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The Best Constants Associated with some Weak Maximal Inequalities in Ergodic Theory

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Abstract. We introduce a new device of measuring the degree of the failure of convergence in the ergodic theorem along subsequences of integers. Relations with other types of bad behavior in ergodic theory and applications to weighted averages are also discussed.

1 Introduction

Let (X, Σ, m) denote a non-atomic probability space and $\tau\colon X \to X$ a measure preserving transformation mapping X to itself. For a given increasing sequence of positive integers (a_k) and $f \in L^1(X)$, we will consider averages of the form $A_n f(x) = \frac{1}{n}\sum_{k=1}^{n} f(\tau^{a_k}x)$. The natural logarithm of a number x will be denoted by $\ln x$, while $[x]$ and $\{x\}$ will stand for the integer and fractional part of x. Given two sequences (a_k) and (b_k), we will use the notation $a_k \gg b_k$ whenever $\alpha \le \frac{a_k}{b_k}$, for some positive constant α. The notation $a_k \asymp b_k$ will mean that $a_k \ll b_k$ and $a_k \gg b_k$ simultaneously. If A is a finite subset of the integers, its cardinality will be denoted by $|A|$. Throughout this paper, unless stated otherwise, all sequences will be meant to be increasing and consist of positive integers. The following two definitions are intrinsic to our discussion, so we present them before we start any investigation.

Definition 1.1 Let $1 \le p \le \infty$. We say that the sequence (a_k) is L^p*-good* for the dynamical system (X, Σ, m, τ), if

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f(\tau^{a_k}x) = \bar{f}(x) \tag{1.1}$$

exists almost everywhere, for all $f \in L^p(X)$. The sequence (a_k) is *universally* L^p*-good,* or *simply* L^p*-good*, if and only if it is L^p*-good* for every dynamical system (X, Σ, m, τ).

Definition 1.2 A sequence (a_k) is called *universally* L^p*-bad,* or *simply* L^p*-bad*, if for all ergodic dynamical systems (X, Σ, m, τ), there is $f \in L^p(X)$ such that the limit (1.1) fails to exist for all x in a set of positive measure.

The celebrated ergodic theorem of Birkhoff [7] asserts that the sequence of positive integers is L^1-good. In the decades following this discovery, it has become an

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interesting problem to characterize the classes of subsequences of integers for which Birkhoff's theorem remains valid. Bourgain has brought a new insight to this issue. In a sequence of papers [8], [9], [10], he proved that sequences such as $a_k = q(k)$, where q is a nonconstant polynomial mapping the natural numbers to themselves, and the sequence of primes, are L^p-good for $p > 1$. Related results based on his techniques can be found in [17], [22].

Equal interest has been shown in recent years in negative results. By using a device employed by Furstenberg [12], Bellow proved in [4] that all lacunary sequences are L^p-bad for every $1 \le p < \infty$. By a different type of argument, based on Bourgain's entropy theorem, Rosenblatt [20] completed the picture by showing that lacunary sequences are also L^∞-bad. Related results can be found in [1], too.

Sparseness seems to be one of the main ingredients which makes a sequence L^p-bad for some $1 \le p < \infty$. In [15] Jones and Wierdl proved using mainly the technique from [4], that a condition weaker than lacunarity,

$$\frac{a_{k+1}}{a_k} > 2^{\varphi(k)(\ln k)^{-1/p}}, \quad k \ge k_0 \tag{1.2}$$

where $\lim_{x\to\infty} \frac{\varphi(x)}{\ln \ln x} = \infty$, is sufficient to conclude that (a_k) is L^p-bad.

Proving that a sequence is universally bad is only one side of the issue. Several concepts have been introduced to measure how dramatic this failure of convergence can be.

Definition 1.3 A sequence (a_k) is called *strongly sweeping out* if for each ergodic dynamical system (X, Σ, m, τ) and each $\epsilon > 0$, there exists $E \in \Sigma$ such that $m(E) < \epsilon$ and

$$\limsup_{n\to\infty} A_n \chi_E(x) = 1 \quad \text{a.e.}$$

where χ_E is the characteristic function of the set E.

If a sequence is strongly sweeping out, it can be proved that the oscillations of the ergodic averages associated with characteristic functions of arbitrarily small norm can occur with maximum amplitude. The existence of L^∞-bad sequences which do not have the strongly sweeping out property was shown by Rosenblatt [20]. In the comprehensive paper [1], several criteria for being strongly sweeping out are established and it is proved that lacunary sequences have this property. See also [20] for other properties related to strongly sweeping out.

This paper is mainly concerned with developing a strategy which will enable us to investigate the universally bad sequences from a different perspective. Instead of looking at the oscillations of the ergodic averages, we can ask how fast is the supremum of the first n such averages growing. The measure for this rate of growth will be provided by the behavior of the best constants associated with some weak maximal inequalities.

We will be particularly interested in the sequences of integers for which the associated constants grow as quickly as possible. In Section 2 we will give a class of superlacunary sequences which have this property and try to understand why the same method of proof does not work for a general lacunary sequence. Part of the frustration is eliminated in the third section, where a positive result is proved for the sequence of powers of 2. Section 4 investigates the relations between different types of bad behavior of sequences. In the last section we will show how we can use these results to answer some questions about weighted averages.

2 Best Constants for Weak Maximal Inequalities

Let (a_k) be an increasing sequence of positive integers. For each $N \geq 1$, $1 \leq p < \infty$ and $f \in L^p(X)$, define

$$f_N^*(x) = \max_{n \leq N} \Big| \frac{1}{n} \sum_{k=1}^{n} f(\tau^{a_k} x) \Big| . \tag{2.1}$$

Denote by $C_{N,p}$ the least constant for which the following inequality is true

$$m\{x \in X : f_N^*(x) > \lambda\} \leq \frac{C_{N,p}}{\lambda^p} \|f\|_p^p \tag{2.2}$$

for every dynamical system (X, Σ, m, τ), $f \in L^p(X)$ and $\lambda > 0$. According to Sawyer's principle [21], if $\lim_{N \to \infty} C_{N,p} = \infty$, then the sequence (a_k) is universally bad for L_p.

The next theorems address the following issue: given p and a sequence (a_k), how fast can the corresponding constants $C_{N,p}$ grow, as $N \to \infty$?

Theorem 2.1 *For any sequence (a_k), positive integer N and $1 \leq p < \infty$, we have that $C_{N,p} \ll \ln N$.*

Proof We start by applying Hölder's inequality

$$\begin{aligned} m\{x \in X : f_N^*(x) > \lambda\} &\leq m\Big\{ x \in X : \max_{n \leq N} \Big(\frac{1}{n} \sum_{k=1}^{n} |f|^p(\tau^{a_k} x) \Big)^{1/p} > \lambda \Big\} \\ &= m\Big\{ x \in X : \max_{n \leq N} \frac{1}{n} \sum_{k=1}^{n} |f|^p(\tau^{a_k} x) > \lambda^p \Big\} \\ &\leq m\Big\{ x \in X : \sum_{k=1}^{N} \frac{1}{k} |f|^p(\tau^{a_k} x) > \lambda^p \Big\} \\ &\leq \frac{1}{\lambda^p} \Big\| \sum_{k=1}^{N} \frac{1}{k} |f|^p \circ \tau^{a_k} \Big\|_1 \\ &= \frac{\|f\|_p^p}{\lambda^p} \Big(1 + \frac{1}{2} + \cdots + \frac{1}{N} \Big). \end{aligned}$$ ■

The main object of our investigation is the class of the sequences for which the growth of the $C_{N,p}$'s is maximal: $C_{N,p} \asymp \ln N$. Next we will prove that this class is

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nonempty, by showing that sequences with a sufficiently high rate of growth belong to it. The main ingredient in the proof is the following lemma, used by Bellow in [4] to prove the L^p-badness of lacunary sequences.

Lemma 2.2 *Let $t \in \mathbb{Z}_+$ and (a_k) satisfy the growth condition*

$$\frac{a_{k+1}}{a_k} \geq t + 1$$

for each $k \in \{1, 2, \ldots, K-1\}$. Then for any finite sequence $(I_k)_{k=1}^K$ of subintervals of $[0, 1]$ of individual length $1/t$, there exists $\theta \in [0, 1)$ such that $\{a_k\theta\} \in I_k$, for each $k \in \{1, 2, \ldots, K\}$.

