# IMPLICATIONS BETWEEN CONDITIONS ON $l$-GROUPS

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**1. Introduction.** The study of lattice-ordered groups ($l$-groups) in recent years has yielded several publications concerning the lattice of all convex $l$-subgroups (denoted by $\Gamma(G)$). As for example, in [**3**], Conrad has shown a direct connection between one possible structure of $\Gamma(G)$ as a lattice and certain finite conditions on the elements of the group $G$. In the process of developing variations of this approach, there has been imposed several different conditions on $G$ and $\Gamma(G)$. For instance, it was proposed by Conrad that if all the minimal prime subgroups have non-zero polars, then the group elements would all be finite-valued. In this paper we show that in general this is not true.

Since the conditions studied in this paper are lengthy and used several times, it is desirable to adopt abbreviations for the conditions, as follows:

MP—polars of all minimal primes are non-zero. (This is equivalent to: each minimal prime is a polar.)

ΓS—all regular elements in $\Gamma(G)$ are special.

FV—all elements of $G$ are finite-valued.

FVB—all elements of $G$ are finite-valued and $G$ has a basis.

ΓE—all regular elements of $\Gamma(G)$ are essential.

F—no positive element of $G$ exceeds an infinite set of pairwise disjoint positive elements.

R$G$—the radical of $G$ is zero.

In case the reader is not familiar with the terms used, § 2 contains definitions. The following diagram illustrates the established implications and in this paper we show that in general none of these implications can be reversed. Theorem 6 and its corollary illustrate situations where they are equivalent.

$$\begin{array}{lcl} \text{(A)} & & \nearrow \text{MP} \longrightarrow \Gamma\text{E} \longrightarrow \text{R}G \\ & \text{F} & \\ \text{(B)} & & \searrow \text{FVB} \longrightarrow \text{FV} \longrightarrow \Gamma\text{E} \end{array}$$

From [**3**, Theorem 3.9], FV is equivalent to ΓS.

**2. Preliminaries.** Let $\Gamma(G)$ denote the set of all convex $l$-subgroups of $G$. $M \subseteq G$ is a prime subgroup if $M \in \Gamma(G)$ and $A, B \in \Gamma(G)$ such that $A \cap B \subseteq M$ implies $M \supseteq A$ or $M \supseteq B$. $M \subset G$ is a regular subgroup if

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$M \in \Gamma(G)$ and $M$ is maximal with respect to not containing some $g \in G$. All regular subgroups are prime; and, moreover, each prime subgroup is the intersection of a chain of regular subgroups [**3**, Theorem 3.2]. The polar for $A \subseteq G$ is given by $A^* = \{x \in G\colon |x| \wedge |y| = 0$ for all $y \in A\}$. The polar is always an element of $\Gamma(G)$; and if $A$ is a prime subgroup, then $A^*$ is totally ordered [**3**]. $\Gamma_0(G)$, the set of all prime subgroups, is closed under intersection of chains [**3**]; thus, a minimal prime will be the intersection of a maximal chain in $\Gamma_0(G)$. Let $\Gamma_1(G)$ denote all regular subgroups of $G$ and let $\Gamma_g$ denote all regular subgroups which are maximal without containing $g$, i.e. values of $g$ [**4**]. $\Delta \subseteq \Gamma_1(G)$ is plenary [**4**] if $\Delta$ contains at least one value for each $0 \neq g \in G$ and $\Delta$ is an upper class in $\Gamma_1(G)$. $\Gamma_1(G)$ is always plenary in itself. $N \in \Gamma_1(G)$ is special if there exists $g \in G$ such that $\Gamma_g = \{N\}$. ($g$ is also called special in this case.) $N \in \Gamma_1(G)$ is essential if there exists $g \in G$ such that $M \in \Gamma_g$ implies $M \subseteq N$. $g \in G$ is finite-valued if $\Gamma_g$ is a finite set. $0 < g \in G$ is a basic element if the convex $l$-subgroup generated by $g$, denoted by $C(g)$, is totally ordered. Basic elements are all single-valued [**3**, Theorem 3.6]. $G$ has a basis if for each $0 < g$ there exists a basic element $b$ such that $b \leqq g$. The radical of $G$ is given by

