

SEMIAMPLENESS FOR CALABI–YAU SURFACES IN POSITIVE AND MIXED CHARACTERISTIC

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Abstract. In this note, we prove the semiample conjecture for Kawamata log terminal Calabi–Yau (CY) surface pairs over an excellent base ring. As applications, we deduce that generalized abundance and Serrano’s conjecture hold for surfaces. Finally, we study the semiample conjecture for CY threefolds over a mixed characteristic DVR.

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§1. Introduction

The abundance conjecture predicts that the canonical divisor K_X of a minimal model X with Kawamata log terminal (in short, klt) or log canonical (lc) singularities is semiample and it is one of the most important conjectures of the minimal model program (MMP). Abundance is known to hold for surfaces over fields of arbitrary characteristic [P], [T5] and threefolds in characteristic 0 by Kawamata and Miyaoka (see [K3] for references), but in higher dimensions, even the effectivity of a multiple of K_X (the so-called nonvanishing

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conjecture) is still an open problem. For threefolds in positive and mixed characteristic, various special cases have been proved in [BBS], [DW], [W1], [XZ], [Z1], [Z2], but the general conjecture is still unanswered.

A generalized form of abundance is expected to hold for K -trivial varieties and their log generalizations (see, e.g., [K2, Conj. 51]).

CONJECTURE 1.1 (Semiampleness conjecture on klt Calabi–Yau pairs). *Let (X, Δ) be a projective klt pair of dimension n over a field k such that $K_X + \Delta \equiv 0$. Let L be a nef \mathbb{Q} -Cartier \mathbb{Q} -divisor on X . Then L is num-semiample, that is, there exists a \mathbb{Q} -Cartier \mathbb{Q} -divisor L' on X such that $L \equiv L'$ and L' is semiample.*

If k is a field of characteristic 0, a thorough discussion of this conjecture and its connections to the MMP can be found in the work of Lazić and Peternell [LP1], [LP2]. Under the same hypothesis on k , Conjecture 1.1 has been proved for projective surfaces [LP1, Th. 8.2], certain classes of Calabi–Yau (CY) threefolds [LOP], [LS], and hyperkähler fourfolds [DHM+]. Moreover, some partial results for compact Kähler surfaces are obtained in [FT].

The aim of this note is to confirm the conjecture for projective surfaces over arbitrary, possibly imperfect, fields of positive characteristic.

THEOREM 1.2 (see Theorem 3.6). *The semiampleness conjecture holds for klt CY surface pairs over a field k of characteristic $p > 0$.*

In [HL], [LP1], the authors propose a further generalization of the abundance and semiampleness conjectures.

CONJECTURE 1.3 (Generalized abundance conjecture). *Let (X, B) be a projective klt pair over a field k such that $K_X + B$ is pseudo-effective, and let M be a nef \mathbb{Q} -Cartier \mathbb{Q} -divisor on X . If $K_X + B + M$ is nef, then it is num-semiample.*

Following, in part, ideas of [LP1], we give a direct proof of the generalized abundance conjecture for excellent surfaces by reducing to the semiampleness conjecture on klt CY surfaces over a field of characteristic $p > 0$.

THEOREM 1.4 (See Theorem 4.2). *Let $\pi: X \rightarrow T$ be a projective R -morphism of quasi-projective integral normal schemes over R . Suppose that (X, B) is a klt surface such that:*

- (a) $K_X + B$ is pseudo-effective over T ;
- (b) M is a nef \mathbb{Q} -Cartier \mathbb{Q} -divisor over T ;
- (c) $L := K_X + B + M$ is nef \mathbb{Q} -Cartier \mathbb{Q} -divisor over T .

Then L is num-semiample over T .

We note that similar results have also been obtained in the context of generalized surface pairs in characteristic 0 by Han–Liu [HL] and following their strategy we characterize when generalized abundance fails if $K_X + B$ is not pseudo-effective (see Proposition 4.5). As an application thereof, together with a careful analysis over imperfect fields, we provide a proof of Serrano’s conjecture [S2] for klt surfaces and suitable threefolds in the positive and mixed setting.

COROLLARY 1.5 (See Theorem 4.6 and Corollary 4.7). *Let R be an excellent ring of finite Krull dimension with dualizing complex. Let $X \rightarrow T$ be a projective contraction of quasi-*

projective R -schemes. Suppose that (X, B) is klt, $-(K_X + B)$ is strictly nef, and further that:

- (a) X has dimension 2; or
- (b) X has dimension 3, $\dim T > 0$, and the closed points of R have residue fields of characteristic $p = 0$ or $p > 5$.

Then $-(K_X + B)$ is ample.

Using Serrano's conjecture, we then show in Theorem 4.8 that the numerical nonvanishing conjecture of [HL] holds for generalized klt surface pairs even if $K_X + B$ is not pseudo-effective. We recall that the numerical nonvanishing conjecture is still open even for threefolds over the complex numbers \mathbb{C} (see [LPT+] for recent progress).

To complete the picture, we also prove that generalized abundance holds for generalized lc pairs if the b-nef part is b-semiample Theorem 4.11 in positive and mixed characteristic. The main difficulty here is the lack of Bertini-type theorems.

Recently, a large part of the MMP for threefolds in mixed characteristic $(0, p > 5)$ has been established in [BMP+], [TY], [W2]. We conclude by showing an application of the semiample conjecture for surfaces to arithmetic klt CY threefolds.

THEOREM 1.6. *Let R be an excellent discrete valuation ring (DVR) with residue field k of positive characteristic $p > 5$. Let $\pi: (X, B) \rightarrow \text{Spec}(R)$ be a projective dominant morphism such that (X, B) is klt and $K_X + B \equiv 0$ over R .*

If L is a nef \mathbb{Q} -Cartier \mathbb{Q} -divisor over R , then L is num-semiample over R .

§2. Preliminaries

2.1 Notation

- (a) In this article, a base ring R will always denote an excellent domain of finite Krull dimension admitting a dualizing complex ω_R^\bullet . We assume that ω_R^\bullet is normalized as explained in [BMP+].
- (b) For a field k , we denote by k^{sep} (resp. \bar{k}) a separable (resp. an algebraic) closure of k .
- (c) For an integral scheme X with generic point η , its function field $k(X)$ is the field $\mathcal{O}_{X, \eta}$.
- (d) If X is an \mathbb{F}_p -scheme, we denote by $F: X \rightarrow X$ its (absolute) Frobenius morphism. We say that X is F -finite if F is a finite morphism.
- (e) For a field k , we say that X is a variety over k or a k -variety if X is an integral scheme that is separated and of finite type over k .
- (f) Given a scheme X , we denote by X_{red} the reduced closed subscheme of X underlying the same topological space (see [TSP, Tag 01IZ]).
- (g) We say that (X, Δ) is a log pair if X is a normal excellent integral pure d -dimensional Noetherian scheme with a dualizing complex, Δ is an effective \mathbb{Q} -divisor, and $K_X + \Delta$ is \mathbb{Q} -Cartier. The dimension of (X, Δ) is the total dimension of X .
- (h) We will follow [K1] and [BMP+, §2.5] for the definition of singularities of log pairs (such as klt and lc).
- (i) We refer to [BMP+, §2.5] and [La] for notions of positivity (such as big, nef, and pseudo-effective) for \mathbb{Q} -Cartier \mathbb{Q} -divisors, relative to a projective morphism of separated Noetherian schemes.

- (j) Let L be a Cartier divisor on an integral scheme X of finite type over R . We denote by $\text{Bs}(L)$ the base locus of L , considered as a closed reduced subscheme of X . We denote by $\mathbf{SB}(L) := \bigcap_{m>0} \text{Bs}(mL)$ the *stable base locus* of L over R .
- (k) A morphism $f: X \rightarrow Y$ of normal schemes is called a *contraction* if f is proper and $f_*\mathcal{O}_X = \mathcal{O}_Y$.

2.2 Num-semiample divisors

In this section, we fix S to be a Noetherian excellent base scheme. Given a proper scheme X over S , we define a *curve* in X over S to be an integral closed subscheme $C \subset X$ of dimension 1 such that C is proper over some closed point $s \in S$. If it is clear from the context, we will omit to mention S .

2.2.1. Nef and num-semiample

The notion of nefness is numerical, whereas semiample is not as the example of a torsion nontrivial line bundle on an elliptic curve shows. The notion of numerical semiample is an interpolation between the two: while remaining a numerical condition, it implies the existence of a contraction morphism to a scheme.

DEFINITION 2.1. Let X be a proper S -scheme. A \mathbb{Q} -Cartier \mathbb{Q} -divisor L on X is said to be *semiample* (resp. *num-semiample*) over S if there exists a proper contraction $f: X \rightarrow Z$ of S -schemes and an ample \mathbb{Q} -Cartier \mathbb{Q} -divisor A on Z such that $L \sim_{\mathbb{Q}} f^*A$ (resp. $L \equiv f^*A$) over S .