Proof For each $k \in \{1, 2, \ldots, K\}$ set

$$\Lambda_k = \bigcup_{m\in\mathbb{Z}} (I_k + m)$$

and let

$$\Lambda_k^* = \{\theta \in [0, 1) : a_k\theta \in \Lambda_k\}.$$

The set Λ_k^* consists of intervals of length $1/ta_k$ repeated periodically with period $1/a_k$. Because of the growth condition on (a_k), it is clear that every interval from Λ_k^* contains some interval from Λ_{k+1}^*. Hence $\bigcap_{k=1}^K \Lambda_k^* \neq \varnothing$ which finishes the proof. ■

We can now prove the following.

Theorem 2.3 *Take a sequence (a_k) satisfying the growth condition*

$$\frac{a_{k+1}}{a_k} \geq t + 1$$

for each $k \geq l_0(t)$, where

$$\sup_{t\in\mathbb{Z}_+} \frac{\ln l_0(t)}{t} < \infty. \tag{2.3}$$

Then (a_k) has $C_{N,p} \asymp \ln N$ for each $1 \leq p < \infty$.

Remark 2.4 Note that the sequences $a_k = k!$ and $a_k = [(\ln k)^k]$ satisfy the requirements of the theorem.

Proof We will follow pretty closely the lines of the proof of Theorem 1 from [4]. Fix $N \in \mathbb{Z}_+$ and $1 \leq p < \infty$. Take $t \in \mathbb{Z}_+$, $t \geq 3$ such that

$$2^t l_0(t) < N \leq 2^{t+1} l_0(t+1)$$

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Find $l_0, l_1, \dots, l_t$ such that

$$l_0 = l_0(t), \quad \frac{l_1}{l_0 + l_1} = \cdots = \frac{l_t}{l_0 + l_1 + \cdots + l_t} = \frac{1}{2}$$

and for each $i \in \{0, 1, \dots, t-1\}$ consider the following block of consecutive integers

$$B_i = \{l_0 + l_1 + \cdots + l_i + 1, l_0 + l_1 + \cdots + l_i + 2, \dots, l_0 + l_1 + \cdots + l_i + l_{i+1}\}.$$

Pick a θ according to Lemma 2.2 such that

$$\{a_k\theta\} \in \Big[\frac{i}{t}, \frac{i+1}{t}\Big]$$

for each $k \in B_i$ and each $i \in \{0, 1, \dots, t-1\}$. Easy computations show that $l_i = l_0(t)2^{i-1}$, hence

$$B_i = \{l_0(t)2^i + 1, l_0(t)2^i + 2, \dots, l_0(t)2^{i+1}\}.$$

Now for the measure preserving transformation $\tau\colon [0,1) \to [0,1)$ defined by $\tau(x) = \{x + \theta\}$ and for $A = [0, \frac{2}{t}]$, we have

$$\tau^{a_k}(x) = \{x + a_k\theta\} \in \Big[\frac{t-i}{t}, \frac{t-i+1}{t}\Big] + \Big[\frac{i}{t}, \frac{i+1}{t}\Big] = A$$

whenever $x \in [\frac{t-i}{t}, \frac{t-i+1}{t}]$ and $k \in B_i$. Hence

$$\frac{1}{l_0(t)2^{i+1}} \sum_{k=1}^{l_0(t)2^{i+1}} \chi_A(\tau^{a_k}x) \geq \frac{1}{2|B_i|} \sum_{k\in B_i} \chi_A(\tau^{a_k}x) = \frac{1}{2}$$

for every $x \in [\frac{t-i}{t}, \frac{t-i+1}{t}]$, which implies

$$\max_{n \leq l_0(t)2^t} \frac{1}{n} \sum_{k=1}^{n} \chi_A(\tau^{a_k}x) \geq \frac{1}{2}$$

for every $x \in [0,1)$. We can conclude that

$$m\{x \in [0,1) : \max_{n\leq N} \frac{1}{n} \sum_{k=1}^{n} \chi_A(\tau^{a_k}x) \geq \frac{1}{2}\} = 1.$$

Since

$$m\{x \in [0,1) : \max_{n\leq N} \frac{1}{n} \sum_{k=1}^{n} \chi_A(\tau^{a_k}x) \geq \frac{1}{2}\} \leq C_{N,p}2^p m(A)$$

it follows easily that

$$C_{N,p} \geq \frac{t}{2^{p+1}}. \tag{2.4}$$

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On the other hand, since $N \leq 2^{t+1} l_0(t+1)$, we must have that

$$\log_2 N \leq t + 1 + \log_2 l_0(t+1) \leq \alpha t$$

for some positive constant α, by relation (2.3). Now (2.4) can be rewritten as $C_{N,p} \gg \ln N$, which finishes the proof. ■

A careful analysis of the proof shows that the condition from (2.3) is optimal if one uses this technique. In particular, the argument does not work for a general lacunary sequence and the best result one can get in this case is $C_{N,p} \gg \frac{\ln N}{(\ln \ln N)^p}$. The explanation is that if a sequence does not grow fast enough, as in the case of $a_k = 2^k$, then Lemma 2.2 does not provide a good control over the $a_k\theta$'s, since the intervals (I_k) are not allowed to be too small. Thus we have to drop to a sufficiently rapidly growing subsequence of (a_k), say (a_{k_i}) and apply Lemma 2.2 to control the $a_{k_i}\theta$'s. Still, by gaining a better control over the $a_{k_i}\theta$'s, we lose all the control over the remaining $a_k\theta$'s. When we compute the usual averages $A_n f(x)$, where f is taken to be the characteristic function of an interval as above, we cannot use the information coming from the terms which contain $a_k\theta$, if a_k is not in the chosen subsequence. The result is, as we mentioned above, that an extra factor of $\ln \ln N$ shows up in the lower bound for $C_{N,p}$.

3 The Sequence $a_k = 2^k$

The purpose of this section is to develop a technique which enables us to conclude that $C_{N,1} \asymp \ln N$ for the sequence of powers of 2. As we explained in the end of the previous section, since (2^k) does not grow fast enough, we should work with its subsequences. If $q \in \mathbb{Z}_+$, $q \geq 2$ and $p = 2^q$, then Lemma 2.2 guarantees that for any finite sequence of intervals $(I_k)_{k=1}^K$ of individual length roughly $1/p$, there is a θ such that $\{2^{qk}\theta\} \in I_k$ for every $k \in \{1, 2, \dots, K\}$. The question is what happens with the remaining $q-1$ subsequences, in other words what is the distribution in $[0, 1)$ of the $2^{qk+t}\theta$'s, for a fixed $t \in \{1, 2, \dots, q-1\}$?

In order to get the desired control, we need to work with some specific θ. Throughout this section $p, q \in \mathbb{Z}_+$, $q \geq 2$ and $p = 2^q$. For any integers $1 \leq a \leq p-2$ and $1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_{a+1}$, with $\alpha_{i+1} - \alpha_i \geq 3$ when $1 \leq i \leq a$, define

$$\theta = \frac{1}{p^{\alpha_1+1}} + \frac{1}{p^{\alpha_1+2}} + \cdots + \frac{1}{p^{\alpha_2}} + \cdots + \frac{i}{p^{\alpha_i+1}} + \frac{i}{p^{\alpha_i+2}} + \cdots + \frac{i}{p^{\alpha_{i+1}}} + \cdots$$
$$+ \frac{a}{p^{\alpha_a+1}} + \frac{a}{p^{\alpha_a+2}} + \cdots + \frac{a}{p^{\alpha_{a+1}}}.$$

The idea is to split the finite sequence $(2^k)_{k=q\alpha_1}^{q\alpha_{a+1}-1}$ into several blocks, so that the $2^k\theta$'s have the same behavior when restricted to a certain block. For each $0 \leq t \leq q-1$ and each $1 \leq i \leq a$ the (i, t) block will be the set

$$B_{i,t} = \{2^t p^{\alpha_i}, 2^t p^{\alpha_i+1}, 2^t p^{\alpha_i+2}, \dots, 2^t p^{\alpha_{i+1}-1}\} = \{2^{qs+t} : \alpha_i \leq s < \alpha_{i+1}\}.$$

The following lemmas will show that, more or less, the $2^k\theta$'s stay very close to each other when the 2^k's stay in a certain block $B_{i,t}$.

