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Weyl Images of Kantor Pairs

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Abstract. Kantor pairs arise naturally in the study of 5-graded Lie algebras. In this article, we introduce and study Kantor pairs with short Peirce gradings and relate them to Lie algebras graded by the root system of type BC_2. This relationship allows us to define so-called Weyl images of short Peirce graded Kantor pairs. We use Weyl images to construct new examples of Kantor pairs, including a class of infinite dimensional central simple Kantor pairs over a field of characteristic $\neq 2$ or 3, as well as a family of forms of a split Kantor pair of type E_6.

1 Introduction

Assume for simplicity in this introduction that $\mathbb{K}$ is a ring of scalars containing $\frac{1}{6}$ (although we will relax this assumption in a few parts of the paper). A *Kantor pair* over $\mathbb{K}$ is a pair $P = (P^-, P^+)$ of $\mathbb{K}$-modules together with two trilinear products $\{\cdot,\cdot,\cdot\}^\sigma : P^\sigma \times P^{-\sigma} \times P^\sigma \to P^\sigma$, $\sigma = \pm$, satisfying two 5-linear identities (K1) and (K2) (see Subsection 4.1 or [AF1]). These structures arise naturally in the study of 5-*graded Lie algebras*, by which we mean $\mathbb{Z}$-graded Lie algebras of the form $L = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$. Indeed, if L is a 5-graded Lie algebra, then the pair (L_{-1}, L_1) has the structure of a Kantor pair, called *the Kantor pair enveloped by* L, where the two products are restrictions of the product $[[x, y], z]$ on L. Conversely, given a Kantor pair P, there exists a 5-graded Lie algebra (not generally unique) that envelops P. This relationship between 5-graded Lie algebras and Kantor pairs, which we view schematically as

$$\text{5-graded Lie algebras} \rightsquigarrow \text{Kantor pairs}, \tag{1.1}$$

is an important tool in the study of each of these structures, and it generalizes the well-known relationship between 3-graded Lie algebras and Jordan pairs [N2, §1.5].

To describe some background, we note that in his foundational paper [K1], Isai Kantor studied a class of triple systems that we call *Kantor triple systems*. He developed the relationship of Kantor triple systems with 5-graded Lie algebras that possess grade-reversing period 2 automorphisms. He used this relationship to obtain a classification of finite dimensional non-polarized (see Subsection 3.1) simple Kantor triple systems over an algebraically closed field of characteristic 0.

Kantor triple systems constitute one of the largest classes of nonassociative objects for which such a classification result has been obtained. The class includes Jordan triple systems as well as triple systems constructed from associative algebras, alternative algebras, Jordan algebras and many other interesting exceptional objects.

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Given a Kantor triple system, one can, as in the Jordan case, construct a Kantor pair by doubling (see Subsection 3.1), but not every Kantor pair arises in this way. So in this sense Kantor pairs are generalizations of Kantor triple systems. Moreover, pairs are more natural objects to consider from the viewpoint of graded Lie algebras, since 5-graded Lie algebras need not possess grade-reversing period 2 automorphisms (see Remark 4.9 (iii)).

Kantor pairs also arise using signed doubling of some analogs of Kantor triple systems called $(1,1)$-Freudenthal–Kantor triple systems, including Freudenthal triple systems (with a suitably modified product). (See Example 4.31.) Freudenthal triple systems have been studied mathematically by many authors and have appeared recently in several important physical models (see [G, BDDRE, MQSTZ], and the references therein).

In this paper, we introduce short Peirce gradings (SP-gradings) of Kantor pairs. We describe a relationship between Lie algebras graded by the root system Δ of type BC_2 and SP-graded Kantor pairs. Using this relationship we define a Weyl image ${}^uP$ of P for each SP-graded Kantor pair P and each element u of the Weyl group of Δ. We develop the properties of Weyl images and use them to construct new examples of Kantor pairs.

Although we obtain our results and examples for the most part without assuming that $\mathbb{K}$ is a field, this article is the beginning of an investigation of central simple Kantor pairs over a field, so we have particular interest in that case. Our results here will be used in a work in progress by the first and third authors that will contain a structure theorem for central simple Kantor pairs over a field of characteristic $\neq 2$, 3 or 5. The theorem asserts that these pairs occur in four classes: a class of Jordan pairs, the class of (finite dimensional) forms of split Kantor pairs of exceptional type (see Subsection 4.7), a new class of Kantor pairs constructed from hermitian forms, and a new class that we introduce in Section 7 using Weyl images.

We conclude this introduction by briefly outlining the contents of the paper. After some preliminaries on root graded Lie algebras in Section 2 and trilinear pairs in Section 3, we recall or prove some basic properties of Kantor pairs in Section 4. One such property is that the relationship (1.1) restricts to a one-to-one correspondence between central simple 5-graded Lie algebras up to graded isomorphism and central simple Kantor pairs up to isomorphism.

In Section 5, we introduce SP-graded Kantor pairs and BC_2-graded Lie algebras. An *SP-grading* of a Kantor pair P is a $\mathbb{Z}$-grading of P whose support is contained in $\{0,1\}$; whereas a BC_2*-graded Lie algebra* is a Lie algebra graded by the root lattice of the root system Δ of type BC_2 with support contained in $\Delta \cup \{0\}$. (The latter definition is convenient for us, but not standard. See Subsection 2.2.) We establish the relationship mentioned above between BC_2-graded Lie algebras and SP-graded Kantor pairs, and deduce some of its properties. (It can be viewed as the rank two version of (1.1), since 5-graded Lie algebras are precisely the same as BC_1-graded Lie algebras.)

In Section 6, we define and study Weyl images of SP-graded Kantor pairs. To describe these briefly, let P be an SP-graded Kantor pair and let u be an element of the Weyl group W_Δ of the root system Δ of type BC_2. Then P is enveloped by a BC_2-graded Lie algebra L, and we use u to adjust the grading of L in an evident fashion to obtain a BC_2-graded Lie algebra ${}^uL$, which in turn envelops a Kantor pair ${}^uP$,

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called a *Weyl image* of P. This gives us a well-defined action of the group W_Δ on the class of SP-graded Kantor pairs, and one sees that Weyl images of central simple SP-graded Kantor pairs are central simple. A particularly interesting case occurs when u is the reflection corresponding to the short basic root, in which case we denote ${}^uP$ by $\check{P}$ and call it simply the *reflection* of P.

The reader may initially suspect that the reflection $\check{P}$ of an SP-graded Kantor pair P is just P with a different SP-grading. However, it turns out that P and $\check{P}$ are not in general isomorphic *even as ungraded pairs*. This suggests a strategy for giving new constructions of Kantor pairs: start with a Kantor pair P, choose an appropriate SP-grading of P, and form the reflection $\check{P}$ of P with that grading. In the last two sections we look in detail at two examples of this strategy where we obtain a pair $\check{P}$ with quite different properties than P.

First, in Section 7, we start with a nondegenerate bilinear form $g\colon V^- \times V^+ \to \mathbb{K}$. Let $\widetilde{g}$ be the symmetric bilinear form on $\widetilde{V} = V^- \oplus V^+$ that extends g and is zero on $V^\sigma \times V^\sigma$, $\sigma = \pm$ and let $\mathfrak{fo}(\widetilde{g})$ be a Lie algebra spanned by endomorphisms of $\widetilde{V}$ of the form $x \mapsto g(x,w)v - g(x,v)w$, where $u, v \in V$. If there exists $e = (e^-, e^+) \in V^- \times V^+$ such that $g(e^-, e^+) = 1$, then $\mathfrak{fo}(\widetilde{g})$ has a natural BC_2-grading, so it envelops an SP-graded Kantor pair $\mathrm{FSkew}(g)$. The pair $\mathrm{FSkew}(g)$ is known to be Jordan [LB], but its reflection $\mathrm{FSkew}(g)\check{}$ is not in general. Moreover, in the case when $\mathbb{K}$ is a field and $\dim(V^\sigma) \geq 3$, Kantor pairs of the form $\mathrm{FSkew}(g)\check{}$ make up the fourth class of central simple Kantor pairs appearing in the structure theorem mentioned above.

In Section 8, we use a non-singular bilinear form $g\colon M^- \times M^+ \to \mathbb{K}$, where each M^σ is a finitely generated projective module over $\mathbb{K}$ of rank 6. Following the approach in [F], we use the exterior algebras $\bigwedge(V^\sigma)$, $\sigma = \pm$, to construct a form $\mathcal{E} = \mathcal{E}(M^-, M^+, g)$ of the split Lie algebra of type E_6. In the case when $\mathbb{K} = \mathbb{C}$, this is a basis-free version, with full proofs, of the construction of the complex Lie algebra E_6 given by Élie Cartan [C, §V.18, pp.89–90]. If there exists $e = (e^-, e^+) \in M^- \times M^+$ such that $g(e^-, e^+) = 1$, then $\mathcal{E}$ has a natural BC_2-grading, which is strikingly similar to the grading of $\mathfrak{fo}(\widetilde{g})$ arising in Section 7. The SP-graded Kantor pair enveloped by $\mathcal{E}$ has the form $\bigwedge_3 = (\bigwedge_3(V^-), \bigwedge_3(V^+))$ with an easily remembered basis-free product and a natural SP-grading that we use to construct the reflection $\check{\bigwedge}_3$. As an example, we see that when $\mathbb{K}$ is a Dedekind domain, the set of isomorphism classes of Kantor pairs of the form $\bigwedge_3$ (resp. $\check{\bigwedge}_3$) is parameterized by the Picard group of $\mathbb{K}$. Suppose finally that $\mathbb{K}$ is a field, in which case $\bigwedge_3$ is central simple. Although the pair $\bigwedge_3$ is not Jordan, it is close to Jordan in a sense that we make precise in Subsection 4.8. In contrast, $\check{\bigwedge}_3$ is not close to Jordan and it turns out that it is a Kantor pair of particular interest in the theory of finite dimensional central simple Kantor pairs (see Remark 8.24). The pair $\check{\bigwedge}_3$ is the double of a Kantor triple system C^2_{55} originally constructed by Kantor using tensors a_{ijk} that are skew-symmetric with respect to i, j, where $i, j = 1, \dots, 5$, $k = 1, 2$ [K2, §4]. Our approach using reflection gives a simple new construction of this interesting Kantor pair.

1.1 Assumptions and Notations

Throughout the rest of the article, we assume that $\mathbb{K}$ is a unital commutative associative ring of scalars. In much of the article, we will also assume that $\frac{1}{6} \in \mathbb{K}$ (and clearly

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state this assumption). We do this because Kantor pairs have not even been defined without the assumption that $\frac{1}{6} \in \mathbb{K}$, and it is not yet clear what the definition should be. However, one place where we do not assume $\frac{1}{6} \in \mathbb{K}$ is in Section 8, where we think the Lie algebra constructions are of independent interest without restriction on $\mathbb{K}$.

We shall require a $\mathbb{K}$-module to be unital; *i.e.*, $1x = x$. Unless otherwise indicated, by a module (resp. an algebra) we will mean a module (resp. an algebra) over $\mathbb{K}$. If V and W are modules, we will often abbreviate $\mathrm{Hom}_{\mathbb{K}}(V, W)$ and $\mathrm{End}_{\mathbb{K}}(V)$ by $\mathrm{Hom}(V, W)$ and $\mathrm{End}(V)$, respectively. Then, as usual, $\mathrm{End}(V)$ is an associative algebra under composition and a Lie algebra under the commutator product (and it will always be clear which is being considered). If V is a module, we use the notation $V^* = \mathrm{Hom}(V, \mathbb{K})$ for the *dual module* of V. If $\mathbb{K}$ is a field, we often abbreviate $\dim_{\mathbb{K}}(V)$ by $\dim(V)$.

If V and W are modules and $g: V \times W \to \mathbb{K}$ is a bilinear form, we say that g is *nondegenerate* (resp. *non-singular*) if the maps $v \to g(v, \cdot)$ from V into W^* and $w \to g(\cdot, w)$ from W into V^* are injective (resp. bijective).

Recall that a module W is said to be *flat* (resp. *faithfully flat*) if for an exact sequence $V' \to V \to V''$ of modules to be exact it is necessary (resp. necessary and sufficient) that the induced sequence $W \otimes_{\mathbb{K}} V' \to W \otimes_{\mathbb{K}} V \to W \otimes_{\mathbb{K}} V''$ be exact [B2, I.2, I.3].

Let $\mathbb{K}\text{-alg}$ denote the category of unital commutative associative $\mathbb{K}$-algebras. We say that $\mathbb{F} \in \mathbb{K}\text{-alg}$ is *flat* (resp. *faithfully flat*) if $\mathbb{F}$ is a flat (resp. faithfully flat) $\mathbb{K}$-module. Note that if $\mathbb{K}$ is a field, then any $\mathbb{F} \in \mathbb{K}\text{-alg}$ is non-trivial and free and hence faithfully flat.

If V is a module and $\mathbb{F} \in \mathbb{K}\text{-alg}$, we write $V_{\mathbb{F}} := \mathbb{F} \otimes_{\mathbb{K}} V$. If V is a $\mathbb{K}$-algebra, then $V_{\mathbb{F}}$ is naturally an $\mathbb{F}$-algebra. If $\varphi: V \to W$ is a homomorphism of modules, we denote the induced homomorphism of $\mathbb{F}$-modules by $\varphi_{\mathbb{F}}: V_{\mathbb{F}} \to W_{\mathbb{F}}$. If $g: V \times W \to \mathbb{K}$ is a bilinear form, we have a unique $\mathbb{F}$-bilinear form $g_{\mathbb{F}}: V_{\mathbb{F}} \times W_{\mathbb{F}} \to \mathbb{F}$, which we say is *induced* by g, such that $g_{\mathbb{F}}(1 \otimes x, 1 \otimes y) = g(x, y)$ for $x \in V$, $y \in W$.

Finally, if $X = \bigoplus_i X_i$ is a direct sum of modules and $\mathbb{F} \in \mathbb{K}\text{-alg}$, then there is a canonical identification of $(X_i)_{\mathbb{F}}$ as an $\mathbb{F}$-submodule of $X_{\mathbb{F}}$ so that $X_{\mathbb{F}} = \bigoplus_i (X_i)_{\mathbb{F}}$. In this way a G-graded algebra $X = \bigoplus_{g \in G} X_g$, where G is an abelian group, yields a G-graded $\mathbb{F}$-algebra $X_{\mathbb{F}} = \bigoplus_{g \in G} (X_g)_{\mathbb{F}}$.

2 Root Graded Lie Algebras

2.1 Root Systems

In this paper, a *root system* will mean a finite root system Δ in a finite dimensional real Euclidean space E_Δ as described in [B3, VI.3]. We use the notation $Q_\Delta := \mathrm{span}_{\mathbb{Z}}(\Delta)$ for the *root lattice* of Δ. The *automorphism group of* Δ, denoted by $\mathrm{Aut}(\Delta)$, is the stabilizer of Δ in $\mathrm{GL}(E_\Delta)$. Using the restriction map, we often identify $\mathrm{Aut}(\Delta)$ with the stabilizer of Δ in $\mathrm{Aut}(Q_\Delta)$. The *Weyl group* of Δ, which is a subgroup of $\mathrm{Aut}(\Delta)$, will be denoted by W_Δ. A root system Δ is said to be *reduced* if $\Delta \cap (2\Delta) = \varnothing$. Recall that for each rank $n \geq 1$, there exists a unique irreducible non-reduced root system of rank n up to isomorphism [B3, VI.1.4, Proposition 14]. This root system is said to have *type* $\mathrm{BC_n}$.

2.2 Root Graded Lie Algebras

Let Δ be a root system. A Δ-*grading* of a Lie algebra L is a Q_Δ-grading of L such that $\operatorname{supp}_{Q_\Delta}(L) \subseteq \Delta \cup \{0\}$, where $\operatorname{supp}_{Q_\Delta}(L)$ denotes the support of L in Q_Δ. In that case we call L together with the Δ-grading a Δ-*graded Lie algebra*. (We note that this definition is less restrictive than the one used in [ABG, BS] and several earlier papers, since we do not assume the existence of a grading subalgebra. Our usage is natural here (see in particular Section 3) and will not cause the reader any confusion.) If Δ is irreducible of type $\mathrm{X_n}$, we often say that L is $\mathrm{X_n}$-*graded*; and we often refer to a Q_Δ-graded isomorphism of $\mathrm{X_n}$-graded Lie algebras as an $\mathrm{X_n}$-*graded isomorphism*.

If we fix a base $\Gamma = \{\alpha_1, \dots, \alpha_n\}$ for Δ, we can identify Q_Δ with $\mathbb{Z}^n$ using the $\mathbb{Z}$-basis Γ for Q_Δ. With this identification, every Δ-graded Lie algebra is a $\mathbb{Z}^n$-graded Lie algebra (but not conversely of course).

2.3 Images of Δ-graded Lie Algebras Under the Left Action of $\operatorname{Aut}(\Delta)$

Suppose that L is a Δ-graded Lie algebra. If $\theta \in \operatorname{Aut}(\Delta)$, we let ${}^{\theta}L$ be the Δ-graded Lie algebra such that ${}^{\theta}L = L$ as Lie algebras and

$$({}^{\theta}L)_\alpha = L_{\theta^{-1}\alpha}$$

for $\alpha \in Q_\Delta$. We call ${}^{\theta}L$ the θ-*image* of L, and if $\theta \in W_\Delta$, we call ${}^{\theta}L$ a *Weyl image* of L. Clearly

$$ {}^{1}L = L \quad \text{and} \quad {}^{\theta_1}({}^{\theta_2}L) = {}^{\theta_1\theta_2}L \tag{2.1}$$

for $\theta_1, \theta_2 \in \operatorname{Aut}(\Delta)$, so we have a left action of $\operatorname{Aut}(\Delta)$ on the class of Δ-graded Lie algebras.

3 Trilinear Pairs

Unless stated to the contrary, we will assume henceforth that $\mathbb{K}$ contains $\frac{1}{6}$.

3.1 Terminology

A *trilinear pair* is a pair $P = (P^-, P^+)$ of modules together with two trilinear maps $\{\cdot,\cdot,\cdot\}^\sigma \colon P^\sigma \times P^{-\sigma} \times P^\sigma \to P^\sigma$, $\sigma = \pm$, which we call the products on P. If needed, we will call $\{\cdot,\cdot,\cdot\}^\sigma$ the σ-*product* on P. We define the *D-operator* $D^\sigma(x^\sigma, y^{-\sigma}) \in \operatorname{End}(P^\sigma)$ for $x^\sigma \in P^\sigma$, $y^{-\sigma} \in P^{-\sigma}$ by $D^\sigma(x^\sigma, y^{-\sigma})z^\sigma = \{x^\sigma, y^{-\sigma}, z^\sigma\}^\sigma$, and we define the *K-operator* $K^\sigma(x^\sigma, z^\sigma) \in \operatorname{Hom}(P^{-\sigma}, P^\sigma)$ for $x^\sigma, z^\sigma \in P^\sigma$ by

$$K^\sigma(x^\sigma, z^\sigma)y^{-\sigma} = \{x^\sigma, y^{-\sigma}, z^\sigma\}^\sigma - \{z^\sigma, y^{-\sigma}, x^\sigma\}^\sigma.$$

When no confusion arises, we usually write $\{\cdot,\cdot,\cdot\}^\sigma$, $D^\sigma(x^\sigma, y^{-\sigma})$, and $K^\sigma(x^\sigma, z^\sigma)$ simply as $\{\cdot,\cdot,\cdot\}$, $D(x^\sigma, y^{-\sigma})$, and $K(x^\sigma, z^\sigma)$, respectively, and we sometimes also omit superscripts in our notation for elements in P^σ.

A *homomorphism* from a trilinear pair P into a trilinear pair P' is a pair $\omega = (\omega^-, \omega^+)$ of linear maps such that

$$\omega^\sigma\{x^\sigma, y^{-\sigma}, z^\sigma\} = \{\omega^\sigma x^\sigma, \omega^{-\sigma} y^{-\sigma}, \omega^\sigma z^\sigma\}$$

for $x^\sigma, z^\sigma \in P^\sigma$, $y^{-\sigma} \in P^{-\sigma}$, $\sigma = \pm$. A homomorphism ω is called an *isomorphism* if each ω^σ is bijective.

If P is a trilinear pair and $Q = (Q^-, Q^+)$, where Q^σ is a submodule of P^σ for $\sigma = \pm$, then Q is called a *subpair*, *ideal* or *left ideal* of P if $\{Q^\sigma, Q^{-\sigma}, Q^\sigma\} \subseteq Q^\sigma$, $\{P^\sigma, P^{-\sigma}, Q^\sigma\} + \{P^\sigma, Q^{-\sigma}, P^\sigma\} + \{Q^\sigma, P^{-\sigma}, P^\sigma\} \subseteq Q^\sigma$, or $\{P^\sigma, P^{-\sigma}, Q^\sigma\} \subseteq Q^\sigma$, respectively for $\sigma = \pm$. A trilinear pair P is said to be *simple* if $\{P^\sigma, P^{-\sigma}, P^\sigma\} \neq \{0\}$ for $\sigma = +$ or $\sigma = -$ and the only ideals of P are P and $\{0\}$.

There are evident notions of *direct sum* and *quotient* for trilinear pairs.

If $\mathbb{K}$ is a field we say that P is *finite dimensional* if each P^σ is finite dimensional, and we call $(\dim(P^-), \dim(P^+))$ the *dimension* of P. If $d = \dim(P^-) = \dim(P^+)$, we say that P has *balanced dimension* d.

The *centroid* of a trilinear pair P is the subalgebra $\mathrm{C}(P)$ of the associative algebra $\mathrm{End}(P^-) \oplus \mathrm{End}(P^+)$ consisting of the pairs of maps $(\omega^-, \omega^+) \in \mathrm{End}(P^-) \oplus \mathrm{End}(P^+)$ such that $\omega^\sigma(\{x^\sigma, y^{-\sigma}, z^\sigma\}) = \{\omega^\sigma(x^\sigma), y^{-\sigma}, z^\sigma\} = \{x^\sigma, \omega^{-\sigma}(y^{-\sigma}), z^\sigma\} = \{x^\sigma, y^{-\sigma}, \omega^\sigma(z^\sigma)\}$. We say P is *central* if the homomorphism $a \mapsto a(\mathrm{id}_{P^-}, \mathrm{id}_{P^+})$ from $\mathbb{K}$ into $\mathrm{C}(P)$ is an isomorphism. If P is simple, then $\mathrm{C}(P)$ is a field.

The *opposite* of a trilinear pair $P = (P^-, P^+)$ is the trilinear pair $P^{\mathrm{op}} = (P^+, P^-)$ whose σ-product is the $-\sigma$-product of P for $\sigma = \pm$.

If $\mathbb{F} \in \mathbb{K}\text{-alg}$ and P is a trilinear pair, then the products on P canonically induce $\mathbb{F}$-trilinear products on $P_\mathbb{F} := (P_\mathbb{F}^-, P_\mathbb{F}^+)$ so that $P_\mathbb{F}$ is a trilinear pair over $\mathbb{F}$.

Suppose that P is a trilinear pair, G is an abelian group (written additively), and $P^\sigma = \bigoplus_{g\in G} P_g^\sigma$ for $\sigma = \pm$, where P_g^σ is a submodule of P^σ for $g \in G$, $\sigma = \pm$. We say that $P = (\bigoplus_{g\in G} P_g^-, \bigoplus_{g\in G} P_g^+)$ is a G-*grading* of P if $\{P_g^\sigma, P_k^{-\sigma}, P_\ell^\sigma\} \subseteq P_{g-k+\ell}^\sigma$ for $g, k, \ell \in G$, $\sigma = \pm$. (Here we follow the terminology in [LN, §8.1]. This notion of grading is equivalent to the usual one if we replace P_g^σ by $P_{\sigma g}^\sigma$ for each σ and g.) We often then write

$$P = \bigoplus_{g\in G} P_g, \tag{3.1}$$

where $P_g := (P_g^-, P_g^+)$ for $g \in G$. Note that each P_g is a subpair of P; however the sum (3.1) is not in general a direct sum of trilinear pairs, since the subpairs P_g need not be ideals. The G-*support* of P is defined to be

$$\mathrm{supp}_G(P) = \{g \in G : P_g^\sigma \neq 0 \text{ for } \sigma = + \text{ or } \sigma = -\}.$$

Finally suppose that X is a *triple system*, by which we mean a module with a trilinear product $\{\cdot,\cdot,\cdot\}: X \times X \times X \to X$. A *polarization* of X is a module decomposition $X = X^- \oplus X^+$ such that $\{X^\sigma, X^{-\sigma}, X^\sigma\} \subseteq X^\sigma$, $\{X^\sigma, X^\sigma, X\} = 0$, and $\{X, X^\sigma, X^\sigma\} = 0$ for $\sigma = \pm$; and we say that X is *non-polarized* if it has no polarizations.

If X is a triple system, then the trilinear pair (X, X) with products defined by $\{x, y, z\}^\sigma = \{x, y, z\}$ (resp. $\{x, y, z\}^\sigma = \sigma\{x, y, z\}$) is called the *double* (resp. the *signed double*) of X. It is easy to check (and well known) that the double (resp. the signed double) of X is simple if and only if X is simple and non-polarized.

Remark 3.1 In the rest of the paper, we will often discuss simplicity and isomorphism of graded algebras and graded trilinear pairs. To be clear, the terms *simple* and *isomorphism* will be used in the ungraded sense as defined above, unless we specify to the contrary.

4 Kantor Pairs and 5-graded (BC_1-graded) Lie Algebras

Throughout this section, we assume that $\Delta = \{-2\alpha_1, -\alpha_1, \alpha_1, 2\alpha_1\}$ is the irreducible root system of type BC_1 with base $\Gamma = \{\alpha_1\}$. We identify $Q_\Delta = \mathbb{Z}$ using the $\mathbb{Z}$-basis Γ for Q_Δ (as in Subsection 2.2). Then a BC_1-grading of a Lie algebra L is merely a 5-*grading* of L. (Recall that if $m \geq 1$, a $2m+1$-*grading of* L is a $\mathbb{Z}$-grading $L = \bigoplus_{i \in \mathbb{Z}} L_i$ with $L_i = 0$ for $|\, i \,| > m$.)

In this section, we recall the definition of a Kantor pair and how Kantor pairs are related to 5-graded Lie algebras.

4.1 Kantor Pairs

A *Kantor pair* is a trilinear pair P such that the following identities hold

$$\text{(K1)}\quad [D(x^\sigma, y^{-\sigma}), D(z^\sigma, w^{-\sigma})] = D(D(x^\sigma, y^{-\sigma})z^\sigma, w^{-\sigma}) - D(z^\sigma, D(y^{-\sigma}, x^\sigma)w^{-\sigma}),$$

$$\text{(K2)}\quad K(x^\sigma, z^\sigma)D(w^{-\sigma}, u^\sigma) + D(u^\sigma, w^{-\sigma})K(x^\sigma, z^\sigma) = K(K(x^\sigma, z^\sigma)w^{-\sigma}, u^\sigma),$$

for $x^\sigma, z^\sigma, u^\sigma \in P^\sigma$, $y^{-\sigma}, w^{-\sigma} \in P^{-\sigma}$, $\sigma = \pm$.