$$\mathrm{R}(G) = \bigcap \{M\colon M \text{ is essential and } M \in \Gamma_1(G)\}$$

or $\mathrm{R}(G) = G$ if no essential elements exist. Let $\Pi = \Pi\{G_i\colon i \in I\}$ be the large direct product of the $l$-groups $G_i$. For the purposes of this paper $\Pi$ is given the cardinal order by defining $0 \leqq x$ if $0 \leqq x_i$ for all $i \in I$. $G$ is a subdirect sum of $\Pi$ if $G$ is an $l$-subgroup of $\Pi$ and the projection maps are all onto. $G$ is a full subdirect sum of $\Pi$ if $G$ is a subdirect sum such that $G$ contains $\Sigma$, the small sum of the $G_i$.

**3. Implications from** (A) **and** (B)**.** Note that we use $+$ as a group operation but we do not assume commutativity.

PROPOSITION 1. *If $G$ is an $l$-group satisfying F, then $G$ satisfies* FVB.

*Proof.* $G$ has a basis [**4**, Theorem 5.2]. Since $G$ is assumed to satisfy F, for $0 \neq g$, $\mathrm{C}(g)$ will have a finite basis. Thus, $g$ will have only a finite number of values in $\mathrm{C}(g)$. Hence $g$ will have only a finite number of values in $G$ [**3**, Theorem 3.5].

As mentioned in the introduction, the condition FV is equivalent to ΓS. It is trivial from the definitions that FVB implies FV, that ΓS implies ΓE, and that ΓE implies R$G$. Thus, with Proposition 1, it follows that the implications of (B) are true.

PROPOSITION 2. *$G$ is an $l$-group satisfying* MP *if and only if every maximal chain $\{G_i\colon i \in I\}$, in $\Gamma_1(G)$ contains a value for a basic element.*

*Proof.* $M = \bigcap \{G_i\colon i \in I\}$ is a minimal prime, where $\{G_i\colon i \in I\}$ is a maximal chain in $\Gamma_1(G)$. If $G$ satisfies MP, then $M^* \neq \{0\}$. Let $0 < b \in M^*$.

$b$ is a basic element since $M^*$ is an o-group. $b \notin M$ implies that there exists $i \in I$ such that $G_i \in \Gamma_b$ since $\{G_i: i \in I\}$ is a maximal chain. For the converse, assume that $M$ is a minimal prime as above. There exists $G_k$ $(k \in I)$, which is a value for a basic element $b > 0$. [**3**, Theorem 3.6] implies that $G_k = C(b)^* + N_k$, where $N_k \subset C(b)$, an o-group; and, moreover, any regular subgroup contained in $G_k$ is also of this form [**4**, Lemma 5.3]. Thus,

$$\bigcap \{G_i: i \in I\} = \bigcap \{G_i: i \in I \text{ and } G_i \subseteq G_k\} = \bigcap \{C(b)^* + N_i\} = C(b)^*.$$

Therefore $0 < b \in M^*$.

COROLLARY 3. *If $G$ is an $l$-group satisfying* MP, *then $G$ has a basis.*

PROPOSITION 4. *If $G$ is an $l$-group satisfying* F, *then $G$ satisfies* MP.

*Proof.* We can assume that $G$ is finite-valued with basis. Denote the basis subgroup by $B = \sum \{B_j: j \in J\}$ [**2**]. Suppose that $G$ does not satisfy MP or that there exists a maximal chain, $\{G_i: i \in I\}$, in $\Gamma_1(G)$ such that no $G_i$ is a value for a basic element. Thus, $B_j \subseteq G_i$ for all $j \in J$ and $i \in I$. $G$ finite-valued implies $G_i = C(x_i)^* + N_i$, where $\{G_i\} = \Gamma_{x_i}, x_i > 0$. Moreover, $N_i$ is the maximal convex $l$-subgroup of $C(x_i)$, and $C(x_i)$ is a lexicographic extension of $N_i$. $B_j \subseteq G_i$ implies