Clearly, a num-semiample \mathbb{Q} -Cartier \mathbb{Q} -divisor is nef, but it is easy to construct nef divisors which are not num-semiample (see [LP1, Exam. 6.1]). Strictly, nef divisors will appear frequently in our proofs.

DEFINITION 2.2. Let X be a projective S -scheme, and let L be a \mathbb{Q} -Cartier \mathbb{Q} -divisor on X . We say that L is *strictly nef* over S if for every curve C over S , we have $L \cdot C > 0$.

Note that the sum of a nef and a strictly nef line bundle is strictly nef. We recall the definition of numerical dimension for nef divisors.

DEFINITION 2.3. Let X be a normal projective variety defined over a field k , and let L be a nef \mathbb{Q} -Cartier \mathbb{Q} -divisor. The *numerical dimension* of L is defined as

$$\nu(L) := \max \{d \in \mathbb{Z}_{\geq 0} \mid L^d \neq 0\}.$$

2.2.2. Descent of relatively numerically trivial divisors

We collect some results on descent of trivial divisors. We recall a descent for divisors which are \mathbb{Q} -linearly trivial on the generic fiber used successfully in [CT], [W1].

PROPOSITION 2.4. *Let $f: X \rightarrow Z$ be a proper contraction between normal projective S -schemes, where Z is one-dimensional. Let L be a nef \mathbb{Q} -Cartier \mathbb{Q} -divisor on X such that $L|_{X_{k(Z)}} \sim_{\mathbb{Q}} 0$, where $X_{k(Z)}$ is the generic fiber. Then there exists a \mathbb{Q} -Cartier \mathbb{Q} -divisor L_Z on Z such that $L \sim_{\mathbb{Q}} f^*L_Z$.*

Proof. It is sufficient to note that hypotheses of [CT, Lem. 2.17] are easily verified when the base has dimension 1. \square

We now discuss numerical descent of numerically trivial nef divisors. In characteristic 0, several results are proved in [Le] and [LP1, Lem. 3.1] and their analogues for threefolds

in positive characteristic are discussed in [BW, §5]. We begin with the case of numerical descent over a curve.

LEMMA 2.5. *Let $f: X \rightarrow Y$ be a projective dominant morphism of integral S -quasi-projective excellent schemes. Let L be a \mathbb{Q} -Cartier \mathbb{Q} -divisor and suppose that:*

- (a) L is nef;
- (b) $L|_{X_{k(Y)}} \equiv 0$; and
- (c) Y has dimension 1.

Then $L \equiv_f 0$.

Proof. We can freely replace Y with its normalization and X by the normalization of the corresponding fiber product. Then, after taking a Stein factorization, we may suppose that Y is regular and $X \rightarrow Y$ is a flat contraction. Let C be a curve contracted to a closed point $y \in Y$. It suffices to show that $L \cdot C = 0$. Since $X \rightarrow Y$ is flat, the fibers are of pure dimension d . If $d = 1$, then $L \cdot C = 0$ by the argument of [BW, Lem. 5.3].

Otherwise, if $d > 2$, we cut with a very ample Cartier divisor and proceed by induction. More precisely, let H be an ample Cartier divisor on X and write \mathcal{I} for the ideal sheaf defining C . Then, for $m \gg 0$, $\mathcal{O}_X(mH) \otimes \mathcal{I}$ is globally generated. Thus, we can find $D \sim_{\mathbb{Q}} mH$ such that D contains C but does not vanish at the generic point of any component of the fiber over y . Then no component of D can be contracted over y , and thus there is some horizontal component Z containing C . Replacing X with Z , we see that the result holds by induction on k . □

PROPOSITION 2.6. *Let X be a normal integral scheme of dimension at most 3, and let $f: X \rightarrow C$ be a projective contraction over S with generic fiber $X_{k(C)}$. Suppose that C is a regular one-dimensional scheme and L is a nef \mathbb{Q} -Cartier \mathbb{Q} -divisor on X such that $L|_{X_{k(C)}} \equiv 0$. Then there exists a \mathbb{Q} -Cartier \mathbb{Q} -divisor D on C such that $L \equiv f^*D$.*

Proof. By Lemma 2.5, $L \equiv_f 0$. If C is not contracted over S , then in fact $L \equiv_S 0$. Otherwise, C is projective over a field and we conclude by the arguments of [BW, Lem. 5.2]. □

2.3 Generalized pairs

Generalized pairs have been introduced in [BZ], and since then, they revealed to be powerful tools in birational geometry over fields of characteristic 0. Here, \mathbb{K} is either \mathbb{Z} or \mathbb{Q} .

DEFINITION 2.7. For an integral normal scheme X , an integral \mathbb{K} -b-divisor is an element

$$\mathbf{D} \in \mathbf{WDiv}(X)_{\mathbb{K}} = \varinjlim_{Y \rightarrow X} \mathbf{WDiv}(Y)_{\mathbb{K}},$$

where $Y \rightarrow X$ run through all possible proper birational morphisms of normal schemes. Given $Y \rightarrow X$ a proper birational morphism of normal schemes, we have a natural map $\text{tr}_Y: \mathbf{WDiv}(X)_{\mathbb{K}} \rightarrow \mathbf{WDiv}(Y)_{\mathbb{K}}$, called the trace. We denote $\mathbf{D}_Y = \text{tr}(\mathbf{D})_Y$.

We say that \mathbf{D} is a \mathbb{K} -b-Cartier \mathbb{K} -b-divisor if there is a model $X' \rightarrow X$ such that $\mathbf{D}_{X'}$ is \mathbb{K} -Cartier and for any $\phi: X'' \rightarrow X'$ the equality $\mathbf{D}_{X''} = \phi^*\mathbf{D}_{X'}$ holds. In this case, we say that \mathbf{D} descends to X' .

Every \mathbb{K} -Cartier \mathbb{K} -divisor D on X induces a natural \mathbb{K} -Cartier \mathbb{K} -b-divisor $\mathbf{D} = \overline{D}$ as follows:

$$(\overline{D})_Y = f^*D \text{ where } f: Y \rightarrow X \text{ is a proper birational morphism.}$$

In particular, $\mathbf{D} = \overline{\mathbf{D}}_{X'}$ if and only if \mathbf{D} descends on X' .

DEFINITION 2.8. Let $X \rightarrow T$ be a proper morphism of quasi-projective normal schemes over R . Let \mathbf{D} be a \mathbb{K} -Cartier \mathbb{K} -b-divisor on X , and let X' be a model on which \mathbf{D} descends. If $\mathbf{D}_{X'}$ is a nef (resp. semiample) \mathbb{K} -Cartier \mathbb{K} -divisor over T , we say that \mathbf{D} is *b-nef* (resp. *b-semiample*) over T .

Clearly, if D is nef (resp. semiample), then \overline{D} is b-nef (resp. b-semiample).

DEFINITION 2.9. A *generalized pair* (or *g-pair*) $(X, B + \mathbf{M})$ over T consists of:

- (a) a projective morphism $f: X \rightarrow T$ of normal quasi-projective R -schemes;
- (b) an effective \mathbb{Q} -divisor B (called the boundary part);
- (c) a b-nef \mathbb{Q} -Cartier \mathbb{Q} -b-divisor \mathbf{M} (called the moduli part);
- (d) $K_X + B + \mathbf{M}_X$ is \mathbb{Q} -Cartier.

If \mathbf{D} is b-nef (resp. b-semiample), then for any X' with $\mathbf{D} = \overline{\mathbf{D}}_{X'}$, the divisor $\mathbf{D}_{X'}$ is nef (resp. semiample). We recall the definition of singularities for generalized pairs.

DEFINITION 2.10. Let $(X, B + \mathbf{M})$ be a *g-pair* over T . For every proper birational morphism $\pi: Y \rightarrow X$ of normal schemes, we can write

$$K_Y + B_Y + \mathbf{M}_Y = \pi^*(K_X + B + \mathbf{M}_X).$$

The *generalized discrepancy* of E is $a(E, X, B + \mathbf{M}) = -\text{coeff}_E(B_Y)$. We say that $(X, B + \mathbf{M})$ is *generalized klt* (resp. *generalized lc*) if $a(E, X, B + \mathbf{M}) > -1$ (resp. $a(E, X, B + \mathbf{M}) \geq -1$) for all divisors E appearing on some birational model.

If $(X, B + \mathbf{M})$ is generalized lc, (X, B) is divisorially log terminal (dlt), and $(X, B + (1 + t)\mathbf{M})$ is generalized lc for some $t > 0$, then we say that the pair is *generalized dlt*.