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Lemma 3.1 (About the $B_{i,0}$'s) *For each* $1 \le i \le a$ *and* $\alpha_i \le k < \alpha_{i+1}$, *we have that* $\{p^k\theta\} \in [\frac{i}{p}, \frac{i+1}{p}]$.

Proof The proof follows easily once it is noted that θ is written in base p. ∎

Lemma 3.2 (About the $B_{i,t}$'s, $t \neq 0$) *Take* $1 \le t \le q-1$ *and* $1 \le i \le a$. *If* $2^t i = m \pmod{p-1}$ *for some* $0 \le m \le p-3$, *then for each* $\alpha_i \le k \le \alpha_{i+1} - 3$ *the following holds*

$$\{2^t p^k \theta\} \in \Big[\frac{m}{p}, \frac{m+1}{p}\Big].$$

Proof We begin with the observation that m cannot be 0, since $p-1$ and 2^t do not share any common factor and $1 \le i \le p-2$. Note that $2^t p^k \theta = \frac{2^t i}{p} + \frac{2^t i}{p^2} + \frac{2^t i}{p^3} + \cdots \pmod 1$, where the remaining terms, if any, have as denominators powers of p, $(p^4, p^5, \dots)$, while the numerators have the form $2^t s$, where s is at least i and at most a. Write now $2^t i = bp + c$, $b, c \in \mathbb{Z}_+$, $0 \le c \le p-1$. Since $2^t \le 2^{q-1} < p-1$ and $i \le a \le p-1$, it follows that $b < p-1$ and so $1 \le b + c < 2p-2$. Now $2^t i = b(p-1) + b + c$, hence by hypothesis $b + c = m \pmod{p-1}$. We distinguish two cases:

(a) If $b + c = m$ then

$$\begin{aligned} 2^t p^k \theta &= \frac{2^t i}{p} + \frac{2^t i}{p^2} + \frac{2^t i}{p^3} + \cdots \pmod 1 \\ &= \frac{bp+c}{p} + \frac{bp+c}{p^2} + \frac{2^t i}{p^3} + \cdots \pmod 1 \\ &= \frac{c+b}{p} + \frac{c}{p^2} + \frac{2^t i}{p^3} + \cdots \pmod 1 \\ &= \frac{m}{p} + \frac{c}{p^2} + \frac{2^t i}{p^3} + \cdots \pmod 1. \end{aligned}$$

If we denote $\gamma_1 = \frac{m}{p} + \frac{c}{p^2} + \frac{2^t i}{p^3} + \cdots$, then

$$\begin{aligned} \frac{m}{p} < \gamma_1 &< \frac{m}{p} + \frac{c}{p^2} + 2^t\Big(\frac{i}{p^3} + \sum_{s=4}^{\infty} \frac{a}{p^s}\Big) \\ &< \frac{m}{p} + \frac{c}{p^2} + 2^t \frac{i+2}{p^3} \le \frac{m}{p} + \frac{c}{p^2} + \frac{bp+c}{p^3} + \frac{p}{p^3} \\ &< \frac{m}{p} + \frac{c+b+2}{p^2} < \frac{m+1}{p}. \end{aligned}$$

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(b) If $b + c = m + p - 1$, then as above

$$\begin{aligned} 2^t p^k \theta &= \frac{2^t i}{p} + \frac{2^t i}{p^2} + \frac{2^t i}{p^3} + \cdots \pmod 1 \\ &= \frac{bp + c}{p} + \frac{bp + c}{p^2} + \frac{2^t i}{p^3} + \cdots \pmod 1 \\ &= \frac{c + b}{p} + \frac{c}{p^2} + \frac{2^t i}{p^3} + \cdots \pmod 1 \\ &= \frac{m - 1}{p} + \frac{c}{p^2} + \frac{2^t i}{p^3} + \cdots \pmod 1. \end{aligned}$$

Similar computations show that if $\gamma_2 = \frac{m-1}{p} + \frac{c}{p^2} + \frac{2^t i}{p^3} + \cdots$, then $\frac{m}{p} \le \gamma_2 < \frac{m+1}{p}$.

In both cases the conclusion is that $\{2^t p^k \theta\} \in [\frac{m}{p}, \frac{m+1}{p}]$. ∎

Remark 3.3 The previous lemmas show that 'almost' all blocks are 'almost' well controlled. The fact that we lack the control in the case $m = p - 2$ or when $k \in \{\alpha_{i+1} - 2, \alpha_{i+1} - 1\}$ will prove to be unimportant.

If for a certain pair (i, t) one has $2^t i = m \pmod{p - 1}$ for some $0 \le m \le p - 3$, the interval $I_m = [\frac{m}{p}, \frac{m+1}{p}]$ will be called the *image* of the block $B_{i,t}$. The next proposition proves that for a fixed i, the blocks $B_{i,t}$ with $0 \le t \le q - 1$ have distinct images, provided q is prime.

Proposition 3.4 *If q is prime and $p = 2^q$ as before, then for each $1 \le i \le a$, $0 \le t$, $t' \le q - 1$ with $t \ne t'$, we have that*

$$2^t i \ne 2^{t'} i \pmod{p - 1}.$$

Proof To prove this we need the following easy result:

Lemma 3.5 *Let g and h be positive integers and let $d = (g, h)$ denote their greatest common divisor. If l is a common divisor of $2^g - 1$ and $2^h - 1$, then $l \mid 2^d - 1$.*

Proof Take $\beta_1, \beta_2 \in \mathbb{Z}_+$ such that $\beta_1 g - \beta_2 h = d$. It's not hard to see that $l \mid 2^{\beta_1 g} - 1$ and $l \mid 2^{\beta_2 h} - 1$, hence $l \mid 2^{\beta_2 h}(2^{\beta_1 g - \beta_2 h} - 1)$, which gives $l \mid 2^d - 1$. ∎

We start the proof of Proposition 3.4 by supposing that $p - 1 \mid 2^t i - 2^{t'} i$ for some $0 \le t' < t \le q - 1$. This certainly implies that $2^q - 1 \mid (2^{t-t'} - 1)i$. Since q is prime, it follows by Lemma 3.5 that $2^q - 1 \mid i$. The contradiction comes now from the fact that $1 \le i \le a \le p - 2$. ∎

For purposes that will become clear throughout the proof of Theorem 3.8, it would be really helpful if the blocks that we are using had distinct images. Part of this plan is easy to fulfill since by Proposition 3.4 the $B_{i,t}$'s have distinct images when

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i is fixed and t varies. On the other hand, the images of $B_{i,t}$ and $B_{j,t'}$ may not be distinct if $i \neq j$. What we will prove in the following is that we can get sufficiently many blocks $B_{i,t}$ with distinct images, provided we don't let i be too large. To be more precise, for each $1 \leq k \leq a$ define $A_k = \{0 \leq m \leq p-2$: there exist $1 \leq i \leq k$, $0 \leq t \leq q-1$ such that $2^t i = m \pmod{p-1}\}$. We will prove that $|A_k| \geq \frac{kq}{8}$ for each $k \leq \frac{q}{\log_2(q)\log_2\log_2(q)}$. Note that kq is the maximum possible cardinality of A_k, so the result is pretty sharp. For each $1 \leq i \leq a$ define

$$D_i = \{0 \leq m \leq p-2 : 2^t i = m \pmod{p-1}, \text{ for some } 0 \leq t \leq q-1\}.$$

Certainly $A_k = \bigcup_{i=1}^k D_i$. From now on, q will be implicitly assumed to be a prime number. Let's first prove the following:

Lemma 3.6 *For every pair of odd numbers $i \neq j$ with $i, j \leq \frac{q}{\log_2(q)\log_2\log_2(q)}$, we have $|D_i \cap D_j| \leq 2\log_2 q$.*

Proof Take i, j as above. If $D_i \cap D_j \neq \varnothing$, consider $0 \leq t, t' \leq q-1$ such that $p-1 \mid 2^t i - 2^{t'} j$. Obviously $t \neq t'$, otherwise $p-1 \mid i-j$ which is certainly impossible. If $t > t'$, then $p-1 \mid 2^{t-t'} i - j$ and since i, j are odd, $2^{t-t'} i - j \neq 0$. So $2^{t-t'} i - j \geq p-1$ and hence $2^{t-t'} i \geq p$. From here we can deduce that

$$2^{t-t'} \geq \frac{p}{i} \geq \frac{2^q \log_2(q)\log_2\log_2(q)}{q} \geq 2^{q-\log_2 q},$$

which automatically gives $t - t' \geq q - \log_2 q$. Similarly, if $t' > t$, then $t' - t \geq q - \log_2 q$. In conclusion, D_i and D_j will have at most $2\log_2 q$ elements in common. ∎

Using the above lemma, we can prove the result we announced before.