$$B_j = (B_j \cap G_i) = (B_j \cap C(x_i)^*) + (B_j \cap N_i).$$

Since each o-subgroup $B_j$ is cardinally indecomposable, either $B_j \subseteq C(x_i)^*$ or $B_j \subseteq N_i$. Thus, the number of $B_j$, $j \in J$, such that $B_j \subseteq N_i$ is exactly the maximal number of disjoint basic elements below $x_i$. Therefore,

$$I_i = \{j \in J: B_j \subseteq N_i\}$$

is finite for each $i \in$ I. Moreover, $G_i \subseteq G_k$ implies $I_i \subseteq I_k$. Thus,

$$T = \bigcap \{I_i: i \in I\}$$

is the intersection of a tower of finite sets and is not empty. Let $t \in T$. If $0 < b \in B_t$, then $b < x_i$ for all $i \in I$, and, moreover, the value for $b$ is contained in $G_i$ for all $i \in I$. This contradicts the assumption.

PROPOSITION 5. *If $G$ is an $l$-group satisfying* MP, *then $G$ satisfies* ΓE.

*Proof.* Proposition 2 implies that each $H \in \Gamma_1(G)$ is comparable to a value for a basic element. The regular subgroups contained in a value for a basic element form a chain and are also values for basic elements [**4**]. Therefore, $H$ contains a value for a basic element and the value for each basic element is special.

From Propositions 2, 3, and 5, the implications of line (A) follow.

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THEOREM 6. *If $G$ is l-isomorphic to a full subdirect sum of* o-*groups, then* F, MP, FVB, $\Gamma$S, *and* $\Gamma E$ *are all equivalent. In fact, under this hypothesis, each is equivalent to $G$ being l-isomorphic to a small cardinal sum of* o-*groups.*

*Proof.* Since the above properties are preserved under $l$-isomorphisms, we may assume that $G$ is a full subdirect sum of $\{G_i: i \in I\}$. From lines (A) and (B), it suffices to show that $\Gamma$E implies F. Let $\Sigma$ denote the small sum of the $G_i$, $i \in I$. Suppose that $0 < g \in G - \Sigma$. Then there exists $H \in \Gamma_g$ such that $\Sigma \subseteq H$. Let $\bar{G}_i$ denote the $l$-subgroup of $G$ which is naturally $l$-isomorphic to $G_i$. $\bar{G}_i \subseteq H$ for all $i \in I$. $\Delta = \{M \in \Gamma_1(G)$: $M$ contains a value for a basic element$\}$ is plenary. Since $H$ is essential, $H \in \Delta$. Therefore, there exists $K$, a value for a basic element $b$, such that $K \subseteq H$. $b \in \bar{G}_j$ for some $j \in I$. $K = \bar{G}_j^* + N$, where $N$ is maximal without $b$ in $\bar{G}_j$. Since $G$ is a full subdirect sum, $G = \bar{G}_j^* + \bar{G}_j$. Thus, $g = g' + g''$, $g' \in \bar{G}_j^*$, $g'' \in \bar{G}_j$. $\bar{G}_j \subseteq H$ implies $g'' \in H$. $\bar{G}_j^* \subseteq K \subseteq H$ implies $g' \in H$. This contradicts $g \notin H$. Therefore, $G = \Sigma$.

COROLLARY 7. *If $G$ is an Archimedean l-group, then* F, MP, FVB, $\Gamma$S, *and* $\Gamma$E *are all equivalent.*

*Proof.* It suffices to show that an Archimedean $l$-group satisfying $\Gamma$E is $l$-isomorphic to a full subdirect sum of o-groups. Since in Archimedean $l$-groups o-subgroups are unbounded, we may apply [**2**, Theorem 7.2] if we show that $G$ has a basis. $\Gamma$E implies that $R(G) = \{0\}$. $G$ Archimedean and $R(G) = \{0\}$ implies that $G$ has a basis [**4**, Theorems 5.4 and 5.7].

**4. Counterexamples.** Examples 1, 4, and 5 indicate that all reverse implications of (A) are in general false. Examples 1, 2, and 3 indicate that all reverse implications of (B) are in general false. Also note that Examples 3 and 4 indicate that MP and FVB are in general non-related.