It is clear that the opposite of a Kantor pair is a Kantor pair.

Special Cases 4.1 (i) A *Jordan pair* P is a Kantor pair satisfying $K(P^\sigma, P^\sigma) = 0$ for $\sigma = \pm$. The structure theory of Jordan pairs is developed in detail in [L], where Jordan pairs are defined in a different way using quadratic operators. (However, since $\frac{1}{6} \in \mathbb{K}$, the two definitions are equivalent [L, Proposition 2.2].)

(ii) Suppose that X is a triple system. Then X is called a *Kantor triple system* if its double is a Kantor pair. Kantor triple systems were introduced by Kantor [K1, K2], where they were called generalized Jordan triple systems of the second order, and where numerous examples can be found. In the literature, Kantor triple systems are also often called $(-1,1)$-*Freudenthal–Kantor triple systems*. (See [YO], as well as the recent papers [EO, EKO] and their references, for information about (ϵ, δ)-*Freudenthal–Kantor triple systems*, where $\epsilon, \delta = \pm 1$.)

(iii) Analogously, a triple system X with product $\{x, y, z\}$ is called a $(1,1)$-*Freudenthal–Kantor triple system* if its signed double is a Kantor pair.

(iv) A *structurable algebra* is a unital algebra with involution $(A, -)$ such that A is a Kantor triple system under the product $\{x, y, z\} = 2((x\overline{y})z + (z\overline{y})x - (z\overline{x})y)$ on A. (Involution here means a period 2 anti-automorphism. Also, the 2 in the expression for $\{\cdot, \cdot, \cdot\}$ is unimportant; it is included for compatibility with the Jordan algebra case.) The double (A, A) of this Kantor triple system is also called the *double* of the structurable algebra A. See [A, Sm] for many examples of structurable algebras, including all unital Jordan algebras and all unital alternative algebras with involution.

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4.2 The Kantor Pair Enveloped by a 5-graded Lie Algebra

If $L = \bigoplus_{i\in\mathbb{Z}} L_i$ is a 5-graded Lie algebra, then $P = (L_{-1}, L_1)$ is a Kantor pair with products defined by

$$\{x^\sigma, y^{-\sigma}, z^\sigma\} = [[x^\sigma, y^{-\sigma}], z^\sigma]$$

for $x^\sigma, z^\sigma \in L_{\sigma 1}$, $y^{-\sigma} \in L_{-\sigma 1}$, $\sigma = \pm$. (See [AF1, Theorem 7], where $-[[x^\sigma, y^{-\sigma}], z^\sigma]$ is used instead of $[[x^\sigma, y^{-\sigma}], z^\sigma]$.) We call P the *Kantor pair enveloped by the* 5*-graded Lie algebra L*, and we say that *the* 5*-graded Lie algebra L envelops P*. (In a similar situation, Jacobson uses the term "enveloping Lie algebra" [J1, §3.1].)

If L is 3-graded, we see using the Jacobi identity that the pair (L_{-1}, L_1) enveloped by L is in fact Jordan.

If P is the Kantor pair enveloped by a 5-graded Lie algebra L, we let

$$T_L(P) := P^- \oplus P^+ = L_{-1} \oplus L_1 \quad \text{in } L.$$

Lemma 4.2 as well as Lemma 4.4 are easily checked.

Lemma 4.2 *Suppose that P is the Kantor pair enveloped by a* 5*-graded Lie algebra L. Then $T_L(P)$ is a triple system under the trilinear product $[[x, y], z]$. Moreover, this triple system depends only on P; specifically, if $x^\tau, y^\tau, z^\tau \in P^\tau$ for $\tau = \pm$, we have*

$$\begin{aligned} &[[x^\sigma, y^\sigma], z^\sigma] = 0, && [[x^\sigma, y^{-\sigma}], z^\sigma] = \{x^\sigma, y^{-\sigma}, z^\sigma\}, \\ &[[x^{-\sigma}, y^\sigma], z^\sigma] = -\{y^\sigma, x^{-\sigma}, z^\sigma\}, && [[x^\sigma, y^\sigma], z^{-\sigma}] = K(x^\sigma, y^\sigma) z^{-\sigma}. \end{aligned} \tag{4.1}$$

Remark 4.3 (i) Despite the conclusion in Lemma 4.2, we include the subscript L in the notation for $T_L(P)$ to emphasize that we are regarding $T_L(P)$ as a submodule of L.

(ii) The triple system $T_L(P)$ is a Lie triple system. Moreover, this triple system is sign-graded, which means that it is $\mathbb{Z}$-graded with support contained in $\{-1, 1\}$. This is the point of view taken in [AF1, §3–4] (and in special cases elsewhere), but for simplicity we wil not make use of Lie triple systems in this work.

Lemma 4.4 *Suppose that P is the Kantor pair enveloped by a* 5*-graded Lie algebra L. Then the subalgebra $\langle T_L(P)\rangle_{\text{alg}}$ of L that is generated by $T_L(P)$ is a* 5*-graded ideal of L that envelops P. Moreover $\langle T_L(P)\rangle_{\text{alg}} = [T_L(P), T_L(P)] \oplus T_L(P)$ and $[T_L(P), T_L(P)] = [L_{-1}, L_{-1}] \oplus [L_{-1}, L_1] \oplus [L_1, L_1]$.*

Definition 4.5 Suppose that L is a 5-graded Lie algebra and P is a Kantor pair. We say that L *tightly envelops* P if L envelops P,

$$\langle T_L(P)\rangle_{\text{alg}} = L \quad \text{and} \quad Z(L) \cap [T_L(P), T_L(P)] = 0, \tag{4.2}$$

where (here and subsequently) $Z(L)$ denotes the *centre* of the Lie algebra L.

Remark 4.6 If L is a 5-graded Lie algebra that envelops a Kantor pair P, it follows easily from Lemma 4.4 that we can replace L by $L' = \langle T_L(P)\rangle_{\text{alg}}$ and then replace L' by $\overline{L'} = L'/(Z(L') \cap [T_{L'}(P), T_{L'}(P)])$ to get a 5-graded Lie algebra $\overline{L'}$ that tightly envelops P (with the evident identifications of P^- and P^+ in $\overline{L'}$).

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Remark 4.7 Suppose that P is the Kantor pair enveloped by a 5-graded Lie algebra L. Since $\mathrm{Aut}(\Delta) = W_\Delta = \{1, -1\}$, we can form the Weyl images ${}^1L$ and ${}^{-1}L$ of L. Clearly the 5-graded Lie algebra ${}^1L = L$ envelops P, whereas the 5-graded Lie algebra ${}^{-1}L$ envelops the Kantor pair P^{op}, which in general in not isomorphic to P. In Section 6, we will look at this phenomenon for BC_2-graded Lie algebras, where the supply of Weyl images is richer.

4.3 The Kantor Construction

To see that any Kantor pair is enveloped by a 5-graded Lie algebra, we now recall from [AF1, §§3–4] the construction of a 5-graded Lie algebra $\mathfrak{K}(P)$ from a Kantor pair P.

Let P be a Kantor pair. Let $\left[\begin{smallmatrix} P^- \\ P^+ \end{smallmatrix}\right]$ be the module of column vectors with entries as indicated, and canonically identify

$$\mathrm{End}\begin{bmatrix} P^- \\ P^+ \end{bmatrix} = \begin{bmatrix} \mathrm{End}(P^-) & \mathrm{Hom}(P^+, P^-) \\ \mathrm{Hom}(P^-, P^+) & \mathrm{End}(P^+) \end{bmatrix}$$

so that the action of $\mathrm{End}\left[\begin{smallmatrix} P^- \\ P^+ \end{smallmatrix}\right]$ on $\left[\begin{smallmatrix} P^- \\ P^+ \end{smallmatrix}\right]$ is by matrix multiplication. Then

$$\mathfrak{S}(P) := \mathrm{span}_{\mathbb{K}}\Big\{ \begin{bmatrix} D(x^-, x^+) & K(y^-, z^-) \\ K(y^+, z^+) & -D(x^+, x^-) \end{bmatrix} : x^\sigma, y^\sigma, z^\sigma \in P^\sigma,\ \sigma = \pm 1 \Big\}$$

is a subalgebra of the Lie algebra $\mathrm{End}\left[\begin{smallmatrix} P^- \\ P^+ \end{smallmatrix}\right]$ under the commutator product. Also

$$\mathfrak{K}(P) := \mathfrak{S}(P) \oplus \begin{bmatrix} P^- \\ P^+ \end{bmatrix},$$

is a Lie algebra under the anti-commutative product $[\cdot, \cdot]$ satisfying

$$[A, B] = AB - BA, \quad \Big[A, \begin{bmatrix} x^- \\ x^+ \end{bmatrix}\Big] = A\begin{bmatrix} x^- \\ x^+ \end{bmatrix},$$

$$\Big[\begin{bmatrix} x^- \\ x^+ \end{bmatrix}, \begin{bmatrix} y^- \\ y^+ \end{bmatrix}\Big] = \begin{bmatrix} D(x^-, y^+) - D(y^-, x^+) & K(x^-, y^-) \\ K(x^+, y^+) & -D(y^+, x^-) + D(x^+, y^-) \end{bmatrix}$$

for $A, B \in \mathfrak{S}(P)$, $x^\sigma, y^\sigma \in P^\sigma$, $\sigma = \pm$. We call $\mathfrak{K}(P)$ the *Kantor Lie algebra* of P.

The Lie algebra $\mathfrak{K}(P)$ is 5-graded with

$$\mathfrak{K}(P)_{-2} = \begin{bmatrix} 0 & K(P^-, P^-) \\ 0 & 0 \end{bmatrix}, \quad \mathfrak{K}(P)_{-1} = \begin{bmatrix} P^- \\ 0 \end{bmatrix},$$

$$\mathfrak{K}(P)_0 = \mathrm{span}_{\mathbb{K}}\Big\{ \begin{bmatrix} D(x^-, x^+) & 0 \\ 0 & -D(x^+, x^-) \end{bmatrix} : x^- \in P^-,\ x^+ \in P^+ \Big\},$$

$$\mathfrak{K}(P)_1 = \begin{bmatrix} 0 \\ P^+ \end{bmatrix}, \quad \mathfrak{K}(P)_2 = \begin{bmatrix} 0 & 0 \\ K(P^+, P^+) & 0 \end{bmatrix}.$$

We call this 5-grading the *standard* 5*-grading* of $\mathfrak{K}(P)$. *Unless mentioned otherwise we will regard $\mathfrak{K}(P)$ as a* 5*-graded algebra with its standard* 5*-grading.*

If P is a Kantor pair, *we identify P^- with $\mathfrak{K}(P)_{-1} = \left[\begin{smallmatrix} P^- \\ 0 \end{smallmatrix}\right]$ and P^+ with $\mathfrak{K}(P)_1 = \left[\begin{smallmatrix} 0 \\ P^+ \end{smallmatrix}\right]$* in the evident fashion. With this identification the following is clear.

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Proposition 4.8 *If P is a Kantor pair, then $\mathfrak{K}(P)$ with its standard 5-grading tightly envelops P.*

We will see in Corollary 4.17 that $\mathfrak{K}(P)$ is the unique 5-graded Lie algebra that tightly envelops the Kantor pair P.

Remark 4.9 Suppose that P is a Kantor pair.

(i) P is Jordan if and only if $\mathfrak{K}(P)_{-2} = \mathfrak{K}(P)_2 = 0$, in which case the 3-graded (*i.e.*, A_1-graded) Lie algebra $\mathfrak{K}(P) = \mathfrak{K}(P)_{-1} \oplus \mathfrak{K}(P)_0 \oplus \mathfrak{K}(P)_1$ is (graded-isomorphic to) the derived algebra of the *Tits–Kantor–Koecher Lie algebra* of P [LN, §9.1].

(ii) P is finitely spanned (as a module) if and only if $\mathfrak{K}(P)$ has the same property.

(iii) Suppose $P = (X, X)$ is the double of a Kantor triple system X. Then $\mathfrak{K}(P)$ is the Lie algebra constructed by Kantor [K1, K2] from X, and it is easy to check that there is a unique grade-reversing period 2 automorphism of $\mathfrak{K}(P)$ which maps $\left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right]$ to $\left[\begin{smallmatrix} 0 \\ x \end{smallmatrix}\right]$ for $x \in X$.

Lemma 4.10 *Let $\mathbb{F} \in \mathbb{K}\text{-alg}$ and let P be a Kantor pair. Assume that either $\mathbb{F}$ is a projective $\mathbb{K}$-module (which holds for example if $\mathbb{K}$ is a field) or that $\mathbb{F}$ is flat and each P^σ is a finitely generated module. Then there is a canonical 5-graded $\mathbb{F}$-algebra isomorphism from $\mathfrak{K}(P)_\mathbb{F}$ onto $\mathfrak{K}(P_\mathbb{F})$.*

Proof Now $\mathfrak{K}(P) = \mathfrak{S}(P) \oplus \left[\begin{smallmatrix} P^- \\ P^+ \end{smallmatrix}\right]$, so $\mathfrak{K}(P)_\mathbb{F} = \mathfrak{S}(P)_\mathbb{F} \oplus \left[\begin{smallmatrix} P^-_\mathbb{F} \\ P^+_\mathbb{F} \end{smallmatrix}\right]$, whereas $\mathfrak{K}(P_\mathbb{F}) = \mathfrak{S}(P_\mathbb{F}) \oplus \left[\begin{smallmatrix} P^-_\mathbb{F} \\ P^+_\mathbb{F} \end{smallmatrix}\right]$. Our isomorphism $\omega \colon \mathfrak{K}(P)_\mathbb{F} \to \mathfrak{K}(P_\mathbb{F})$, is the direct sum $\omega' \oplus \omega''$, where ω'' is the identity map and ω' is the composition

$$\mathfrak{S}(P)_\mathbb{F} \to \operatorname{End}\Big(\begin{bmatrix} P^- \\ P^+ \end{bmatrix}\Big)_\mathbb{F} \to \operatorname{End}_\mathbb{F}\Big(\begin{bmatrix} P^-_\mathbb{F} \\ P^+_\mathbb{F} \end{bmatrix}\Big). \tag{4.3}$$

Here the first map in (4.3) is induced by inclusion and is injective since $\mathbb{F}$ is flat; whereas the second map in (4.3) is the canonical homomorphism, which is injective because of our assumptions on $\mathbb{F}$ and P (see [B1, II.5.3, Proposition 7] and [B2, I.2.10, Proposition 11]). It is easy to check that the image of ω' is in fact $\mathfrak{S}(P_\mathbb{F})$, and that ω is a graded $\mathbb{F}$-algebra homomorphism. ■

4.4 Simplicity and Centrality

The following proposition is proved in [GLN, Proposition 2.7(iii)].

Proposition 4.11 *If P is a Kantor pair, then P is simple if and only if $\mathfrak{K}(P)$ is simple.*

Recall that the *centroid* of an algebra L is the subalgebra $\mathrm{C}(L)$ of the associative algebra $\operatorname{End}(L)$ consisting of the endomorphisms of L that commute with all left and right multiplication operators. We say that L is *central* if the homomorphism $a \mapsto a\,\mathrm{id}_L$ from $\mathbb{K}$ into $\mathrm{C}(L)$ is an isomorphism. If L is simple, then $\mathrm{C}(L)$ is a field. If L is G-graded, where G is an abelian group, then

$$\mathrm{C}(L, G) := \{\chi \in \mathrm{C}(L) : \chi(L_g) \subseteq L_g \text{ for } g \in G\}$$

is a subalgebra of $\mathrm{C}(L)$.

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Lemma 4.12 *Suppose that P is a Kantor pair. Then the restriction map $\chi \mapsto (\chi|_{P^-}, \chi|_{P^+})$ is an isomorphism of $\mathrm{C}(\mathfrak{K}(P), \mathbb{Z})$ onto the centroid $\mathrm{C}(P)$ of P.*

Proof All but surjectivity are clear. For surjectivity, suppose that $\omega = (\omega^-, \omega^+) \in \mathrm{C}(P)$. Define $\chi: \mathfrak{K}(P) \to \mathfrak{K}(P)$ by $\chi(X) = \left[\begin{smallmatrix} \omega^- & 0 \\ 0 & \omega^+ \end{smallmatrix}\right] X$. Then one checks easily that $\chi \in \mathrm{C}(\mathfrak{K}(P), \mathbb{Z})$ and clearly χ restricts to ω. ∎

Proposition 4.13 *Suppose that P is a simple Kantor pair over $\mathbb{K}$. Then we have $\mathrm{C}(\mathfrak{K}(P), \mathbb{Z}) = \mathrm{C}(\mathfrak{K}(P))$. Furthermore, the restriction map is an isomorphism of $\mathrm{C}(\mathfrak{K}(P))$ onto $\mathrm{C}(P)$, so P is central if and only if $\mathfrak{K}(P)$ is central.*

Proof In view of Lemma 4.12, it is enough to show the first statement. This follows from [Z2, Lemma 1.6 (a)] when $\mathbb{K}$ is a field. In general, note that $\mathfrak{K}(P)$ is simple by Proposition 4.11, so $\mathfrak{K}(P)$ is a finitely generated module for its multiplication algebra. Therefore by [BN, (2.15)], $\mathrm{C}(\mathfrak{K}(P))$ is naturally $\mathbb{Z}$-graded with $\mathrm{C}(\mathfrak{K}(P))_0 = \mathrm{C}(\mathfrak{K}(P), \mathbb{Z})$. But this grading is trivial since $\mathrm{C}(\mathfrak{K}(P))$ is a field. ∎

Corollary 4.14 *If P is a Kantor pair, then P is central simple if and only if $\mathfrak{K}(P)$ is central simple.*

The next proposition lists facts about Kantor pairs that are analogues of well-known facts for algebras. The first two of these tell us that the study of simple Kantor pairs over a field $\mathbb{K}$ is reduced to the study of central simple Kantor pairs over extension fields of $\mathbb{K}$.

Proposition 4.15 *Suppose that P is a Kantor pair over $\mathbb{K}$.*

(i) *If P is simple, then P is a central simple Kantor pair over the field $\mathrm{C}(P)$.*
(ii) *If $\mathbb{K}$ is a field and P is a central simple Kantor pair over a field $\mathbb{F}$ containing $\mathbb{K}$, then P is a simple Kantor pair over $\mathbb{K}$ with centroid $\mathbb{F}$.*
(iii) *If $\mathbb{K}$ is a field, then P is a central simple Kantor pair over $\mathbb{K}$ if and only if $P_{\mathbb{F}}$ is simple over $\mathbb{F}$ for all fields $\mathbb{F}$ containing $\mathbb{K}$.*

Proof Using Proposition 4.11, Corollary 4.14, and Lemma 4.10, all these statements follow from the corresponding statements for Lie algebras. The statement corresponding to (i) is [Mc, Theorem II.1.6.3 (2)]; the statement corresponding to (ii) follows from the second part of [J3, Theorem X.3] (with $\Gamma = \Delta = \mathbb{F}$ and $\Phi = \mathbb{K}$); and the statement corresponding to (iii) is [Mc, Theorem II.1.6.3 (2)]. ∎

4.5 5-graded Lie Algebras Enveloping a Kantor Pair

In this subsection, we use, as usual, the standard 5-grading on each Kantor Lie algebra.

Lemma 4.16 *Suppose that P and P' are Kantor pairs and L is a 5-graded Lie algebra that tightly envelops P. Let $\chi: P \to P'$ be a surjective homomorphism of Kantor pairs. Then there exists a unique 5-graded algebra homomorphism $\varphi: L \to \mathfrak{K}(P')$ that extends*

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$\widetilde{\chi} := \chi^- \oplus \chi^+ \colon T_L(P) \to T_{\mathfrak{K}(P')}(P')$. *Furthermore, φ is surjective and*

$$\ker(\varphi) = \left\{ d \in [T_L(P), T_L(P)] : [d, T_L(P)] \subseteq \ker(\widetilde{\chi}) \right\} + \ker(\widetilde{\chi}). \tag{4.4}$$

Proof Let $T = T_L(P)$. By assumption, $L = \langle T \rangle_{\text{alg}}$; so uniqueness in the lemma is clear and, by Lemma 4.4, we have $L = [T, T] \oplus T$. Next let $L' = \mathfrak{K}(P')$, $T' = T_{L'}(P')$. Using Lemma 4.2, one sees that $\widetilde{\chi}([[x, y], z]) = [[\widetilde{\chi}(x), \widetilde{\chi}(y)], \widetilde{\chi}(z)]$ for $x, y, z \in T$.

Since $L = [T, T] \oplus T$ and $\varphi|_T = \widetilde{\chi}$, we only need to define φ on $[T, T]$. So we consider $\sum_i [x_i, y_i] \in [T, T]$ for $x_i, y_i \in T$ and set $\varphi(\sum_i [x_i, y_i]) = \sum_i [\widetilde{\chi}(x_i), \widetilde{\chi}(y_i)]$. If $\sum_i [x_i, y_i] = 0$, then $[\sum_i [\widetilde{\chi}(x_i), \widetilde{\chi}(y_i)], \widetilde{\chi}(z)] = \widetilde{\chi}([\sum_i [x_i, y_i], z]) = 0$ for any $z \in T$. It follows that $\sum_i [\widetilde{\chi}(x_i), \widetilde{\chi}(y_i)] \in Z(L') \cap [T', T'] = 0$ since $\widetilde{\chi}$ is surjective and T' generates L'. Thus φ is well defined.

It is clear that φ is $\mathbb{Z}$-graded, and one checks directly that φ is a homomorphism of Lie algebras and that (4.4) holds. ∎

Applying Lemma 4.16 with $P' = P$ and $\chi = (\mathrm{id}_{P^-}, \mathrm{id}_{P^+})$, we obtain the following corollary.

Corollary 4.17 *If L is a 5-graded Lie algebra that tightly envelops a Kantor pair P, then there exists a unique 5-graded algebra isomorphism $\varphi \colon L \to \mathfrak{K}(P)$ that restricts to the identity map on $T_L(P)$.*

Also, applying Lemma 4.16 with $L = \mathfrak{K}(P)$, we obtain the following corollary.

Corollary 4.18 *Suppose that P and P' are Kantor pairs. If $\varphi \colon \mathfrak{K}(P) \to \mathfrak{K}(P')$ is a 5-graded isomorphism, then $(\varphi|_{P^-}, \varphi|_{P^+})$ is an isomorphism of P onto P'. Conversely, if $\chi = (\chi^-, \chi^+)$ is an isomorphism of P onto P', there exists a unique 5-graded isomorphism $\varphi \colon \mathfrak{K}(P) \to \mathfrak{K}(P')$ that extends $\chi^- \oplus \chi^+ \colon T_L(P) \to T_{\mathfrak{K}(P')}(P')$.*

Proof The first statement is clear. The converse follows from Corollary 4.8 and Lemma 4.16 (with $L = \mathfrak{K}(P)$). ∎

Proposition 4.19 *Suppose P is a nonzero Kantor pair and L is a simple 5-graded Lie algebra that envelops P. Then L tightly envelops P.*

Proof By Lemma 4.4, $\langle T_L(P) \rangle_{\text{alg}}$ is an ideal of L, which is nonzero since $P \neq 0$. Also $Z(L) \cap [T_L(P), T_L(P)]$ is an ideal of L, which is proper since $P \neq 0$. ∎

The next theorem will be among the basic tools in our study of simple Kantor pairs in this paper and in our future work. It tells us in particular that each simple Kantor pair is enveloped by a unique simple 5-graded Lie algebra.

Theorem 4.20 *Suppose that P is a nonzero Kantor pair.*

(i) *The following statements are equivalent.*
   (a) *P is simple.*
   (b) *There exists a simple 5-graded Lie algebra L that envelops P.*

(ii) *If L is a simple 5-graded Lie algebra that envelops P, then there exists a unique 5-graded isomorphism of L onto $\mathfrak{K}(P)$ that extends the identity on $T_L(P)$.*

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(iii) *Statements* (a) *and* (b) *in* (i) *with "simple" replaced by "central simple" are equivalent.*

Proof If (i) (a) holds, we know from Propositions 4.8 and 4.11 that $\mathfrak{K}(P)$ is a simple 5-graded Lie algebra that envelops P. Conversely, suppose (b) holds. Then by Proposition 4.19 and Corollary 4.17 we have a unique 5-graded isomorphism as indicated in (ii). So P is simple by Proposition 4.11. This proves (i) and (ii); and (iii) is proved by the same argument using Corollary 4.14. ∎

Remark 4.21 Suppose that $\mathbb{K}$ is an algebraically closed field of characteristic 0. Here we sketch an argument due to Kantor [K1] for the classification of finite dimensional simple Kantor pairs in terms of weighted Dynkin diagrams. (Kantor worked with Kantor triple systems.) We fix a finite dimensional simple Lie algebra $\mathcal{G}$ of type $\mathrm{X_n}$ with Cartan subalgebra $\mathcal{H}$, root system Σ, root spaces $\mathcal{G}_\alpha$ for $\alpha \in Q_\Sigma$, base $\Pi = \{\mu_1, \dots, \mu_n\}$ for Σ, and highest root μ^+. We view Π as usual as a Dynkin diagram. If $\mathbf{p} = (p_1, \dots, p_n)$ is an n-tuple of non-negative integers, we call $(\Pi, \mathbf{p})$ a *weighted Dynkin diagram* and we define $\chi_\mathbf{p}: Q_\Sigma \to \mathbb{Z}$ by $\chi_\mathbf{p}(\sum_{j=1}^n k_j\mu_j) = \sum_{j=1}^n k_j p_j$.

If $(\Pi, \mathbf{p})$ is a weighted Dynkin diagram, then $\mathcal{G} = \oplus_{i\in\mathbb{Z}}\mathcal{G}_i$ is a $\mathbb{Z}$-grading, where $\mathcal{G}_i$ is the sum of all $\mathcal{G}_\mu$ for $\mu \in Q_\Sigma$ with $\chi_\mathbf{p}(\mu) = i$. Moreover it is well known that any $\mathbb{Z}$-graded finite dimensional simple Lie algebra of type $\mathrm{X_n}$ is graded isomorphic to one obtained in this way (see [K1, Proposition 12], [D, Theorem 1] or [GOV, § 3.3.5]). Clearly this grading $\mathcal{G} = \oplus_{i\in\mathbb{Z}}\mathcal{G}_i$ is a 5-grading with $(\mathcal{G}_{-1}, \mathcal{G}_1) \neq 0$ if and only if

$$\chi_\mathbf{p}(\mu^+) \le 2 \quad \text{and} \quad p_j = 1 \text{ for some } j. \tag{4.5}$$

We see, using Theorem 4.20 (i) and (ii), that if (4.5) holds, then $(\mathcal{G}_{-1}, \mathcal{G}_1)$ is a simple Kantor pair whose Kantor Lie algebra has type $\mathrm{X_n}$, and conversely any such Kantor pair is isomorphic to one that arises in this way for some $\mathbf{p}$ satisfying (4.5). We will discuss two examples for type $\mathrm{E_6}$ in Remark 8.19 (i).