If (X, B) is klt/lc/dlt and N is a nef \mathbb{K} -Cartier \mathbb{K} -divisor, then $(X, B + \mathbf{N})$ is always generalized klt/lc/dlt for $\mathbf{N} = \overline{N}$.

We will often use the following result on singularities of surfaces.

LEMMA 2.11. *Let (X, Δ) be a dlt surface pair. Then X has rational and \mathbb{Q} -factorial singularities.*

Proof. By [K1, Prop. 2.28], dlt surface singularities are rational and rational surface singularities are \mathbb{Q} -factorial by [K1, Prop. 10.9]. □

For surfaces, the MMP for generalized pairs is immediate from the usual MMP as the moduli part \mathbf{M} is nef on every model.

PROPOSITION 2.12. *Let $(X, B + \mathbf{M})$ be a generalized dlt projective surface over T . Then we can run a $(K_X + B + \mathbf{M}_X)$ -MMP over T which terminates.*

Proof. As X is \mathbb{Q} -factorial, \mathbf{M}_X is a \mathbb{Q} -Cartier \mathbb{Q} -divisor. We start by proving that \mathbf{M}_X is nef. If $\pi: Y \rightarrow X$ is a proper birational morphism on which \mathbf{M}_Y is nef, then by projection formula, we have that for every curve $C \subset X$,

$$\mathbf{M}_X \cdot C = \pi_* \mathbf{M}_Y \cdot C = \mathbf{M}_Y \cdot \pi^* C \geq 0.$$

This shows that \mathbf{M}_X is nef. As \mathbf{M}_X is a nef, then a step of a $(K_X + B + \mathbf{M}_X)$ -MMP is thus also a step of a $(K_X + B)$ -MMP, which we know exists and terminates by [T4]. \square

2.4 Birational geometry of klt Calabi–Yau pairs

Let S be a quasi-projective normal integral scheme over R . We study birational properties of CY pairs over S .

DEFINITION 2.13. We say that (X, Δ) is a *klt CY pair* over S if (X, Δ) is a klt pair, proper over S such that $K_X + \Delta \equiv 0$ over S .

If S is clear from the context, we will simply omit it. We discuss some results on the birational geometry of klt CY pairs.

LEMMA 2.14. Let (X, Δ) be a klt CY pair over S . Let $\pi: X \rightarrow Y$ be a proper birational contraction between normal proper S -schemes. Then $(Y, \pi_*\Delta)$ is crepant birational to (X, Δ) . In particular, $(Y, \pi_*\Delta)$ is a klt CY pair over S .

Proof. As $K_X + \Delta \equiv 0$, then clearly $K_Y + \pi_*\Delta \equiv 0$. Then, by negativity lemma [BMP+, Lemma 2.14], we conclude that $K_X + \Delta \sim_{\mathbb{Q}} \pi^*(K_Y + \pi_*\Delta)$. \square

LEMMA 2.15. Let k be an algebraically closed field. Let X be a projective surface over k such that $K_X \equiv 0$. Suppose that:

- (a) If $k = \overline{\mathbb{F}}_p$, X is klt.
- (b) If $k \neq \overline{\mathbb{F}}_p$, X is \mathbb{Q} -factorial (e.g., X is klt).

If the singularities of X are worse than canonical, then X is birational to \mathbb{P}_k^2 .

Proof. Let $\pi: Y \rightarrow X$ be the minimal resolution. We have $K_Y + E = \pi^*K_X = 0$, where $E > 0$. Suppose for contradiction that Y is not a rational surface. Therefore, there exists $f: Y \rightarrow B$ where B is a curve of genus $g(B) \geq 1$. We now claim that all irreducible components of E are rational curves. This is guaranteed by (a) and (b): the minimal resolution of a klt singularity is a tree of rational curves (by the classification in [K1, 3.40, page 123]) and in the case where $k \neq \overline{\mathbb{F}}_p$ and X is \mathbb{Q} -factorial we apply [T1, Th. 3.20] to conclude. Therefore, each irreducible component of E must be contained in a fiber of f . Let F be a general fiber of f . As $E \cdot F = 0$, by adjunction, we know $(K_Y + E) \cdot F = K_Y \cdot F = \deg K_F = -2$, contradicting $K_Y + E \equiv 0$. \square

We recall a straightforward application of the abundance theorem to the semiample conjecture. We say that a \mathbb{Q} -Cartier \mathbb{Q} -divisor D on X is \mathbb{Q} -effective if there exists $n > 0$ such that $H^0(X, \mathcal{O}_X(nD)) \neq 0$.

PROPOSITION 2.16. Let (X, Δ) be a klt CY pair over S , and let L be a nef \mathbb{Q} -Cartier \mathbb{Q} -divisor on X . Suppose that:

- (a) $\dim X = 2$; or
- (b) $\dim X = 3$, the image of X is positive-dimensional, and the residue field of any closed point is of characteristic $p > 5$.

If L is \mathbb{Q} -effective, then L is semiample. Moreover, if $L \equiv E$ over S where E is an effective \mathbb{Q} -divisor, then L is num-semiample.

Proof. By assumption, there exists an effective \mathbb{Q} -Cartier \mathbb{Q} -divisor E such that $L \sim_{\mathbb{Q}} E$. If we consider $0 < \varepsilon \ll 1$, the pair $(X, \Delta + \varepsilon E)$ is klt and $K_X + \Delta + \varepsilon E \sim_{\mathbb{Q}} \varepsilon E$. By the

abundance theorem for klt surfaces and threefolds (see [BBS, Ths. 1.1 and 3.1]), the divisor E , and thus L , is semiample. The last assertion now follows immediately. \square

§3. Semiampleness for klt Calabi–Yau surfaces

In this section, we prove the semiampleness conjecture for klt CY surface pairs. Our principal tools are the MMP for excellent surfaces [T4], the classification of Bombieri–Mumford of smooth varieties with trivial canonical class over an algebraically closed field [BM1] and the abundance theorem for surfaces [T5]. To treat the case of imperfect fields, we use the base change formula of [PW], [T3]. The main difficulty in the proof lies in ruling out the existence of strictly nef divisors which are not ample on klt CY surfaces.

3.1 Semiampleness for canonical K -trivial surfaces

We fix k to be a field of characteristic $p > 0$. We prove the semiampleness conjecture for K -trivial projective surfaces with canonical singularities over algebraically closed k . The result is probably well known to the experts, but we include a proof for the sake of completeness. We start with the following lemma on abelian varieties.

LEMMA 3.1. *Let k be an algebraically closed field, and let A be an abelian variety over k . If L is a strictly nef line bundle on A , then L is ample.*

Proof. The proof of [S2, Prop. 1.4] works over any algebraically closed field of arbitrary characteristic. \square

PROPOSITION 3.2. *Let k be an algebraically closed field of characteristic $p > 0$, and let X be a smooth projective surface over k such that $K_X \equiv 0$. If L is a strictly nef Cartier divisor on X , then L is ample.*

Proof. By the Bombieri–Mumford classification (see [BM1, p. 1]), we have $\chi(X, \mathcal{O}_X) \geq 0$. If $\chi(X, \mathcal{O}_X) > 0$, then we conclude that $h^0(X, L) > 0$ by the Riemann–Roch theorem. Thus, $L^2 > 0$ and so L is ample by the Nakai–Moishezon criterion.

If $\chi(X, \mathcal{O}_X) = 0$, then by [BM1], [BM2] X is either an abelian surface, a hyperelliptic surface, or a quasi-hyperelliptic surface. If X is hyperelliptic, we consider a finite étale cover $f: Y \rightarrow X$ where Y is an abelian variety. Then f^*L is strictly nef and we conclude that f^*L , and thus L , is ample by Lemma 3.1. If X is quasi-hyperelliptic, by [BM1, Th. 1], there exists a finite morphism $f: E \times \mathbb{P}^1 \rightarrow X$, where E is an elliptic curve. Since f^*L is strictly nef and $\overline{NE}(E \times \mathbb{P}^1) = NE(E \times \mathbb{P}^1)$, we conclude by Kleiman’s criterion (see [La, Th. 1.4.29]) that f^*L , and thus L , is ample. \square

LEMMA 3.3. *Let X be a normal projective surface over a field k . Suppose that L is a nef line bundle with $\nu(L) = 1$ and C a curve with $L \cdot C = 0$. Then $C^2 \leq 0$. Moreover, $C^2 = 0$ if and only if $L \equiv tC$ for some $t > 0$.*

Proof. If $C^2 > 0$, then C is big and we write $C \sim_{\mathbb{Q}} A + E$ for A ample and $E \geq 0$. Then $L \cdot C = L \cdot (A + E) \geq L \cdot A > 0$ since L is nef, and hence pseudo-effective, but not numerically trivial. If $C^2 = 0$, then we conclude that $L \equiv tC$ by the Hodge index theorem. As C is effective and $\nu(L) = 1$, then $t > 0$. \square

PROPOSITION 3.4. *Let k be an algebraically closed field of characteristic $p > 0$, and let X be a projective surface over k with canonical singularities such that $K_X \equiv 0$. If L is a nef Cartier divisor, then it is num-semiample.*

Proof. By passing to the minimal resolution and the base point-free-theorem [B, Prop. 2.1.a], it is sufficient to discuss the case of smooth surfaces with numerically trivial canonical class. If $\nu(L) = 0$, the claim is obvious, and if $\nu(L) = 2$, we conclude by the base-point-free theorem [T4, Th. 4.2].