Proposition 3.7 *For every sufficiently large q and every positive integer $k \leq \frac{q}{\log_2(q)\log_2\log_2(q)}$, we have $|A_k| \geq \frac{kq}{8}$.*

Proof We start by noting that

$$|A_k| \geq \Big|\bigcup_{i=0}^{[\frac{k-1}{2}]} D_{2i+1}\Big| \geq \sum_{i=0}^{[\frac{k-1}{2}]} \Big| D_{2i+1} \setminus \bigcup_{\substack{j=0\\ j\neq i}}^{[\frac{k-1}{2}]} D_{2j+1}\Big|.$$

From the previous lemma it follows that

$$\Big| D_{2i+1} \setminus \bigcup_{\substack{j=0\\ j\neq i}}^{[\frac{k-1}{2}]} D_{2j+1}\Big| \geq |D_{2i+1}| - \sum_{\substack{j=0\\ j\neq i}}^{[\frac{k-1}{2}]} |D_{2i+1} \cap D_{2j+1}|$$
$$\geq q - k\log_2 q \geq \frac{q}{2}$$

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for q large enough. Consequently

$$|A_k| \geq \sum_{i=0}^{[\frac{k-1}{2}]} \frac{q}{2} \geq \frac{kq}{8}. \qquad \blacksquare$$

We have now all the machinery needed for proving the main result of this section.

Theorem 3.8 *The sequence $a_k = 2^k$ has the associated constant $C_{N,1} \asymp \ln N$.*

Proof Take an arbitrary $N \in \mathbb{Z}_+$ and choose consecutive primes $\hat{q}_N, \bar{q}_N$ such that

$$2\hat{q}_N^2 \leq N < 2\bar{q}_N^2. \tag{3.1}$$

For simplicity of notation we will denote $\hat{q}_N$ by q. Take θ corresponding to $a = [\log_2 q]$ and $\alpha_i = 2^{i+1} - 1$. As before $p = 2^q$ and $A = [0, \frac{2}{p}]$. For every $m \in A_a \setminus \{p-2\}$, there exist $1 \leq i \leq a$ and $0 \leq t \leq q-1$ such that $2^t i = m \pmod{p-1}$. By Lemma 3.1 and Lemma 3.2,

$$\{2^t p^k \theta\} \in \Big[\frac{m}{p}, \frac{m+1}{p}\Big]$$

for each $2^{i+1}-1 \leq k \leq 2^{i+2}-4$. As before, if $x \in [\frac{p-m}{p}, \frac{p-m+1}{p}]$, then $\{x+2^{t+qk}\theta\} \in A$ for each $2^{i+1} - 1 \leq k \leq 2^{i+2} - 4$ and hence

$$\frac{1}{t + q(2^{i+2} - 4)} \sum_{s=1}^{t+q(2^{i+2}-4)} \chi_A(x + 2^s\theta) \geq \frac{2^{i+1} - 2}{q + q(2^{i+2} - 4)} \geq \frac{1}{4q}.$$

Note that

$$t + q(2^{i+2} - 4) \leq q(2^{i+2} - 3) \leq 2q2^{\log_2 q+1} = 4q^2.$$

By combining the results from the last lines with Proposition 3.7 we get:

$$\begin{aligned} m\Big\{ x \in [0,1) : \max_{n\leq 4q^2} \frac{1}{n} \sum_{s=1}^{n} \chi_A(x + 2^s\theta) \geq \frac{1}{4q} \Big\} &\geq \frac{1}{p}(|A_a| - 1) \\ &\geq \frac{1}{p}(a\frac{q}{8} - 1) \geq \frac{aq}{16p}. \end{aligned}$$

That's where we needed A_a to be fairly large. Using the maximal inequality (2.2) with $f = \chi_A$, $\lambda = 1/4q$ and $N = 4q^2$ we find

$$\frac{aq}{16p} \leq \frac{8q}{p} C_{4q^2,1}$$

and so

$$C_{4q^2,1} \geq \frac{a}{128} = \frac{[\log_2 q]}{128} \geq \gamma \log_2(4q^2) \tag{3.2}$$

for some appropriate $\gamma > 0$. A classical estimate on consecutive primes (see [13]) forces $\bar{q}_N \leq 2q$. This together with (3.1) proves that

$$\lim_{N\to\infty} \frac{\log_2 N}{\log_2(2\hat{q}_N^2)} = 1.$$

Since $C_{N,1} > C_{4q^2,1}$, (3.2) easily implies that

$$\liminf_{N\to\infty} \frac{C_{N,1}}{\log_2 N} > 0$$

which finishes the proof. ■

Remark 3.9 The proof of Theorem 2.3 does not work for the other L_p spaces. The question whether $C_{N,p} \asymp \ln N$ for the sequence of powers of 2 when $p > 1$ is open.

Remark 3.10 The argument in Theorem 3.8 relies heavily on the arithmetic properties of the sequence of powers of 2, rather than on its rate of growth and we could not extend the proof to general lacunary sequences. Whether or not $C_{N,p} \asymp \ln N$ in general for a lacunary sequence remains an open question, but the next theorem gives us some interesting related information. The probabilistic argument shows that lacunarity provides the lowest degree of sparseness of a sequence for which we can hope that $C_{N,p} \asymp \ln N$.

Theorem 3.11 *Let (σ_n) be a decreasing sequence of real numbers from $(0,1)$ with $\lim_{n\to\infty} n\sigma_n = \infty$ and define $A \in \mathbb{Z}_+$ to be the random sequence obtained by including each positive integer n in the set A with probability σ_n. Then almost surely the sequence A has $C_{N,p} = o(\ln N)$, for each $1 < p < \infty$.*

Proof Let (X, Σ, m, τ) be a dynamical system and $f \in L^p(X)$. The proof will be based on a fundamental inequality derived by Bourgain in Lemma 8.9 from [8], which asserts that with probability one, for all N

$$\Big\| \frac{1}{|S_N|}\Big(\sum_{n\in S_N} \tau^n f\Big) - \frac{1}{\sum_{n\leq N}\sigma_n}\Big(\sum_{n\leq N}\sigma_n\tau^n f\Big)\Big\|_p \leq c_p\Big(\frac{\ln N}{\sum_{n\leq N}\sigma_n}\Big)^{a(p)} \tag{3.3}$$

Here c_p is a constant which does not depend on the dynamical system, $S_N = A \cap [1, N]$, while $a(p)$ equals either $(p-1)/p$ if $p \leq 2$, or $1/p$ if $p > 2$. As Bourgain pointed out, we also have

$$\Big\| \sup_N \frac{1}{\sum_{n\leq N}\sigma_n}\Big|\sum_{n\leq N}\sigma_n\tau^n f\Big|\Big\|_p \leq c'_p\|f\|_p. \tag{3.4}$$

Since by the law of large numbers (see [18]) $\lim_{N\to\infty} |S_N|/\sum_{n\leq N}\sigma_n = 1$, it follows that almost surely $S_{2N}/S_N \ll 1$. So there is a universal constant α, such that with

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probability one, for $f \geq 0$

$$\sup_{n\leq N}\frac{1}{|S_n|}\sum_{k\in S_n} f(\tau^k x) \leq \alpha \sup_{\substack{n=2^i\\ i\leq \log_2 N}}\frac{1}{|S_n|}\sum_{k\in S_n} f(\tau^k x). \tag{3.5}$$