In [AF2], the method just described is extended to study other types of root gradings of $\mathcal{G}$ (in particular $\mathrm{BC_2}$-gradings), and use this to obtain a Dynkin diagram interpretation of reflection.

4.6 Split Lie Algebras of Type $\mathrm{X_n}$ and Their Forms

In this subsection, we do *not* assume that $\mathbb{K}$ contains $\frac{1}{6}$. Let $\mathrm{X_n}$ be one of the types of an irreducible reduced root system.

Lemma 4.22 *Suppose L is a Lie algebra that is finitely generated as a module, and suppose $\mathbb{F} \in \mathbb{K}$-alg is flat. Then $Z(L)_\mathbb{F} \simeq Z(L_\mathbb{F})$ and $(L/Z(L))_\mathbb{F} \simeq L_\mathbb{F}/Z(L_\mathbb{F})$ as Lie algebras over $\mathbb{F}$.*

Proof Let $x_1, \dots, x_\ell$ generate L as a module. The sequence

$$0 \to Z(L) \xrightarrow{\iota} L \xrightarrow{\eta} L^\ell$$

is exact, where ι is the inclusion map and $\eta(y) = ([y, x_1], \dots, [y, x_\ell])$. So

$$0 \to Z(L)_\mathbb{F} \xrightarrow{\iota_\mathbb{F}} L_\mathbb{F} \xrightarrow{\eta_\mathbb{F}} L_\mathbb{F}^\ell$$

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is exact. Thus $\iota_{\mathbb{F}}$ is an isomorphism of $Z(L)_{\mathbb{F}}$ onto $Z(L_{\mathbb{F}})$ (as $\mathbb{F}$-modules and hence as $\mathbb{F}$-algebras). Moreover, the sequence

$$0 \to Z(L) \xrightarrow{\iota} L \xrightarrow{\pi} L/Z(L) \to 0$$

is exact and thus so is

$$0 \to Z(L)_{\mathbb{F}} \xrightarrow{\iota_{\mathbb{F}}} L_{\mathbb{F}} \xrightarrow{\pi_{\mathbb{F}}} (L/Z(L))_{\mathbb{F}} \to 0.$$

Hence $(L/Z(L))_{\mathbb{F}} \simeq L_{\mathbb{F}}/\iota_{\mathbb{F}}(Z(L)_{\mathbb{F}}) = L_{\mathbb{F}}/Z(L_{\mathbb{F}})$ as $\mathbb{F}$-algebras. ∎

Definition 4.23 Let $\mathfrak{g}(\mathbb{C})$ be a finite dimensional simple Lie algebra of type X_n over the complex field $\mathbb{C}$, and choose a Chevalley basis B for $\mathfrak{g}(\mathbb{C})$ [H, §25]. Then the $\mathbb{Z}$-span $\mathfrak{g}(\mathbb{Z})$ of B is a Lie algebra over $\mathbb{Z}$ which depends up to isomorphism only on the type X_n. A Lie algebra isomorphic to $\mathfrak{g}(\mathbb{K}) := \mathfrak{g}(\mathbb{Z})_{\mathbb{K}}$ is called the *Chevalley algebra of type X_n* over $\mathbb{K}$, and a Lie algebra isomorphic to the quotient algebra $\mathfrak{g}(\mathbb{K})/Z(\mathfrak{g}(\mathbb{K}))$ is called the *split Lie algebra of type X_n* over $\mathbb{K}$.

Remark 4.24 If $\mathbb{K}$ is a field of characteristic $\neq 2$ or 3, the (finite dimensional) split Lie algebra of type $\mathrm{X_n}$ is central simple and studied in detail in [Sel], where it is called the *classical simple Lie algebra* of type $\mathrm{X_n}$. Furthermore, if $\mathbb{K}$ is a field of characteristic 0, the split Lie algebra of type $\mathrm{X_n}$ is the *split simple Lie algebra of type* $\mathrm{X_n}$ defined and studied in [J3, Chapter IV].

Definition 4.25 Suppose that L is a Lie algebra. We say that L is a *form of the Chevalley algebra of type* $\mathrm{X_n}$ (resp. a *form of the split Lie algebra of type* $\mathrm{X_n}$*)* if for some faithfully flat $\mathbb{F} \in \mathbb{K}$-alg, $L_{\mathbb{F}}$ is the Chevalley algebra of type $\mathrm{X_n}$ (resp. the split Lie algebra of type $\mathrm{X_n}$) over $\mathbb{F}$.

Remark 4.26 If L is a form of the Chevalley algebra of type $\mathrm{X_n}$, then by Lemma 4.22, $L/Z(L)$ is a form of the split Lie algebra of type $\mathrm{X_n}$.

Remark 4.27 (i) If L is the Chevalley algebra of type $\mathrm{X_n}$ and $\mathbb{F} \in \mathbb{K}$-alg, then $L_{\mathbb{F}}$ is the Chevalley algebra of type $\mathrm{X_n}$ over $\mathbb{F}$. Hence, by Lemma 4.22, if $\mathbb{F}$ is flat and L is the split Lie algebra of type $\mathrm{X_n}$, then $L_{\mathbb{F}}$ is the split Lie algebra of type $\mathrm{X_n}$ over $\mathbb{F}$.

(ii) If $\mathbb{K}$ is a field and L is a form of the Chevalley algebra of type $\mathrm{X_n}$ (resp. a form of the split Lie algebra of type $\mathrm{X_n}$), then it is easy to see using (i) that $L_{\mathbb{F}}$ is the Chevalley algebra of type $\mathrm{X_n}$ (resp. the split Lie algebra of type $\mathrm{X_n}$) over $\mathbb{F}$ *for some field* $\mathbb{F}$ *containing* $\mathbb{K}$.

4.7 Split Kantor Pairs of Type $\mathrm{X_n}$ and Their Forms

We return to our assumption that $\frac{1}{6} \in \mathbb{K}$.

Definition 4.28 Suppose P is a Kantor pair. We say P is a *split Kantor pair of type* $\mathrm{X_n}$ if $\mathfrak{K}(P)$ is the split Lie algebra of type $\mathrm{X_n}$; and we say that P is a *form of a split Kantor pair of type* $\mathrm{X_n}$ if for some faithfully flat $\mathbb{F} \in \mathbb{K}$-alg, $P_{\mathbb{F}}$ is a split Kantor pair of type $\mathrm{X_n}$ over $\mathbb{F}$.

The reader should keep in mind that, unlike the situation for Lie algebras, there can be non-isomorphic split Kantor pairs of type $\mathrm{X_n}$. This is already true for Jordan pairs, and the same type can even encompass Jordan as well as non-Jordan Kantor pairs (see Example 7.12).

Lemma 4.29 *Suppose P is a Kantor pair. Then P is a form of a split Kantor pair of type $\mathrm{X_n}$ if and only if $\mathfrak{K}(P)$ is a form of the split Lie algebra of type $\mathrm{X_n}$.*

Proof Let $\mathbb{F} \in \mathbb{K}\text{-alg}$ be faithfully flat and let L be the split Lie algebra of type $\mathrm{X_n}$ over $\mathbb{F}$. It is sufficient to show that $\mathfrak{K}(P_{\mathbb{F}}) \simeq L$ if and only if $\mathfrak{K}(P)_{\mathbb{F}} \simeq L$.

If $\mathfrak{K}(P_{\mathbb{F}}) \simeq L$, then $\mathfrak{K}(P_{\mathbb{F}})$ is a finitely generated $\mathbb{F}$-module and hence so is each $P^{\sigma}_{\mathbb{F}}$ (see Remark 4.9 (ii)). Thus each P^{σ} is a finitely generated $\mathbb{K}$-module [B2, I.3.6, Proposition 11], so $\mathfrak{K}(P_{\mathbb{F}}) \simeq \mathfrak{K}(P)_{\mathbb{F}}$ by Lemma 4.10.

Conversely, if $\mathfrak{K}(P)_{\mathbb{F}} \simeq L$, then $\mathfrak{K}(P)_{\mathbb{F}}$ is a finitely generated $\mathbb{F}$-module. Hence $\mathfrak{K}(P)$ is a finitely generated $\mathbb{K}$-module [B2, I.3.6, Proposition 11], and therefore so is each P^{σ}. Again $\mathfrak{K}(P_{\mathbb{F}}) \simeq \mathfrak{K}(P)_{\mathbb{F}}$ by Lemma 4.10. ∎

Remark 4.30 In view of Remark 4.27 and Lemma 4.29, we see the following.

(i) If P is a split Kantor pair of type $\mathrm{X_n}$ and $\mathbb{F} \in \mathbb{K}\text{-alg}$, then $P_{\mathbb{F}}$ is a split Kantor pair of type $\mathrm{X_n}$ over $\mathbb{F}$.

(ii) If $\mathbb{K}$ is a field and P is a form of a Kantor pair type $\mathrm{X_n}$, then $P_{\mathbb{F}}$ is a split Kantor pair algebra of type $\mathrm{X_n}$ over $\mathbb{F}$ *for some field* $\mathbb{F}$ *containing* $\mathbb{K}$.

It turns out that forms of split Kantor pairs of type $\mathrm{D_4}$, $\mathrm{E_6}$, $\mathrm{E_7}$, $\mathrm{E_8}$, $\mathrm{G_2}$, and $\mathrm{F_4}$ make up one of the four classes of central simple Kantor pairs that appear in the structure theorem mentioned in the introduction.

4.8 The Jordan Obstruction and the 2-dimension of a Kantor Pair

Let P be a Kantor pair. Note first that $\mathfrak{K}(P)_{-2} \oplus \mathfrak{K}(P)_0 \oplus \mathfrak{K}(P)_2$ is a 3-graded Lie algebra (with the grading scaled in the obvious way), so

$$J(P) := (\mathfrak{K}(P)_{-2}, \mathfrak{K}(P)_2)$$

is a Jordan pair with products $[[x,y],z]$ calculated in $\mathfrak{K}(P)$. We call $J(P)$ the *Jordan obstruction* of P. We use this term because $J(P)$ *is trivial if and only if P is Jordan* (see Remark 4.9(i)).

If L is a 5-graded Lie algebra that tightly envelops P, then by Corollary 4.17,

$$J(P) \simeq (L_{-2}, L_2) \tag{4.6}$$

with the products $[[x,y],z]$ calculated in L.

Suppose that $\mathbb{K}$ is a field. If $J(P)$ is finite dimensional, we call its dimension the 2-*dimension* of P. Further, if $J(P)$ has balanced dimension k, we say that P has *balanced* 2-*dimension* k. So 2-dimension and balanced 2-dimension (when they are defined) can be viewed as numerical measures of the distance of the pair P from Jordan theory. In particular, Kantor pairs of balanced 2-dimension 1 can be thought of as being *close to Jordan*, and they are, after Jordan pairs, the most studied and best understood Kantor pairs in the literature (see the following example).

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Example 4.31 Suppose $\mathbb{K}$ is a field. A *symplectic* triple system is defined as a triple system $\mathcal{T}$ (with product $[x, y, z]$) together with a nonzero skew-symmetric bilinear form $(\cdot \mid \cdot)$ on $\mathcal{T}$ satisfying a list of axioms [E, §4] [EK, §6.4], [YA, §2]. These structures are a variation on Freudenthal triple systems, which have been studied by many authors (see [M, FF, KS]) going back to the work of Freudenthal on exceptional Lie algebras. Indeed given a symplectic triple $\mathcal{T}$, the product $xyz = [x, y, z] - (x|z)y - (y|z)x$ endows $\mathcal{T}$ with the structure of a *Freudenthal triple system*, and this process can be reversed [E, Theorem 4.7].

Given a symplectic triple system $\mathcal{T}$, one can construct a 5-graded Lie algebra $\mathfrak{g}(\mathcal{T})$ with $\dim(\mathfrak{g}(\mathcal{T})_{\pm 2}) = 1$ [E, §4], [EK, §6.4], [YA, §2]. It turns out that the Kantor pair $\mathcal{P}(\mathcal{T})$ enveloped by $\mathfrak{g}(\mathcal{T})$ is the signed double of a $(1,1)$-Freudenthal–Kantor triple system (see Special Case 4.1 (iii)) with product $\{x, y, z\} = [x, y, z] - (x \mid y)z$. Moreover, one easily checks that $\mathfrak{g}(\mathcal{T})$ tightly envelops $\mathcal{P}(\mathcal{T})$ and hence, using Corollary 4.17, $\mathcal{P}(\mathcal{T})$ is close to Jordan in the above sense.

5 Short Peirce Gradings and BC_2-gradings

In this section, we assume that Δ is an irreducible root system of type BC_2. We realize Δ in the standard way as $\Delta = \{\ell_1\varepsilon_1 + \ell_2\varepsilon_2 : \ell_i \in \mathbb{Z},\ |\ell_1| + |\ell_2| \le 2\} \smallsetminus \{0\}$, where $\{\varepsilon_1, \varepsilon_2\}$ is an orthonormal basis in a 2-dimensional real Euclidean space E_Δ. (See [B3, VI.4.14], although there is an evident typo in the last line there.) We *fix the base* $\Gamma = \{\alpha_1, \alpha_2\}$ for Δ, where $\alpha_1 = \varepsilon_1$, $\alpha_2 = -\varepsilon_1 + \varepsilon_2$, in which case

$$\Delta = \left\{ \pm\alpha_1, \pm\alpha_2, \pm(\alpha_1 + \alpha_2), \pm 2\alpha_1, \pm(2\alpha_1 + \alpha_2), \pm(2\alpha_1 + 2\alpha_2) \right\}. \tag{5.1}$$

We identify $Q_\Delta = \mathbb{Z}^2$ using the $\mathbb{Z}$-basis Γ for Q_Δ, so that $(\ell_1, \ell_2) = \ell_1\alpha_1 + \ell_2\alpha_2$ for $\ell_i \in \mathbb{Z}$. (*Caution*: We are not using the $\mathbb{Z}$-basis $\{\varepsilon_1, \varepsilon_2\}$ of Q_Δ for this identification.) So, as in Subsection 2.2, any BC_2-graded Lie algebra is a $\mathbb{Z}^2$-graded Lie algebra.

In this section, we introduce Kantor pairs with short Peirce gradings and see how they are related to BC_2-graded Lie algebras.

5.1 Short Peirce Gradings

Definition 5.1 A *short Peirce grading* (or *SP-grading*) of a Kantor pair P is a $\mathbb{Z}$-grading $P = \bigoplus_{i \in \mathbb{Z}} P_i$ of P such that $\mathrm{supp}_{\mathbb{Z}}(P) \subseteq \{0, 1\}$. In that case we have $P = P_0 \oplus P_1$. A Kantor pair P together with a short Peirce grading of P is called a *short Peirce graded* (or *SP-graded*) Kantor pair.

Note that if P is an SP-graded Kantor pair, then

$$\{P_i^\sigma, P_{1-i}^{-\sigma}, P_i^\sigma\} = 0. \tag{5.2}$$

for $i = 0, 1$ and $\sigma = \pm$. Every Kantor pair P has at least two SP-gradings, the *zero SP-grading*, $P = P_0$ with $P_1 = 0$, and the *one SP-grading*, $P = P_1$ with $P_0 = 0$. We call these two SP-gradings *trivial*.

The following example explains our use of the term short Peirce grading.

Example 5.2 (The Jordan case) Suppose that P is a Jordan pair. Neher [N1] defined a *Peirce grading* of P to be a $\mathbb{Z}$-grading $P = \bigoplus_{i \in \mathbb{Z}} P_i$ of P such that $\mathrm{supp}_{\mathbb{Z}}(P) \subseteq \{0, 1, 2\}$

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and

$$\{P_2^\sigma, P_0^{-\sigma}, P_0^\sigma\} = \{P_0^\sigma, P_2^{-\sigma}, P_2^\sigma\} = 0 \tag{5.3}$$

for $\sigma = \pm$. (See also [LN, §8], where Peirce gradings are used in the study of groups associated with Jordan pairs.) Clearly SP-gradings of P are precisely the same as Peirce gradings of P satisfying $P_2 = 0$. The motivating example in [N1] of a Peirce grading is the Peirce decomposition $P = P_0 \oplus P_1 \oplus P_2$ relative to an idempotent c in P [L, Theorem 5.4]. This Peirce grading is never short (if $c \neq 0$) since $c \in P_2$. However, a very important case in the structure theory of Jordan pairs occurs when $P_0 = 0$ [L, Theorem 8.2]. In that case, one obtains an SP-grading $P = \widetilde{P}_0 \oplus \widetilde{P}_1$, where $\widetilde{P}_i = P_{2-i}$ for $i \in \mathbb{Z}$.

Remark 5.3 Let P be an SP-graded Kantor pair. There are two simple procedures for modifying P to obtain another SP-graded Kantor pair.

(i) Let $P^{\mathrm{op}} = (P^+, P^-)$ be the opposite pair of P and set $(P^{\mathrm{op}})_i^\sigma = P_i^{-\sigma}$ for $\sigma = \pm$, $i \in \mathbb{Z}$. Then P^{op} is an SP-graded Kantor pair called the *opposite* of P (as an SP-graded Kantor pair).

(ii) Let $\overline{P} = P$ as a Kantor pair and define $\overline{P}_i = P_{1-i}$ for $i \in \mathbb{Z}$. Then $\overline{P}$ is an SP-graded Kantor pair called the *shift* of P. (We use this terminology because, if we view degrees modulo 2, we have shifted the SP-grading of P by 1 to obtain $\overline{P}$.) Note that if P has the zero SP-grading, then $\overline{P}$ has the one SP-grading and vice-versa.

We will see in Subsection 6.1 that the SP-graded Kantor pairs P^{op} and $\overline{P}$ are examples of Weyl images of P.

5.2 Component Gradings

If $L = \bigoplus_{(i,j)\in\mathbb{Z}^2} L_{(i,j)}$ is a $\mathbb{Z}^2$-graded module, we often write $L_{(i,j)}$ as $L_{i,j}$ for brevity. Then the *first component grading* of L is the $\mathbb{Z}$-grading $L = \bigoplus_{i\in\mathbb{Z}} L_{i,*}$, where $L_{i,*} = \bigoplus_{j\in\mathbb{Z}} L_{i,j}$. Similarly, we have the *second component* grading $L = \bigoplus_{j\in\mathbb{Z}} L_{*,j}$ with $L_{*,j} = \bigoplus_{i\in\mathbb{Z}} L_{i,j}$. Of course, if $L = \bigoplus_{(i,j)\in\mathbb{Z}^2} L_{i,j}$ is an algebra grading, then so are its component gradings.

5.3 The SP-graded Kantor Pair Enveloped by a BC_2-graded Lie Algebra

Suppose that L is a BC_2-graded Lie algebra. Then by (5.1), the first component grading $L = \bigoplus_{i\in\mathbb{Z}} L_{i,*}$ of L is a 5-grading, and we have $L_{-1,*} = L_{-1,0} \oplus L_{-1,-1}$ and $L_{1,*} = L_{1,0} \oplus L_{1,1}$. Let P be the Kantor pair enveloped by L with this 5-grading. So

$$P = (L_{-1,*}, L_{1,*}) = (L_{-1,0} \oplus L_{-1,-1}, L_{1,0} \oplus L_{1,1})$$

with products $\{x^\sigma, y^{-\sigma}, z^\sigma\} = [[x^\sigma, y^{-\sigma}], z^\sigma]$. For $i \in \mathbb{Z}$, let $P_i = (P_i^-, P_i^+)$, where

$$P_i^\sigma = L_{\sigma 1, \sigma i} \tag{5.4}$$

for $\sigma = \pm$. Then $P = P_0 \oplus P_1$ is an SP-grading of P since

$$\{P_i^\sigma, P_j^{-\sigma}, P_k^\sigma\} = [[L_{\sigma 1,\sigma i}, L_{-\sigma 1,-\sigma j}], L_{\sigma 1,\sigma k}] \subseteq L_{\sigma 1,\sigma(i-j+k)} = P_{i-j+k}^\sigma$$

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for $\sigma = \pm$, $i, j, k \in \mathbb{Z}$. We call P together with this SP-grading the *SP-graded Kantor pair enveloped by the* BC_2*-graded Lie algebra* L, and we say that *the* BC_2*-graded Lie algebra* L *envelops the SP-graded Kantor pair* P. If, in addition, (4.2) holds, we say that the BC_2-graded Lie algebra L *tightly envelops* P.

Remark 5.4 It follows that every BC_2-graded Lie algebra enveloping an SP-graded Kantor pair P is also a 5-graded Lie algebra, using the first component grading, enveloping the pair P when considered without its SP-grading. The former is tight if and only if the latter is tight.

5.4 The Standard BC_2-grading of $\mathfrak{K}(P)$

We now see that any SP-graded Kantor pair is enveloped by some BC_2-graded Lie algebra.

Proposition 5.5 *Suppose that P is an SP-graded Kantor pair. Then there exists a unique* BC_2*-grading of* $\mathfrak{K}(P)$*, which we call the* standard BC_2-grading of $\mathfrak{K}(P)$*, such that*

$$\mathfrak{K}(P)_{\sigma 1, \sigma i} = P_i^\sigma \tag{5.5}$$

for $\sigma = \pm$, $i \in \mathbb{Z}$. Moreover, for $\sigma = \pm$,

$$\begin{aligned} \mathfrak{K}(P)_{\sigma 2, \sigma 2i} &= [P_i^\sigma, P_i^\sigma] \text{ for } i = 0, 1, \quad \mathfrak{K}(P)_{\sigma 2, \sigma 1} = [P_0^\sigma, P_1^\sigma], \\ \mathfrak{K}(P)_{0, \sigma 1} &= [P_1^\sigma, P_0^{-\sigma}], \quad \mathfrak{K}(P)_{0,0} = \textstyle\sum_{j=0,1} [P_j^\sigma, P_j^{-\sigma}]. \end{aligned} \tag{5.6}$$

The first component grading of the standard BC_2*-grading is the standard* 5*-grading of* $\mathfrak{K}(P)$*, and with the standard* BC_2*-grading,* $\mathfrak{K}(P)$ *tightly envelops* P*.*

Proof Let $\mathfrak{K} = \mathfrak{K}(P)$ and $T = T_{\mathfrak{K}}(P)$. Since $\mathfrak{K}$ is generated by T (by Proposition 4.8), uniqueness in the first statement is clear. For existence, define a $\mathbb{Z}^2$-grading of the module T by setting

$$T_{i,j} = \begin{cases} P_{\sigma j}^\sigma & \text{if } i = \sigma 1 \text{ with } \sigma = \pm, \\ 0 & \text{otherwise.} \end{cases}$$

Then one checks directly using (4.1) that the trilinear product $[[x,y],z]$ on T is $\mathbb{Z}^2$-graded. Also, since the $\mathbb{Z}^2$-grading of T has finite support, it induces a natural $\mathbb{Z}^2$-grading of the Lie algebra $\mathrm{End}(T)$ under the commutator product. Furthermore, since $\mathfrak{S}(P) = [T,T]$, we see that $\mathfrak{S}(P)$ is a $\mathbb{Z}^2$-graded subalgebra of this Lie algebra. Next, we give $\mathfrak{K} = \mathfrak{S}(P) \oplus T$ the direct sum $\mathbb{Z}^2$-grading. It then follows that $\mathfrak{K}$ is a $\mathbb{Z}^2$-graded Lie algebra. But, since $\mathfrak{K} = [T,T] \oplus T$, we have

$$\mathrm{supp}_{\mathbb{Z}^2}(\mathfrak{K}) \subseteq (\mathrm{supp}_{\mathbb{Z}^2}(T) + \mathrm{supp}_{\mathbb{Z}^2}(T)) \cup \mathrm{supp}_{\mathbb{Z}^2}(T). \tag{5.7}$$

So, since $\mathrm{supp}_{\mathbb{Z}^2}(T) \subseteq \{\pm(1,0), \pm(1,1)\}$, we have $\mathrm{supp}_{\mathbb{Z}^2}(\mathfrak{K}) \subseteq \Delta$. Thus $\mathfrak{K}$ is Δ-graded. Moreover, the union in (5.7) is disjoint, so we obtain (5.5). The second statement follows immediately from this proof, and the last statement is clear using Proposition 4.8 and Remark 5.4. ∎

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If P is an SP-graded Kantor pair, then, unless mentioned to the contrary, we will regard $\mathfrak{K}(P)$ as BC_2-graded with its standard BC_2-grading.

Using Remark 5.4, it is easy to write down BC_2-graded versions of many of the 5-graded results shown in Section 4. We content ourselves now with recording the results of this type that will be needed in this paper or in [AF2].

Proposition 5.6 *Let P and P' be SP-graded Kantor pairs.*

(i) *P and P' are SP-graded isomorphic if and only if $\mathfrak{K}(P)$ and $\mathfrak{K}(P')$ are BC_2-graded isomorphic.*
(ii) *If L is a BC_2-graded Lie algebra that tightly envelops P, then there exists a unique BC_2-graded Lie algebra isomorphism $\varphi: L \to \mathfrak{K}(P)$ that restricts to the identity map on $T_L(P)$.*
(iii) *If $P \neq 0$ and L is a simple BC_2-graded Lie algebra that envelops P, then L tightly envelops P and so we have the conclusion in* (ii).

Proof (i) Since $T_{\mathfrak{K}(P)}(P)$ generates the algebra $\mathfrak{K}(P)$, the implication "$\Rightarrow$" follows from Corollary 4.18, and the reverse implication is clear.

(ii) By Remark 5.4 and Corollary 4.17, there exists a unique 5-graded Lie algebra isomorphism $\varphi: L \to \mathfrak{K}(P)$ that restricts to the identity map on $T_L(P)$. If we use φ to transfer the BC_2-grading from L to $\mathfrak{K}(P)$, we obtain a grading that must coincide with the standard BC_2-grading by uniqueness in Proposition 5.5.

(iii) This follows from (ii) using Remark 5.4 and Proposition 4.19. ∎

6 Weyl Images of SP-graded Kantor Pairs

In this section (except in Subsection 6.5), we continue with the assumptions of Section 5. In particular, Δ is the irreducible root system of type BC_2 with base $\Gamma = \{\alpha_1, \alpha_2\}$, where α_1 is the short basic root and α_2 is the long basic root. Let $s_\alpha \in W_\Delta$ be the reflection through the hyperplane orthogonal to α for $\alpha \in \Delta$, and put $s_i = s_{\alpha_i}$ for $i = 1, 2$. The generators s_1 and s_2 of W_Δ satisfy $s_1 s_2 s_1 s_2 = s_2 s_1 s_2 s_1 = -1$, and

$$\mathrm{Aut}(\Delta) = W_\Delta = \{1,\ s_1,\ s_2,\ s_2 s_1,\ -1,\ -s_1,\ -s_2,\ -s_2 s_1\}$$

is the dihedral group of order 8. In particular, since $\mathrm{Aut}(\Delta) = W_\Delta$, all images of a BC_2-graded Lie algebra are Weyl images. (See Subsection 2.3.)

