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A LIPSCHITZ METRIC FOR THE CAMASSA–HOLM EQUATION

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Abstract

We analyze stability of conservative solutions of the Cauchy problem on the line for the Camassa–Holm (CH) equation. Generically, the solutions of the CH equation develop singularities with steep gradients while preserving continuity of the solution itself. In order to obtain uniqueness, one is required to augment the equation itself by a measure that represents the associated energy, and the breakdown of the solution is associated with a complicated interplay where the measure becomes singular. The main result in this paper is the construction of a Lipschitz metric that compares two solutions of the CH equation with the respective initial data. The Lipschitz metric is based on the use of the Wasserstein metric.

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1. Introduction

We study the Cauchy problem for weak conservative solutions of the Camassa–Holm (CH) equation, which reads as

$$u_t + uu_x + p_x = 0, \tag{1.1a}$$

$$\mu_t + (u\mu)_x = (u^3 - 2pu)_x, \tag{1.1b}$$

where $p(t, x)$ is given by

$$p(t, x) = \frac{1}{4}\int_{\mathbb{R}} e^{-|x-y|}u^2(t, y)\,dy + \frac{1}{4}\int_{\mathbb{R}} e^{-|x-y|}\,d\mu(t, y), \tag{1.2}$$

with initial data $(u, \mu)|_{t=0} = (u_0, \mu_0)$. In this case, the natural solution space consists of all pairs (u, μ) such that

$$u(t,\,\cdot\,) \in H^1(\mathbb{R}), \quad \mu(t,\,\cdot\,) \in \mathcal{M}_+(\mathbb{R}), \quad \text{and} \quad d\mu_{\mathrm{ac}} = (u^2 + u_x^2)\,dx,$$

where $\mathcal{M}_+(\mathbb{R})$ denotes the set of all positive and finite Radon measures on $\mathbb{R}$. Our main goal is to prove the existence of a metric d such that

$$d((u_1(t), \mu_1(t)), (u_2(t), \mu_2(t))) \leqslant \alpha(t) d((u_{1,0}, \mu_{1,0}), (u_{2,0}, \mu_{2,0})),$$

for two weak conservative solutions $(u_j(t), \mu_j(t))$ $j = 1, 2$ of (1.1) with initial data $(u_{j,0}, \mu_{j,0})$ $j = 1, 2$. Here $\alpha(t)$ depends on the total energy of the solutions and $\alpha(0) = 1$.

The CH equation has been introduced in the seminal paper [10]; see also [11]. Originally derived in the context of models for shallow water (see also [27, 58]), it also turns up in models for hyperelastic rods [24, 28, 55]. Since it captures the nonlinear effects that give insight into important phenomena such as breaking waves and breaking rods, the CH equation has been intensively studied. The intricate behavior of solutions of the Cauchy problem will be the focus of this paper. We will not discuss further properties of the CH equation, for example, the fact that the equation is completely integrable and allows for a geometric interpretation. With the latter, we mean that the CH equation is a re-expression of a geodesic flow on the diffeomorphism group of the line or the circle [26, 31, 60]. Several extensions and generalizations exist, but we will focus on (1.1).

The intriguing aspect of solutions to the Cauchy problem is the generic development of singularities in finite time, irrespective of the smoothness of the initial data. A solution may develop steep gradients, but in contrast to, say, hyperbolic conservation laws, the solution itself remains continuous. A finer analysis reveals that the energy density $u^2 + u_x^2$ develops singularities, and at breakdown, which is often referred to as wave breaking, energy concentrates on sets of measure zero. Thus it becomes useful to introduce a measure, here denoted by μ, that encodes the energy, and away from breakdowns this measure should coincide with the energy density $u^2 + u_x^2$. In technical terms, we consider a nonnegative Radon measure μ with an absolutely continuous part $d\mu_{\mathrm{ac}} = (u^2 + u_x^2)\,dx$. This measure μ satisfies equation (1.1b) (and as can easily be verified, $\mu = u^2 + u_x^2$ will satisfy the same equation in the case of smooth solutions). An illustrating example of how intricate the structure of the points of wave breaking may be can be found in [40]. The behavior in the proximity of the point of wave breaking, and, in particular, the prolongation of the solution past wave breaking, has been extensively studied. See, for example, [6, 7, 21, 22, 25, 32, 33, 39, 43–54, 56, 57] and references therein. The key point here is that past wave breaking uniqueness fails, and there is a continuum of distinct solutions [47], with two extreme cases called dissipative and conservation solutions. To understand this conundrum, it turns out to be advantageous to rewrite the equation in a different set of variables where the solution remains smooth.