If we denote $\frac{\ln N}{\sum_{n\leq N}\sigma_n}$ by x_n then obviously $\lim_{n\to\infty} x_n = 0$. As a consequence of (3.3), (3.4), (3.5) we easily get

$$\begin{aligned}
\Big\|\sup_{n\leq N}\frac{1}{|S_n|}\sum_{k\in S_n}\tau^k f\Big\|_p &\leq \alpha\Big\|\sup_n \frac{1}{\sum_{k\leq n}\sigma_k}\sum_{k\leq n}\sigma_k\tau^k f\Big\|_p \\
&\quad + \alpha\Big\{\sum_{\substack{n=2^i\\ i\leq\log_2 N}}\Big\|\frac{1}{|S_n|}\sum_{k\in S_n}\tau^k f - \frac{1}{\sum_{k\leq n}\sigma_k}\sum_{k\leq n}\sigma_k\tau^k f\Big\|_p^p\Big\}^{1/p} \\
&\leq \alpha c_p\|f\|_p + \alpha\Big\{\sum_{\substack{n=2^i\\ i\leq\log_2 N}}(x_n)^{pa(p)}\Big\}^{1/p}\|f\|_p \\
&= o\big((\ln N)^{1/p}\big)\,\|f\|_p. \qquad \blacksquare
\end{aligned}$$

Remark 3.12 It is known (see for example the proof of Theorem B from [14]) that if $\sigma_n = 1/n$, then almost every random sequence $A(\omega)$ contains a lacunary subsequence $B(\omega)$ which has positive density in $A(\omega)$. One can similarly see that if $\lim_{n\to\infty} n\sigma_n = \infty$, then almost every random sequence $A(\omega)$ contains a subsequence $B(\omega)$ which has positive density in $A(\omega)$ and has some sublacunary rate of growth, depending on σ_n. In any case, the rate of growth of $C_{N,p}$ is the same for the sequence and its subsequence having positive relative density, which justifies the conclusion of Remark 3.10.

Remark 3.13 The sequences from Theorem 3.11 are known to be almost surely L^p-bad if $n\sigma_n$ grows sufficiently slowly [14] and almost surely L^p-good if $n\sigma_n$ grows sufficiently fast [8].

4 The Relation Between Different Types of Bad Behavior in Ergodic Theory

In the previous sections we discussed about two types of extreme in the bad behavior of the ergodic averages along subsequences. One was measured by the strongly sweeping out property, the other one was characterized by the maximal growth of the sequence $(C_{N,p})_{N=1}^{\infty}$. In the following we will show that these properties are distinct, in the sense that there are sequences which satisfy one condition but fail to satisfy the other one. The relation between the asymptotic behavior of the sequences $(C_{N,p})_{N=1}^{\infty}$ for different values of p is also discussed. The first result is quite surprising: it shows that there are strongly sweeping out sequences with arbitrarily small rate of growth of the constants $C_{N,p}$.

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Proposition 4.1 *Given $1 \le p < \infty$ and a nondecreasing function $\varphi\colon \mathbb{Z}_+ \to \mathbb{R}$ with $\lim_{n\to\infty} \varphi(n) = \infty$, there exists an increasing sequence of positive integers (b_k) such that*

(i) *(b_k) is strongly sweeping out;*
(ii) *the associated constants satisfy $C_{N,p} \le \varphi(N)$, for each N large enough.*

Proof Take $c_k = [(k-1)\varphi^{1/p}(k-1)]$ for k large enough, so that $\varphi(k-1)$ is larger than 1. If we denote $\psi(k) = c_{k+1} - c_k - 1$, then obviously $\psi(k) \ge 0$ and $\lim_{k\to\infty} \psi(k) = \infty$. By [15, Corollary 2.4], there exists $0 \le \delta_k \le \psi(k)$ such that the sequence defined by $b_k = c_k + \delta_k$ is strongly sweeping out. Note that $c_k \le b_k < c_{k+1}$. Now for each dynamical system (X, Σ, m, τ), $f \in L^p(X)$, $\lambda > 0$ and $N \in \mathbb{Z}_+$ we have

$$
\begin{aligned}
m\Big\{ x \in X : \max_{n\le N} \Big| \frac{1}{n} \sum_{k=1}^{n} f(\tau^{b_k} x) \Big| > \lambda \Big\}
&\le m\Big\{ x \in X : \max_{n\le N} \frac{1}{b_n} \sum_{k=1}^{n} |f|(\tau^{b_k} x) > \lambda \min_{n\le N} \frac{n}{b_n} \Big\} \\
&\le m\Big\{ x \in X : \max_{n\le N} \frac{1}{b_n} \sum_{k=1}^{b_n} |f|(\tau^{k} x) > \lambda \min_{n\le N} \frac{n}{b_n} \Big\} \\
&\le m\Big\{ x \in X : \max_{m\in\mathbb{Z}_+} \frac{1}{m} \sum_{k=1}^{m} |f|(\tau^{k} x) > \lambda \min_{n\le N} \frac{n}{b_n} \Big\} \\
&\le \frac{\|f\|_p^p}{\lambda^p} \max_{n\le N} \Big(\frac{b_n}{n} \Big)^p < \frac{\|f\|_p{}^p}{\lambda^p} \max_{n\le N} \Big(\frac{c_{n+1}}{n} \Big)^p \\
&= \frac{\|f\|_p^p}{\lambda^p} \max_{n\le N} \Big(\frac{[n\varphi^{1/p}(n)]}{n} \Big)^p \le \frac{\|f\|_p^p}{\lambda^p} \varphi(N). \qquad \blacksquare
\end{aligned}
$$

We have the following immediate corollary.

Corollary 4.2 *There exists a strongly sweeping out sequence with $C_{N,p} = o(\ln N)$.*

The converse is also true, but we postpone the proof until after a preliminary discussion.

Proposition 4.3 *There exists a sequence (c_k) with $C_{N,p} \asymp \ln N$, which is not strongly sweeping out.*

The result of Jones and Wierdl [15] used in the proof of Proposition 4.1, shows that strongly sweeping out sequences can grow arbitrarily slowly. This is not the case for sequences with $C_{N,p} \asymp \ln N$, as Proposition 4.5 will show. We first prove the following lemma.

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Lemma 4.4 *If the sequence (a_k) has $C_{N,p} \asymp \ln N$ for some $1 \leq p < \infty$, then for each sequence of positive real numbers (u_k) with $\lim_{k\to\infty} u_k = \infty$ and each ergodic dynamical system (X, Σ, m, τ), there exists $f \in L^p(X)$ for which*

$$V_n f(x) = \frac{u_n}{n(\ln n)^{1/p}} \sum_{k=1}^{n} f(\tau^{a_k} x)$$

fails to converge almost everywhere.

Proof Based on Sawyer's principle, it is enough to prove the failure of the maximal inequality for $V^\star f(x) = \sup_{n\in\mathbb{Z}_+} |V_n f(x)|$. Take C positive such that $C_{N,p} > \frac{\ln N}{C}$, for all $N \in \mathbb{Z}_+$. Choose $\gamma = \gamma_N > 0$ and $g = g_N \in L^p(X)$ such that

$$m\Big\{ x \in X : \max_{n\leq N} \Big| \frac{1}{n} \sum_{k=1}^{n} g(\tau^{a_k} x) \Big| > \gamma \Big\} \geq \frac{\|g\|_p^p \ln N}{C\gamma^p}.$$

Now

$$\begin{aligned} &m\Big\{ x \in X : \max_{n\leq N} \Big| \frac{1}{n} \sum_{k=1}^{n} g(\tau^{a_k} x) \Big| > \gamma \Big\} \\ &\quad\leq m\Big\{ x \in X : \max_{n\leq N} \Big| \frac{u_n}{n(\ln n)^{1/p}} \sum_{k=1}^{n} g(\tau^{a_k} x) \Big| > \gamma \min_{n\leq N} \frac{u_n}{(\ln n)^{1/p}} \Big\} \\ &\quad\leq \frac{D\|g\|_p^p \max_{n\leq N} \frac{\ln n}{(u_n)^p}}{\gamma^p} \end{aligned}$$

where D is the best constant in the inequality

$$m\{x \in X : V^\star f(x) > \lambda\} \leq D \frac{\|f\|_p^p}{\lambda^p}.$$

Hence $D \geq \frac{1}{C}(\ln N) \min_{n\leq N} \frac{(u_n)^p}{\ln n}$ for each N. An easy argument shows that $\lim_{N\to\infty} (\ln N) \min_{n\leq N} \frac{(u_n)^p}{\ln n} = \infty$ which proves that $D = \infty$. ∎

We can now easily prove the following.