6.1 Weyl Images

Definition 6.1 Suppose that P is an SP-graded Kantor pair and $u \in W_\Delta$. Choose a BC_2-graded Lie algebra L that envelops P. Then ${}^uL$ is a BC_2-graded Lie algebra, which therefore envelops an SP-graded Kantor pair ${}^uP$ (see Subsection 5.3). We call ${}^uP$ the *u-image* (or a *Weyl image*) of P.

In parts (i)–(iii) of the next proposition, we give an internal characterization of the Weyl image ${}^uP$. It follows in part (iv) that ${}^uP$ is well defined.

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Proposition 6.2 *Suppose that P is an SP-graded Kantor pair, $u \in W_\Delta$, L is a* BC_2*-graded Lie algebra that envelops P, and ${}^uP$ is defined as above.*

(i) *$T_{{}^uL}({}^uP) = T_L(P)$; so if L tightly envelopes P, then ${}^uL$ tightly envelopes ${}^uP$.*

(ii) *$u^{-1}(\alpha_1) = \pi(\alpha_1 + a\alpha_2)$ and $u^{-1}(\alpha_1 + \alpha_2) = \rho(\alpha_1 + b\alpha_2)$ for some $\pi = \pm$, $\rho = \pm$, $a, b \in \{0,1\}$, in which case $({}^uP)_0^\sigma = P_a^{\pi\sigma}$ and $({}^uP)_1^\sigma = P_b^{\rho\sigma}$.*

(iii) *If $\sigma = \pm$ and $i, j, k \in \{0,1\}$, the σ-product on ${}^uP$ restricted to $({}^uP)_i^\sigma \times ({}^uP)_j^{-\sigma} \times ({}^uP)_k^\sigma$ is given by $[[x,y],z]$ in L, which can be expressed in terms of products in P using* (ii) *and* (4.1).

(iv) *${}^uP$ does not depend on the choice of L. (This justifies our notation ${}^uP$.)*

Proof First $T_L(P) = \sum_{\alpha\in S^-\cup S^+} L_\alpha$, where $S^\sigma = \{\sigma\alpha_1, \sigma(\alpha_1+\alpha_2)\}$, and similarly $T_{{}^uL}({}^uP) = \sum_{\alpha\in S^-\cup S^+} ({}^uL)_\alpha = \sum_{\alpha\in u^{-1}(S^-\cup S^+)} L_\alpha$. But $S^-\cup S^+$ is the set of short roots of Δ and hence this set is stabilized by W. So we have (i) and the expressions for $u^{-1}(\alpha_1)$ and $u^{-1}(\alpha_1+\alpha_2)$ in (ii). Furthermore $({}^uP)_0^\sigma = ({}^uL)_{\sigma 1,0} = ({}^uL)_{\sigma\alpha_1} = L_{u^{-1}(\sigma\alpha_1)} = L_{\pi\sigma(\alpha_1+a\alpha_2)} = L_{\pi\sigma 1,\pi\sigma a} = P_a^{\pi\sigma}$, and similarly $({}^uP)_1^\sigma = P_b^{\rho\sigma}$, so we have (ii). Next (iii) follows from (ii), and (iv) follows from (ii) and (iii). ∎

Corollary 6.3 *If P is an SP-graded Kantor pair and $u \in W_\Delta$, then $\mathfrak{K}({}^uP)$ with its standard* BC_2*-grading is graded isomorphic to the u-image ${}^u\mathfrak{K}(P)$ of $\mathfrak{K}(P)$ with its standard* BC_2*-grading. So $\mathfrak{K}({}^uP)$ is isomorphic to $\mathfrak{K}(P)$ (as algebras).*

Proof By Proposition 5.5, $\mathfrak{K}(P)$ with its standard BC_2-grading tightly envelops P. So by Proposition 6.2 (i), the BC_2-graded Lie algebra ${}^u\mathfrak{K}(P)$ tightly envelops ${}^uP$, which completes the proof by Proposition 5.6 (ii). ∎

If P is an SP-graded Kantor pair, we see using Proposition 6.2 (iv) and (2.1) that

$$ {}^1P = P \quad \text{and} \quad {}^{u_1}({}^{u_2}P) = {}^{u_1u_2}P \tag{6.1}$$

for $u_1, u_2 \in W_\Delta$. In other words, we have a left action of W_Δ on the class of SP-graded Kantor pairs.

Proposition 6.4 *Let P be an SP-graded Kantor pair and $u \in W_\Delta$. Then P is simple (resp. central simple) if and only if ${}^uP$ is simple (resp. central simple). Also P is a split Kantor pair of type* $\mathrm{X_n}$ *(resp. a form of a split Kantor pair of type* $\mathrm{X_n}$*) if and only if the same is true for ${}^uP$.*

Proof In view of Corollary 6.3, the first statement follows from Proposition 4.11 and Corollary 4.14, while the second statement follows from Lemma 4.29. ∎

If P is an SP-graded Kantor pair, two of the Weyl images of P (besides ${}^1P$) are already familiar to us. Indeed, using the notation of Remark 5.3, we have

$$ {}^{-1}P = P^{\mathrm{op}} \quad \text{and} \quad {}^{s_2}P = \overline{P}. \tag{6.2}$$

The first of these follows immediately from (ii) and (iii) in Proposition 6.2, as does the second, since s_2 exchanges α_1 and $\alpha_1 + \alpha_2$.

6.2 Reflection

If P is an SP-graded Kantor pair, the Weyl image ${}^{s_1}P$ of P is of particular interest in the theory. For convenience we introduce the following notation:

$$\check{P} := {}^{s_1}P. \tag{6.3}$$

Since s_1 is the reflection in W_Δ corresponding to the short basic root α_1, we call $\check{P}$ the *reflection of P corresponding to the short basic root*, or more simply the *reflection* of P. We have the following explicit description of $\check{P}$.

Proposition 6.5 *Suppose that P is an SP-graded Kantor pair. Then* $\check{P}_0 = (P_0)^{\mathrm{op}}$ *and* $\check{P}_1 = P_1$ *as Kantor pairs, and the σ-product* $\{\cdot,\cdot,\cdot\}^{\vee}$ *on* $\check{P}$ *is given by*

$$\begin{aligned}\{a_0^{-\sigma}+a_1^{\sigma}, b_0^{\sigma}+b_1^{-\sigma}, c_0^{-\sigma}+c_1^{\sigma}\}^{\vee} = \{a_0^{-\sigma}, b_0^{\sigma}, c_0^{-\sigma}\} - \{b_1^{-\sigma}, a_1^{\sigma}, c_0^{-\sigma}\} + K(a_0^{-\sigma}, b_1^{-\sigma})c_1^{\sigma}\\ + \{a_1^{\sigma}, b_1^{-\sigma}, c_1^{\sigma}\} - \{b_0^{\sigma}, a_0^{-\sigma}, c_1^{\sigma}\} + K(a_1^{\sigma}, b_0^{\sigma})c_0^{-\sigma},\end{aligned}$$

where $a_i^{\tau}, b_i^{\tau}, c_i^{\tau} \in P_i^{\tau}$ *in each case.*

Proof Since $s_1(\alpha_1) = -\alpha_1$ and $s_1(\alpha_1+\alpha_2) = \alpha_1+\alpha_2$, the conclusions follow easily using parts (ii) and (iii) of Proposition 6.2 and (4.1). ■

Corollary 6.6 *Suppose P is an SP-graded pair. If the SP-grading on P is the one SP-grading, then* $\check{P} = P$ *as SP-graded Kantor pairs. On the other hand, if the SP-grading on P is the zero SP-grading, then* $\check{P} = P^{\mathrm{op}}$ *as SP-graded Kantor pairs.*

Suppose that P is an SP-graded Kantor pair. By (6.1), (6.2), and (6.3), the eight elements $1, s_1, s_2, s_2s_1, -1, -s_1, -s_2, -s_2s_1$ of W_Δ yield in order the following eight SP-graded Weyl images of P:

$$P,\quad \check{P},\quad \overline{P},\quad \overline{\check{P}},\quad P^{\mathrm{op}},\quad (\check{P})^{\mathrm{op}},\quad (\overline{P})^{\mathrm{op}},\quad (\overline{\check{P}})^{\mathrm{op}}. \tag{6.4}$$

Since shifting does not change the underlying ungraded Kantor pair, our list includes, in general, four non-isomorphic Kantor pairs: $P, \check{P}, P^{\mathrm{op}}, (\check{P})^{\mathrm{op}}$. This suggests the central role that reflection plays in our study of Weyl images.

6.3 A Geometric Interpretation of $\check{P}$

First (using the notation at the beginning of Section 5) $\{\varepsilon_1,\varepsilon_2\}$ and $\{\alpha_1,\alpha_2\}$ are $\mathbb{Z}$-bases of Q_Δ, with $\alpha_1 = \varepsilon_1$ and $\alpha_2 = -\varepsilon_1+\varepsilon_2$. To avoid conflict with our ongoing notation $(\ell_1,\ell_2) = \ell_1\alpha_1+\ell_2\alpha_2$, we set $\langle \ell_1,\ell_2\rangle = \ell_1\varepsilon_1+\ell_2\varepsilon_2$ for $\ell_1.\ell_2 \in \mathbb{Z}$. Note that $(k_1,k_2) = \langle k_1-k_2, k_2\rangle$ and $s_1(\langle \ell_1,\ell_2\rangle) = \langle -\ell_1,\ell_2\rangle$. In the figures below, we will use the coordinates $\langle \ell_1,\ell_2\rangle$ to plots points in Q_Δ

Suppose now that L is a BC_2-graded Lie algebra enveloping an SP-graded Kantor pair P. Then the first component grading of L is $L = \bigoplus_{i\in\mathbb{Z}} L_{i,*}$, where

$$L_{i,*} = \bigoplus_{\ell_2\in\mathbb{Z}} L_{i,\ell_2} = \bigoplus_{\ell_2\in\mathbb{Z}} L_{\langle i-\ell_2,\ell_2\rangle} = \bigoplus_{\substack{\ell_1,\ell_2\in\mathbb{Z}\\ \ell_1+\ell_2=i}} L_{\langle \ell_1,\ell_2\rangle}. \tag{6.5}$$

Thus, the first component grading for L is pictured by the dashed lines in Figure 1 below. (More precisely the support of $L_{i,*}$ in Q_Δ is contained in the set of points

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labelled by circles, filled or unfilled, on the dashed line labelled as i.) Hence the Kantor pair P is pictured by the dashed lines labelled as -1 and 1 in Figure 1. (In this case we have filled the support circles for emphasis.) In fact, using (5.4), we see that $P_0^\sigma = L_{\langle \sigma 1, 0\rangle}$ and $P_1^\sigma = L_{\langle 0, \sigma 1\rangle}$, so the SP-grading of P is pictured as well.

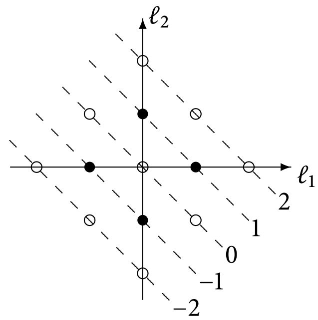

Figure 1. L and P

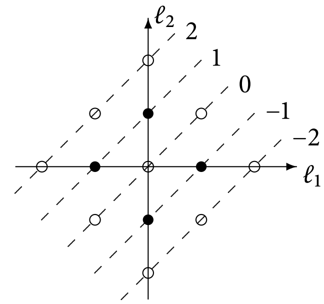

Figure 2. $\check{L}$ and $\check{P}$

On the other hand, with $\check{L} = {}^{s_1}L$, the first component grading of $\check{L}$ is $\check{L} = \bigoplus_{i\in\mathbb{Z}} \check{L}_{i,*}$, where, using (6.5) applied to $\check{L}$, we have

$$\check{L}_{i,*} = \bigoplus_{\substack{\ell_1,\ell_2\in\mathbb{Z},\\ \ell_1+\ell_2=i}} \check{L}_{\langle \ell_1,\ell_2\rangle} = \bigoplus_{\substack{\ell_1,\ell_2\in\mathbb{Z},\\ -\ell_1+\ell_2=i}} L_{\langle \ell_1,\ell_2\rangle}.$$

Thus the first component grading for $\check{L}$ is pictured by the dashed lines in Figure 2. Next, by definition $\check{P}$ is enveloped by $\check{L}$, so the Kantor pair $\check{P}$ is pictured by the dashed lines labelled as -1 and 1 in Figure 2. Also we have $\check{P}_0^\sigma = L_{\langle -\sigma 1, 0\rangle}$ and $\check{P}_1^\sigma = L_{\langle 0, \sigma 1\rangle}$, so the SP-grading of $\check{P}$ is pictured as well.

Evidently, Figure 2 is obtained from Figure 1 by orthogonal reflection through the vertical axis.

6.4 The Jordan Obstruction of a Weyl Image

Suppose P is an SP-graded Kantor pair.

Recall from Subsection 4.8 that the Jordan obstruction $J(P)$ of P is the Jordan pair $(\mathfrak{K}(P)_{-2,*}, \mathfrak{K}(P)_{2,*})$. So

$$J(P) = (\mathfrak{K}(P)_{-2,0} \oplus \mathfrak{K}(P)_{-2,-1} \oplus \mathfrak{K}(P)_{-2,-2}, \mathfrak{K}(P)_{2,0} \oplus \mathfrak{K}(P)_{2,1} \oplus \mathfrak{K}(P)_{2,2}).$$

It is interesting to note that $J(P) = \bigoplus_{i=0}^2 J(P)_i$ is Peirce graded (see Subsection 4.8) with $J(P)_i^\sigma = \mathfrak{K}(P)_{\sigma 2, \sigma i}$, since $\mathfrak{K}(P)_{0,\sigma 2} = 0$. However, we will view $J(P)$ as an ungraded Jordan pair.

If L is a BC_2-graded Lie algebra that tightly envelops P, then

$$J(P) \simeq (L_{-2,*}, L_{2,*}) = (L_{-2,0} \oplus L_{-2,-1} \oplus L_{-2,-2}, L_{2,0} \oplus L_{2,1} \oplus L_{2,2}) \tag{6.6}$$

by Proposition 5.6 (ii).

In view of (6.4), the following proposition tells us how to compute the Jordan obstruction of any Weyl image of P.

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Proposition 6.7 *If P is an SP-graded Kantor pair, and L is a* BC_2*-graded Lie algebra that tightly envelops P, then $J(\overline{P}) \simeq J(P)$, $J(P^{\mathrm{op}}) \simeq J(P)^{\mathrm{op}}$ and*

$$J(\check{P}) \simeq (L_{2,0} \oplus L_{0,-1} \oplus L_{-2,-2}, L_{-2,0} \oplus L_{0,1} \oplus L_{2,2}) \tag{6.7}$$

under the products $[[x,y],x]$ calculated in L.

Proof If $u \in W$, the BC_2-graded Lie algebra ${}^uL$ tightly envelops ${}^uP$, by Proposition 6.2 (i). Thus, by (6.6), $J({}^uP) \simeq (({}^uL)_{-2,*}, ({}^uL)_{2,*})$. If $u = s_2$, $u = -1$, or $u = s_1$, then $u^{-1} = u$ maps $\{2\alpha_1, 2\alpha_1+\alpha_2, 2\alpha_1+2\alpha_2\}$ onto $\{2\alpha_1, 2\alpha_1+\alpha_2, 2\alpha_1+2\alpha_2\}$, $\{-2\alpha_1, -2\alpha_1-\alpha_2, -2\alpha_1-2\alpha_2\}$, and $\{-2\alpha_1, \alpha_2, 2\alpha_1+2\alpha_2\}$, respectively. ■

If follows from Proposition 6.7 (or from the definitions) that if P is Jordan, then so are $\overline{P}$ and P^{op}. We now see that this fails for the reflection $\check{P}$ of P.

Lemma 6.8 *$\check{P}$ is a Jordan pair if and only if P_0 and P_1 are left ideals of P which are Jordan pairs. Suppose further that P is Jordan. Then $\check{P}$ is Jordan if and only if P_0 and P_1 are ideals of P, in which case $P = P_0 \oplus P_1$ and $\check{P} = (P_0)^{\mathrm{op}} \oplus P_1$ are direct sums of ideals.*

Proof Now $\check{P}$ is Jordan if and only $J(\check{P}) = 0$, and, by (6.7) (with $L = \mathfrak{K}(P)$) and (5.6), this holds if and only if $[P_i^\sigma, P_i^\sigma] = 0$ and $[P_1^\sigma, P_0^{-\sigma}] = 0$ in $\mathfrak{K}(P)$ for $\sigma = \pm$ and $i = 0,1$. Moreover, this holds if and only if $K(P_i^\sigma, P_i^\sigma)P^{-\sigma} = 0$ and $\{P_i^\sigma, P_{1-i}^{-\sigma}, P^\sigma\} = 0$ for $\sigma = \pm$, $i = 0,1$. Then, using (5.2), we see that $\check{P}$ is Jordan if and only if each P_i is Jordan and $\{P_{1-i}^\sigma, P_i^{-\sigma}, P_i^\sigma\} = 0$ for $\sigma = \pm$, $i = 0,1$. But one checks easily using (5.2) that this last condition holds if and only if P_i is a left ideal of P for $i = 0,1$.

Suppose next that P is Jordan. If $\check{P}$ is Jordan, then we know that P_i is a left and right ideal of P for $i = 0,1$; and hence, by (5.2), P_i is an ideal for $i = 0,1$. Conversely, suppose that each P_i is an ideal of P. Then $P = P_0 \oplus P_1$ is a direct sum of ideals; so by Proposition 6.5, $\check{P} = (P_0)^{\mathrm{op}} \oplus P_1$ is a direct sum of ideals and hence $\check{P}$ is Jordan. ■

Corollary 6.9 *Suppose that P is a simple Jordan pair with non-trivial SP-grading. Then $\check{P}$ is not Jordan.*

There are many well-understood examples of simple Jordan pairs with non-trivial SP-gradings (for example simple Jordan pairs with a maximal non-invertible idempotent). In this way, reflection produces many examples of simple SP-graded Kantor pairs that are not Jordan. We consider one such example in Subsection 7.3.

In the same spirit, we will give an example in Subsection 8.8 of an SP-graded Kantor pair of balanced 2-dimension 1 whose reflection has balanced 2-dimension 5. So reflection produces a Kantor pair that is far from Jordan starting from a pair that is close to Jordan.

6.5 Remarks on Weyl Images Using Other Rank 2 Root Systems

Up to this point, we have discussed how BC_2-graded Lie algebras arise in the theory of SP-graded Kantor pairs. With this in mind, it is natural to wonder how Lie algebras graded by other irreducible rank 2 root systems fit into this picture. We discuss this briefly in this subsection (with few details), and explain why we have focused on the BC_2-case.

Suppose that Δ is an irreducible root system with base $\Gamma = \{\alpha_1, \alpha_2\}$, and denote the type of Δ by X_2. Let $m_1\alpha_1 + m_2\alpha_2$ be the highest root of Δ and suppose that $m_1 = 1$ or 2, and let $\alpha_1 + n_2\alpha_2$ be the highest root in $(\alpha_1 + \mathbb{Z}\alpha_2) \cap \Delta$. Then we have the following possible cases:

(1) $X_2 = A_2, m_1 = 1, n_2 = 1$; (2) $X_2 = B_2, \alpha_1$ long, $m_1 = 1, n_2 = 2$;
(3) $X_2 = B_2, \alpha_1$ short, $m_1 = 2, n_2 = 1$; (4) $X_2 = G_2, \alpha_1$ long, $m_1 = 2, n_2 = 3$;
(5) $X_2 = BC_2, \alpha_1$ long, $m_1 = 2, n_2 = 2$; (6) $X_2 = BC_2, \alpha_1$ short, $m_1 = 2, n_2 = 1$.

Suppose that L is a Δ-graded Lie algebra. Then, since $m_1 \leq 2$, the first component grading $L = \bigoplus_{i\in\mathbb{Z}} L_{i,*}$ of L is a 5-grading. Let $P = (L_{-1,*}, L_{1,*})$ be the Kantor pair enveloped by L with this 5-grading. Then $P = \bigoplus_{j\in\mathbb{Z}} P_j = \bigoplus_{j=0}^{n_2} P_j$, where $P_j^\sigma = L_{\sigma 1,\sigma j}$, is a $\mathbb{Z}$-graded Kantor pair which we say is *enveloped by the Δ-graded Lie algebra L*. Furthermore, in each case we have $[P_i^\sigma, P_j^{-\sigma}] = 0$ in L if $(i-j)\alpha_2 \notin \Delta \cup \{0\}$ and $[P_i^\sigma, P_j^\sigma] = 0$ in L if $2\alpha_1 + (i+j)\alpha_2 \notin \Delta \cup \{0\}$, and these facts translate into relations in P which we call *Peirce* relations. For example, the Peirce relations in Case 2 are the relations (5.3) in Example 5.2 as well as the relations $K(P_i^\sigma, P_j^\sigma) = 0$ for all i, j. In Case 6, the set of relations is empty.

If we are in Case ℓ, where $1 \leq \ell \leq 6$, we are led to introduce the class $\mathfrak{C}(\ell)$ of $\mathbb{Z}$-graded Kantor pairs that by definition have support contained in $\{0, \dots, n_2\}$, and satisfy the Peirce relations mentioned above.

For example $\mathfrak{C}(1)$, $\mathfrak{C}(2)$, and $\mathfrak{C}(3)$ are, respectively, the class of SP-graded Jordan pairs, the class of Peirce graded Jordan pairs (see Example 5.2), and the class of SP-graded Kantor pairs with P_0 and P_1 Jordan. We leave it to the interested reader to work out $\mathfrak{C}(4)$ and $\mathfrak{C}(5)$. Of course, $\mathfrak{C}(6)$ is the class of SP-graded Kantor pairs that we have been studying.

So in each case, any Δ-graded Lie algebra envelops a $\mathbb{Z}$-graded Kantor pair in $\mathfrak{C}(\ell)$. And conversely one sees exactly as in Proposition 5.5 that any $\mathbb{Z}$-graded Kantor pair in $\mathfrak{C}(\ell)$ is enveloped by a Δ-graded Lie algebra.

If P is in $\mathfrak{C}(\ell)$ and $u \in W_\Delta$, one can define the Weyl image ${}^uP$ in $\mathfrak{C}(\ell)$ as in Definition 6.1. However, to see that ${}^uP$ is well defined as in Proposition 6.2, one needs the fact that $S^- \cup S^+$ is invariant under W_Δ, where $S^\sigma := \{\sigma\alpha_1 + \sigma j\alpha_2\}_{j=1}^{n_2}$. Since this is not true in general, it is natural to consider Weyl images ${}^uP$ only for u in the stabilizer V of $S^- \cup S^+$ in W_Δ.

Now because of our work in this article, we understand Weyl images in $\mathfrak{C}(6)$, and we therefore also understand them in $\mathfrak{C}(3)$, since $\mathfrak{C}(3)$ is a subclass of $\mathfrak{C}(6)$ that is closed under Weyl images in $\mathfrak{C}(6)$. In the remaining four Cases 1, 2, 4 and 5, it turns out that we have $u(S^+) = S^+$ or S^- for $u \in V$ and hence ${}^uP \simeq P$ or P^{op} *as ungraded Kantor pairs* for $P \in \mathfrak{C}(\ell)$ and $u \in V$. So in those four cases, Weyl images cannot be used to produce new ungraded Kantor pairs. For this reason, we have only considered Case 6 in this paper.

Nevertheless, it seems that all of the classes $\mathfrak{C}(\ell)$, as well as analogous classes for higher rank root systems, are of interest in their own right and may play a role in the development of a theory of grids for Kantor pairs following the lead of Neher in the Jordan case (see [N2] and the references therein).

7 Kantor Pairs of Skew Transformations

In this section, let V^- and V^+ be modules and let $g\colon V^- \times V^+ \to \mathbb{K}$ be a nondegenerate bilinear form. If $v^+ \in V^+$ and $v^- \in V^-$, we set $g(v^+, v^-) = g(v^-, v^+)$ for convenience.

7.1 3-graded Lie Algebras of Skew Transformations

Let $\widetilde{V} = V^- \oplus V^+$, and define a nondegenerate symmetric bilinear form $\widetilde{g}\colon \widetilde{V} \times \widetilde{V} \to \mathbb{K}$ by $\widetilde{g}(v^- + v^+, w^- + w^+) = g(v^-, w^+) + g(v^+, w^-)$.

For $\sigma, \tau = \pm$, set $\operatorname{End}(\widetilde{V})^{\sigma,\tau} = \{A \in \operatorname{End}(\widetilde{V}) : AV^{-\tau} = 0,\ AV^{\tau} \subseteq V^{\sigma}\}$ and identify this module with $\operatorname{Hom}(V^\tau, V^\sigma)$ in the evident fashion. Then we have $\operatorname{End}(\widetilde{V}) = \bigoplus_{\sigma,\tau=\pm} \operatorname{End}(\widetilde{V})^{\sigma,\tau}$ with $\operatorname{End}(\widetilde{V})^{\kappa,\lambda} \operatorname{End}(\widetilde{V})^{\sigma,\tau} \subseteq \delta_{\lambda,\sigma} \operatorname{End}(\widetilde{V})^{\kappa,\tau}$. Hence, the associative algebra $\operatorname{End}(\widetilde{V}) = \bigoplus_{i\in\mathbb{Z}} \operatorname{End}(\widetilde{V})_i$ is $\mathbb{Z}$-graded with

$$\operatorname{End}(\widetilde{V})_{\sigma 1} = \operatorname{End}(\widetilde{V})^{\sigma,-\sigma}, \quad \operatorname{End}(\widetilde{V})_0 = \operatorname{End}(\widetilde{V})^{-,-} \oplus \operatorname{End}(\widetilde{V})^{+,+}$$

and $\operatorname{End}(\widetilde{V})_i = 0$ otherwise. So $\operatorname{End}(\widetilde{V}) = \bigoplus_{i\in\mathbb{Z}} \operatorname{End}(\widetilde{V})_i$ is a 3-graded Lie algebra under the commutator product.