We now suppose that $\nu(L) = 1$. As L is not strictly nef by Proposition 3.2, then there exists a curve C such that $L \cdot C = 0$. By Lemma 3.3, either $C^2 = 0$ and $L \equiv tC$ for some $t > 0$ or $C^2 < 0$. Take $\varepsilon > 0$ with $(X, \varepsilon C)$ klt. If $C^2 = 0$, we conclude that $K_X + \varepsilon C \equiv \varepsilon C$ is semiample by Proposition 2.16, and hence L is num-semiample. Otherwise, $C^2 < 0$, then we can contract C as a step of $(K_X + \varepsilon C)$ -MMP which is L -trivial. We thus reduce to CY surface with canonical singularities by Lemma 2.14 with smaller Picard rank. After finitely many steps, as L is not strictly nef by Proposition 3.2, there is a curve C with $C^2 = 0$ and we conclude. \square

3.2 Klt CY surfaces

We now prove the semiampleness conjecture for klt CY surface pairs (X, Δ) over an arbitrary field k . We start by discussing the case where the boundary divisor Δ is empty.

PROPOSITION 3.5. *Let k be a field of characteristic $p > 0$, and let X be a klt projective surface over k such that $K_X \equiv 0$. Let L be a nef Cartier divisor on X with $\nu(L) = 1$. Then L is num-semiample.*

Proof. Without loss of generality, we can suppose that $k = H^0(X, \mathcal{O}_X)$. We divide the proof in several steps.

STEP 1. We can suppose that for all irreducible curves $C \subset X$ such that $L \cdot C = 0$, then $C^2 = 0$. \square

Proof. We first note that $C^2 \leq 0$ by Lemma 3.3. Let us consider a sufficiently small rational number $\varepsilon > 0$ such that $(X, \varepsilon C)$ is klt. If $C^2 < 0$, by [T4, Theorem 4.4], there exists a birational morphism $\varphi: X \rightarrow Y$ such that $\text{Ex}(\varphi) = C$ and a Cartier divisor L_Y such that $L \sim \varphi^* L_Y$. Thus, it is sufficient to prove that L_Y is num-semiample. Since Y is a klt CY surface by Lemma 2.14, this process will terminate after a finite number of steps as the Picard number decreases by 1 at each step. \square

STEP 2. If there exists a curve C such that $L \cdot C = 0$, then L is num-semiample.

Proof. By Lemma 3.3, we have $L \equiv tC$ for some $t > 0$. However, $(X, \varepsilon C)$ is klt for some $\varepsilon > 0$ and so we conclude that L is num-semiample by Proposition 2.16. \square

Suppose now that L is strictly nef. To conclude the proof, it is sufficient to show that L is ample.

STEP 3. We can suppose that X is geometrically normal.

Proof. Suppose X is not geometrically normal. Let Y be the normalization of $(X \times_k \bar{k})_{\text{red}}$, and let $f: Y \rightarrow X$ be the natural morphism. Since X is not geometrically normal and $k = H^0(X, \mathcal{O}_X)$ is algebraically closed in $k(X)$, by [T3, Th. 1.1], there exists an effective divisor $E > 0$ such that

$$K_Y + E \sim f^* K_X \equiv 0.$$

By Serre duality, we have $H^2(Y, \mathcal{O}_Y(f^* mL)) \simeq H^0(Y, \mathcal{O}_Y(K_Y - f^* mL))^*$, which vanishes as $(K_Y - f^* mL) \cdot A = (-E - f^* mL) \cdot A < 0$ for an ample Cartier divisor A and $m > 0$. Let $\pi: Z \rightarrow Y$ be the minimal resolution and write $K_Z + F = \pi^* K_Y$, for some effective

π -exceptional \mathbb{Q} -divisor F . By the Riemann–Roch theorem [T4, Th. 2.10] and the projection formula, we deduce

$$\begin{aligned} h^0(Z, \mathcal{O}_Z(m\pi^* f^* L)) &\geq \chi(Z, \mathcal{O}_Z) + m\pi^* f^* L \cdot (m\pi^* f^* L - K_Z) \\ &= \chi(Z, \mathcal{O}_Z) + m f^* L \cdot E, \end{aligned}$$

where the last inequality follows from $L^2 = 0$, $\pi^* f^* L \cdot F = 0$, and $K_Y \equiv -E$. Since L is strictly nef, we have $f^* L \cdot E > 0$ and thus $h^0(Y, \mathcal{O}_Y(mf^* L)) > 0$ for m sufficiently large. As $f^* L$ is strictly nef and effective, then $f^* L$ is ample. Therefore, L is ample, concluding. \square

We now prove the assertion of the theorem when X is geometrically normal. Note that if $\bar{k} \simeq \overline{\mathbb{F}_p}$ (resp. $\bar{k} \neq \overline{\mathbb{F}_p}$), then $X_{\bar{k}}$ is klt (resp. \mathbb{Q} -factorial). Indeed, $X_{k^{\text{sep}}}$ is klt by [K1, Prop. 2.15] and thus \mathbb{Q} -factorial by Lemma 2.11. Therefore, by [T3, Lem. 2.5], $X_{\bar{k}}$ is \mathbb{Q} -factorial. We now divide the proof in two cases according to the singularities of $X_{\bar{k}}$. If $X_{\bar{k}}$ has canonical singularities, we conclude by Proposition 3.4. If the singularities of $X_{\bar{k}}$ are worse than canonical, then Lemma 2.15 guarantees that $X_{\bar{k}}$ is a rational surface. Let us consider the minimal resolution $\pi: Y \rightarrow X_{\bar{k}}$. Since Y is a smooth rational surface, $H^1(Y, \mathcal{O}_Y) = 0$ and thus by the Leray spectral sequence we deduce that $H^1(X_{\bar{k}}, \mathcal{O}_{X_{\bar{k}}}) = 0$. By flat base change, we conclude that $H^1(X, \mathcal{O}_X) = 0$ and thus $\chi(X, \mathcal{O}_X) \geq 1$. Since $\nu(L) \geq 1$, we have $H^2(X, \mathcal{O}_X(L)) \simeq H^0(X, \mathcal{O}_X(K_X - L))^* = 0$ and by Riemann–Roch we deduce that $h^0(X, \mathcal{O}_X(L)) \geq \chi(X, \mathcal{O}_X) \geq 1$. Therefore, L is ample, concluding the proof.

THEOREM 3.6. *Let k be a field, and let (X, Δ) be a projective klt surface pair such that $K_X + \Delta$ is \mathbb{Q} -Cartier and $K_X + \Delta \equiv 0$. If L is a nef \mathbb{Q} -Cartier \mathbb{Q} -divisor on X , then L is num-semiample.*

Proof. If the characteristic of k is 0, this is [LP1, Th. 8.2]. So we suppose that the characteristic is $p > 0$ and we subdivide the proof according to the numerical dimension of L . If $\nu(L) = 0$, the claim is obvious. If $\nu(L) = 2$, then L is big and nef and we conclude by the base-point-free theorem (see [T4, Th. 4.2]).

The only case we thus need to study in detail is when $\nu(L) = 1$. We can suppose L is Cartier. By Serre duality, $h^2(X, \mathcal{O}_X(L)) = h^0(X, \mathcal{O}_X(K_X - L))$, which vanishes as $\nu(L) \geq 1$. The strategy is to reduce to Proposition 3.5. For this, we subdivide the proof in various steps.

STEP 1. If $L \cdot \Delta > 0$, then L is semiample. \square

Proof. Let $\pi: W \rightarrow X$ be the minimal resolution, and let $K_W + \Delta_W = \pi^*(K_X + \Delta)$. By the Riemann–Roch theorem, we have

$$\begin{aligned} h^0(W, \mathcal{O}_W(m\pi^* L)) &\geq \chi(W, \mathcal{O}_W) + m\pi^* L \cdot (m\pi^* L - K_W) \\ &= \chi(W, \mathcal{O}_W) + m\pi^* L \cdot \Delta_W = \chi(W, \mathcal{O}_W) + mL \cdot \Delta. \end{aligned}$$

In particular, for $m \gg 0$, we have $h^0(X, \mathcal{O}_X(mL)) \neq 0$ and we conclude that L is semiample by Proposition 2.16. \square

STEP 2. We can assume that every irreducible curve C contained in the support of Δ satisfies $C^2 \leq 0$.

