ON A FUNCTIONAL EQUATION

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1. We have recently discussed in (**1**) the general solution of a certain functional equation arising in statistical thermodynamics, and we propose in this note to deal with another functional equation arising from the same source (**2**).

The problem is to obtain the most general function $f(x)$ which, for all positive integral values of m, n, satisfies the functional equation

$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i y_j) = \sum_{i=1}^{m} f(x_i) + \sum_{j=1}^{n} f(y_j), \quad \ldots\ldots\ldots\ldots\ldots(1.1)$$

where

$$\sum_{i=1}^{m} x_i = 1, \quad \sum_{j=1}^{n} y_j = 1, \quad 0 \leq x_i \leq 1 \ (i = 1, \ldots, m), \quad 0 \leq y_j \leq 1 \ (j = 1, \ldots, n).$$

It is easily verified by direct substitution that a solution is

$$f(x) = Ax \log x,$$

where A is an arbitrary constant. It is the object of this note to prove that this is the only continuous solution.

It is of particular interest physically to consider infinite values of m, n in (1.1). The proof given below of the uniqueness of the form of $f(x)$ does not require that (1.1) should hold for infinite values of m, n; it is sufficient that it should hold for all finite values. But it is readily verified that the solution

$$f(x) = Ax \log x$$

does indeed satisfy (1.1) for infinite values of m, n, provided that x_i, y_j are such that one side or other of the functional equation converges.

2. Let p, q, r, s be any integers for which $1 \leq r \leq p$, $1 \leq s \leq q$. In the functional equation (1.1), we take

$$m = p - r + 1, \quad n = q - s + 1,$$

$$x_i = 1/p \ (i = 1, \ldots, p-r), \quad x_{p-r+1} = r/p,$$

$$y_j = 1/q \ (j = 1, \ldots, q-s), \quad x_{q-s+1} = s/q,$$

and obtain

$$(p-r)(q-s)f\left(\frac{1}{pq}\right) + (p-r)f\left(\frac{s}{pq}\right) + (q-s)f\left(\frac{r}{pq}\right) + f\left(\frac{rs}{pq}\right)$$

$$= (p-r)f\left(\frac{1}{p}\right) + f\left(\frac{r}{p}\right) + (q-s)f\left(\frac{1}{q}\right) + f\left(\frac{s}{q}\right) \ldots\ldots\ldots\ldots(2.1)$$

Putting

$$xf\left(\frac{1}{x}\right) = \phi(x),$$

we then find that multiplication of (2.1) by pq yields

$$(p-r)(q-s)\phi(pq)+s(p-r)\phi\left(\frac{pq}{s}\right)+r(q-s)\phi\left(\frac{pq}{r}\right)+rs\phi\left(\frac{pq}{rs}\right)$$
$$= q(p-r)\phi(p)+qr\phi\left(\frac{p}{r}\right)+p(q-s)\phi(q)+ps\phi\left(\frac{q}{s}\right). \quad\ldots\ldots\ldots\ldots(2.2)$$

If in (2.2) we take $r = s = 1$, we obtain

$$\phi(pq) = \phi(p)+\phi(q). \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.3)$$

If in (2.2) we take only $s = 1$, leaving r undetermined, we obtain

$$q(p-r)\phi(pq)+qr\phi\left(\frac{pq}{r}\right) = q(p-r)\phi(p)+qr\phi\left(\frac{p}{r}\right)+pq\phi(q),$$

which, on substitution for $\phi(pq)$ from (2.3), yields

$$\phi(pq/r) = \phi(p/r)+\phi(q),$$

and so, symmetrically,

$$\phi(pq/s) = \phi(p)+\phi(q/s).$$

Finally, substituting now for $\phi(pq)$, $\phi(pq/r)$, $\phi(pq/s)$ in (2.2), we obtain, on clearing,

$$\phi(pq/rs) = \phi(p/r)+\phi(q/s),$$

i.e.
$$\phi(xy) = \phi(x)+\phi(y), \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.4)$$

where $x, y \geqq 1$ and x, y are rational. Continuity then extends (2.4) to all $x, y \geqq 1$.

Equation (2.4) is a well-known functional equation, of which the most general continuous solution is

$$\phi(x) = -A \log x,$$

where A is an arbitrary constant. Referring back to the definition of $\phi(x)$, we see that

$$f(1/x) = \frac{1}{x}\,\phi(x),$$

i.e.
$$f(x) = x\phi(1/x)$$
$$= Ax \log x,$$

as required.

REFERENCES

(1) T. W. Chaundy and J. B. McLeod, On a functional equation, *Quart. J. of Math.* (Oxford) (2), **9** (1958), 202-6.

(2) W. B. Brown, *American J. of Phys.*, (to appear.)

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