Let $\mathfrak{o}(\widetilde{g}) = \{A \in \operatorname{End}(\widetilde{V}) : \widetilde{g}(Av, w) + \widetilde{g}(v, Aw) = 0 \text{ for } v, w \in \widetilde{V}\}$. Then $\mathfrak{o}(\widetilde{g})$ is a graded subalgebra of the Lie algebra $\operatorname{End}(\widetilde{V})$. Thus $\mathfrak{o}(\widetilde{g}) = \bigoplus_{i\in\mathbb{Z}} \mathfrak{o}(\mathfrak{g})_i$ is a 3-graded Lie algebra with $\mathfrak{o}(\widetilde{g})_i = \mathfrak{o}(\widetilde{g}) \cap \operatorname{End}(\widetilde{V})_i$ for $i \in \mathbb{Z}$. We call $\mathfrak{o}(\widetilde{g})$ the *orthogonal Lie algebra of* $\widetilde{g}$ or sometimes the *Lie algebra of skew transformations of* $\widetilde{g}$.

For $v, w \in \widetilde{V}$, define $\zeta(v,w) \in \operatorname{End}(\widetilde{V})$ by $\zeta(v,w)x = \widetilde{g}(x,w)v - \widetilde{g}(x,v)w$, in which case $\zeta(v,w) = -\zeta(w,v)$. Also $\zeta(v,w) \in \mathfrak{o}(\widetilde{g})$ and

$$[A, \zeta(v,w)] = \zeta(Av, w) + \zeta(v, Aw) \text{ for } A \in \mathfrak{o}(\widetilde{g}).$$

Thus $\mathfrak{fo}(\widetilde{g}) := \zeta(\widetilde{V}, \widetilde{V}) = \operatorname{span}_{\mathbb{K}}\{\zeta(v,w) : v, w \in \widetilde{V}\}$ is an ideal of $\mathfrak{o}(\widetilde{g})$ which is graded since $\zeta(V^\sigma, V^\sigma) \subseteq \operatorname{End}(\widetilde{V})_{\sigma 1}$ and $\zeta(V^+, V^-) \subseteq \operatorname{End}(\widetilde{V})_0$. So $\mathfrak{fo}(\widetilde{g}) = \bigoplus_{i\in\mathbb{Z}} \mathfrak{fo}(\mathfrak{g})_i$ is a 3-graded Lie algebra with

$$\mathfrak{fo}(\widetilde{g})_{\sigma 1} = \zeta(V^\sigma, V^\sigma) \quad \text{and} \quad \mathfrak{fo}(\widetilde{g})_0 = \zeta(V^+, V^-).$$

If $\mathbb{K}$ is a field, it is well known (and easy to check) that $\mathfrak{fo}(\widetilde{g})$ is the Lie algebra of all finite rank homomorphisms in $\mathfrak{o}(\widetilde{g})$.

Lemma 7.1 *Let $\mathbb{F} \in \mathbb{K}$-alg. Suppose that either $\mathbb{F}$ is a projective-$\mathbb{K}$-module, or else $\mathbb{F}$ is flat and each V^σ is a finitely generated $\mathbb{K}$-module. Then $g_{\mathbb{F}}\colon V^-_{\mathbb{F}} \times V^+_{\mathbb{F}} \to \mathbb{F}$ is nondegenerate, and the canonical homomorphism $\operatorname{End}(\widetilde{V})_{\mathbb{F}} \to \operatorname{End}(\widetilde{V}_{\mathbb{F}})$ restricts to a 3-graded $\mathbb{F}$-algebra isomorphism from $\mathfrak{fo}(\widetilde{g})_{\mathbb{F}}$ onto $\mathfrak{fo}(\widetilde{g}_{\mathbb{F}})$. (Here we can identify $\mathfrak{fo}(\widetilde{g})_{\mathbb{F}}$ as an $\mathbb{F}$-submodule of $\operatorname{End}(\widetilde{V})_{\mathbb{F}}$ since $\mathbb{F}$ is flat.)*

Proof As in Lemma 4.10, we know from our assumptions that the canonical $\mathbb{F}$-module homomorphisms $\operatorname{Hom}(V^\sigma, \mathbb{K})_{\mathbb{F}} \to \operatorname{Hom}(V^\sigma_{\mathbb{F}}, \mathbb{F})$ and $\operatorname{End}(\widetilde{V})_{\mathbb{F}} \to \operatorname{End}_{\mathbb{F}}(\widetilde{V}_{\mathbb{F}})$ are injective. Using this we can easily verify both statements. ∎

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If $\{v_i^\sigma\}_{i\in I}$ is a basis for the module W^σ for $\sigma = \pm$, we say that the bases $\{v_i^-\}_{i\in I}$ and $\{v_i^+\}_{i\in I}$ are *dual with respect to* g if $g(v_i^-, v_j^+) = \delta_{ij}$ for all $i, j \in I$. Of course, if W^- or W^+ is not free, such dual bases cannot exist. In fact, even if $\mathbb{K}$ is a field, such dual bases need not exist [J2, §IV.5].

Lemma 7.2 *Let $\mathbb{K}$ be a field.*

(i) *Suppose W^σ is a finite dimensional subspace of V^σ for $\sigma = \pm$ such that $g|_{W^-\times W^+}$ is nondegenerate. Then there exist dual bases for W^- and W^+ relative to $g|_{W^-\times W^+}$. Moreover $V^\sigma = W^\sigma \oplus (W^{-\sigma})^\perp$ for $\sigma = \pm$, where*

$$(W^{-\sigma})^\perp = \{v^\sigma \in V^\sigma : g(v^\sigma, W^{-\sigma}) = 0\}.$$

(ii) *Suppose that X^σ is a finite subset of V^σ for $\sigma = \pm$. Then there exists a finite dimensional subspace W^σ of V^σ containing X^σ for $\sigma = \pm$ such that $g|_{W^-\times W^+}$ is nondegenerate.*

Proof (i) is a well-known fact from elementary linear algebra. (ii) is a special case of [LB, Proposition 3.18], or it can be checked, using (i), by induction on

$$\dim(\operatorname{span}_{\mathbb{K}}\{X^-\}) + \dim(\operatorname{span}_{\mathbb{K}}\{X^+\}). \qquad \blacksquare$$

Proposition 7.3 *If $\mathbb{K}$ is a field and $\dim(V^\sigma) \geq 3$, then $\mathfrak{fo}(\widetilde{g})$ is central simple.*

Proof Simplicity is well known. One way to show it is to reduce to the finite dimensional case using Lemma 7.2, in which case simplicity is a classical fact which is easily checked. Thus, by Lemma 7.1, $\mathfrak{fo}(\widetilde{g})_{\mathbb{F}}$ is a simple $\mathbb{F}$-algebra for each field $\mathbb{F}$ containing $\mathbb{K}$. So $\mathfrak{fo}(\widetilde{g})$ is central simple by [Mc, Theorem II.1.6.3(2)]. $\blacksquare$

Remark 7.4 If there exist dual bases for V^- and V^+ relative to g, then the converse of Proposition 7.3 is true [GN, §3.8]. The same remark applies to the corresponding Jordan and Kantor pair results below. (See Proposition 7.5 (iii) and the first statement in Theorem 7.10 (iii).)

7.2 Jordan Pairs of Skew Transformations

Recall that $H = (H^-, H^+)$, where $H^\sigma = \operatorname{Hom}(V^{-\sigma}, V^\sigma)$, is a Jordan pair under the products $\{A^\sigma, B^{-\sigma}, C^\sigma\} = A^\sigma B^{-\sigma} C^\sigma + C^\sigma B^{-\sigma} A^\sigma$. Indeed, H is the Jordan pair determined by the 3-graded Lie algebra $\operatorname{End}(\widetilde{V})$ discussed in Subsection 7.1.

Proposition 7.5 (i) *Let $\operatorname{Skew}(g) = (\operatorname{Skew}(g)^-, \operatorname{Skew}(g)^+)$, where*

$$\operatorname{Skew}(g)^\sigma = \{A^\sigma \in H^\sigma : g(A^\sigma v^{-\sigma}, w^{-\sigma}) + g(v^{-\sigma}, A^\sigma w^{-\sigma}) = 0\}$$

for $\sigma = \pm$. Then $\operatorname{Skew}(g)$ is a subpair of H, and $\operatorname{Skew}(g)$ is enveloped by the 3-graded Lie algebra $\mathfrak{o}(\widetilde{g})$.

(ii) *Let $\operatorname{FSkew}(g) = (\operatorname{FSkew}(g)^-, \operatorname{FSkew}(g)^+)$, where $\operatorname{FSkew}(g)^\sigma = \zeta(V^\sigma, V^\sigma)$. Then $\operatorname{FSkew}(g)$ is an ideal of $\operatorname{Skew}(g)$, and $\operatorname{FSkew}(g)$ is enveloped by the 3-graded Lie algebra $\mathfrak{fo}(\widetilde{g})$.*

(iii) *If $\mathbb{K}$ is a field and $\dim(V^\sigma) \geq 2$, the Jordan pair $\operatorname{FSkew}(g)$ is central simple.*

Proof (i) and (ii) follow from the discussion in Subsection 7.1. For (iii), suppose $\mathbb{K}$ is a field and $\dim(V^\sigma) \geq 2$. If $\dim(V^\sigma) = 2$, $\mathrm{FSkew}(g)$ is one-dimensional with non-trivial products, so it is central simple. If $\dim(V^\sigma) \geq 3$, then $\mathrm{FSkew}(g)$ is central simple by Theorem 4.20 (iii) and Proposition 7.3. ∎

Remark 7.6 The Jordan pairs $\mathrm{Skew}(g)$ and $\mathrm{FSkew}(g)$ are special cases of Jordan pairs studied in [LB, Z1]. Part (iii) in the proposition is a special case of [LB, Theorem 3.9 and Proposition 4.1 (3)] or [Z1, Lemma 5 (2)].

7.3 Kantor Pairs of Skew Transformations

Assume now that $e = (e^-, e^+) \in V^- \times V^+$ satisfies $g(e^-, e^+) = 1$. We will use e to construct a BC_2-grading of $\mathfrak{fo}(\widetilde{g})$ and hence an SP-grading of $\mathrm{FSkew}(g)$.

Let $U^\sigma = (e^{-\sigma})^\perp$ in V^σ relative to g, in which case we have $V^\sigma = \mathbb{K}e^\sigma \oplus U^\sigma$.

Proposition 7.7 *$\mathfrak{fo}(\widetilde{g}) = \bigoplus_{(i_1,i_2)\in\mathbb{Z}^2} \mathfrak{fo}(\widetilde{g})_{i_1,i_2}$ is a BC_2-grading of the Lie algebra $\mathfrak{fo}(\widetilde{g})$, where*

$$\mathfrak{fo}(\widetilde{g})_{\sigma 1,0} = \zeta(U^\sigma, U^\sigma), \quad \mathfrak{fo}(\widetilde{g})_{\sigma 1,\sigma 1} = \zeta(e^\sigma, U^\sigma),$$
$$\mathfrak{fo}(\widetilde{g})_{0,0} = \zeta(U^-, U^+) + \mathbb{K}\zeta(e^{-\sigma}, e^\sigma), \quad \mathfrak{fo}(\widetilde{g})_{0,\sigma 1} = \zeta(e^\sigma, U^{-\sigma})$$

for $\sigma = \pm$, and $\mathfrak{fo}(\widetilde{g})_{i_1,i_2} = 0$ for all other (i_1, i_2) in $\mathbb{Z}^2$. Moreover, the first component grading of this grading is the 3-grading of $\mathfrak{fo}(\widetilde{g})$ in Subsection 7.1.

Proof Note that it suffices to show that $\mathfrak{fo}(\widetilde{g}) = \bigoplus_{(i_1,i_2)\in\mathbb{Z}^2} \mathfrak{fo}(\widetilde{g})_{i_1,i_2}$ is a $\mathbb{Z}^2$-grading, as the rest is then clear. There are a number of ways to see this including a direct case-by-case check. We give an argument that we can use again in Subsection 8.

Suppose first that there exists a unit $t \in \mathbb{K}$ such that

$$(t^i - t^j)x = 0,\ i, j \in \mathbb{Z},\ x \in \mathfrak{fo}(\widetilde{g}) \implies x = 0 \text{ or } i = j. \tag{7.1}$$

For $\sigma = \pm$, we define $\theta^\sigma \in \mathrm{GL}(V^\sigma)$ by $\theta^\sigma(e^\sigma) = t^{\sigma 1} e^\sigma$ and $\theta^\sigma|_{U^\sigma} = \mathrm{id}_{U^\sigma}$. Then $g(\theta^\sigma x^\sigma, \theta^{-\sigma} x^{-\sigma}) = g(x^\sigma, x^{-\sigma})$, so $\widetilde{\theta} := \theta^- \oplus \theta^+ \in \mathrm{GL}(\widetilde{V})$ preserves the form $\widetilde{g}$. Hence

$$\widetilde{\theta}\zeta(x, y)\widetilde{\theta}^{-1} = \zeta(\widetilde{\theta}x, \widetilde{\theta}y) \tag{7.2}$$

for $x, y \in \widetilde{V}$, so we can define $\psi_\theta \in \mathrm{Aut}(\mathfrak{fo}(\widetilde{g}))$ by $\psi_\theta(X) = \widetilde{\theta}X\widetilde{\theta}^{-1}$ For $(i_1, i_2) \in \mathbb{Z}^2$, let $\mathcal{F}_{i_1,i_2} = \{x \in \mathfrak{fo}(\widetilde{g})_{i_1} : \psi_\theta(x) = t^{i_2}x\}$. (We will see that $\mathcal{F}_{i_1,i_2} = \mathfrak{fo}(\widetilde{g})_{i_1,i_2}$.)

Note that the sum $\sum_{(i_1,i_2)\in\mathbb{Z}^2} \mathcal{F}_{i_1,i_2}$ in $\mathfrak{fo}(\widetilde{g})$ is direct. Indeed, to show this it suffices to show that $\sum_{i_2\in\mathbb{Z}} \mathcal{F}_{i_1,i_2}$ is direct for $i_1 \in \mathbb{Z}$, and this is checked by a standard argument in linear algebra using (7.1). Note also that $\mathfrak{fo}(\widetilde{g})_{i_1,i_2} \subseteq \mathcal{F}_{i_1,i_2}$ for each (i_1, i_2), which follows from (7.2) and the definition of the modules $\mathcal{F}_{i_1,i_2}$.

Since $\sum_{(i_1,i_2)\in\mathbb{Z}^2} \mathfrak{fo}(\widetilde{g})_{i_1,i_2} = \mathfrak{fo}(\widetilde{g})$ we see from the preceding paragraph that

$$\mathfrak{fo}(\widetilde{g}) = \bigoplus_{(i_1,i_2)\in\mathbb{Z}^2} \mathfrak{fo}(\widetilde{g})_{i_1,i_2}$$

as modules and that $\mathfrak{fo}(\widetilde{g})_{i_1,i_2} = \mathcal{F}_{i_1,i_2}$ for each (i_1, i_2). Thus, since ψ_θ is an automorphism, $\mathfrak{fo}(\widetilde{g}) = \bigoplus_{(i_1,i_2)\in\mathbb{Z}^2} \mathfrak{fo}(\widetilde{g})_{i_1,i_2}$ is an algebra grading.

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Finally, consider the general case (without assuming the existence of t satisfying (7.1)). Let $\mathbb{T} = \mathbb{K}[t, t^{-1}]$ be the algebra of Laurent polynomials. Then, by Lemma 7.1, we have a canonical 3-graded $\mathbb{T}$-algebra isomorphism $\varphi: \mathfrak{fo}(\widetilde{g})_{\mathbb{T}} \to \mathfrak{fo}(\widetilde{g_{\mathbb{T}}})$. So $\mathfrak{fo}(\widetilde{g_{\mathbb{T}}})$ satisfies (7.1) (over $\mathbb{T}$), and hence $\mathfrak{fo}(\widetilde{g_{\mathbb{T}}}) = \bigoplus_{(i_1,i_2)\in\mathbb{Z}^2} \mathfrak{fo}(\widetilde{g_{\mathbb{T}}})_{i_1,i_2}$ is a $\mathbb{Z}^2$-grading. Moreover, since $\mathbb{T}$ is flat, we can identify $(\mathfrak{fo}(\widetilde{g})_{i_1,i_2})_{\mathbb{T}}$ as a $\mathbb{T}$-submodule of $\mathfrak{fo}(\widetilde{g})_{\mathbb{T}}$ and we see that φ maps $(\mathfrak{fo}(\widetilde{g})_{i_1,i_2})_{\mathbb{T}}$ onto $\mathfrak{fo}(\widetilde{g_{\mathbb{T}}})_{i_1,i_2}$ for each (i_1, i_2). So $\mathfrak{fo}(\widetilde{g})_{\mathbb{T}} = \bigoplus_{(i_1,i_2)\in\mathbb{Z}^2} (\mathfrak{fo}(\widetilde{g})_{i_1,i_2})_{\mathbb{T}}$ is a $\mathbb{Z}^2$-grading. Since $\mathbb{T}$ is faithfully flat, this easily implies our result. ∎

Corollary 7.8 *Let* $\mathrm{FSkew}(g)$ *be the Kantor pair in Proposition 7.5* (ii) *and let*

$$\mathrm{FSkew}(g)_0^\sigma = \zeta(U^\sigma, U^\sigma) \quad \textit{and} \quad \mathrm{FSkew}(g)_1^\sigma = \zeta(e^\sigma, U^\sigma).$$

Then $\mathrm{FSkew}(g) = \mathrm{FSkew}(g)_0 \oplus \mathrm{FSkew}(g)_1$ *is an SP-graded Jordan pair, which is enveloped by the* BC_2*-graded Lie algebra in Proposition 7.7.*

Remark 7.9 In a similar fashion one obtains a BC_2-grading of $\mathfrak{o}(\widetilde{g})$ and hence an SP-grading of $\mathrm{Skew}(g)$ with $\mathrm{Skew}(g)_0^\sigma = \{A^\sigma \in \mathrm{Skew}(g)^\sigma : A^\sigma e^{-\sigma} = 0\}$ and $\mathrm{Skew}(g)_1^\sigma = \{A^\sigma \in \mathrm{Skew}(g)^\sigma : A^\sigma U^{-\sigma} \subseteq \mathbb{K}e^\sigma\}$. We leave the details of this to the interested reader.

Theorem 7.10 *Suppose* $\mathbb{K}$ *is a unital commutative ring containing* $\frac{1}{6}$, $g: V^- \times V^+ \to \mathbb{K}$ *is a nondegenerate bilinear form, and* $e = (e^-, e^+) \in V^- \times V^+$ *satisfies* $g(e^-, e^+) = 1$. *Let* $\mathrm{FSkew}(g)$ *be the SP-graded Jordan pair in Corollary 7.8.*

(i) $\mathrm{FSkew}(g)\check{}$ *is an SP-graded Kantor pair.*

(ii) *Let* $U = (U^-, U^+)$ *be the Jordan pair with products*

$$\{u^\sigma, v^{-\sigma}, w^\sigma\}^\sigma = g(u^\sigma, v^{-\sigma})w^\sigma + g(w^\sigma, v^{-\sigma})u^\sigma,$$

let U' *be the ideal of* U *with* $(U')^\sigma = \zeta(U^\sigma, U^\sigma)U^{-\sigma}$, *and let* U'' *be the ideal of* U' *with* $(U'')^\sigma = \{u^\sigma \in (U')^\sigma : g(u^\sigma, (U')^{-\sigma}) = 0\}$. *Then the Jordan obstruction* $J(\mathrm{FSkew}(g)\check{})$ *of* $\mathrm{FSkew}(g)\check{}$ *is isomorphic to* $(U'/U'')^{\mathrm{op}}$.

(iii) *If* $\mathbb{K}$ *is a field and* $\dim(V^-) \geq 2$, *then* $\mathrm{FSkew}(g)\check{}$ *is a central simple SP-graded Kantor pair. Moreover, if* $\mathbb{K}$ *is a field and* $\dim(V^-) \geq 3$, *then* $J(\mathrm{FSkew}(g)\check{}) \simeq U^{\mathrm{op}}$, *so* $\mathrm{FSkew}(g)\check{}$ *is not Jordan.*

Proof (i) is immediate from Corollary 7.8.

(ii) Let $P = \mathrm{FSkew}(g)$ with the SP-grading in Corollary 7.8 and let $L = \mathfrak{fo}(\widetilde{g})$ with the BC_2-grading in Proposition 7.7. Let $L' = \langle T_L(P)\rangle_{\mathrm{alg}}$, $L'' = Z(L') \cap [T_{L'}(P), T_{L'}(P)]$ and $\overline{L'} = L'/L''$. Then L' is a BC_2-graded ideal of L and L'' is a BC_2-graded ideal of L', so $\overline{L'}$ is BC_2-graded. Moreover, by Remarks 4.6 and 5.4, $\overline{L'}$ tightly envelopes the SP-graded Kantor pair P (suitably identified in $\overline{L'}$). So by (6.7) and Proposition 7.7, $J(\check{P}) \simeq (\overline{L'}_{0,-1}, \overline{L'}_{0,1})$ under the products $[[x, y], z]$ in $\overline{L'}$. Thus $J(\check{P})^{\mathrm{op}} \simeq (\overline{L'}_{0,1}, \overline{L'}_{0,-1}) \simeq (L'_{0,1}, L'_{0,-1})/(L''_{0,1}, L''_{0,-1})$.

We now calculate $(L'_{0,1}, L'_{0,-1})$ and $(L''_{0,1}, L''_{0,-1})$, leaving some of the details to the reader. First $L_{0,\sigma1} = \zeta(U^\sigma, e^{-\sigma})$, and we define $\lambda^\sigma: U^\sigma \to L_{0,\sigma1}$ by $\lambda^\sigma(u^\sigma) = \sigma\zeta(u^\sigma, e^{-\sigma})$. One checks that $\lambda = (\lambda^-, \lambda^+): U \to (L_{0,1}, L_{0,-1})$ is an isomorphism of Jordan pairs. Next we have $L'_{0,-\sigma1} = [L_{\sigma1,0}, L_{-\sigma1,-\sigma1}] = [\zeta(U^\sigma, U^\sigma), \zeta(U^{-\sigma}, e^{-\sigma})] =$

$\zeta(\zeta(U^\sigma, U^\sigma)U^{-\sigma}, e^{-\sigma})$. So λ maps U' onto $(L'_{0,1}, L'_{0,-1})$. Finally $L''_{0,-\sigma 1}$ is the centralizer of $L_{-1,*} + L_{1,*} = \zeta(V^-, V^-) + \zeta(V^+, V^+)$ in $L'_{0,-\sigma 1}$, and one checks using this that λ maps U'' onto $(L''_{0,1}, L''_{0,-1})$.

(iii) The first statement is a consequence of Propositions 6.4 and 7.5 (iii). For the second statement, we know by Lemma 7.2 that there exist $e_i^\sigma \in U^\sigma$ for $\sigma = \pm$ and $i = 1, 2$ such that $g(e_i^-, e_j^+) = \delta_{ij}$. Thus, $\zeta(e_1^\sigma, e_2^\sigma)e_2^{-\sigma} = e_1^\sigma$ and

$$\zeta(u^\sigma, e_1^\sigma)e_1^{-\sigma} = u^\sigma - g(e_1^{-\sigma}, u^\sigma)e_1^\sigma$$

for $u^\sigma \in U^\sigma$, so $U' = U$ and $U'' = 0$. Hence we are done by (ii). (See also Corollary 6.9.) ∎

It turns out that the Kantor pairs in Theorem 7.10 (iii) make up one of the four classes of central simple Kantor pairs that appear in the structure theorem mentioned in the introduction.

Remark 7.11 Suppose $\mathbb{K}$ is a field. If $f = (f^-, f^+) \in V^- \times V^+$ is another pair of vectors satisfying $g(f^-, f^+) = 1$, then it is easy to see, using Lemma 7.2, that there exists an *isometry* $(\varphi^-, \varphi^+) \in \mathrm{GL}(V^-) \times \mathrm{GL}(V^+)$ of g (satisfying $g(\varphi^-(v^-), \varphi^+(v^+)) = g(v^-, v^+)$) such that $\varphi^\sigma(e^\sigma) = f^\sigma$ for $\sigma = \pm$. Using this fact it is also easy to see that the BC_2-graded Lie algebra $\mathfrak{o}(\widetilde{g})$ and the Kantor pairs $\mathrm{FSkew}(g)$ and $\mathrm{FSkew}(g)^\smallsmile$ described in this subsection are independent up to graded isomorphism of the choice of e.

Example 7.12 (The finite dimensional case) Suppose that $\mathbb{K}$ is a field and V^σ has finite dimension n, where $n \geq 3$. Then $\mathfrak{fo}(\widetilde{g}) = \mathfrak{o}(\widetilde{g})$ and $\mathrm{FSkew}(g) = \mathrm{Skew}(g)$. Choose dual basis $\{v_i^-\}_{i=1}^n$ and $\{v_i^+\}_{i=1}^n$ for V^- and V^+ relative to g with $v_1^- = e^-$ and $v_1^+ = e^+$, and use these bases to identify $\mathrm{End}(\widetilde{V})^{\sigma,\tau}$ with $M_n(\mathbb{K})$ for $\sigma, \tau = \pm$. Then one sees directly using Proposition 7.5 (ii) and Corollary 7.8 that $\mathrm{FSkew}(g)$ is the double of the Jordan triple system $\mathrm{A}_n(\mathbb{K})$ of alternating $n \times n$-matrices with product $ABC + CBA$, and that

$$\mathrm{FSkew}(g)_0^\sigma = \sum_{i,j=2}^n \mathbb{K}(E_{ij} - E_{ji}) \quad \text{and} \quad \mathrm{FSkew}(g)_1^\sigma = \sum_{i=2}^n \mathbb{K}(E_{i1} - E_{1i}).$$

(Here E_{ij} is the (i, j)-matrix unit.) Also, $\mathfrak{fo}(\widetilde{g})$ is the split central simple Lie algebra of type $\mathrm{D_n}$ (interpreting $\mathrm{D_3}$ as $\mathrm{A_3}$) [Sel, §IV.3]. So $\mathrm{FSkew}(g)$ and $\mathrm{FSkew}(g)^\smallsmile$ are split central simple Kantor pairs of type $\mathrm{D_n}$ by Proposition 6.4. Note that $\mathrm{FSkew}(g)$ (being Jordan) has balanced 2-dimension 0; whereas, by Proposition Theorem 7.10 (iii), $\mathrm{FSkew}(g)^\smallsmile$ has balanced 2-dimension $n - 1$.

If n is odd in Example 7.12, the SP-grading of $\mathrm{FSkew}(g)$ arises as in Example 5.2 from an idempotent c with trivial Peirce 0-component [L, §8.16].

Remark 7.13 If $\mathbb{K}$ is algebraically closed of characteristic 0, the Kantor pair $\mathrm{FSkew}(g)^\smallsmile$ in Example 7.12 is the double of a Kantor triple system that appears in Kantor's classification (mentioned in the introduction), where it is represented by the notation $\mathrm{C_{n-1,n-1}}$ — $\mathrm{A_{n-1,1}}$ [K1, (5.27)]. In fact this notation displays the SP-grading on $\mathrm{FSkew}(g)^\smallsmile$.