Proposition 4.5 *A sequence (a_k) for which $C_{N,p} \asymp \ln N$ cannot satisfy $a_k = o(k \ln k)$.*

Proof Take an arbitrary dynamical system (X, Σ, m, τ). Suppose by contradiction that $a_k = o(k \ln k)$ and define $u_k = (\frac{k \ln k}{a_k})^{1/p}$. Then, using the notation from

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Lemma 4.4 and Hölder's inequality, we get for each positive λ and each $f \in L^p(X)$

$$m\{x \in X : V^\star f(x) > \lambda\} \le m\Big\{x \in X : \sup_n \frac{(u_n)^p}{n \ln n} \sum_{k=1}^{n} |f|^p(\tau^{a_k} x) > \lambda^p\Big\}$$
$$\le m\Big\{x \in X : \sup_n \frac{1}{a_n} \sum_{k=1}^{a_n} |f|^p(\tau^k x) > \lambda^p\Big\}$$
$$\le \frac{\|f\|_p^p}{\lambda^p}.$$

On the other hand, since $\lim_{k\to\infty} \frac{u_k}{k(\ln k)^{1/p}} = 0$, it follows easily that $V_n f(x)$ converges to 0 almost everywhere for L^∞ functions. As usual, almost everywhere convergence on a dense class together with the maximal inequality gives almost everywhere convergence for all L^p functions. But this contradicts Lemma 4.4, since $\lim_{k\to\infty} u_k = \infty$. ■

The result of Proposition 4.5 turns out to be optimal in the following sense:

Proposition 4.6 *There exists a sequence $(a_k) \asymp k \ln k$ with $C_{N,p} \asymp \ln N$ for each $1 \le p < \infty$.*

Proof The sequence will be constructed by putting together sets H_k, $k \ge 1$, each H_k being a finite union of sets: $H_k = \bigcup_{i=0}^{2^{k-1}-1} H_{k,i}$. We will start the inductive construction of the H_k's by first taking $H_1 = \{2, 3, 4, 5\}$. Assume we have constructed H_1 through H_{k-1}. In order to construct H_k we need to define its components $H_{k,i}$. The set $H_{k,0}$ will be placed at the right of the set H_{k-1} and will consist of the first $2^{2^{k-1}}$ such integers congruent to $0 \pmod{2^{k-1}}$. After the completion of the set $H_{k,i}$, the set $H_{k,i+1}$ will be placed at the right of $H_{k,i}$ and will consist of the first $2^{2^{k-1}+i}$ such integers congruent to $i \pmod{2^{k-1}}$. The construction of H_k will end when i reaches the value $2^{k-1} - 1$. By this procedure we get a sequence that will be denoted by (a_k). Note that $|H_k| = 2^{2^k} - 2^{2^{k-1}}$ for each $k \ge 2$, hence $L_N = |\bigcup_{k=1}^N H_k| = 2^{2^N}$ for each $N \ge 1$.

We first prove that the constant $C_{N,p}$ corresponding to (a_k) satisfies $C_{N,p} \asymp \ln N$. It will suffice to show that $C_{L_N,p} \gg 2^N$. In order to underestimate the value of $C_{L_N,p}$ we use only the information from the set H_N. We will work with $X = [0, 1)$, $A = A_N = [0, \frac{1}{2^{N-1}})$, $\theta = \frac{1}{2^{N-1}}$ and $\tau(x) = \{x + \theta\}$. Obviously for each element $h \in H_{N,i}$, $h\theta = \frac{i}{2^{N-1}} \pmod{2^{N-1}}$, hence if $x \in [\frac{2^{N-1}-i}{2^{N-1}}, \frac{2^{N-1}-i+1}{2^{N-1}}]$ we get that $\tau^h x \in A$. If we define

$$E_{N,i} = \bigcup_{k=1}^{N-1} H_k \cup \bigcup_{j=0}^{i} H_{N,j}$$

then it's easy to see that $|E_{N,i}| = 2\,|H_{N,i}|$. Consequently, we have for each x in $[\frac{2^{N-1}-i}{2^{N-1}}, \frac{2^{N-1}-i+1}{2^{N-1}}]$,

$$\frac{1}{|E_{N,i}|}\sum_{k=1}^{|E_{N,i}|}\chi_A(\tau^{a_k}x) \geq \frac{1}{2|H_{N,i}|}\sum_{a_k\in H_{N,i}}\chi_A(\tau^{a_k}x) = \frac{1}{2}.$$

This together with the fact that $|E_{N,i}| \leq L_N$ leads to

$$m\Big\{x \in [0,1) : \max_{n\leq L_N}\frac{1}{n}\sum_{k=1}^{n}\chi_A(\tau^{a_k}x) \geq \frac{1}{2}\Big\} = 1.$$

From here we easily get

$$C_{L_N,p} \geq \frac{1}{2^p m(A)} = \frac{2^N}{2^{p+1}}$$

which finishes the first part of the proof.

To see that $a_k \ll k\ln k$, note first that by construction

$$\sum_{k=1}^{N}(|H_k| - 2^{k-1})2^{k-1} \leq a_{L_N} \leq \sum_{k=1}^{N}|H_k|2^{k-1}. \tag{4.1}$$

Using the fact that

$$\sum_{k=1}^{N}|H_k|\,2^{k-1} = \sum_{k=1}^{N}(2^{2^k} - 2^{2^{k-1}})2^{k-1} + 2$$

we easily get

$$(2^{2^N} - 2^{2^{N-1}})2^{N-1} \leq \sum_{k=1}^{N}|H_k|\,2^{k-1} \leq 2^{2^N}2^{N-1}. \tag{4.2}$$

By combining (4.1) with (4.2) we find that

$$\lim_{N\to\infty}\frac{a_{L_N}}{L_N\log_2 L_N} = \frac{1}{2}. \tag{4.3}$$

For each $1 \leq l \leq L_{N+1} - L_N$, a_{L_N+l} is in the block H_{N+1}, hence

$$\begin{aligned} a_{L_N+l} &\leq a_{L_N} + 2^N l \\ &\leq L_N\log_2 L_N + 2^N l \text{ (for } N \text{ large enough, from (4.3))} \\ &= (L_N + l)\log_2 L_N \leq (L_N + l)\log_2(L_N + l). \end{aligned}$$

This together with (4.3) immediately gives that $a_k \ll k\ln k$. By using the same kind of argument one can also prove that $a_k \gg k\ln k$. ■

We can use this construction to prove Proposition 4.3.