Proof. Suppose that there exists a curve C in the support of Δ such that $C^2 > 0$. Then $\Delta = aC + \Gamma$ and $K_X + \Gamma = -aC$ is a big and nef \mathbb{Q} -Cartier \mathbb{Q} -divisor. In particular, (X, Γ) is klt and $L - (K_X + \Gamma)$ is a big and nef \mathbb{Q} -Cartier \mathbb{Q} -divisor. Thus, we conclude that L is semiample by the base-point-free theorem. \square

From now on, we assume that $L \cdot \Delta = 0$ and all curves C in the support of Δ satisfy $C^2 \leq 0$.

STEP 3. We can suppose that each curve C in Δ satisfies $C^2 = 0$. In particular, Δ is a nef \mathbb{Q} -Cartier \mathbb{Q} -divisor.

Proof. Suppose that there exists a curve C such that $\Delta = aC + \Gamma$, C is not contained in the support of Γ and $C^2 < 0$. Then we have

$$(K_X + C) \cdot C \leq (K_X + \Delta + (1 - a)C) \cdot C = (1 - a)C^2 < 0.$$

Thus, by [T5, Th. 2.10], there exists a birational map $\pi: X \rightarrow Y$ such that $\text{Ex}(\pi) = C$ and $L = \pi^*L_Y$ for some \mathbb{Q} -Cartier \mathbb{Q} -divisor L_Y on Y . Moreover, $(Y, \pi_*\Delta)$ is a klt CY pair by Lemma 2.14. After a finite number of such contractions, we end up with a klt CY pair (Z, Δ_Z) such that all irreducible curves C in Δ_Z satisfy $C^2 = 0$. \square

STEP 4. We can suppose $\Delta = 0$.

Proof. If $\Delta \neq 0$, then Δ is an effective nef \mathbb{Q} -Cartier \mathbb{Q} -divisor with numerical dimension $\nu(\Delta) \geq 1$. Since it is not big by Step 1, $\nu(\Delta) = 1$. By Proposition 2.16, Δ is semiample, and we denote by $g: X \rightarrow B$ the induced contraction. Since $L \cdot \Delta = 0$, by Proposition 2.6, there exists an ample \mathbb{Q} -Cartier \mathbb{Q} -divisor A on B such that $L \equiv g^*A$. \square

Therefore, we reduced to prove the theorem in the case where X is a projective surface with klt singularities and $K_X \equiv 0$, which we proved in Proposition 3.5.

§4. Generalized abundance for surfaces

In [LP1, Th. B], the authors show that, over fields of characteristic 0, Conjecture 1.3 is implied by the standard conjectures of the MMP and the semiample conjecture. Their arguments heavily rely on the canonical bundle formula and therefore do not extend to positive or mixed characteristic. Despite this obstacle, in this section, we show the generalized abundance conjecture (and variants thereof) in the case of excellent surfaces.

4.1 Surfaces over a field

For excellent klt surfaces, the base-point-free theorem (resp. abundance) has been proved in [T4, Th. 4.2] (resp. in [T5] and [BBS, Th. 3.1]). These results together with the MMP and Theorem 3.6 are sufficient to prove Conjecture 1.3 for surfaces over fields. Our strategy is similar to [LP1].

PROPOSITION 4.1. *Let (X, B) be a projective klt surface pair over a field k . Suppose that $K_X + B$ is a pseudo-effective \mathbb{Q} -Cartier, and let M be a nef \mathbb{Q} -Cartier \mathbb{Q} -divisor such that $L := K_X + B + M$ is nef. Then L is num-semiample.*

Proof. Note first that if L is big, then $2L - (K_X + B) = L + M$ is big and nef; hence, L is semiample by the base-point-free theorem (see [T4, Th. 4.2]).

As X is a surface, by running a $(K_X + B)$ -MMP and abundance [T4], there is a birational contraction $g: X \rightarrow X_{\min}$ together with a contraction morphism $h: X_{\min} \rightarrow Z := \text{Proj}_k R(X, K_X + B)$ and an ample \mathbb{Q} -Cartier \mathbb{Q} -divisor A on Z such that $K_{X_{\min}} + g_*B = h^*A$. We denote by $f: X \rightarrow Z$ the composition $h \circ g$. We divide the proof according to the dimension of Z .

CASE 1. Suppose that $\dim Z = 2$. This means that $K_X + B$ is big and therefore also L , concluding. \square

CASE 2. Suppose that $\dim Z = 1$. If $L \cdot F = 0$ for a general fiber F of f , by Proposition 2.6, we have $L \equiv f^*N$ for some \mathbb{Q} -Cartier \mathbb{Q} -divisor N on Z . Note that N must be an ample \mathbb{Q} -Cartier \mathbb{Q} -divisor, since if C is a sufficiently general curve on X , we have $L \cdot C \geq (K_X + B) \cdot C > 0$ as $f_*C = Z$.

Suppose now that $L \cdot F > 0$, that is, L is relatively big over Z . It is sufficient to show that L big to conclude. By the negativity lemma, we deduce that $K_X + B = g^*(K_{X_{\min}} + g_*B) + E$, where E is an effective \mathbb{Q} -divisor contracted by f . In particular, $K_X + B \sim_{\mathbb{Q}} f^*A + E$ where A is an ample \mathbb{Q} -Cartier \mathbb{Q} -divisor on Z . As $L = (f^*A + E) + M$ is relatively big and $f^*A + E$ is numerical trivial on the generic fiber, then M is relatively big. Note that $f^*A + M$ is nef and $(f^*A + M)^2 \geq 2f^*A \cdot M > 0$ and thus big. Therefore, L is big, concluding.

CASE 3. Finally, suppose that $\dim Z = 0$. Let $\pi: X \rightarrow Y$ be the minimal model of a $(K_X + B)$ -MMP. Then $K_Y + B_Y \sim_{\mathbb{Q}} 0$ and $L' = \pi_*L \sim_{\mathbb{Q}} \pi_*M$ is nef on Y , and hence num-semiample by Theorem 3.6. Choose $0 < t < 1$ sufficiently small such that $X \rightarrow Y$ is the end product of a $(K_X + B + tM)$ -MMP. Then, by the negativity lemma, $K_X + B + tM \equiv t\pi^*L' + E$ for $E \geq 0$. If L' is big, then so too is $K_X + B + tM$ and also L , and the result follows as above.

Suppose now that $\nu(L') = 1$. As L' is num-semiample, there exists a contraction $g: Y \rightarrow W$ with $\dim W = 1$ such that $L' \equiv g^*A$ for an ample \mathbb{Q} -Cartier \mathbb{Q} -divisor A on W . Denote by $f: X \rightarrow W$ the composition $g \circ \pi$. As $L|_{X_k(W)} \equiv \pi^*L'|_{X_k(W)} \equiv 0$, we conclude that $L \equiv f^*D$ for some D on W by Proposition 2.6. As $\nu(L') = 1$, we conclude that D is ample and thus L is num-semiample.

Finally, if L' is numerically trivial, then $L \sim_{\mathbb{Q}} \pi^*L' + E \equiv E$, where E is π -exceptional. As L is nef, then $E \leq 0$ by the negativity lemma [BMP+, Lem. 2.14]. As $L \equiv E$ is nef, we conclude that $E = 0$.

4.2 Excellent case

We are now ready to prove Conjecture 1.3 for projective surfaces over R .

THEOREM 4.2. *Let $\pi: X \rightarrow T$ be a projective R -morphism of quasi-projective integral normal schemes over R . Suppose that (X, B) is a klt surface such that:*

- (a) $K_X + B$ is pseudo-effective over T ;
- (b) M is a nef \mathbb{Q} -Cartier \mathbb{Q} -divisor over T ;
- (c) $L := K_X + B + M$ is nef \mathbb{Q} -Cartier \mathbb{Q} -divisor over T .

Then L is num-semiample over T .

Proof. Without loss of generality, we can suppose that π is a surjective contraction between normal schemes. Suppose first that L is big, in which case $2L - (K_X + B)$ is big and nef, so L is semiample by [T4, Th. 4.2]. In particular, if $\dim(T) = 2$, we conclude. From now on, we suppose that L is not big over T , or equivalently that $L|_{X_k(T)}$ is not big.