Remark 7.14 Example 7.12 can be formulated somewhat more generally. Indeed, suppose we have the assumptions and notation of Theorem 7.10, suppose each V^σ is a finitely generated projective module of rank n over $\mathbb{K}$ with $n \geq 3$, and suppose $g: V^- \times V^+ \to \mathbb{K}$ is non-singular. (See Subsection 8.1 below to recall this terminology.) Then $Z(\mathfrak{fo}(\widetilde{g})) = 0$ and $\mathfrak{fo}(\widetilde{g})$ tightly envelops $\mathrm{FSkew}(g)$. Furthermore, $\mathrm{FSkew}(g)$ is a form of an SP-graded split Jordan pair of type $\mathrm{D_n}$ which is split if each V^σ is free; $\mathrm{FSkew}(g)^{\check{}}$ is a form of a split SP-graded Kantor pair of type $\mathrm{D_n}$ which is split if each V^σ is free; and finally the Jordan obstruction of $\mathrm{FSkew}(g)^{\check{}}$ is isomorphic to U^{op}. The verifications of these facts, which we omit, follow the methods (faithfully flat base ring extension to the free case) used in the next section, where an example of type $\mathrm{E_6}$ is treated in detail.

8 Construction of $\mathrm{E_6}$ and Kantor Pairs From the Exterior Algebra

It is well known that over an algebraically closed field of characteristic 0, the simple Lie algebra $\mathcal{E}$ of type $\mathrm{E_6}$ has a 5-grading with components $\wedge^6(V^*)$, $\wedge^3(V^*)$, $\mathrm{sl}(V) \oplus \mathbb{K}h$, $\wedge^3(V)$ and $\wedge^6(V)$ as $\mathbb{K}$-spaces, and with natural actions of $\mathrm{sl}(V)$ on each component, where V is a six-dimensional space [C, §V.18, pp. 89–90], [GOV, §3.3.5]. Recently this fact has been used in work on gradings of $\mathcal{E}$ [ADG, §3.4], [EK, §6.4].

In this section, we take this point of view in a more general context to construct 5-graded Lie algebras, SP-graded Kantor pairs, and their reflections. Throughout the section, we assume only that $\mathbb{K}$ is a unital commutative associative ring (not necessarily containing $\frac{1}{6}$). We will add further assumptions on $\mathbb{K}$ as needed.

8.1 Finitely Generated Projective Modules and Non-singular Forms

Recall that a module M is *finitely generated projective* (FGP) if and only if M is a direct summand of a free module of finite rank. If M and N are FGP, then M^* and $M \otimes N$ are FGP. Moreover, we may identify M with M^{**}, where $m(\varphi) = \varphi(m)$ for $m \in M$ and $\varphi \in M$, and the linear map $M \otimes M^* \to \mathrm{End}(M)$ with $m \otimes \varphi \to m\varphi$, where $(m\varphi)(m') = \varphi(m')m$ is bijective. The *trace* function tr on $\mathrm{End}(M)$ is the unique linear map with $\mathrm{tr}(m\varphi) = \varphi(m)$. If $A, B \in \mathrm{End}(M)$, then $\mathrm{tr}(AB) = \mathrm{tr}(BA)$. For more details see [B1, II.4.3] and [F, §2].

If $\mathfrak{p} \in \mathrm{Spec}(\mathbb{K})$, the set of prime ideals of $\mathbb{K}$, let $\mathbb{K}_\mathfrak{p} = (\mathbb{K}\backslash\mathfrak{p})^{-1}\mathbb{K}$ be the localization of $\mathbb{K}$ at $\mathfrak{p}$. If M is a finitely generated projective module, then $M_{\mathbb{K}_\mathfrak{p}}$ is a free $\mathbb{K}_\mathfrak{p}$-module of finite rank [B2, II.5.2]. If $M_{\mathbb{K}_\mathfrak{p}}$ has rank n for all $\mathfrak{p} \in \mathrm{Spec}(\mathbb{K})$, we say M has *rank* n, in which case we have $\mathrm{tr}(\mathrm{id}_M) = n1_{\mathbb{K}}$.

8.2 Assumptions and Notation

Henceforth we assume that

(8.1) $g: M^- \times M^+ \to \mathbb{K}$ is a non-singular bilinear form, where M^- and M^+ are FGP modules of rank n.

(To check this condition, using [B2, II.5.3, (4)] and [B1, II.2.7, Cor. 4] it is easy to see that it suffices to show that M^+ is FGP of rank n and the map $v^- \to g(v^-, \ )$ from V^-

into $(V^+)^*$ is bijective.) If $v^+ \in M^+$ and $v^- \in M^-$, we again set $g(v^+, v^-) = g(v^-, v^+)$ for convenience.

Remark 8.1 (i) If (M'^-, M'^+, g') is another triple satisfying (8.1), an *isomorphism* of (M^-, M^+, g) onto (M'^-, M'^+, g') is a pair $\theta = (\theta^-, \theta^+)$, where $\theta^\sigma: M^\sigma \to M'^\sigma$ is a linear isomorphism and $g'(\theta^- v^-, \theta^+ v^+) = g(v^-, v^+)$ for $v^\sigma \in M^\sigma$.

(ii) If N^+ is an FGP module of rank n and can: $(N^+)^* \times N^+ \to \mathbb{K}$ is the canonical map given by $\mathrm{can}(x^-, y^+) = x^-(y^+)$, then $((N^+)^*, N^+, \mathrm{can})$ satisfies (8.1); any triple satisfying (8.1) is isomorphic to one obtained in this way. So in this sense we are really just starting with an FGP module of rank n. However, the more symmetric point of view taken here is very convenient for us.

(iii) The reference [B1, III] works with triples $((N^+)^*, N^+, \mathrm{can})$ as in (ii). In view of (ii), we can use facts from that reference.

If $A^\sigma \in \mathrm{End}(M^\sigma)$, then, since g is non-singular, there exists a unique $(A^\sigma)^* \in \mathrm{End}(M^{-\sigma})$, called the *adjoint* of A^σ, such that $g(v^\sigma, (A^\sigma)^* v^{-\sigma}) = g(A^\sigma v^\sigma, v^{-\sigma})$ for $v^\sigma \in M^\sigma, v^{-\sigma} \in M^{-\sigma}$

For $\sigma = \pm$, we form the exterior algebra $\wedge(M^\sigma)$ with the natural $\mathbb{Z}$-grading

$$\wedge(M^\sigma) = \bigoplus_{k \geq 0} \wedge_k(M^\sigma),$$

where $\wedge_k(M^\sigma) = 0$ if $k < 0$. For convenience, we write the product in $\wedge(M^\sigma)$ as uv (rather than the usual $u \wedge v$).

In the next three subsections, we record the facts about $\wedge(M^\sigma)$ that we will need for our constructions. For more details, the reader can consult [B1, III.7, III.10, and III.11] or [F, §2].

8.3 The $\cdot$ Action of $\wedge(M^{-\sigma})$ on $\wedge(M^\sigma)$

Recall that an endomorphism D of the graded algebra $\wedge(M^\sigma)$ is called an *anti-derivation of degree* -1 of $\wedge(M^\sigma)$ if $D(\wedge_k(M^\sigma)) \subseteq \wedge_{k-1}(M^\sigma)$ and

$$D(x^\sigma y^\sigma) = D(x^\sigma) y^\sigma + (-1)^k x^\sigma D(y^\sigma)$$

for $k \geq 0$, $x^\sigma \in \wedge_k(M^\sigma)$ and $y^\sigma \in \wedge(M^\sigma)$ [B1, III.10.3].

For $v^{-\sigma} \in M^{-\sigma}$, there is a unique anti-derivation $\Delta_{v^{-\sigma}}$ of $\wedge(M^\sigma)$ of degree -1 with $\Delta_{v^{-\sigma}}(v^\sigma) = g(v^{-\sigma}, v^\sigma)$ for $v^\gamma \in M^\gamma$ [B1, III.10.9, Example 2]. Since $\Delta^2_{v^{-\sigma}} = 0$, the map $v^{-\sigma} \to \Delta_{v^{-\sigma}}$ extends to a homomorphism $a \to \Delta_a$ of $\wedge(M^{-\sigma})$ into the associative algebra $\mathrm{End}(\wedge(M^\sigma))$, and hence we can view $\wedge(M^\sigma)$ as a left module for the associative algebra $\wedge(M^{-\sigma})$. We write the action as $a \cdot x = \Delta_a(x)$ for $a \in \wedge(M^{-\sigma})$, $x \in \wedge(M^\sigma)$.

Remark 8.2 If we identify $M^{-\sigma}$ with $(M^\sigma)^*$ via the pairing g, then it follows from [B1, III.11.9, (68)] and [B1, III.11.8, Proposition 10] that $a \cdot x$ is the *left inner product* of x by a that is studied in [B1, III.11.9].

Note that $\wedge_k(M^{-\sigma}) \cdot \wedge_\ell(M^\sigma) \subseteq \wedge_{\ell-k}(M^\sigma)$. In particular if $a \in \wedge_k(M^{-\sigma})$ and $x \in \wedge_k(M^\sigma)$, then $a \cdot x$ is a scalar in $\mathbb{K}$ and, by [F, Lemma 3 (i)], we have $x \cdot a = a \cdot x$.

Note also that

$$x \in \bigwedge_k(M^\sigma),\ x \cdot \bigwedge_k(M^{-\sigma}) = 0 \implies x = 0 \tag{8.2}$$

by [B1, III.11.5, Proposition 7]. Furthermore, if $v^\sigma \in M^\sigma$ and $v^{-\sigma} \in M^{-\sigma}$, then $v^\sigma \cdot v^{-\sigma} = g(v^\sigma, v^{-\sigma})$, so there is no confusion of notation in case $M^- = M^+ = \mathbb{K}^n$ and g is the usual dot product.

If $p \in \bigwedge_n(M^\sigma)$, $q \in \bigwedge_n(M^{-\sigma})$ (recall that n is the rank of M^σ) and $a \in \bigwedge(M^{-\sigma})$, then

$$(a \cdot p) \cdot q = (p \cdot q)a, \tag{8.3}$$

by [B1, III.11.11, Proposition 12 (i)] or [F, Lemma 4 (i)].

8.4 The Lie Algebra $\mathcal{S}$ and its $\circ$ Action on $\bigwedge(M^\sigma)$

We consider the following Lie subalgebra

$$\begin{aligned}\mathcal{S} &= \mathcal{S}(M^-, M^+, g)\\ &= \{(A^-, A^+) \in \mathrm{End}(M^-) \oplus \mathrm{End}(M^+) : g(A^- v^-, v^+) + g(v^-, A^+ v^+) = 0\}.\end{aligned}$$

of $\mathrm{End}(M^-) \oplus \mathrm{End}(M^+)$. Note that for $\sigma = \pm$ we have the Lie algebra isomorphism $\iota^\sigma := \iota^\sigma(M^-, M^+, g) \colon \mathrm{End}(M^\sigma) \to \mathcal{S}$ given by

$$\iota^+(A^+) = (-(A^+)^*, A^+) \quad \text{and} \quad \iota^-(A^-) = (A^-, -(A^-)^*),$$

in which case the inverse of ι^σ is projection onto the σ-factor restricted to $\mathcal{S}$.

If $A^\sigma \in \mathrm{End}(M^\sigma)$, there is a unique extension of A^σ to a derivation D_{A^σ} of $\bigwedge(M^\sigma)$ stabilizing each $\bigwedge_k(M^\sigma)$, and it is easy to see that $[D_{A^\sigma}, D_{B^\sigma}] = D_{[A^\sigma, B^\sigma]}$ [B1, III.10.9, Example 1]. So each $\bigwedge_k(M^\sigma)$, and therefore also $\bigwedge(M^\sigma)$, is a module for the Lie algebra $\mathrm{End}(M^\sigma)$ with $A^\sigma \circ x^\sigma = D_{A^\sigma}(x^\sigma)$ for $x^\sigma \in \bigwedge_k(M^\sigma)$. We also view $\bigwedge(M^\sigma)$ as a module for $\mathcal{S}$ with $A \circ x^\sigma := A^\sigma \circ x^\sigma$ for $A = (A^-, A^+) \in \mathcal{S}$.

The $\cdot$ action and the $\circ$ action are related by the identity

$$A \circ (a \cdot x) = (A \circ a) \cdot x + a \cdot (A \circ x) \tag{8.4}$$

which, by [F, (2)], holds for $A \in \mathcal{S}$, $a \in \bigwedge(M^{-\sigma})$, $x \in \bigwedge(M^\sigma)$, $\sigma = \pm$.

Note that if $A^\sigma \in \mathrm{End}(M^\sigma)$ and $p^\sigma \in \bigwedge_n(M^\sigma)$, we have

$$A^\sigma \circ p^\sigma = \mathrm{tr}(A^\sigma) p^\sigma. \tag{8.5}$$

Indeed, this holds if M^σ is free [B1, III.10.9, Proposition 15]; and hence it holds in general by an easy localization argument [F, §2, ¶3].

8.5 The Elements $E(x^\sigma, x^{-\sigma})$ in $\mathcal{S}$

For $\sigma = \pm$, $x \in \bigwedge_k(M^\sigma)$, $a \in \bigwedge_k(M^{-\sigma})$, and $1 \le k \le n$, we define $E^\sigma(x, a) \in \mathrm{End}(M^\sigma)$ by

$$E^\sigma(x, a)(v^\sigma) = (v^\sigma \cdot a) \cdot x.$$

In that case we have

$$E^\sigma(x, a)^* = E^{-\sigma}(a, x) \quad \text{and} \quad \mathrm{tr}(E^\sigma(x, a)) = k\, x \cdot a, \tag{8.6}$$

by [F, Lemma 3 (ii) (iii)], where $E^\sigma(x, a)$ is denoted by $e(x, a)$.

If $x^- \in \bigwedge_k(M^-)$, $x^+ \in \bigwedge_k(M^+)$, and $1 \le k \le n$, we define

$$E(x^-, x^+) = \big(E^-(x^-, x^+), -E^+(x^+, x^-)\big) \quad \text{and} \quad E(x^+, x^-) = -E(x^-, x^+).$$

Then by the first equation in (8.6), $E(x^\sigma, x^{-\sigma}) \in \mathcal{S}$ for $\sigma = \pm$. Observe also that

(8.7) $$E(x^\sigma, x^{-\sigma}) \circ z^\sigma = E^\sigma(x^\sigma, x^{-\sigma}) \circ z^\sigma$$

for $z^\sigma \in \bigwedge(M^\sigma)$.

8.6 The 5-graded Lie Algebra $\widetilde{\mathcal{E}}$

Hereafter suppose that $n = 6$. So

(8.8) $g\colon M^- \times M^+ \to \mathbb{K}$ is a non-singular bilinear form, where M^- and M^+ are FGP modules of rank 6.

Let $\widetilde{\mathcal{S}}$ be the Lie algebra direct sum

$$\widetilde{\mathcal{S}} = \widetilde{\mathcal{S}}(M^-, M^+, g) = \mathcal{S}(M^-, M^+, g) \oplus \mathbb{K}h_0,$$

where $\mathbb{K}h_0$ is free with basis $h_0 = h_0(M^-, M^+, g)$ and $\mathbb{K}h_0$ has trivial product. We extend the action of $\mathcal{S}$ on the subalgebra

$$\textstyle\bigwedge_{(3)}(M^\sigma) := \mathbb{K}1 \oplus \bigwedge_3(M^\sigma) \oplus \bigwedge_6(M^\sigma)$$

of $\bigwedge(M^\sigma)$ generated by $\bigwedge_3(M^\sigma)$ to an action of $\widetilde{\mathcal{S}}$ on $\bigwedge_{(3)}(M^\sigma)$ by letting

$$h_0 \circ p = \sigma k p \quad \text{for } p \in \textstyle\bigwedge_{3k}(M^\sigma).$$

Thus, $\widetilde{\mathcal{S}}$ acts as derivations of $\bigwedge_{(3)}(M^\sigma)$.

We define a $\mathbb{Z}$-graded module $\widetilde{\mathcal{E}} = \widetilde{\mathcal{E}}(M^-, M^+, g) := \bigoplus_{i\in\mathbb{Z}} \widetilde{\mathcal{E}}_i$, where the modules $\widetilde{\mathcal{E}}_i = \widetilde{\mathcal{E}}_i(M^-, M^+, g)$ are given by

$$\widetilde{\mathcal{E}}_0 := \widetilde{\mathcal{S}}, \quad \widetilde{\mathcal{E}}_{\sigma k} := \textstyle\bigwedge_{3k}(M^\sigma) \text{ for } k = 1,2, \text{ and } \widetilde{\mathcal{E}}_{\sigma k} := 0 \text{ for } k > 2.$$

Define a $\mathbb{Z}$-graded skew-symmetric product on $\widetilde{\mathcal{E}}$ by

(8.9)
$$\begin{aligned} &[A, B] \text{ is the above product in } \widetilde{\mathcal{S}}, && [A, p_i] = A \circ p_i, \text{ for } i = \pm1, \pm2,\\ &[p_{-1}, q_1] = E(p_{-1}, q_1) + (p_{-1} \cdot q_1)h_0, && [p_{-2}, q_2] = -(p_{-2} \cdot q_2)(h_M - 2h_0),\\ &[p_i, q_i] = p_i q_i \text{ for } i = \pm1, && [p_i, q_{-2i}] = p_i \cdot q_{-2i} \text{ for } i = \pm1, \end{aligned}$$

for $A, B \in \widetilde{\mathcal{S}}$, $p_j, q_j \in \widetilde{\mathcal{E}}_j$, where $h_M := \iota^+(\mathrm{id}_{M^+}) = (-\mathrm{id}_{M^-}, \mathrm{id}_{M^+}) \in \mathcal{S}$.

Remark 8.3 If $i \in \mathbb{Z}$ and $p_i \in \widetilde{\mathcal{E}}_i$, then

(8.10) $$[h_0, p_i] = i p_i \quad \text{and} \quad [h_M, p_i] = 3i p_i,$$

so $h_M - 3h_0 \in Z(\widetilde{\mathcal{E}})$. Also, if $x^\sigma \in \widetilde{\mathcal{E}}_{\sigma 1}$, and $y^{-\sigma} \in \widetilde{\mathcal{E}}_{-\sigma 1}$, then

(8.11) $$[x^\sigma, y^{-\sigma}] = E(x^\sigma, y^{-\sigma}) - \sigma(x^\sigma \cdot y^{-\sigma})h_0;$$

so, if $z^\sigma \in \widetilde{\mathcal{E}}_{\sigma i}$ with $i = 1, 2$, we have

(8.12) $$[[x^\sigma, y^{-\sigma}], z^\sigma] = E^\sigma(x^\sigma, y^{-\sigma}) \circ z^\sigma - i(x^\sigma \cdot y^{-\sigma})z^\sigma$$

by (8.7) and (8.10). Moreover, if $p^\sigma \in \widetilde{\mathcal{E}}_{\sigma 2}$ and $q^{-\sigma} \in \widetilde{\mathcal{E}}_{-\sigma 2}$, then

(8.13) $$[p^\sigma, q^{-\sigma}] = \sigma(p^\sigma \cdot q^{-\sigma})(h_M - 2h_0);$$

so, if $r^\sigma \in \widetilde{\mathcal{E}}_{\sigma 2}$, we have, using (8.10) and (8.3),

$$[[p^\sigma, q^{-\sigma}], r^\sigma] = 2(p^\sigma \cdot q^{-\sigma})r^\sigma = (p^\sigma \cdot q^{-\sigma})r^\sigma + (r^\sigma \cdot q^{-\sigma})p^\sigma. \tag{8.14}$$

Theorem 8.4 *Suppose $\mathbb{K}$ is a unital commutative ring and (M^-, M^+, g) satisfies (8.8). Then $\widetilde{\mathcal{E}} = \widetilde{\mathcal{E}}(M^-, M^+, g)$ is a 5-graded Lie algebra.*

Proof It suffices to check the Jacobi identity

$$J(z_1, z_2, z_3) := [[z_1, z_2], z_3] + [[z_2, z_3], z_1] + [[z_3, z_1], z_2] = 0$$

for homogeneous $z_1 \in \widetilde{\mathcal{E}}_{d_1}, z_2 \in \widetilde{\mathcal{E}}_{d_2}, z_3 \in \widetilde{\mathcal{E}}_{d_3}$, where $|d_i| \le 2$. Moreover, since the product is skew-symmetric, $J(z_1, z_2, z_3) = 0$ implies $J(z_{\pi 1}, z_{\pi 2}, z_{\pi 3}) = 0$ for any $\pi \in S_3$. Also, since $\widetilde{\mathcal{E}}_{\sigma k} = 0$ for $k > 2$, we can assume $|d_1 + d_2 + d_3| \le 2$. With these observations, we are reduced to considering the following cases for (d_1, d_2, d_3):

$$(0,0,0),\ (0,0,\sigma 1),\ (0,0,\sigma 2),\ (0,\sigma 1,\sigma 1),\ (0,2,-2),\ (0,\sigma 1,-\sigma 2),\ (0,-1,1),$$
$$(\sigma 2,-\sigma 2,\sigma 2),\ (\sigma 1,\sigma 1,-\sigma 2), (\sigma 1,\sigma 1,-\sigma 1),\ (\sigma 1,\sigma 2,-\sigma 2),\ (\sigma 1,\sigma 2,-\sigma 1),$$

where in each case $\sigma = \pm$.

Now the case $(0,0,0)$ holds since $\widetilde{\mathcal{S}}$ is a Lie algebra; the cases $(0,0,\sigma 1)$, $(0,0,\sigma 2)$ hold since $\bigwedge(M^\sigma)$ is an $\widetilde{\mathcal{S}}$-module under the $\circ$ action; the case $(0,\sigma 1,\sigma 1)$ holds since $\widetilde{\mathcal{S}}$ acts by derivations on $\bigwedge(M^\sigma)$ under $\circ$, and the cases $(0,2,-2)$ and $(0,\sigma 1,-\sigma 2)$ follow from (8.4). This leaves the following cases.

Case $(0,-1,1)$: For $A \in \widetilde{\mathcal{S}}$, $x^- \in \bigwedge_3(M^-)$, and $x^+ \in \bigwedge_3(M^+)$, we have

$$\begin{aligned}
J(A, x^-, x^+) &= [[A, x^-], x^+] - [A, [x^-, x^+]] + [x^-, [A, x^+]]\\
&= E(A \circ x^-, x^+) + ((A \circ x^-) \cdot x^+)h_0 - [A, E(x^-, x^+) + (x^- \cdot x^+)h_0]\\
&\qquad + E(x^-, A \circ x^+) + (x^- \cdot (A \circ x^+))h_0\\
&= E(A \circ x^-, x^+) + E(x^-, A \circ x^+) - [A, E(x^-, x^+)],
\end{aligned}$$

since, by (8.4), $(A \circ x^-) \cdot x^+ + x^- \cdot (A \circ x^+) = A \circ (x^- \cdot x^+) = 0$. For $v \in M^-$,

$$\begin{aligned}
(E(A \circ x^-, x^+) + E(x^-, A \circ x^+) - [A, E(x^-, x^+)]) \circ v &= (v \cdot x^+) \cdot (A \circ x^-)\\
&\quad + (v \cdot (A \circ x^+)) \cdot x^- - A \circ ((v \cdot x^+) \cdot x^-) + ((A \circ v) \cdot x^+) \cdot x^-.
\end{aligned}$$

This is 0 by (8.4), so $J(A, x^-, x^+) = 0$.

Case $(\sigma 2, -\sigma 2, \sigma 2)$: For $p, r \in \widetilde{\mathcal{E}}_{\sigma 2}$ and $q \in \widetilde{\mathcal{E}}_{-\sigma 2}$, we have

$$J(p, r, q) = 0 + [[r, q], p] - [[p, q], r],$$

which is 0 by (8.14).

Case $(\sigma 1, \sigma 1, -\sigma 2)$: For $x, y \in \widetilde{\mathcal{E}}_{\sigma 1}$ and $q \in \widetilde{\mathcal{E}}_{-\sigma 2}$, we have, using (8.13),

$$\begin{aligned}
J(x, y, q) &= [[x, y], q] + [[y, q], x] - [[x, q], y]\\
&= \sigma((xy) \cdot q)(h_M - 2h_0) + E(y \cdot q, x) + \sigma(x \cdot (y \cdot q))h_0\\
&\qquad - E(x \cdot q, y) - \sigma(y \cdot (x \cdot q))h_0\\
&= \sigma((xy) \cdot q)h_M + E(y \cdot q, x) - E(x \cdot q, y).
\end{aligned}$$

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For $v \in M^{-\sigma}$, we have

$$\begin{aligned}(E(y\cdot q,x)-E(x\cdot q,y))\circ v &= (v\cdot x)\cdot(y\cdot q)-(v\cdot y)\cdot(x\cdot q)\\ &= ((v\cdot x)y)\cdot q-((v\cdot y)x)\cdot q\\ &= (v\cdot(xy))\cdot q = ((xy)\cdot q)v\end{aligned}$$

by (8.3). Thus, $E(y\cdot q,x)-E(x\cdot q,y) = -\sigma((xy)\cdot q)h_M$, so $J(x,y,q)=0$.

Case $(\sigma 1,\sigma 1,-\sigma 1)$: For $x,y\in\widetilde{\mathcal{E}}_{\sigma 1}$ and $a\in\widetilde{\mathcal{E}}_{-\sigma 1}$, we have

$$\begin{aligned}J(x,y,a) &= -[a,[x,y]]+[[y,a],x]-[[x,a],y]\\ &= -a\cdot(xy)+E^\sigma(y,a)\circ x-(y\cdot a)x-E^\sigma(x,a)\circ y+(x\cdot a)y,\end{aligned}$$

using (8.12). This is 0 by [F, Lemma 3 (vi)].

Case $(\sigma 1,\sigma 2,-\sigma 2)$: For $x\in\widetilde{\mathcal{E}}_{\sigma 1}$, $p\in\widetilde{\mathcal{E}}_{\sigma 2}$, $q\in\widetilde{\mathcal{E}}_{-\sigma 2}$, we have

$$\begin{aligned}J(x,p,q) &= [[x,p],q]-[[q,p],x]-[[x,q],p]\\ &= 0+\sigma[(q\cdot p)(h_M-2h_0),x]-(x\cdot q)\cdot p &&\text{by (8.13)}\\ &= (q\cdot p)x-(x\cdot q)\cdot p &&\text{by (8.10)}.\end{aligned}$$

This is 0 by (8.3).