*Example* 1. This is an example of an Abelian $l$-group which satisfies both FVB and MP, but not F. Let $A = \Sigma\{I_i$: $I_i$ is the integers with their natural order for $i = 1, 2, 3, \ldots\}$ and let $G$ be an extension of $A$ by the integers, lexicographically ordered. Clearly $\Gamma_1(G)$ has one maximal element, namely $A$, and the minimal primes of $G$ are the minimal primes $M_i$ of $A$, where $M_i = \{f \in A: f(i) = 0\}$. Thus, no polar of a minimal prime is zero; and, moreover, $G$ is finite-valued with basis. Any positive element not zero in the maximal coordinate will exceed infinitely many disjoint basic elements.

*Example* 2. See [**4**, p. 166, example 5]. Instead of the $G$ given in Example 5, let $G$ denote the set of all elements which are finitely non-zero. $G$ will be a finite-valued Abelian $l$-group with no basic elements.

*Example* 3. This is an example of an Abelian $l$-group which satisfies MP but not FV. This $l$-group also satisfies $\Gamma$E but not $\Gamma$S. Let $K_i$ be the lexicographic direct sum of the integers $I$ with the integers $I$ (using natural order on $I$). Let $K = \Pi\{K_i: i = 1, 2, \ldots\}$. The elements of $K_i$ will be written

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as $(x_{2i}, x_{1i})$; and, thus, the elements of $K$ will be written as $(x_{2i}, x_{1i})$, $i \in N$. $G$ is the set of all elements of $K$ with the following properties:

(1) $x_{2i} = x_{21}$ for all but finitely many $i$; and

(2) $x_{1i}$ are finitely non-zero.

The verification that $G$ is an $l$-group is straightforward. Let

$$G_{2j} = \{x \in G: x_{2j} = 0\} \quad \text{and} \quad G_{1j} = \{x \in G: x_{1j} = 0 \text{ and } x_{2j} = 0\}.$$

It suffices to show that

$$\Gamma_1(G) = \{G_{ij}: i = 1 \text{ or } 2 \text{ and } j \in N\},$$

since the $G_{1j}$ are all minimal primes with non-zero polars. Suppose that $H$ is a regular subgroup not of the form $G_{ij}$. It easily follows that $H$ must contain, for each $2j$, an element non-zero in the $2j$ position. Using the element non-zero in the 21 position and a finite number of elements to fill in the coordinates which may be zero, one can construct an element positive in every $2j$ position. The smallest convex $l$-subgroup containing such an element is all of $G$; thus, a contradiction.

*Example* 4. This is an example of an Abelian $l$-group $G$ which satisfies FVB but not MP. It also illustrates an $l$-group which satisfies FV but not MP. Let $\Delta$ be the negative integers partially ordered as follows:

(1) The odd integers assume their natural order;

(2) If $\delta$ is even, then $\delta$ is less than $\delta + 1$, (and $\delta + 3, \ldots, -1$) and is non-comparable to all other elements of $\Delta$.

$\Delta$ is a root system [**4**] and $G$, the set of all finitely non-zero real-valued functions on $\Delta$ is an $l$-group with order given as in the $V$-group of [**4**]. With reference to the $V_\delta$ given in [**4**, p. 162], the $V_\delta$ will form all the elements of $\Gamma_1(G)$. Moreover, $\{V_\delta: \delta \text{ is odd}\}$ will form a maximal chain, and $M = \bigcap \{V_\delta: \delta \text{ odd}\}$ will have a zero polar. The fact that $G$ is finite-valued with basis follows from [**3**, Theorem 4.1].

*Example* 5. This is an example of an Abelian $l$-group for which $R(G)$ is zero but which has one non-essential regular subgroup. Let $A = \Pi\{I_i: I_i$ is the integers for $i = 1, 2, \ldots\}$, where each $I_i$ has the natural order. Let $G$ be the set of all elements of $A$ which eventually form constant sequences. $G$ is an $l$-group and it is not hard to verify that $\Sigma$, the set of all finitely non-zero elements, is the only non-essential regular subgroup.

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