Proof Let $A = (a_k)$ be the sequence constructed in Proposition 4.6. For each $N \in \mathbb{Z}_+$ let $A_N = |\{k : a_k \le N\}|$. By using standard arguments, one can easily deduce that

$$A_N \asymp \frac{N}{\ln N}. \tag{4.4}$$

Our next goal is to construct another sequence (b_k), having the same asymptotic density as (a_k), its support disjoint from the support of (a_k) and which is L^∞-good. The sequence (c_k) resulting from gluing together (a_k) and (b_k) will have the desired properties. A short analysis of the proof of Proposition 8.2 from [8], shows that the result remains true if we choose $\sigma_n = \frac{1}{\ln n}$, for $n \ge 2$. According to this proposition, the random sequence $D = (d_k)$ obtained by including each positive integer n in the set D with probability $\sigma_n = \frac{1}{\ln n}$ is almost surely L^p-good for every $p > 1$, in particular for $p = \infty$. By the law of large numbers

$$\lim_{N\to\infty} \frac{D_N}{\sum_{n=2}^N \sigma_n} = 1$$

for almost every ω in the induced probability space Ω. Here $D_N = D_N(\omega) = |\{k : d_k \le N\}|$. Since

$$\lim_{N\to\infty} \frac{\ln N}{N} \sum_{n=2}^{N} \frac{1}{\ln n} = 1 \tag{4.5}$$

it follows that

$$\lim_{N\to\infty} \frac{D_N \ln N}{N} = 1 \quad \text{a.e. } \omega. \tag{4.6}$$

We need to modify the sequence (d_k) in order to make it disjoint from (a_k). For each ω, let $B = (b_k)$ be the sequence obtained by removing all the terms of (d_k) which lie in A. If as before $B_N = B_N(\omega) = |\{k : b_k \le N\}|$, then by the law of large numbers

$$\lim_{N\to\infty} \frac{1}{B_N} \sum_{\substack{n=2 \\ n\notin A}}^{N} \frac{1}{\ln n} = 1 \quad \text{a.e. } \omega.$$

Since

$$\sum_{n=2}^{N} \frac{1}{\ln n} \ge \sum_{\substack{n=2 \\ n\notin A}}^{N} \frac{1}{\ln n} \ge \sum_{n=2}^{N} \frac{1}{\ln n} - \sum_{n=2}^{A_N} \frac{1}{\ln n}$$

using (4.4) and (4.5) we obtain

$$\lim_{N\to\infty} \frac{B_N \ln N}{N} = 1 \quad \text{a.e. } \omega. \tag{4.7}$$

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We start now the deterministic part of the argument. Pick an ω for which both (4.6) and (4.7) hold. Moreover, we can make our choice such that the corresponding $(d_k) = (d_k(\omega))$ is L^∞-good. The claim is that the sequence $(b_k(\omega))$, which for the future will be denoted by (b_k), is L^∞-good, too. To see this, note first that for each dynamical system (X, Σ, m, τ), $f \in L^\infty(X)$ and positive integer N we have

$$\frac{1}{N}\sum_{k=1}^{N} f(\tau^{d_k}x) - \frac{D_N - B_N}{N}\|f\|_\infty \le \frac{1}{N}\sum_{k=1}^{N} f(\tau^{b_k}x) \le \frac{1}{N}\sum_{k=1}^{N} f(\tau^{d_k}x) + \frac{D_N - B_N}{N}\|f\|_\infty$$

then use (4.6) and (4.7).

Let now $C = \{a_k : k \in \mathbb{Z}_+\} \cup \{b_k : k \in \mathbb{Z}_+\}$ and enumerate C as an increasing sequence (c_k). We will show in the following that $C_{N,p} \asymp \ln n$ for the sequence (c_k). Note first that for each positive function f and each positive integer n

$$\begin{aligned}\frac{1}{n}\sum_{k=1}^{n} f(\tau^{c_k}x) &= \frac{1}{n_1}\sum_{k=1}^{n_1} f(\tau^{a_k}x)\frac{n_1}{n} + \frac{1}{n_2}\sum_{k=1}^{n_2} f(\tau^{b_k}x)\frac{n_2}{n} \\ &\ge \frac{1}{n_1}\sum_{k=1}^{n_1} f(\tau^{a_k}x)\frac{n_1}{n}\end{aligned} \tag{4.8}$$

where $n_1 = |\{k : a_k \le c_n\}|$, $n_2 = |\{k : b_k \le c_n\}|$. From (4.4) and (4.7) we can assume the existence of positive constants γ, δ such that for each $n \in \mathbb{Z}_+$

$$\gamma \le \frac{n_1}{n_2} \le \delta. \tag{4.9}$$

Since $n = n_1 + n_2$ it follows that

$$\frac{\delta}{\delta+1} \ge \frac{n_1}{n} \ge \frac{\gamma}{\gamma+1}. \tag{4.10}$$

Using (4.8) and (4.9) we have for each positive integer N and each positive f

$$m\Big\{x \in X : \max_{n \le N} \frac{1}{n}\sum_{k=1}^{n} f(\tau^{c_k}x) > \lambda\Big\} \ge m\Big\{x \in X : \max_{n_1 \le \frac{\gamma}{\gamma+1}N} \frac{1}{n_1}\sum_{k=1}^{n_1} f(\tau^{a_k}x) > \lambda\frac{\gamma+1}{\gamma}\Big\}.$$

Now, the fact that $C_{N,p} \asymp \ln N$ for (c_k) follows immediately from the analogous result for (a_k).

To prove that (c_k) is not strongly sweeping out we proceed as in [20]. Note first that for each $E \in \Sigma$, the following limit exists almost everywhere

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_E(\tau^{b_k} x) = f^\star(x).$$

Moreover, by the dominated convergence theorem, $\int_X f^\star(x)\, dm(x) = m(E)$. From (4.10) we can now write

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_E(\tau^{c_k} x) &= \limsup_{n\to\infty} \Big(\frac{n_1}{n} \frac{1}{n_1} \sum_{k=1}^{n_1} \chi_E(\tau^{a_k} x) + \frac{n_2}{n} \frac{1}{n_2} \sum_{k=1}^{n_2} \chi_E(\tau^{b_k} x) \Big) \\ &\le \frac{\delta}{\delta+1} + \frac{1}{\gamma+1} f^\star(x) \quad \text{a.e.}\end{aligned}$$

By integration we find

$$\int_X \limsup_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_E(\tau^{c_k} x)\, dm(x) \le \frac{\delta}{\delta+1} + \frac{1}{\gamma+1} m(E).$$

Since the measure of E can be made arbitrarily small, the sequence (c_k) cannot be strongly sweeping out. ■

We close this section with an analysis of the relation between bad behavior of sequences with respect to different L_p spaces.

Theorem 4.7 *If a given sequence (a_k) has $C_{N,p} \asymp \ln N$ for some $1 < p < \infty$, then $C_{N,p'} \asymp \ln N$ for all $1 \le p' < \infty$.*

Proof Assume $C_{N,p} \asymp \ln N$. Take arbitrary $1 \le p' < p < p'' < \infty$ and choose $\theta \in (0,1)$ such that

$$\frac{1}{p} = \frac{\theta}{p'} + \frac{1-\theta}{p''}.$$

For each positive integer N, consider the operator T_N defined for each $f \in L^1(X)$ by $T_N f(x) = f_N^\star(x)$, where $f_N^\star$ was introduced in (2.1). Obviously, for each $1 \le q < \infty$, $(C_{N,q})^{1/q}$ is nothing else than the weak (q,q) norm of the operator $T_N \colon L_q \to L_q^\star$, where $L_q^\star$ is the weak L^q space. The strong (q,q) norm of T_N defined as $\sup_{\|f\|_q \le 1} \|T_N f\|_q$ is always at least as large as the weak (q,q) norm. Using this and Marcinkiewicz interpolation theorem [6] applied to T_N, we deduce the existence of a constant ρ independent of N such that

$$(C_{N,p})^{1/p} \le \rho (C_{N,p'})^{\theta/p'} (C_{N,p''})^{1-\theta/p''}.$$

This can be rewritten as

$$\Big(\frac{C_{N,p}}{\ln N}\Big)^{1/p} \le \rho \Big(\frac{C_{N,p'}}{\ln N}\Big)^{\theta/p'} \Big(\frac{C_{N,p''}}{\ln N}\Big)^{1-\theta/p''}.$$

Since by Theorem 2.1 $\frac{C_{N,p'}}{\ln N} \ll 1$ and $\frac{C_{N,p''}}{\ln N} \ll 1$, it follows immediately that $C_{N,p'} \asymp \ln N$ and $C_{N,p''} \asymp \ln N$. ■

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Remark 4.8 From Theorem 4.7 we cannot conclude anything about the L^∞ behavior of the sequence (a_k). Interpolation is also not useful when we try to get information about the growth of the $C_{N,p}$'s for $p > 1$, knowing that $C_{N,1} \asymp \ln N$. These issues are interesting in the light of the constructions from [19], [5] of sequences which are L^p-good for values of p larger then p_0 and L^p-bad for values of p smaller then p_0, where $1 \le p_0 \le \infty$.