If $\dim(T) = 1$, then $X_{k(T)}$ is a curve. As $L|_{X_k(T)}$ is not big, then $L|_{X_k(T)} \equiv 0$ and we conclude by Proposition 2.6 that $L \equiv \pi^*A$ where A is a nef \mathbb{Q} -divisor on T . As nef divisors on a curve are num-semiample, we conclude.

If $\dim(T) = 0$, we apply Proposition 4.1. \square

In particular, the semiample conjecture holds for klt CY excellent surface pairs.

REMARK 4.3. Note that the klt assumption is necessary in Conjectures 1.1 and 1.3 as there are counterexamples if (X, B) is allowed to be lc (or even log smooth), even if L is supposed to be big (see [MNW, Exams. 7.1 and 7.2]).

REMARK 4.4. The semiample conjecture and generalized abundance are false for \mathbb{R} -divisors as shown by the examples on projective K3 surfaces constructed in [FT, Th. 1.5.b].

The pseudo-effectivity of $K_X + B$ is necessary as shown in [LP1, Exam. 6.2]. In fact, we can completely characterize when generalized abundance holds in the case of surfaces (cf. [HL, Th. 3.13]).

PROPOSITION 4.5. *Let $(X, B + \mathbf{M})$ be a generalized klt surface over T and suppose that \mathbf{M}_X and $L = K_X + B + \mathbf{M}_X$ are nef \mathbb{Q} -Cartier \mathbb{Q} -divisors over T . If L is not num-semiample, then:*

- (a) T is the spectrum of a field k ;
- (b) $K_X + B$ is not pseudo-effective;
- (c) $-(K_X + B) \equiv t\mathbf{M}_X$ for some $0 < t \leq 1$;
- (e) $\nu(\mathbf{M}_X) = 1$.

Proof. We first suppose $\dim(T) \geq 1$. If L is big, then it is num-semiample by the base-point-free theorem. Therefore, the only interesting case is L not big and $\dim(T) = 1$. Then $L|_{X_{k(T)}} \equiv 0$ and therefore $L \equiv g^*M$ by Proposition 2.6, proving (a).

We can thus suppose that $\dim(T) = 0$, that is, it is the spectrum of a field. We first note that $K_X + B$ is not pseudo-effective by Theorem 4.2. We now follow some of the ideas of [HL, Th. 3.13]. If $K_X + B + 2\mathbf{M}_X$ is big, then $2L - (K_X + B)$ is big and nef and thus L is semiample by the base-point-free theorem.

If not, then $(K_X + B + 2\mathbf{M}_X)^2 = 0$, which implies that $(K_X + B + \mathbf{M}_X) \cdot \mathbf{M}_X = 0$ and $\mathbf{M}_X^2 = 0$. By an application of the Hodge index theorem (see [HL, Lem. 3.2]), we conclude that there exists $a \in \mathbb{Q}_{\geq 0}$ such that $a\mathbf{M}_X \equiv K_X + B + \mathbf{M}_X$. Suppose that $a \geq 1$. Then $K_X + B \equiv (a - 1)\mathbf{M}_X$ is pseudo-effective and thus L is num-semiample by Proposition 4.1, contradicting the hypothesis. Therefore, $0 \leq a < 1$, which implies (c) as $-(K_X + B) \equiv (1 - a)\mathbf{M}_X$.

We are only left to check that $\nu(\mathbf{M}_X) = 1$. As \mathbf{M}_X is not big, then $\nu(\mathbf{M}_X) < 2$. If $\nu(\mathbf{M}_X) = 0$, then $K_X + B$ is pseudo-effective, and thus L is num-semiample by Proposition 4.1, contradiction. \square

4.3 Serrano's conjecture and numerical nonvanishing

As a consequence, we obtain a version of Serrano's conjecture for excellent klt surface pairs (see [HL, Cor. 1.8] for a similar result in characteristic 0). We note that the case where the base is an imperfect field needs a careful analysis.

THEOREM 4.6. *Let $\pi: X \rightarrow T$ be a projective morphism of quasi-projective integral normal schemes over R and suppose that (X, B) is a klt surface pair. If M is a strictly nef Cartier divisor, then $L_t := K_X + B + tM$ is ample for $t > 4$.*

Proof. We can suppose π is a contraction, by taking the Stein factorisation. By the cone theorem for excellent surfaces (see [BMP+, Th. 2.40]), L_t is strictly nef for $t > 4$. By

Proposition 4.5, we see that either L_t is num-semiample or $L_t \equiv -s(K_X + B)$ for some $s > 0$. In the first case, L_t is necessarily ample.

In the latter case, $H^0(X, \mathcal{O}_X) = k$ is a field, $T = \text{Spec}(k)$, and we are left to show that the strictly nef \mathbb{Q} -divisor $L := -(K_X + B)$ is ample. Suppose for contradiction it is not: in particular, $L^2 = 0$. It is sufficient to show that L is \mathbb{Q} -effective. We first show that $H^2(\mathcal{O}_X(nL)) = 0$ for n sufficiently large. Let $n > 0$ such that nL is Cartier, so that by Serre duality there is an isomorphism $H^2(X, \mathcal{O}_X(nL)) \simeq H^0(X, \mathcal{O}_X(K_X - nL))^*$. If $H^0(X, \mathcal{O}_X(K_X - nL)) \neq 0$, for an ample Cartier divisor H , we have $0 < (K_X - nL) \cdot H \leq K_X \cdot H - n$, which gives a contradiction for n sufficiently large.

Let $f: Y := (X \times_k \bar{k})_{\text{red}}^n \rightarrow X$ be the normalization of the base change to the algebraic closure, and let $E \geq 0$ be the \mathbb{Z} -divisor for which $K_Y + E = f^*K_X$. First suppose that $E + f^*B > 0$, which implies that

$$f^*L \cdot (-K_Y) = f^*L \cdot (E - f^*K_X) = f^*L \cdot (E + f^*B) > 0,$$

where we used the condition $L^2 = 0$ in the last equality. Let $\pi: Z \rightarrow Y$ be the minimal resolution. Thus, by the Riemann–Roch formula, we have

$$h^0(Z, \mathcal{O}_Z(\pi^*f^*nL)) \geq \chi(Z, \mathcal{O}_Z) - \frac{1}{2}n f^*L \cdot K_Y > 0$$

for sufficiently large $n > 0$ for which nL is Cartier. In particular, f^*nL is effective, and thus ample. In particular, L is \mathbb{Q} -effective and thus ample.

If $E + f^*B = 0$, then $B = 0$ and X is geometrically normal by [T3, Th. 1.1]. We run a K_X -MMP which ends with a Mori fiber space $\pi: Y \rightarrow C$ and $-K_Y$ is strictly nef. If $\dim(C) = 0$, then Y is a klt del Pezzo surface. If $\dim(C) = 1$, then $\rho(Y) = 2$ and thus the cone theorem [BMP+, Th. 2.40] implies that Y is a klt del Pezzo surface. In both cases, Y is a geometrically normal del Pezzo surface and by [S1] we know $H^1(Y, \mathcal{O}_Y) = 0$. As X and Y have both rational singularities, by Lemma 2.11, we deduce that $H^1(X, \mathcal{O}_X) = 0$. By Riemann–Roch, we thus have

$$h^0(X, \mathcal{O}_X(nL)) \geq \chi(X, \mathcal{O}_X) + \frac{1}{2}nL \cdot (nL - K_X) \geq 1,$$

which shows L is ample. □

We can generalize this result immediately to threefolds in the setting of [BMP+].

COROLLARY 4.7. *Let $\pi: X \rightarrow T$ be a projective morphism of quasi-projective integral normal schemes over R , and let (X, B) be a klt threefold pair. Suppose that the closed points of R have residue fields of characteristic 0 or $p > 5$. Let M be a strictly nef Cartier divisor. If $\dim(\pi(X)) \geq 1$, then $L_t := K_X + B + tM$ is ample for $t > 4$.*

Proof. We can suppose π to be a contraction. Let F be the generic fiber of $X \rightarrow T$. If F is a surface, then $L_t|_F$ is ample by Theorem 4.6. If instead F is a curve, then $L_t|_F$ is ample since it has positive degree. In particular, L_t is big.

Then, by [BMP+, Th. H], L_t is strictly nef and moreover $2L_t - (K_X + B) = M + L_t$ is nef and big. Thus, we conclude that L_t is semiample by [BMP+, Th. G], and hence it is ample as claimed. □

We can also show the numerical nonvanishing conjecture for generalized surface pairs. The strategy of the proof is similar to [HL]. However, due to some complications over

imperfect field, we apply Serrano’s conjecture to solve the nonpseudo-effective case, instead of referring to the explicit classification of [S3, Exam. 1.1].