Case $(\sigma 1,\sigma 2,-\sigma 1)$: For $x\in\widetilde{\mathcal{E}}_{\sigma 1}$, $p\in\widetilde{\mathcal{E}}_{\sigma 2}$, $a\in\widetilde{\mathcal{E}}_{-\sigma 1}$, we have

$$\begin{aligned}J(x,p,a) &= [[x,p],a]-[[a,p],x]-[[x,a],p]\\ &= 0-(a\cdot p)x-E^\sigma(x,a)\circ p+2(x\cdot a)p &&\text{by (8.12)}\\ &= -(a\cdot p)x-\operatorname{tr}(E^\sigma(x,a))p+2(x\cdot a)p &&\text{by (8.5)}\\ &= -(a\cdot p)x-3(x\cdot a)p+2(x\cdot a)p &&\text{by (8.6)}\\ &= -(a\cdot p)x-(x\cdot a)p.\end{aligned}$$

But if $q\in\widetilde{\mathcal{E}}_{-\sigma 2}$, we have

$$\begin{aligned}-((a\cdot p)x)\cdot q = (x(a\cdot p))\cdot q = x\cdot((a\cdot p)\cdot q) &= x\cdot((p\cdot q)a) \quad\text{by (8.3)}\\ &= (x\cdot a)(p\cdot q) = ((x\cdot a)p)\cdot q,\end{aligned}$$

so $-(a\cdot p)x = (x\cdot a)p$ by (8.2). ∎

Remark 8.5 For $i=-2,-1,0,1,2$, the module $\widetilde{\mathcal{E}}_i$ is FGP of rank 1, 20, 37, 20, 1, respectively. Indeed this holds since $\wedge_k(M^\sigma)$ is FGP of rank $\binom{6}{k}$ for $0\le k\le 6$ and $\operatorname{End}(M^\sigma)$ is FGP of rank 36 [B1, II.5.3]. So $\widetilde{\mathcal{E}}$ is FGP of rank 79.

Remark 8.6 Suppose that $\mathbb{F}\in\mathbb{K}$-alg. One sees using [B2, II.5.3, II.7.5] and [B1, II.5.3] that $(M_{\mathbb{F}}^-, M_{\mathbb{F}}^+.g_{\mathbb{F}})$ satisfies (8.8) (over $\mathbb{F}$). We now show that *there exists a canonical $\mathbb{Z}$-graded $\mathbb{F}$-algebra isomorphism* $\omega\colon\widetilde{\mathcal{E}}_{\mathbb{F}}\to\widetilde{\mathcal{E}}(M_{\mathbb{F}}^-,M_{\mathbb{F}}^+,g_{\mathbb{F}})$. We define ω by defining its restriction ω_i to the i-th graded component $(\widetilde{\mathcal{E}}_i)_{\mathbb{F}}$ of $\widetilde{\mathcal{E}}_{\mathbb{F}}$ for $-2\le i\le 2$. First

$$\begin{aligned}(\widetilde{\mathcal{E}}_0)_{\mathbb{F}} &= \mathcal{S}_{\mathbb{F}}\oplus\mathbb{F}(1\otimes h_0),\\ \widetilde{\mathcal{E}}_0(M_{\mathbb{F}}^-,M_{\mathbb{F}}^+,g_{\mathbb{F}}) &= \mathcal{S}(M_{\mathbb{F}}^-,M_{\mathbb{F}}^+,g_{\mathbb{F}})\oplus h_0(M_{\mathbb{F}}^-,M_{\mathbb{F}}^+,g_{\mathbb{F}}).\end{aligned}$$

We define ω_0 on $\mathcal{S}_{\mathbb{F}}$ as the composite $\mathbb{F}$-algebra isomorphism $\mathcal{S}_{\mathbb{F}} \to \mathrm{End}(M^+)_{\mathbb{F}} \to \mathrm{End}(M^+_{\mathbb{F}}) \to \mathcal{S}(M^-_{\mathbb{F}}, M^+_{\mathbb{F}}, g_{\mathbb{F}})$, where the first isomorphism is induced by $(\iota^+)^{-1}$ (§8.4), the second is canonical [B1, II.5.4], and the third is $\iota^+(M^-_{\mathbb{F}}, M^+_{\mathbb{F}}, g_{\mathbb{F}})$; we define $\omega_0(1 \otimes h_0) = h_0(M^-_{\mathbb{F}}, M^+_{\mathbb{F}}, g_{\mathbb{F}})$. Lastly, we have the canonical $\mathbb{F}$-module isomorphism

$$\omega_{\sigma k} \colon (\widetilde{\mathcal{E}}_{\sigma k})_{\mathbb{F}} = (\wedge_{3k}(M^\sigma))_{\mathbb{F}} \to \wedge_{3k}(M^\sigma_{\mathbb{F}}) = \widetilde{\mathcal{E}}_{\sigma k}(M^-_{\mathbb{F}}, M^+_{\mathbb{F}}, g_{\mathbb{F}})$$

for $\sigma = \pm$, $k = 1, 2$ [B1, III.7.4, Proposition 8]. One checks that the direct sum ω of these maps is in fact an $\mathbb{F}$-algebra isomorphism as desired.

8.7 The 5-graded Lie Algebra $\mathcal{E}$

We now introduce an ideal $\mathcal{E}$ of $\widetilde{\mathcal{E}}$ that, as we will see in Proposition 8.12, is actually the derived algebra of $\widetilde{\mathcal{E}}$. For this we define a linear map $\lambda = \lambda(M^-, M^+, g) \colon \widetilde{\mathcal{E}}_0 \to \mathbb{K}$ by $\lambda((A^-, A^+) + ah_0) = \mathrm{tr}(A^+) + 3a$ for $(A^-, A^+) \in \mathcal{S}$ and $a \in \mathbb{K}$. Let

$$\mathcal{E} = \mathcal{E}(M^-, M^+, g) := \bigoplus_{i \in \mathbb{Z}} \mathcal{E}_i \ \text{ in } \ \widetilde{\mathcal{E}},$$

where the submodules $\mathcal{E}_i = \mathcal{E}_i(M^-, M^+, g)$ of $\widetilde{\mathcal{E}}$ are given by $\mathcal{E}_i := \widetilde{\mathcal{E}}_i$ for $i \neq 0$ and $\mathcal{E}_0 := \{X \in \widetilde{\mathcal{E}}_0 : \lambda(X) = 0\}$. Using (8.9), $\mathrm{tr}(\mathrm{id}_{M^+}) = 6(1_{\mathbb{K}})$, and $\mathrm{tr}([A^+, B^+]) = 0$, one checks easily that $\lambda([\widetilde{\mathcal{E}}_{-i}, \widetilde{\mathcal{E}}_i]) = 0$ for $i = 0, 1, 2$. So $\mathcal{E}$ contains the derived algebra $[\widetilde{\mathcal{E}}, \widetilde{\mathcal{E}}]$ and is hence a 5-graded ideal of $\widetilde{\mathcal{E}}$.

If $\mathbb{F} \in \mathbb{K}\text{-alg}$, one checks that the $\mathbb{F}$-algebra isomorphism

$$\omega_0 \colon (\widetilde{\mathcal{E}}_0)_{\mathbb{F}} \to \widetilde{\mathcal{E}}_0(M^-_{\mathbb{F}}, M^+_{\mathbb{F}}, g_{\mathbb{F}})$$

in Remark 8.6 satisfies

$$\lambda(M^-, M^+, g)_{\mathbb{F}} = \lambda(M^-_{\mathbb{F}}, M^+_{\mathbb{F}}, g_{\mathbb{F}}) \circ \omega_0. \tag{8.15}$$

We have a (non-canonical) direct sum decomposition of $\widetilde{\mathcal{E}}_0$.

Lemma 8.7 *The map $\lambda \colon \widetilde{\mathcal{E}}_0 \to \mathbb{K}$ is surjective, and if $X_0 \in \widetilde{\mathcal{E}}_0$ is chosen so that $\lambda(X_0) = 1$, then $\widetilde{\mathcal{E}}_0 = \mathcal{E}_0 \oplus \mathbb{K}X_0$ and $\mathbb{K}X_0$ is free of rank 1 with basis X_0.*

Proof For $\mathfrak{p} \in \mathrm{Spec}(\mathbb{K})$, $M^+_{\mathbb{K}_\mathfrak{p}}$ is free of rank 6, so there is $A^+ \in \mathrm{End}(M^+_{\mathbb{K}_\mathfrak{p}})$ with $\mathrm{tr}(A^+) = 1$. Hence $\lambda(M^-_{\mathbb{K}_\mathfrak{p}}, M^+_{\mathbb{K}_\mathfrak{p}}, M^-_{\mathbb{K}_\mathfrak{p}})$ is surjective, so $\lambda_{\mathbb{K}_\mathfrak{p}}$ is surjective by (8.15) (with $\mathbb{F} = \mathbb{K}_\mathfrak{p}$). Since this holds for all $\mathfrak{p} \in \mathrm{Spec}(\mathbb{K})$ and since $\widetilde{\mathcal{E}}_0$ is FGP, it follows that $\lambda \colon \widetilde{\mathcal{E}}_0 \to \mathbb{K}$ is surjective [B2, II.3.3]. The rest is clear. ∎

Remark 8.8 It follows from Remark 8.5 and Lemma 8.7 that for $i = -2, -1, 0, 1, 2$, the module $\mathcal{E}_i$ is FGP of rank 1, 20, 36, 20, 1, respectively. So $\mathcal{E}$ is FGP of rank 78.

If $\mathbb{F} \in \mathbb{K}\text{-alg}$, it follows from Lemma 8.7 that the canonical homomorphism $\mathcal{E}_{\mathbb{F}} \to \widetilde{\mathcal{E}}_{\mathbb{F}}$ induced by inclusion is injective. Using this map we will identify $\mathcal{E}_{\mathbb{F}}$ as a $\mathbb{Z}$-graded subalgebra of $\widetilde{\mathcal{E}}_{\mathbb{F}}$.

Lemma 8.9 *If $\mathbb{F} \in \mathbb{K}\text{-alg}$, the restriction of the isomorphism ω in Remark 8.6 is a $\mathbb{Z}$-graded $\mathbb{F}$-algebra isomorphism from $\mathcal{E}_{\mathbb{F}}$ onto $\mathcal{E}(M^-_{\mathbb{F}}, M^+_{\mathbb{F}}, g_{\mathbb{F}})$.*

Proof One checks using the direct sum decomposition $\widetilde{\mathcal{E}}_0 = \mathcal{E}_0 \oplus \mathbb{K}X_0$ in Lemma 8.7 that $(\mathcal{E}_0)_{\mathbb{F}} = \{X \in (\widetilde{\mathcal{E}}_0)_{\mathbb{F}} : \lambda_{\mathbb{F}}(X) = 0\}$. It follows from this and (8.15) that $\omega_0((\mathcal{E}_0)_{\mathbb{F}}) = \mathcal{E}_0(M^-_{\mathbb{F}}, M^+_{\mathbb{F}}, g_{\mathbb{F}})$ as needed. ∎

Remark 8.10 Suppose M^σ, $\sigma = \pm$, is free of rank 6. We now introduce some notation that will be useful in our calculations. Since g is non-singular, we can choose bases $B^\sigma = \{v^\sigma_1, \dots, v^\sigma_6\}$ for M^σ, $\sigma = \pm$, that are dual relative to g. We use these bases to identify $\mathrm{End}(M^-)$ and $\mathrm{End}(M^+)$ with $\mathrm{M}_6(\mathbb{K})$, in which case $(A^\sigma)^*$ is the transpose of A^σ for $A^\sigma \in \mathrm{End}(M^\sigma)$. Then $E(v^+_i, v^-_j) = \iota^+(E_{ij})$ for $i \neq j$, where E_{ij} is the standard matrix unit in $\mathrm{M}_6(\mathbb{K})$ and $\iota^+ = \iota^+(M^-, M^+, g)$. For $S = \{i_1 < \cdots < i_\ell\} \subseteq [1,6] := \{1,2,3,4,5,6\}$, let $v^-_S = v^-_{i_\ell} \cdots v^-_{i_1}$ and $v^+_S = v^+_{i_1} \cdots v^+_{i_\ell}$. So $\{v^-_S : |S| = k\}$ and $\{v^+_S : |S| = k\}$ are dual bases for $\bigwedge_k(M^-)$ and $\bigwedge_k(M^+)$ relative to the bilinear form $\cdot$. Note that if $S = \{i < j < k\}$ and $|T| = 3$, then $E^+(v^+_S, v^-_T) = E^+(v^+_i, (v^+_j v^+_k) \cdot v^-_T) + E^+(v^+_j, (v^+_k v^+_i) \cdot v^-_T) + E^+(v^+_k, (v^+_i v^+_j) \cdot v^-_T)$ by [F, Lemma 3 (v)], so by (8.11) we have

$$[v^+_S, v^-_T] = \begin{cases} \iota^+(E_{ii} + E_{jj} + E_{kk}) - h_0 & \text{if } T = S, \\ \pm\iota^+(E_{pq}) & \text{if } S\backslash T = \{p\}, T\backslash S = \{q\}, \\ 0 & \text{if } |T \cap S| \le 1. \end{cases} \tag{8.16}$$

Finally let $h_i = \iota^+(E_{ii} - E_{i+1,i+1})$ for $1 \le i \le 5$ and $h_6 = \iota^+(E_{44} + E_{55} + E_{66}) - h_0$ in $\mathcal{E}_0$.

Lemma 8.11 *Suppose that M^σ, $\sigma = \pm$, is free. With the above notation, the set*

$$B_{\mathcal{E}_0} = \{h_i : 1 \le i \le 6\} \cup \{\iota^+(E_{ij}) : 1 \le i \neq j \le 6\}$$

is a basis for $\mathcal{E}_0(M^-, M^+, g)$, and

$$B_{\mathcal{E}} = B_{\mathcal{E}_0} \cup \{v^\sigma_S : \sigma = \pm, |S| = 3\} \cup \{v^\sigma_{[1,6]} : \sigma = \pm\}$$

is a basis for $\mathcal{E}(M^-, M^+, g)$, where $[1,6] := \{1,2,3,4,5,6\}$.

Proof Since $\{v^\sigma_S : |S| = k\}$ is a basis for $\bigwedge_k(M^\sigma)$, it suffices to show $B_{\mathcal{E}_0}$ is a basis for $\mathcal{E}_0$. Since $\widetilde{\mathcal{E}}_0 = \mathcal{S} \oplus \mathbb{K}h_0 = \mathcal{S} \oplus \mathbb{K}h_6$ and $B_{\mathcal{E}_0}\backslash\{h_6\}$ is independent in $\mathcal{S}$, we see that $B_{\mathcal{E}_0}$ is independent. If $X = (A^-, A^+) + ah_0 \in \mathcal{E}_0$ with $(A^-, A^+) \in \mathcal{S}$ and $a \in \mathbb{K}$, then $X + ah_6 \in \mathcal{E}_0$ and $X + ah_6 = (B^-, B^+) \in \mathcal{S}$ for some $B^\sigma \in M_6(\mathbb{K})$. Thus $\mathrm{tr}(B^+) = 0$ and hence $(B^-, B^+) \in \mathrm{span}_{\mathbb{Z}}\{h_1, \dots, h_5\}$. ∎

Proposition 8.12 *$\mathcal{E}_0 = [\mathcal{E}_{-1}, \mathcal{E}_1]$; $\mathcal{E}$ is generated by $\mathcal{E}_{-1} \cup \mathcal{E}_1$ and $\mathcal{E} = [\widetilde{\mathcal{E}}, \widetilde{\mathcal{E}}] = [\mathcal{E}, \mathcal{E}]$.*

Proof It suffices to show $[\mathcal{E}_i, \mathcal{E}_j] = \mathcal{E}_{i+j}$ for $(i,j) = (-1,1), (\sigma 1, \sigma 1)$, and $(-\sigma 1, \sigma 2)$. First we assume that M^σ is free of rank 6, $\sigma = \pm$, and we use the notation of Remark 8.10. Then (8.16) and (8.11) show that $\iota^+(E_{ij})$ and $\iota^+(E_{ii} + E_{jj} + E_{jj}) - h_0$ are in $[\mathcal{E}_{-1}, \mathcal{E}_1]$ for distinct i, j, k. Thus $B_{\mathcal{E}_0} \subseteq [\mathcal{E}_{-1}, \mathcal{E}_1]$, so $\mathcal{E}_0 = [\mathcal{E}_{-1}, \mathcal{E}_1]$. Also, if $|S| = |T| = 3$, then

$$[v^\sigma_S, v^\sigma_T] = v^\sigma_S v^\sigma_T = \begin{cases} \pm v^\sigma_{[1,6]} & \text{if } S \cup T = [1,6], \\ 0 & \text{if } S \cup T \neq [1,6], \end{cases} \tag{8.17}$$

$$[v^{-\sigma}_S, v^\sigma_{[1,6]}] = v^{-\sigma}_S \cdot v^\sigma_{[1,6]} = \pm v^\sigma_{[1,6]\backslash S}, \tag{8.18}$$

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so $[\mathcal{E}_{\sigma 1}, \mathcal{E}_{\sigma 1}] = \mathcal{E}_{\sigma 2}$ and $[\mathcal{E}_{-\sigma 1}, \mathcal{E}_{\sigma 2}] = \mathcal{E}_{\sigma 1}$.

In the general case, it follows from Lemma 8.9 and the preceding paragraph that for $\mathfrak{p} \in \mathrm{Spec}(\mathbb{K})$ we have $[(\mathcal{E}_i)_{\mathbb{K}_\mathfrak{p}}, (\mathcal{E}_j)_{\mathbb{K}_\mathfrak{p}}] = (\mathcal{E}_{i+j})_{\mathbb{K}_\mathfrak{p}}$ for $(i,j) = (-1,1)$, $(\sigma 1, \sigma 1)$, and $(-\sigma 1, \sigma 2)$. But by Remark 8.8, $\mathcal{E}_i$, $\mathcal{E}_j$, $\mathcal{E}_{i+j}$, and $\mathcal{E}_i \otimes \mathcal{E}_j$ are FGP with $[\mathcal{E}_i, \mathcal{E}_j] \subseteq \mathcal{E}_{i+j}$. Then a localization argument using the multiplication map $\mathcal{E}_i \otimes \mathcal{E}_j \to \mathcal{E}_{i+j}$ shows that $[\mathcal{E}_i, \mathcal{E}_j] = \mathcal{E}_{i+j}$. ■

Remark 8.13 For use in the next proof (and again in Proposition 8.20), we describe here two isomorphisms involving the Lie algebras $\widetilde{\mathcal{E}}$ and $\mathcal{E}$. We omit some of the details which are easy to fill in.

(i) First suppose (M'^-, M'^+, g') is another triple satisfying (8.8) and $\theta = (\theta^-, \theta^+)$ is an isomorphism of (M^-, M^+, g) onto (M'^-, M'^+, g'). It is easy to see that θ^σ extends to a graded algebra isomorphism $\wedge(M^\sigma) \to \wedge(M'^\sigma)$ and, by conjugation, induces an isomorphism $\mathrm{End}(M^\sigma) \to \mathrm{End}(M'^\sigma)$ such that the products $\circ$ and $\cdot$ are preserved. Thus θ induces a graded isomorphism $\psi_\theta : \widetilde{\mathcal{E}}(M^-, M^+, g) \to \widetilde{\mathcal{E}}(M'^-, M'^+, g')$ that maps $\widetilde{\mathcal{S}}(M^-, M^+, g)$ onto $\widetilde{\mathcal{S}}(M'^-, M'^+, g')$ and h_0 to $h_0(M'^-, M'^+, g')$. It is clear that ψ_θ maps $\mathcal{E}(M^-, M^+, g)$ onto $\mathcal{E}(M'^-, M'^+, g')$.

(ii) We consider now the opposite triple $(M^-, M^+, g)^{\mathrm{op}} = (M^+, M^-, g)$. It is easy to see that for $x \in \wedge_3(M^\sigma)$ and $a \in \wedge_3(M^{-\sigma})$, we have $(E^{op})^{-\sigma}(x,a) = E^\sigma(x,a)$, so $E^{\mathrm{op}}(x_{-1}, x_1) = (-E^+(x_1, x_{-1}), E^-(x_{-1}, x_1))$, for $x_i \in \mathcal{E}_i$. Then the map $\zeta : \widetilde{\mathcal{E}}(M^-, M^+, g) \to \widetilde{\mathcal{E}}(M^+, M^-, g)$ defined by

$$\zeta(x_{-2} + x_{-1} + (A^-, A^+) + ah_0 + x_1 + x_2) = x_2 + x_1 + (A^+, A^-) - ah_0^{\mathrm{op}} + x_{-1} + x_{-2}$$

for $x_i \in \mathcal{E}_i$, $(A^-, A^+) \in \mathcal{S}(M^-, M^+, g)$, and $a \in \mathbb{K}$, is a graded algebra isomorphism. For the proof of this, we check two cases (the others being easily checked). We have

$$\begin{aligned}[\zeta(x_{-1}), \zeta(x_1)]^{\mathrm{op}} &= [x_{-1}, x_1]^{\mathrm{op}} = E^{\mathrm{op}}(x_{-1}, x_1) - (x_{-1} \cdot x_1) h_0^{\mathrm{op}} \\ &= \zeta(-E(x_1, x_{-1}) + (x_{-1} \cdot x_1) h_0) = \zeta([x_{-1}, x_1])\end{aligned}$$

and

$$\begin{aligned}[\zeta(x_{-2}), \zeta(x_2)]^{\mathrm{op}} &= [x_{-2}, x_2]^{\mathrm{op}} = (x_{-2} \cdot x_2)((-\mathrm{id}_{M^+}, \mathrm{id}_{M^-}) - 2h_0^{\mathrm{op}}) \\ &= \zeta((x_{-2} \cdot x_2)((\mathrm{id}_{M^-}, -\mathrm{id}_{M^+}) + 2h_0) = \zeta([x_{-2}, x_2]).\end{aligned}$$

Evidently ζ maps $\mathcal{E}(M^-, M^+, g)$ onto $\mathcal{E}(M^+, M^-, g)$.

Remark 8.14 By [B2, II.5, Exercise 8], there exists a faithfully flat $\mathbb{F} \in \mathbb{K}$-alg such that M^σ is free of rank 6 for $\sigma = \pm$. So, using Remarks 8.6, 8.10, and 8.13 (i), we have 5-graded isomorphisms $\mathcal{E}_\mathbb{F} \simeq \mathcal{E}(\mathbb{F}^6, \mathbb{F}^6, \cdot) \simeq \mathcal{E}(\mathbb{K}^6, \mathbb{K}^6, \cdot)_\mathbb{F}$. In other words, $\mathcal{E}$ is an $\mathbb{F}/\mathbb{K}$-form of $\mathcal{E}(\mathbb{K}^6, \mathbb{K}^6, \cdot)$ as defined in [Ser, III.1.1]. A similar remark, which we leave to the reader, holds for the Kantor pairs $\wedge_3$ and the SP-graded Kantor pairs $\breve{\wedge}_3$ described in Subsections 8.8 and 8.9 below.

Proposition 8.15 *If $\mathbb{K} = \mathbb{C}$, then $\mathcal{E} = \mathcal{E}(\mathbb{C}^6, \mathbb{C}^6, \cdot)$ is a simple Lie algebra of type* E_6 *and $B_\mathcal{E}$ is a Chevalley basis of $\mathcal{E}$.*

Proof We use the notation of Remark 8.10 relative to the standard dual bases.

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We first note that $\eta\colon (A^-, A^+) + ah_0 \to A^+ + \frac{1}{3}a\,\mathrm{id}_{\mathbb{C}^6}$ is a homomorphism $\eta\colon \widetilde{\mathcal{S}} = \widetilde{\mathcal{S}}(\mathbb{C}^6, \mathbb{C}^6, \cdot) \to \mathrm{End}(M^+) = \mathrm{M}_6(\mathbb{C})$ with $\eta(X) \circ x = X \circ x$ for $X \in \widetilde{\mathcal{S}}$, $x \in \mathcal{E}_k$, $k = 1, 2$. Since $\mathrm{tr}(\eta(h_6)) = 1$, η maps $B_{\mathcal{E}_0}$ to a basis for $\mathrm{M}_6(\mathbb{C})$, so η restricts to an isomorphism $\mathcal{E}_0 \to \mathrm{M}_6(\mathbb{C})$.

Using (8.17), we see that if I is an ideal of $\mathcal{E}$ and $x \in I$ with nonzero component in $\mathcal{E}_{\sigma 1} \oplus \mathcal{E}_{\sigma 2}$, then some $y \in [x, \mathcal{E}_{\sigma 1}] \cup \mathbb{C}x$ has component $v^\sigma_{[1,6]}$ in $\mathcal{E}_{\sigma 2}$. So (fixing $S \subseteq [1,6]$ with $|S| = 3$), some $z \in [y, \mathcal{E}_{-\sigma 1}]$ has component v^σ_S in $\mathcal{E}_{\sigma 1} \oplus \mathcal{E}_{\sigma 2}$, and hence $[[[z, \mathcal{E}_{-\sigma 2}], \mathcal{E}_{-\sigma 1}], \mathcal{E}_{\sigma 1}] = \mathcal{E}_{-\sigma 1}$. Thus, an ideal I is either contained in $\mathcal{E}_0$ or contains $\mathcal{E}_{\sigma 1}$, $\sigma = \pm$, and hence $\mathcal{E}$. If $I \subseteq \mathcal{E}_0$, then $\eta(I)$ is an ideal of the Lie algebra $\mathrm{M}_6(\mathbb{C})$, so $\eta(I) = 0$, $\mathbb{C}\,\mathrm{id}_{\mathbb{C}^6}$, $[\mathrm{M}_6(\mathbb{C}), \mathrm{M}_6(\mathbb{C})]$, or $\mathrm{M}_6(\mathbb{C})$. On the other hand, $\eta(I) \circ \mathcal{E}_1 = [I, \mathcal{E}_1] \subseteq I \cap \mathcal{E}_1 = 0$, so $I = 0$. Thus, $\mathcal{E}$ is simple.

