum_{\alpha\beta} c_{\gamma\delta\beta}c_{\beta i\alpha}c_{\alpha} = \sum_{j\alpha} c_{\delta ij}c_{j\gamma\alpha}c_{\alpha} + \sum_{\beta\alpha} c_{\delta i\beta}c_{\beta\gamma\alpha}c_{\alpha} + \sum_{\alpha\beta} c_{i\gamma\beta}c_{\beta\delta\alpha}c_{\alpha} + \sum_{j\alpha} c_{i\gamma j}c_{j\delta\alpha}c_{\alpha}$$

$$\sum_{\alpha} c_{\beta\gamma\alpha}c_{\alpha ii} = -\sum_{k} c_{\gamma ik}c_{k\beta i} - \sum_{k} c_{i\beta k}c_{k\gamma i} = \sum_{k} c_{\beta ki}c_{\gamma ik} - \sum_{k} c_{\beta ki}c_{\gamma ik} = 0, \tag{4.8}$$

and so we get

$$\begin{aligned}\sum_{\beta} A_{i\beta}c_{\gamma\delta\beta} &= \sum_{j\alpha} c_{\delta ij}c_{j\gamma\alpha}c_{\alpha} + \sum_{j\alpha} c_{i\gamma j}c_{j\delta\alpha}c_{\alpha} \\ &= -\sum_{j\alpha} c_{\gamma ij}c_{j\delta\alpha}c_{\alpha} + \sum_{j\alpha} c_{\delta ij}c_{j\gamma\alpha}c_{\alpha} = -\sum_{j} c_{\gamma ij}A_{j\delta} + \sum_{j} c_{\delta ij}A_{j\gamma}.\end{aligned}$$

As for (4.7) we can proceed as in **3.3** if $n>2$ and $i\neq k$ but if $n>1$ and $i=k$ we proceed as follows. Let $i\neq j$ then we have by (4.2) (2.19)

$$\begin{aligned}\sum_{\beta} A_{i\beta}c_{\gamma i\beta} &= \sum_{\alpha\beta} c_{\gamma i\beta}c_{i\beta\alpha}c_{\alpha jj} = \sum_{\alpha\beta} c_{\gamma i\beta}c_{j\beta\alpha}c_{\alpha ij} \\ &= -\sum_{\alpha\beta} c_{\gamma j\beta}c_{i\beta\alpha}c_{\alpha ji} = -\sum_{\alpha\beta} c_{\gamma j\beta}c_{j\beta\alpha}c_{\alpha ii} = -\sum_{\beta} A_{j\beta}c_{\gamma j\beta}.\end{aligned}$$

Hence

$$\sum_{\beta} A_{i\beta}c_{\gamma i\beta} + \sum_{\beta} A_{j\beta}c_{\gamma j\beta} = 0.$$

From the equations obtained from this by putting j, k and k, i in place of i, j we get (4.7), and so (4.7) holds good if $n>2$. In the case $n=2$ we get only

$$\sum_{\beta\alpha} c_{\gamma 1\beta}c_{1\beta\alpha}c_{\alpha} + \sum_{\beta\alpha} c_{\gamma 2\beta}c_{2\beta\alpha}c_{\alpha} = 0$$

and in order that each term reduces to 0 it is necessary and sufficient that

$$\sum_{\alpha\beta} c_{\gamma 1\beta}c_{1\beta\alpha}c_{\alpha} = \sum_{\alpha\beta} c_{\gamma 2\beta}c_{2\beta\alpha}c_{\alpha}. \tag{4.9}$$

When $n=1$ we have

$$\sum_{\alpha\beta} c_{\gamma 1\beta}c_{1\beta\alpha}c_{\alpha 11} = 0.$$

As to $\pi_{ii} = -\omega_{00} = -\sum_{\alpha} c_{\alpha}\omega_{\alpha}$ we have

$$d\omega_{00} = \sum_{\alpha\beta k} c_{\alpha ii}c_{\beta k\alpha}[\omega_{\beta}\omega_{k}] + \sum_{\alpha(\beta\gamma)} c_{\alpha ii}c_{\beta\gamma\alpha}[\omega_{\beta}\omega_{\gamma}] = -\sum_{\beta k} A_{k\beta}[\omega_{\beta}\omega_{k}] + \sum_{\alpha(\beta\gamma)} c_{\beta\gamma\alpha}c_{\alpha ii}[\omega_{\beta}\omega_{\gamma}].$$

and we get $d\omega_{00} = \sum_{k}[\omega_{k}\omega_{k0}]$ by virtue of (4.8).

Thus if $n>2$ and the relations

$$c_{ijk} = 0, \quad c_{ij\alpha} = 0 \tag{I}$$

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(II) $$c_{\alpha ij} = -c_{\alpha ji} \quad (i \neq j), \quad c_{\alpha ii} = c_\alpha$$

are satisfied we get (1.11) by putting

$$\pi_{ij} = -\sum_\alpha c_{\alpha ij}\omega_\alpha, \quad \omega_{00} = -\pi_{ii} = \sum_\alpha c_\alpha \omega_\alpha, \quad \omega_{i0} = \sum_{\alpha\beta} c_{i\beta\alpha} c_\alpha \omega_\beta .$$

In the case $n = 2$ we must add one more condition (4.9). The case of dimension one reduces to theorem 3.2.

Hence we get the following.

THEOREM 4.1 *A homogeneous space of projective type whose dimension is greater than 2 and whose linear group of isotropy is conformal can be locally imbedded into a conformal space of the same dimension preserving the group-theoretical structure. If the dimension of the space of projective type is 2 and the linear group of isotropy is conformal and (4.9) holds, the space can be imbedded into the conformal space of dimension 2.*

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