THEOREM 4.8 (Numerical nonvanishing for generalized klt surfaces). *If $(X, B + \mathbf{M})$ is a generalized klt surface pair over T and $K_X + B + \mathbf{M}_X$ is pseudo-effective, then $K_X + B + \mathbf{M}_X$ is num-effective.*

Proof. Let $r \in \mathbb{Q}_{>0}$ such that $L := r(K_X + B + \mathbf{M}_X)$ is Cartier. By running a $(K_X + B + \mathbf{M}_X)$ -MMP over T , we reduce to prove the statement in the case $K_X + B + \mathbf{M}_X$ is nef. By Proposition 4.5, we are only left to prove the case when $-(K_X + B)$ is nef with $\nu(-(K_X + B)) = 1$ and $-(K_X + B) \equiv t\mathbf{M}_X$.

By Theorem 4.6, $-(K_X + B)$ is not strictly nef. Thus, there exists an irreducible curve $C \subset X$ such that $-(K_X + B) \cdot C = 0$. By Lemma 3.3, we have $C^2 \leq 0$ with equality if and only if $(-K_X + B) \equiv \lambda C$ for $\lambda > 0$. Thus, we may suppose that $C^2 < 0$. Hence, we can contract it $X \rightarrow X'$ by running a $(K_X + B + \varepsilon C)$ -MMP for $\varepsilon > 0$ sufficiently small. This MMP is clearly $(K_X + B)$ -trivial. Therefore, (X', B') is a klt pair with $-(K_{X'} + B')^2 = 0$ and $\nu(-(K_{X'} + B')) = 1$ and we can repeat then the same procedure. After a finite number of steps, this process must terminate. Replacing (X, B) with the output, either L is strictly nef or we find a curve C such that $-(K_X + B) \cdot C = 0$ and $C^2 = 0$. In either case, we conclude. \square

4.4 Semiample for lc pairs

In this section, we study the generalized abundance for generalized lc surface pairs under the assumption that the b-nef part is b-semiample. In characteristic 0, this is immediate from the Bertini theorem and the abundance for lc surfaces. We will overcome the lack of Bertini in positive and mixed characteristic via Keel–Witaszek’s base-point-free theorem. For a nef line bundle L on X over S , we define the exceptional locus $\mathbb{E}(L)$ to be the union of all closed integral subschemes $Z \subset X$ such that $L|_Z$ is not relatively big over S . We recall the following semiample criterion for line bundles.

THEOREM 4.9 [W2]. *Let L be a nef line bundle on a scheme X projective over an excellent Noetherian base scheme S . Then L is semiample over S if and only if both $L|_{\mathbb{E}(L)}$ and $L|_{X_{\mathbb{Q}}}$ are semiample.*

We first need an adjunction result.

LEMMA 4.10. *Let $X \rightarrow T$ be a projective contraction of integral, excellent, normal quasi-projective schemes over R . Let X be a surface and C is an irreducible curve on X contracted over T . Suppose that (X, C) is an lc surface and C is an irreducible curve over T such that $(K_X + C) \cdot C = 0$. Then $(K_X + C)|_C \sim_{\mathbb{Q}} 0$.*

Proof. As C is contracted over T , it is a curve over some field k . The same proof of [T5, Th. 2.13] assures that $(K_X + C)|_C \sim_{\mathbb{Q}} 0$ as claimed. \square

The following theorem should be well known if T is a scheme over $\text{Spec}(\mathbb{Q})$. We include a proof as we lack a suitable reference and some arguments do not clearly carry over to the case that T is not of finite type over a field.

THEOREM 4.11. *Let $\pi: X \rightarrow T$ be a projective R -morphism of quasi-projective integral normal schemes over R . Suppose that $(X, B + \mathbf{M})$ is a generalized lc surface pair over T such that $\dim T > 0$, \mathbf{M} is b-semiample, and $L = K_X + B + \mathbf{M}_X$ nef. Then L is semiample.*

Proof. By taking Stein factorization, we may assume that π is a contraction. If $\dim(T) = 1$ and $L|_{X_{k(z)}} \sim_{\mathbb{Q}} 0$, then we conclude by Proposition 2.4.

We may assume that L is big. After taking a dlt modification, we may suppose that (X, B) is a \mathbb{Q} -factorial dlt pair. Let C be an integral curve on X such that $L \cdot C = 0$. As L is big, we deduce that $C^2 < 0$. We write $B = B_C + t_C C$ for some $t_C \in [0, 1]$ such that the support B_C does not contain C . As $C^2 < 0$, we deduce that

$$(K_X + C) \cdot C \leq (K_X + t_C C) \cdot C \leq (K_X + B) \cdot C \leq L \cdot C = 0$$

since \mathbf{M}_X is nef. Moreover, this is a chain of equalities if and only if $t_C = 1$ and $B_C \cdot C = \mathbf{M}_X \cdot C = 0$.

If $(K_X + C) \cdot C < 0$, by [T3, Th. 4.4], there exists a contraction $X \rightarrow Y$ such that its exceptional locus is C and Y is \mathbb{Q} -factorial. In this way, we contract all the curves satisfying $(K_X + C) \cdot C < 0$ and $C^2 < 0$. As during this process we contract only curves C with $(K_X + B) \cdot C \leq 0$, the pair (X, B) remains lc. Moreover, if $(K_X + B) \cdot C \leq 0$, then either $t_C = 1$ and (X, C) is lc or $t_C < 1$ and $(X, t_C C)$ is klt, so X remains klt and \mathbb{Q} -factorial by Lemma 2.11.

We may suppose that every curve C_i for which $L \cdot C_i = 0$ satisfies $C_i^2 < 0$ as L is assumed to be big and the following equalities hold:

$$(K_X + C_i) \cdot C_i = \mathbf{M}_X \cdot C_i = B_{C_i} \cdot C_i = 0.$$

We can write $B = \Delta + \sum_i C_i$ where C_i are the curves with $L \cdot C_i = 0$. As observed above, $\text{Supp}(\Delta)$ is disjoint from $C := \sum C_i$ and $C_i \cap C_j = \emptyset$ if $i \neq j$. Recall by construction that we have $\mathbb{E}[L] = C$ and $\mathbf{M}_X \cdot C = 0$. Since \mathbf{M}_X is b-semiample, there is a model $\pi: Y \rightarrow X$ with \mathbf{M}_Y semiample on Y and $\pi_* \mathbf{M}_Y = \mathbf{M}_X$.

Let $D \sim_{\mathbb{Q}} \mathbf{M}_Y$ be an effective section which does not contain any irreducible component of the strict transform of C . Then $\pi_* D \sim_{\mathbb{Q}} \mathbf{M}_X$ does not contain any irreducible component of C . As $\mathbf{M}_X \cdot C = 0$, we deduce that $\mathbf{M}_X|_C \sim_{\mathbb{Q}} 0$ and in particular $L|_C \sim_{\mathbb{Q}} (K_X + C)|_C$. We then have $L|_C \sim_{\mathbb{Q}} 0$ by Lemma 4.10.

The final argument differs depending on the characteristic. We may assume that T is local, and suppose first that the closed point has characteristic $p > 0$. For dimension reasons, $L|_{X_{K(z)}}$ is ample, and therefore by Theorem 4.9 L is semiample if and only if $L|_{\mathbb{E}[L]}$ is so. As $\mathbb{E}[L]$ is the disjoint union of curves C_i with $L \cdot C_i = 0$ and $L|_{C_i} \sim_{\mathbb{Q}} 0$ for every i from above, we conclude.

Now, suppose that the closed point has a residue field of characteristic 0 instead. Take $k \in \mathbb{N}$ such that $L_k = kL - C$ is big, and in particular effective. Then L_k intersects only finitely many curves negatively, call them $\gamma_j \subseteq \mathbf{SB}(L_k)$. By construction, $C \cdot C_i < 0$ for each i , since C is a disjoint union of irreducible components and every component has negative self-intersection. Hence, $L \cdot \gamma_j > 0$ for each j , so increasing k we may assume that L_k is strictly nef. Since it is big, L_k is ample. In particular, the stable base locus has $\mathbf{SB}(L) \subseteq C$.

Now, we have a short exact sequence

$$0 \rightarrow \mathcal{O}_X((k+1)L - C) \rightarrow \mathcal{O}_X((k+1)L) \rightarrow \mathcal{O}_C((k+1)L|_C) \rightarrow 0,$$

where $(k+1)L - C = K_X + B + \mathbf{M}_X + L_k$. Then $H^i(X, \mathcal{O}_X((k+1)L - C)) = 0$ by Kawamata–Viewhweg vanishing [T4, Th. 3.3], since L_k is ample, (X, B) is lc, and X is klt. As $L|_C \sim_{\mathbb{Q}} 0$, we have $(k+1)L|_C \sim 0$ for $(k+1)$ sufficiently large and divisible. As $H^0(X, \mathcal{O}_X((k+1)L)) \rightarrow H^0(C, \mathcal{O}_C((k+1)L))$ is surjective, we conclude that $\mathbf{SB}(L)$ is empty, concluding. \square

We prove a variant of the semiample conjecture for lc pairs.