Since η maps $\mathcal{H} := \mathrm{span}_{\mathbb{C}}\{h_1, \dots, h_6\}$ to the set of diagonal matrices, $\mathcal{H}$ is an abelian Cartan subalgebra of $\mathcal{E}_0$. Also, $t = \iota^+(\mathrm{id}_{\mathbb{C}^6}) - 2h_0 \in \mathcal{H}$ has $\mathrm{ad}(t) = \mathrm{ad}(h_0)$ on $\mathcal{E}$, so the normalizer of $\mathcal{H}$ in $\mathcal{E}$ is contained in $\mathcal{E}_0$ and hence equals $\mathcal{H}$. Thus, $\mathcal{H}$ is an abelian Cartan subalgebra of $\mathcal{E}$. It is clear that $\mathrm{ad}(h)$ is diagonalizable on $\mathcal{E}$ for $h \in \mathcal{H}$, so we have $\mathcal{E} = \bigoplus_{\mu \in \mathcal{H}^*} \mathcal{E}(\mu)$, where $\mathcal{E}(\mu) = \{x \in \mathcal{E} : [h, x] = \mu(h)x \text{ for } h \in \mathcal{H}\}$ for $\mu \in \mathcal{H}^*$. Let $\Sigma = \{\mu \in \mathcal{H}^* : \mu \neq 0, \mathcal{E}(\mu) \neq 0\}$, so $\mathcal{E} = \bigoplus_{\mu \in \Sigma \cup \{0\}} \mathcal{E}(\mu)$ with $\mathcal{E}(0) = \mathcal{H}$. Now let $\varepsilon_i \in \mathcal{H}^*$ with $\varepsilon_i(h) = a_i$, where $\eta(h) = \mathrm{diag}(a_1, \dots, a_6)$. It is easy to see that the elements $\mu \in \Sigma$ and the corresponding root spaces $\mathcal{E}(\mu) = \mathbb{K}x_\mu$ are

(8.19) $\mu = \varepsilon_i - \varepsilon_j, \quad i \neq j,$ with $x_\mu = \iota^+(E_{ij})$,

(8.20) $\mu = \sigma(\varepsilon_i + \varepsilon_j + \varepsilon_k), \quad i < j < k, \sigma = \pm,$ with $x_\mu = v^\sigma_{\{i,j,k\}}$,

(8.21) $\mu = \sigma(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 + \varepsilon_6), \quad \sigma = \pm,$ with $x_\mu = v^\sigma_{[1,6]}$.

To show that $\mathcal{E}$ has type E_6, let $\mu_i = \varepsilon_i - \varepsilon_{i+1}$ for $1 \le i \le 5$ and $\mu_6 = \varepsilon_4 + \varepsilon_5 + \varepsilon_6$. Let $\Pi = \{\mu_1, \dots, \mu_6\}$ and let A_{ij} be the Cartan integer of the pair (μ_i, μ_j) for $1 \le i, j \le 6$. An examination of the μ_j-string through μ_i shows that $A = (A_{ij})$ is the Cartan matrix of type E_6.

Let $h_\mu = [x_\mu, x_{-\mu}]$ for $\mu \in \Sigma$. To show that $B_{\mathcal{E}}$ is a Chevalley basis, we need to show [H, p. 147]

(a) $[h_\mu, x_\mu] = 2x_\mu$ for $\mu \in \Sigma$;
(b) $h_{\mu_i} = h_i$, $i = 1, \dots, 6$;
(c) the linear map with $x_\mu \to -x_{-\mu}$, $h_i \to -h_i$ is an automorphism of $\mathcal{E}$.

By (8.16), we have

$$
\begin{aligned}
h_\mu &= \iota^+(E_{ii} - E_{jj}) && \text{for } \mu \text{ as in (8.19)},\\
h_\mu &= \sigma\iota^+(E_{ii} + E_{jj} + E_{kk}) - \sigma h_0 && \text{for } \mu \text{ as in (8.20)},\\
h_\mu &= \sigma\iota^+(\mathrm{id}_{\mathrm{M}_6(\mathbb{C})}) - \sigma 2h_0 && \text{for } \mu \text{ as in (8.21)},
\end{aligned}
$$

and (a) and (b) follow. For (c), let $\theta = (\theta^-, \theta^+)$ where $\theta^\sigma\colon M^\sigma \to M^{-\sigma}$ with $\theta^\sigma(v^\sigma_i) = v^{-\sigma}_i$, so that ψ_θ and ζ, as described in Remark 8.13 (i) and (ii), respectively, are isomorphisms $\widetilde{\mathcal{E}}(M^-, M^+, g) \to \widetilde{\mathcal{E}}(M^+, M^-, g)$. For $A \in \mathrm{M}_6(\mathbb{C})$, we have $\zeta^{-1}\psi_\theta(-A^*, A) = \zeta^{-1}(-A^*, A) = (A, -A^*)$ and $\zeta^{-1}\psi_\theta(h_0) = \zeta^{-1}(h_0^{op}) = -h_0$. Thus, $\zeta^{-1}\psi_\theta(x_\mu) = -x_{-\mu}$ for $\mu = \varepsilon_i - \varepsilon_j$ and $\zeta^{-1}\psi_\theta(h_i) = -h_i$. Also, if $S = \{i < j < k\}$, then

ψ_θ interchanges $v_S^+ = v_i^+ v_j^+ v_k^+$ with $v_i^- v_j^- v_k^- = -v_k^- v_j^- v_i^- = -v_S^-$, so $\zeta^{-1}\psi_\theta(v_S^\sigma) = -v_S^{-\sigma}$. Similarly, $\zeta^{-1}\psi_\theta(v_{[1,6]}^\sigma) = -v_{[1,6]}^{-\sigma}$ and (c) holds. ∎

Theorem 8.16 *Suppose $\mathbb{K}$ is a unital commutative ring and (M^-, M^+, g) satisfies* (8.8). *Let $\mathcal{E} = \mathcal{E}(M^-, M^+, g)$.*

(i) *$\mathcal{E}$ is a form of the Chevalley algebra of type* E_6 *and, if M^- and M^+ are free, $\mathcal{E}$ is the Chevalley algebra of type* E_6.

(ii) *$\mathcal{E}/Z(\mathcal{E})$ is a form of the split Lie algebra of type* E_6 *which is split if M^- and M^+ are free.*

(iii) *If $\frac{1}{3} \in \mathbb{K}$, then $Z(\mathcal{E}) = 0$.*

(iv) *If $\mathbb{K}$ is a field of characteristic $\neq 3$, $\mathcal{E}$ is central simple.*

Proof (i) By Remarks 8.10 and 8.14, we can assume that $(M^-, M^+, g) = (\mathbb{K}^6, \mathbb{K}^6, \cdot)$ and show that $\mathcal{E}$ is the Chevalley algebra of type E_6. Then it is clear that the $\mathbb{Z}$-linear map taking elements of the basis $B_\mathcal{E}$ in $\mathcal{E}(\mathbb{Z}^6, \mathbb{Z}^6, \cdot)$ to the corresponding elements of the basis $B_\mathcal{E}$ in $\mathcal{E}(\mathbb{C}^6, \mathbb{C}^6, \cdot)$ is an injective $\mathbb{Z}$-algebra homomorphism. We use this map to view $\mathcal{E}(\mathbb{Z}^6, \mathbb{Z}^6, \cdot)$ as the $\mathbb{Z}$-span of the Chevalley basis $B_\mathcal{E}$ of $\mathcal{E}(\mathbb{C}^6, \mathbb{C}^6, \cdot)$. Since $\mathcal{E}(\mathbb{K}^6, \mathbb{K}^6, \cdot) \cong \mathcal{E}(\mathbb{Z}^6, \mathbb{Z}^6, \cdot)_\mathbb{K}$ by Lemma 8.9, $\mathcal{E}(\mathbb{K}^6, \mathbb{K}^6, \cdot)$ is the Chevalley algebra.

(ii) follows from (i) and Remark 4.26.

(iii) By Lemma 4.22 and Remark 8.14, we can assume that

$$(M^-, M^+, g) = (\mathbb{K}^6, \mathbb{K}^6, \cdot).$$

Then a direct calculation, which we leave to the reader, shows that

$$Z(\mathcal{E}) = \{ch_M : c \in \mathbb{K}, 3c = 0\},$$

which is 0 when $\frac{1}{3} \in \mathbb{K}$.

(iv) By (i) and (iii), $\mathcal{E}$ is the Chevalley algebra of type E_6 and $Z(\mathcal{E}) = 0$. Hence, by [St, 2.6 (5)], $\mathcal{E}$ is simple. Moreover, if $\mathbb{F}$ is a field containing $\mathbb{K}$, then $\mathcal{E}_\mathbb{F} \simeq \mathcal{E}(M_\mathbb{F}^-, M_\mathbb{F}^+, g_\mathbb{F})$ by Lemma 8.9, so $\mathcal{E}_\mathbb{F}$ is simple over $\mathbb{F}$. Thus, $\mathcal{E}$ is central simple by [Mc, Theorem II.1.7.1]. ∎

Remark 8.17 Faulkner [F] also used exterior algebras to construct forms of Chevalley algebras of exceptional type. However, the Lie algebras obtained there have natural $\mathbb{Z}_3$-gradings rather than the 5-gradings and BC_2-gradings that we need in order to construct Kantor pairs.

8.8 The Kantor Pair $\wedge_3$

We now use the results of Subsection 8.7 to construct a Kantor pair $\wedge_3$ with simply described underlying modules and products.

Theorem 8.18 *Suppose $\mathbb{K}$ is a unital commutative ring containing $\frac{1}{6}$ and (M^-, M^+, g) satisfies* (8.8). *Let $\wedge_3 = \wedge_3(M^-, M^+, g) := (\wedge_3(M^-), \wedge_3(M^+))$ with trilinear products*

$$\{x^\sigma y^{-\sigma} z^\sigma\}^\sigma = E^\sigma(x^\sigma, y^{-\sigma}) \circ z^\sigma - (x^\sigma \cdot y^{-\sigma}) z^\sigma.$$

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(See Subsections 8.3–8.5 for the notation used here.)

(i) *$\bigwedge_3$ is a form of a split Kantor pair of type* E_6, *which is split if each M^σ is free. Also $\bigwedge_3$ is tightly enveloped by the 5-graded Lie algebra $\mathcal{E} = \mathcal{E}(M^-, M^+, g)$, so $\mathfrak{K}(\bigwedge_3) \simeq \mathcal{E}$ as 5-graded Lie algebras.*

(ii) *The Jordan obstruction of $\bigwedge_3$ is isomorphic to $(\bigwedge^6(M^-), \bigwedge^6(M^+))$ with products $\{p,q,r\}^\sigma = 2(p\cdot q)r = (p\cdot q)r + (r\cdot q)p$.*

(iii) *If $\mathbb{K}$ is a field, $\bigwedge_3$ is a central simple split Kantor pair of type* E_6 *of balanced dimension* 20 *and balanced 2-dimension* 1.

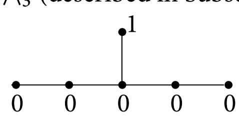
Proof (i) It follows from (8.12) (with $i = 1$) that the trilinear pair $\bigwedge_3$ is the Kantor pair enveloped by $\mathcal{E}$, and, in particular, it is a Kantor pair. The fact that $\mathcal{E}$ tightly envelops $\bigwedge_3$ follows from Proposition 8.12 and Theorem 8.16 (ii). So $\mathfrak{K}(\bigwedge_3) \simeq \mathcal{E}$ as 5-graded Lie algebras by Corollary 4.17. Hence by Lemma 4.29 and Theorem 8.16 (ii), $\bigwedge_3$ is a form of a split Kantor pair of type E_6, and this form is split if each M^σ is free by Theorem 8.16 (i).

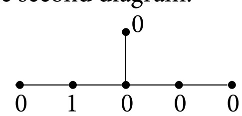
(ii) By (i) and (4.6), $J(\bigwedge_3) \simeq (\mathcal{E}_{-2}, \mathcal{E}_2) = (\bigwedge_6(M^-), \bigwedge_6(M^+))$ under the products $[[X,Y],Z]$ in $\mathcal{E}$. The formulas for the products now follow from (8.14).

(iii) $\bigwedge_3$ is central simple by Theorem 8.16 (iii) and Theorem 4.20 (iii). The dimension statements follow from the definition of $\bigwedge_3$ and (ii). ∎

Remark 8.19 Suppose that $\mathbb{K}$ is an algebraically closed field of characteristic 0.

(i) The weighted Dynkin diagram corresponding to $\bigwedge_3$ in Kantor's classification (see Remark 4.21) is the first diagram below, whereas the one corresponding to its reflection $\check{\bigwedge}_3$ (described in Subsection 8.9) is the second diagram.





(ii) The Kantor pair $\bigwedge_3$ is a simplified and basis-free version of the double of the KTS that was described by Kantor without full proofs in [K1, (6.11) and §6.6].

$\bigwedge_3$ is also isomorphic to the signed double of a $(1,1)$-Freudenthal–Kantor triple system. Indeed, Elduque and Kochetov have defined the structure of a symplectic triple system on $\mathcal{T} = \bigwedge^3(V)$ (although it appears that the scalar -24 used in this definition should be replaced by -2), where V is a 6-dimensional space [EK, §6.4] . The signed double of the corresponding $(1,1)$-Freudenthal triple system is a Kantor pair $\mathcal{P}(\mathcal{T})$ (see Example 4.31) that can be shown directly to be isomorphic to $\bigwedge_3$.

Finally, it can be seen from Kantor's classification that $\bigwedge_3$ is isomorphic to the double of the structurable algebra $\left[\begin{smallmatrix}\mathbb{K} & J\\ J & \mathbb{K}\end{smallmatrix}\right]$, where J is the Jordan algebra of 3×3-matrices over $\mathbb{K}$ (see [K2, §2] and [A, §8]). However from this matrix point of view a natural non-trivial SP-grading is not apparent to us.

8.9 The Kantor Pair $\check{\bigwedge}_3$

Suppose henceforth that (M^-, M^+, g, e) is a quadruple satisfying

(8.22) $g\colon M^- \times M^+ \to \mathbb{K}$ is a non-singular bilinear form, M^- and M^+ are FGP modules of rank 6, and $e = (e^-, e^+) \in M^- \times M^+$ satisfies $g(e^-, e^+) = 1$.

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As in Subsection 7.7, we now use e to define a BC_2-grading on $\mathcal{E}$.

Once again, we have $M^\sigma = \mathbb{K}e^\sigma \oplus U^\sigma$, where $U^\sigma = (e^{-\sigma})^\perp$ relative to g in M^σ. We write M^σ as $\left[\begin{smallmatrix} \mathbb{K}e^\sigma \\ U^\sigma \end{smallmatrix}\right]$, and correspondingly identify

$$\operatorname{End}(M^\sigma) = \begin{bmatrix} \operatorname{End}(\mathbb{K}e^\sigma) & \operatorname{Hom}(U^\sigma, \mathbb{K}e^\sigma) \\ \operatorname{Hom}(\mathbb{K}e^\sigma, U^\sigma) & \operatorname{End}(U^\sigma) \end{bmatrix}.$$

Proposition 8.20 *$\mathcal{E} = \bigoplus_{(i_1,i_2)\in\mathbb{Z}^2} \mathcal{E}_{i_1,i_2}$ is a BC_2-grading of the Lie algebra $\mathcal{E}$, where*

$$\mathcal{E}_{\sigma 1,0} = \wedge_3(U^\sigma), \quad \mathcal{E}_{\sigma 1,\sigma 1} = \wedge_2(U^\sigma)e^\sigma, \quad \mathcal{E}_{\sigma 2,\sigma 1} = \wedge_6(M^\sigma),$$
$$\mathcal{E}_{0,0} = \mathcal{E}_0 \cap \big(\iota^+(\operatorname{End}(\mathbb{K}e^\sigma) \oplus \operatorname{End}(U^\sigma)) + \mathbb{K}h_0\big),$$
$$\mathcal{E}_{0,1} = \iota^+(\operatorname{Hom}(U^+, \mathbb{K}e^+)), \quad \mathcal{E}_{0,-1} = \iota^+(\operatorname{Hom}(\mathbb{K}e^+, U^+)),$$

and $\mathcal{E}_{i_1,i_2} = 0$ for all other pairs $(i_1,i_2) \in \mathbb{Z}^2$. Moreover the first component grading of this BC_2-grading is the 5-grading of $\mathcal{E}$ in Subsection 8.7.

Proof We follow the proof of Proposition 7.7 with $\mathfrak{fo}(\widetilde{g})$ replaced by $\mathcal{E}$. As in the last paragraph of that proof, we can (by a base ring extension argument using Lemma 8.9) assume that there exists a unit $t \in \mathbb{K}$ such that $(t^i - t^j)x = 0$, $x \in \mathcal{E}$ implies $x = 0$ or $i = j$. Let $\theta^\sigma \in \mathrm{GL}(M^\sigma)$ be left multiplication by $\left[\begin{smallmatrix} t^{\sigma 1} & 0 \\ 0 & 1 \end{smallmatrix}\right]$. Since $\theta = (\theta^-, \theta^+)$ is an automorphism of (M^-, M^+, g), it induces an automorphism ψ_θ of $\mathcal{E}(M^-, M^+, g)$ as in Remark 8.13 (i). The proof now proceeds as in Proposition 7.7. ∎

By Proposition 8.20 and Remark 5.4, we have the following result which describes an SP-grading on the Kantor pair $\wedge_3$ constructed in Theorem 8.18.

Proposition 8.21 *Suppose $\frac{1}{6} \in \mathbb{K}$, and let $(\wedge_3)_0^\sigma = \wedge_3(U^\sigma)$ and $(\wedge_3)_1^\sigma = \wedge_2(U^\sigma)e^\sigma$. Then $\wedge_3 = (\wedge_3)_0 \oplus (\wedge_3)_1$ is an SP-graded Kantor pair, which is tightly enveloped by the BC_2-graded Lie algebra $\mathcal{E}$ described in Proposition 8.20.*

We now have the main result about Kantor pairs in this section.

Theorem 8.22 *Suppose $\mathbb{K}$ is a unital commutative ring containing $\frac{1}{6}$ and assume (M^-, M^+, g, e) satisfies (8.22).*

(i) *The reflection $\check{\wedge}_3$ of the SP-graded Kantor pair $\wedge_3$ described in Proposition 8.21 is an SP-graded form of a split Kantor pair of type E_6 which is split if each M^σ is free.*
(ii) *The Jordan obstruction J of $\check{\wedge}_3$ is isomorphic to U^{op}, where $U = (U^-, U^+)$ with products $\{u^\sigma, v^{-\sigma}, w^\sigma\}^\sigma = g(u^\sigma, v^{-\sigma})w^\sigma + g(w^\sigma, v^{-\sigma})u^\sigma$.*
(iii) *If $\mathbb{K}$ is a field, $\check{\wedge}_3$ is a central simple split Kantor pair of type E_6 of balanced dimension 20 and balanced 2-dimension 5.*

Proof (i) follows from Theorem 8.18 and Proposition 6.4.

(ii) By Proposition 6.7, Theorem 8.18 (i), and Proposition 8.20,

$$J^{\mathrm{op}} \simeq (\mathcal{E}_{0,1}, \mathcal{E}_{0,-1}) = \big(\iota^+(\operatorname{Hom}(U^+, \mathbb{K}e^+)), \iota^+(\operatorname{Hom}(\mathbb{K}e^+, U^+))\big)$$

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under the products $[[X,Y],Z]$ in $\mathcal{E}_{0,*}$. Since there are natural module isomorphisms $U^- \simeq \mathrm{Hom}(U^+, \mathbb{K}e^+)$ and $U^+ \simeq \mathrm{Hom}(\mathbb{K}e^+, U^+)$ (induced by g in the first case), our conclusion is easily checked.

(iii) $\breve{\wedge}_3$ is a central simple split Kantor pair of type E_6 by Corollary 8.18 (iii) and Proposition 6.4. Also, since $(\breve{\wedge}_3)^\sigma = \wedge_3(U^{-\sigma}) \oplus \wedge_2(U^\sigma)e^\sigma$, $\breve{\wedge}_3$ has balanced dimension 10 + 10, while $\breve{\wedge}_3$ has balanced 2-dimension 5 by (ii). ∎

If (M'^-, M^+, g', e') is another quadruple satisfying (8.22), an isomorphism of (M^-, M^+, g, e) onto (M'^-, M'^+, g', e') is an isomorphism of $(M^-, M+, g)$ onto (M'^-, M^+, g'), which maps e onto e'. If such an isomorphism exists, one sees, using Remark 8.13 (i), that the BC_2-graded Lie algebras $\mathcal{E}$ and $\mathcal{E}'$ constructed above are graded-isomorphic, and, if $\frac{1}{6} \in \mathbb{K}$, the SP-graded Kantor pairs $\wedge_3$ and $\wedge_3'$ are graded isomorphic, as are the SP-graded pairs $\breve{\wedge}_3$ and $(\wedge_3')\breve{}$.

Remark 8.23 Suppose $\mathbb{K}$ is a field. Then one easily checks that any two quadruples satisfying (8.22) are isomorphic. Hence the BC_2-graded Lie algebra $\mathcal{E}$ constructed above is independent up to graded isomorphism of the choice of (M^-, M^+, g, e), and if $\frac{1}{6} \in \mathbb{K}$, so too are the SP-graded Kantor pairs $\wedge_3$ and $\breve{\wedge}_3$.

Remark 8.24 Suppose $\mathbb{K}$ is an algebraically closed field of characteristic 0. The construction of $\breve{\wedge}_3$ given above is a simple new basis-free construction of the double of the KTS C^2_{55} constructed by Kantor without full proofs in [K2, §4] and [K1, §6.6]. The pair $\breve{\wedge}_3$ is of particular interest since it is one of the two split simple Kantor pairs of exceptional type that do not arise by doubling a structurable algebra—the other being the famous Jordan pair $(M_{1,2}(C), M_{1,2}(C^{op}))$ determined by a Cayley algebra C [L, §8.15].

8.10 An Example Using Rank 1 Modules

If I is an FGP module of rank 1, set $M_I = \mathbb{K}^5 \oplus I$ and $e_I = (e_I^-, e_I^+) \in (M_I^*, M_I)$, where $e_I^+ = (1,0,0,0,0)$, $e_I^-(a_1,\dots,a_5) = a_1$, and $e_I^-|_I = 0$. Then $(M_I^*, M_I, \mathrm{can})$ satisfies (8.8) and $(M_I^*, M_I, \mathrm{can}, e_I)$ satisfies (8.22) (see Remark 8.1 (ii)). Let $\mathcal{E}_I$ be the BC_2-graded Lie algebra constructed from $(M_I^*, M_I, \mathrm{can}, e_I)$ in Proposition 8.20. We also regard $\mathcal{E}_I$ as a 5-graded Lie algebra with the first component grading. If $\frac{1}{6} \in \mathbb{K}$, let $\wedge_{3,I}$ be the SP-graded Kantor pair constructed from $(M_I^*, M_I, \mathrm{can}, e_I)$ in Proposition 8.21.

Suppose that I and I' are FGP modules of rank 1. We now show that

$$\mathcal{E}_I \simeq_{\mathrm{BC}_2} \mathcal{E}_{I'} \iff \mathcal{E}_I \simeq_{5\text{-gr}} \mathcal{E}_{I'} \iff I \simeq I', \tag{8.23}$$

where $\simeq_{\mathrm{BC}_2}$ and $\simeq_{5\text{-gr}}$ indicate isomorphisms as BC_2-graded and 5-graded Lie algebras. Indeed, denoting these statements by (a), (b), and (c) in order, it is clear that (a) implies (b). Suppose (b) holds. Then $(\mathcal{E}_I)_2 \simeq (\mathcal{E}_{I'})_2$ as $\mathbb{K}$-modules, so $\wedge_6(M_I) \simeq \wedge_6(M_{I'})$. But $\wedge_6(M_I) = \wedge_6(\mathbb{K}^5 \oplus I) \simeq \wedge_5(\mathbb{K}^5) \otimes \wedge_1(I) \simeq R \otimes I \simeq I$ using [B2, III.7.7, Proposition 10], and we have (c). Suppose finally that (c) holds. Then the quadruples $(M_I^*, M_I, \mathrm{can}, e_I)$ and $(M_{I'}^*, M_{I'}, \mathrm{can}, e_{I'})$ are isomorphic, so (a) holds (as noted after Theorem 8.22).

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Suppose next that $\frac{1}{6} \in \mathbb{K}$ and I, I' are FGP modules of rank 1. We now show that

$$(8.24) \qquad \begin{aligned} (\bigwedge_{3,I})\check{} \simeq (\bigwedge_{3,I'})\check{} &\iff (\bigwedge_{3,I})\check{} \simeq_{\mathrm{SP}} (\bigwedge_{3,I'})\check{} \iff \bigwedge_{3,I} \simeq_{\mathrm{SP}} \bigwedge_{3,I'},\\ &\iff \bigwedge_{3,I} \simeq \bigwedge_{3,I'} \iff I \simeq I', \end{aligned}$$

where $\simeq_{\mathrm{SP}}$ indicates isomorphism as SP-graded Kantor pairs. To see this, we denote these statements by (α), (β), (γ), (δ), and (ε) in order. Then (γ) (resp. (δ)) is equivalent to (a) (resp. (b)) in (8.23). Also the equivalence of (β) and (γ) is clear (for example, from Proposition 6.5). So (β), (γ), (δ), and (ε) are equivalent. Since (β) implies (α), it is enough now to check that (α) implies (ε). But if (α) holds, then $J((\bigwedge_{3,I})\check{}) \simeq J((\bigwedge_{3,I'})\check{})$, so $\mathbb{K}^4 \oplus I \simeq \mathbb{K}^4 \oplus I'$ by Theorem 8.22 (ii). Hence,

$$\bigwedge_5(\mathbb{K}^4 \oplus I) \simeq \bigwedge_5(\mathbb{K}^4 \oplus I'),$$

so $I \simeq I'$ as above.

Suppose hereafter that $\mathbb{K}$ is a Dedekind domain. Let $\mathrm{Pic}(\mathbb{K})$ (the Picard group of $\mathbb{K}$) be the group of all isomorphism classes of FGP modules of rank 1 under the product induced from the tensor product. Recall that $\mathrm{Pic}(\mathbb{K})$ is naturally isomorphic to the ideal class group $\mathfrak{C}(\mathbb{K})$ of $\mathbb{K}$ [B2, II.5.7, Proposition 12].

Recall further that any FGP module of rank $m \geq 1$ is isomorphic to $\mathbb{K}^{m-1} \oplus I$ for some FGP module I of rank 1 (see [Na, §1.3]). Using this fact with $m = 6$ (resp. $m = 5$), it is easy to see that any triple satisfying (8.8) (resp. any quadruple satisfying (8.22)) is isomorphic to $(M_I^*, M_I, \mathrm{can})$ (resp. $(M_I^*, M_I, \mathrm{can}, e_I)$) for some I.

Therefore all of the following algebraic structures are obtained from some I as above: (i) 5-graded Lie algebras $\mathcal{E}$ in Theorem 8.16, (ii) (ungraded) Kantor pairs $\bigwedge_3$ in Theorem 8.18, (iii) BC_2-graded Lie algebras $\mathcal{E}$ in Proposition 8.20, (iv) SP-graded Kantor pairs $\bigwedge_3$ in Proposition 8.21, (v) SP-graded pairs $\check{\bigwedge}_3$ in Theorem 8.22, and (vi) (ungraded) Kantor pairs $\check{\bigwedge}_3$ in Theorem 8.22. So, by (8.23) and (8.24), the sets of graded-isomorphism classes for each of the families (i), (iii), (iv), and (v), as well as the sets of isomorphism classes for the families (ii) and (vi), are each in one-to-one correspondence with $\mathrm{Pic}(\mathbb{K})$.

Since any abelian group arises as the ideal class group of some Dedekind domain [L-G, Theorem 1.4], we see that we have many examples of the indicated structures.

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