5 Divergence of Weighted Sums

Let T be a positively dominated contraction of $L^p(X)$, $1 < p < \infty$ and let (x_k) be an r-Besicovitch sequence, where $r > q$ and $\frac{1}{p} + \frac{1}{q} = 1$. Baxter and Olsen proved in [3] that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} x_k T^k f(x)$$

exists a.e. for all $f \in L^p(X)$. On the other hand if T is a Dunford-Schwartz operator, the result holds for $1 \le p \le \infty$ and for $r \ge q$ (see [16]). It was not clear in these papers whether the restriction $r \ge q$ was really necessary or just an artifact of the proof. The following theorem constructs a counterexample, hence showing that duality cannot be broken.

Theorem 5.1 *Let (X, Σ, m, τ) be an ergodic dynamical system. For each $1 \le p < \infty$ and each $r < q$, where q is the conjugate exponent of p, there exist a function $f \in L_p(X)$ and a sequence (x_k) of positive numbers with the following two properties:*

(i) $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} x_k^r = 0$ *(hence (x_k) is r-Besicovitch).*
(ii) *The averages $\frac{1}{n} \sum_{k=1}^{n} x_k f(\tau^k x)$ fail to converge almost everywhere.*

The proof will be based on the following version of Lemma 1 from [23], which relates weighted averages like those from Theorem 5.1 with the weighted sums of Lemma 4.4. We include the proof for completeness.

Lemma 5.2 *Take (a_k) an increasing sequence of integers, (t_k) an increasing sequence of positive reals, (d_k) and (g_k) sequences of positive real numbers. Define $D_n = \sum_{k=1}^{n} d_k$ and $G_n = \sum_{k=1}^{n} g_k$. Assume the following are true:*

(i) $\sup_n \frac{g_n D_n}{d_n G_n} = c < \infty$.
(ii) $\frac{g_n}{d_n}$ *is nonincreasing.*
(iii) $\lim_{n\to\infty} G_n = \infty$

If $\frac{1}{D_n t_n} \sum_{k=1}^{n} d_k f(\tau^{a_k} x) \to 0$ almost everywhere then $\frac{1}{G_n t_n} \sum_{k=1}^{n} g_k f(\tau^{a_k} x) \to 0$ almost everywhere.

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Proof The proof will follow closely the lines of Wierdl's argument. Define $B_n f(x) = \sum_{k=1}^n d_k f(\tau^{a_k} x)$, while $B_0 = 0$. Then

$$\begin{aligned}\frac{1}{G_n t_n}\sum_{k=1}^n g_k f(\tau^{a_k} x) &= \frac{1}{G_n t_n}\sum_{k=1}^n \frac{g_k}{d_k} d_k f(\tau^{a_k} x) = \frac{1}{G_n t_n}\sum_{k=1}^n \frac{g_k}{d_k}(B_k - B_{k-1})\\ &= \frac{1}{G_n t_n}\sum_{k=1}^{n-1} B_k\Big(\frac{g_k}{d_k} - \frac{g_{k+1}}{d_{k+1}}\Big) + \frac{g_n B_n}{G_n d_n t_n}\\ &= \frac{1}{G_n t_n}\sum_{k=1}^{n-1} \frac{B_k}{t_k D_k} t_k D_k\Big(\frac{g_k}{d_k} - \frac{g_{k+1}}{d_{k+1}}\Big) + \frac{B_n}{D_n t_n}\frac{g_n D_n}{G_n d_n} = (\star).\end{aligned}$$

We write for $1 < m < n$

$$\begin{aligned}(*) &= \frac{1}{G_n t_n}\sum_{k=1}^{m-1} + \frac{1}{G_n t_n}\sum_{k=m}^{n-1} + \frac{B_n}{D_n t_n}\frac{g_n D_n}{G_n d_n}\\ &= \quad (1) \quad + \quad (2) \quad + \quad (3).\end{aligned}$$

Choose m so large that $\frac{B_k}{t_k D_k} < \epsilon$ for each $k \geq m$, then let n_1 be so large that if $n > n_1$ then $(1) < \epsilon$. For $n > n_1$ we get

$$\begin{aligned}|(1)+(2)+(3)| &< \epsilon + \epsilon\frac{1}{G_n t_n}\sum_{k=m}^{n-1} t_k D_k\Big(\frac{g_k}{d_k} - \frac{g_{k+1}}{d_{k+1}}\Big) + c\epsilon\\ &\leq \epsilon + \epsilon\frac{1}{G_n}\sum_{k=1}^{n-1}\Big(\frac{g_k}{d_k} - \frac{g_{k+1}}{d_{k+1}}\Big)D_k + c\epsilon\\ &= (c+1)\epsilon + \epsilon\frac{1}{G_n}\sum_{k=1}^{n}\frac{g_k}{d_k}(D_k - D_{k-1}) - \epsilon\frac{D_n g_n}{d_n G_n}\\ &< (c+1)\epsilon + \frac{\epsilon}{G_n}\sum_{k=1}^n g_k\\ &= (c+2)\epsilon\end{aligned}$$

■

Proof of Theorem 5.1 We denote by (a_k) the sequence constructed in Proposition 4.6. Take $r < t < q$ and define the sequence (x_k) as having value $d_i = (\ln i)^{1/t}$ if $k = a_i$ and 0 otherwise. Note that (x_k) satisfies condition (i) from Theorem 5.1. Define also

$$u_k = \frac{\sum_{i=1}^k d_i}{k(\ln k)^{1/q}} \quad \text{and} \quad t_k = \frac{k \ln k}{\sum_{i=1}^k d_i}.$$

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It will suffice to show that $M_n f(x) = \frac{1}{n \ln n} \sum_{k=1}^{n} f(\tau^{a_k}x)d_k$ fails to converge almost everywhere for some $f \in L^p(X)$. Since $u_k \to \infty$, we know from Lemma 4.4 that there is an $f_0 \in L^p(X)$ for which $V_n f_0(x)$ diverges on a positive measure set. Note that

$$M_n f_0(x) = \frac{1}{t_n \sum_{k=1}^{n} d_k} \sum_{k=1}^{n} f_0(\tau^{a_k}x)d_k.$$

If by contradiction $M_n f_0(x)$ converged almost everywhere, then an easy approximation argument using L^∞ functions, shows that the limit would have to be 0 almost everywhere. But then by Lemma 5.2 applied to $f = f_0$, $a_k = a_k$, $d_k = d_k$, $t_k = t_k$ and $g_k = 1$,

$$V_n f_0(x) = \frac{1}{nt_n} \sum_{k=1}^{n} f_0(\tau^{a_k}x) = \frac{u_n}{n(\ln n)^{1/p}} \sum_{k=1}^{n} f_0(\tau^{a_k}x)$$

would also have to converge to 0 almost everywhere. The contradiction is now obvious. ■

Remark 5.3 Theorem 5.1 is interesting because of its connections with the return times theorem. This latter result was proved in [11] for conjugate exponents, but it is still unknown whether duality can be broken in this case.

We close this section with an application of the techniques developed in Section 3. In [2] it is proved that the weighted sums $\frac{1}{L_n} \sum_{k=1}^{n} f(\tau^{a_k}x)$ may diverge for some $f \in L^1(X)$ and some sequence (a_k), whenever L_k is an expression in which the logarithmic form is expanded, like $L_k = \ln k$, $L_k = \ln(k) \ln \ln(k)$, *etc.* In the end of [2] it is asked whether we can choose $a_k = 2^k$ in this theorem and still get a negative result. The answer is yes, as provided by the following:

Theorem 5.4 *Let (L_k) be any of the sequences we mentioned above. For each ergodic dynamical system (X, Σ, m, τ), there exists $f \in L^1(X)$ with the property that the averages*

$$\frac{1}{L_n} \sum_{k=1}^{n} f(\tau^{2^k}x)$$

fail to converge almost everywhere.

Proof All the elements of the proof are contained in the proof of Proposition 3.13 from [2] and in Section 3. The argument is quite computational so we omit it. ■

Remark 5.5 If in the above theorem we choose (L_k) to be a sequence which is $o(k \ln k)$, then we get a slightly weaker result, which on the other hand has an immediate proof, based on Lemma 4.4 and Theorem 3.8.

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