COROLLARY 4.12. *Let $\pi: X \rightarrow T$ be a projective R -morphism of quasi-projective integral normal schemes over R . Let $(X, B + \mathbf{M})$ be a generalized lc surface over T with \mathbf{M} b -semiample. If $K_X + B + \mathbf{M}_X$ is nef over T , then it is semiample over T .*

Proof. We can suppose π to be surjective. Otherwise, if $\dim T > 0$, then this is Theorem 4.11. Suppose then that $\dim T = 0$, so it is the spectrum of a field k . If the characteristic of k is 0, then we conclude by the Bertini theorem and [F, Th. 6.1]. If k is finite, we consider the base change to \bar{k} . Then $(X_{\bar{k}}, B_{\bar{k}} + M_{\bar{k}})$ is a generalized lc pair and since $k \rightarrow \bar{k}$ is faithfully flat, it is sufficient to prove $L_{\bar{k}}$ is semiample. Suppose now that k is an infinite F -finite field of characteristic $p > 0$. Then there exists an effective \mathbb{Q} -divisor such that $D \sim_{\mathbb{Q}} \mathbf{M}$ and $(X, B + D)$ is lc by [T2, Th. 1]. Hence, $L \sim_{\mathbb{Q}} K_X + B + D$ is semiample by [T5, Th. 1].

If k is not F -finite, then by standard arguments (cf. [DW]), there are an F -finite subfield $l \subseteq k$ and a generalized klt pair $(X_l, B_l + \mathbf{M}_{X_l})$ such that:

- $(X, B + M) = (X_{\times_{\text{Spec}(l)} \text{Spec}(k)}, B_l \times_{\text{Spec}(l)} \text{Spec}(k) + \mathbf{M}_{X_l} \times_{\text{Spec}(l)} \text{Spec}(k))$; and
- \mathbf{M}_{X_l} is b -semiample.

As $K_{X_l} + B_l + \mathbf{M}_{X_l}$ is semiample, so is $K_X + B + \mathbf{M}_X$. □

§5. Semiample for CY threefolds

In this section, we show the semiample conjecture for threefolds in mixed characteristic (or over positive-dimensional bases of positive characteristic). We fix R to be an excellent DVR with maximal ideal \mathfrak{m} , residue field $k := R/\mathfrak{m}$ of characteristic $p > 0$, and fraction field K . We recall the following well-known fact on divisor class groups of projective morphisms over DVR.

LEMMA 5.1. *Let $\pi: X \rightarrow \text{Spec}(R)$ be a projective contraction of integral normal schemes. We write $(X_k)_{\text{red}} = \sum_i D_i$ as a Weil divisor. Then the following sequence is exact:*

$$\bigoplus_i \mathbb{Q}[D_i] \xrightarrow{j} \text{Cl}(X)_{\mathbb{Q}} \rightarrow \text{Cl}(X_K)_{\mathbb{Q}} \rightarrow 0.$$

Define $\text{Pic}_{X_k}(X)_{\mathbb{Q}} := j(\bigoplus_i \mathbb{Q}[D_i] \cap \text{Pic}(X)_{\mathbb{Q}})$ and denote by $N^1_{X_k}(X)_{\mathbb{Q}}$ its quotient by the numerical equivalence relation. Then the following sequences are exact:

$$\text{Pic}_{X_k}(X)_{\mathbb{Q}} \rightarrow \text{Pic}(X/R)_{\mathbb{Q}} \rightarrow \text{Pic}(X_K)_{\mathbb{Q}},$$

$$N^1_{X_k}(X)_{\mathbb{Q}} \rightarrow N^1(X/R)_{\mathbb{Q}} \rightarrow N^1(X_K)_{\mathbb{Q}}.$$

If X is \mathbb{Q} -factorial, then the second and third sequences are also surjective on the right.

Proof. The first sequence is obtained from [H, Prop. 6.5] by tensoring with \mathbb{Q} . The second exact sequence follows immediately from the first one by considering the natural injections $\text{Pic}(X/R) \rightarrow \text{Cl}(X)$ and $\text{Pic}(X_K) \rightarrow \text{Cl}(X_K)$. The third exact sequence follows from the second and the fact that $L \equiv L'$ over R if and only if $L|_{X_K} \equiv L'|_{X_K}$ by Proposition 2.6. If X is \mathbb{Q} -factorial, then we have a natural isomorphism $\text{Pic}(X_K) \simeq \text{Cl}(X)_{\mathbb{Q}}$, so the second sequence is surjective on the right. Then the third sequence must also be surjective on the right as $N^1(X/R)_{\mathbb{Q}}$ is a quotient of $\text{Pic}(X/R)_{\mathbb{Q}}$. □

We now prove the semiample conjecture for CY threefolds over a DVR of residue characteristic $p > 5$. Using the semiample conjecture for surfaces, we show a numerical nonvanishing on the threefold fibration. This allows to conclude by the abundance theorem for klt threefolds [BBS].

THEOREM 5.2. *Let R be an excellent DVR with residue characteristic $p > 5$, and let $\pi: (X, B) \rightarrow \text{Spec}(R)$ be a projective klt CY pair of dimension 3. If π is surjective and L is a nef \mathbb{Q} -Cartier \mathbb{Q} -divisor, then L is num-semiample.*

Proof. By taking a \mathbb{Q} -factorialization [BBS], we can suppose X to be \mathbb{Q} -factorial. If $\nu(L|_{X_K}) = 2$, then we conclude by the base-point-free theorem [BMP+, Th. 9.15]. If $\nu(L|_{X_K}) = 0$, then $L|_{X_K} \equiv 0$ and by Proposition 2.6, there exists a \mathbb{Q} -Cartier \mathbb{Q} -divisor D on $\text{Spec}(R)$ such that $L \equiv \pi^*D$, concluding.

Suppose $\nu(L|_{X_K}) = 1$. Thanks to Proposition 2.16, it is sufficient to show that there exists an effective \mathbb{Q} -divisor E such $L \equiv_{\pi} E$. By Theorem 3.6, there exists $\varphi: X_K \rightarrow W$ such that $L_K \equiv \varphi^*A$ for some A ample \mathbb{Q} -Cartier \mathbb{Q} -divisor on W . Take a Nagata compactification \overline{W} of W over $\text{Spec}(R)$ (see [TSP, Tag 0F41]), and let $g: Z \rightarrow \overline{W}$ be a projective resolution of singularities [BMP+, Prop. 2.12]. Resolving the indeterminacy of $t: X \dashrightarrow Z$ and applying again [BMP+, Prop. 2.12], there exist projective birational morphisms $\pi: Y \rightarrow X$ and $\psi: Y \rightarrow Z$ of normal projective R -schemes.

Write $(Y_k)_{\text{red}} = \sum_i E_i$ and $(Z_k)_{\text{red}} = \sum_j D_j$ as Weil divisors. By Lemma 5.1, we have the following commutative diagram of exact sequences:

$$\begin{CD} \bigoplus \mathbb{Q}[D_j] @>>> N^1(Z/R)_{\mathbb{Q}} @>>> N^1(Z_K)_{\mathbb{Q}} @>>> 0 \\ @VV\psi^*V @VV\psi^*V @VV\psi_K^*V \\ N^1_{X_k}(X)_{\mathbb{Q}} @>>> N^1(Y/R)_{\mathbb{Q}} @>>> N^1(Y_K)_{\mathbb{Q}}, \end{CD}$$

where the surjectivity of the top row comes from Z being regular. If we write $N := \pi^*L$, we know that $N_K = \pi|_K^* \varphi^*A$. Let $H := g_K^* \varphi^*A$ be the pullback of A via the morphism $Z_K \rightarrow W$. By the above sequence, there exists a \mathbb{Q} -Cartier \mathbb{Q} -divisor D on Z such that $D|_{Z_K} \sim_{\mathbb{Q}} H$. Note, in particular, that D is big over R .

By the exact sequence, we deduce that on Z we have $N - \psi^*D \equiv \sum b_i E_i$ for certain $b_i \in \mathbb{Z}$. By adding a sufficiently high multiple of the central fiber X_k , we see that $N \equiv \psi^*D + F$, where F is effective. As D is big over R , we conclude that $N \equiv E$ for some E effective. As $N = \pi^*L$, we conclude that $L \equiv \pi_*E$. □

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