The explicit peakon–antipeakon solution [41], which illustrates this problem, is given by

$$u(t,x) = \begin{cases} -\alpha(t)e^{x}, & x \leqslant -\gamma(t), \\ \beta(t)\sinh(x), & -\gamma(t) < x < \gamma(t), \\ \alpha(t)e^{-x}, & \gamma(t) \leqslant x, \end{cases} \tag{1.3}$$

where

$$\alpha(t) = \frac{E}{2}\sinh\left(\frac{E}{2}t\right), \quad \beta(t) = E\frac{1}{\sinh(\frac{E}{2}t)}, \quad \gamma(t) = \ln\left(\cosh\left(\frac{E}{2}t\right)\right),$$

where $E = \|u(t)\|_{H^1}$ for all $t \neq 0$. This function is a weak conservative solution which consists of a 'peak' moving to the right and an 'antipeak' moving to the left; see Figure B.3. At $t = 0$ the 'peak' and 'antipeak' collide, and the solution vanishes, yet the solution is highly nontrivial before and after the collision time. Clearly the trivial solution will coincide with this solution at $t = 0$, yet the trivial solution and (1.3) are very different at any other time. Thus it is not clear how to derive a metric comparing two solutions, that is stable under the time evolution. This is the task of the present work.

To be more precise, we are here presenting a metric d with the property that

$$d((u_1(t), \mu_1(t)), (u_2(t), \mu_2(t))) \leqslant \alpha(t) d((u_{1,0}, \mu_{1,0}), (u_{2,0}, \mu_{2,0})),$$

for two weak conservative solutions $(u_j(t), \mu_j(t))$, $j = 1, 2$, of (1.1) with initial data $(u_{j,0}, \mu_{j,0})$, $j = 1, 2$. Here $\alpha(t)$ is a continuous function with $\alpha(0) = 1$, which may depend on the total energies involved, but not on the particular solutions. We stress that no standard Sobolev norm nor Lebesgue space norm will work. There exist alternative metrics for solutions of the CH equation; see [8, 42, 43]. In [8] the periodic case is treated by approximating the solution by multipeakons. The metric is defined by optimizing over a class of functions. The approach in [42, 43] depends on a reformulation of the CH equation in terms of Lagrangian variables. An intrinsic problem in this formulation is that of *relabeling*, where there will be many different parametrizations in Lagrangian coordinates, corresponding to one and the same solution $(u(t), \mu(t))$ in Eulerian variables. Thus one has to compute the distance between equivalence classes, which is not transparent. In the present approach, the key idea is to introduce a new set of variables, where one variable plays a role similar to a characteristic, while the remaining variables are linked to (u, μ) with the help of the 'characteristic'. As we will outline next, there is no

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need to resort to equivalence classes or to optimize over classes of functions, and in spite of this proof being longer, we consider this approach to be more natural.

Our approach is based on the fact that a natural metric for measuring distances between Radon measures (with the same total mass) is given through the Wasserstein (or Monge–Kantorovich) distance d_W, which in one dimension is defined with the help of pseudoinverses; see [67]. Given a nonnegative measure μ of finite mass $M > 0$, we define the cumulative distribution function associated to μ as

$$F(x) = \mu((-\infty, x)),$$

which is a nondecreasing function from $\mathbb{R}$ onto $[0, M]$, left continuous and with limit from the right at any point $x \in \mathbb{R}$. The pseudoinverse associated to μ denoted by $\mathcal{X}$ is the function from $[0, M]$ onto $\mathbb{R}$ given by

$$\mathcal{X}(\eta) = \sup\{x \mid F(x) < \eta\}.$$

The pseudoinverse of F is a nondecreasing function from $[0, M]$ onto $\mathbb{R}$, left continuous and with limit from the right (caglad) at any point $x \in \mathbb{R}$. Notice the different convention adopted here with respect to the usual one in probability theory defining cumulative distribution functions continuous from the right and with limits from the left (cadlag) at every point. We prefer to have caglad instead of cadlag functions due to the use of the methods from [54] developed under the present convention. Wasserstein distances between nonnegative measures with the same mass can be defined via L^p-norms of the difference between their associated pseudoinverses; see [19, 20, 61, 67] and the references therein.

The approach of using Wasserstein distances to control the expansion of solutions of evolutionary PDEs leading to curves of probability measures goes back to the proofs of the mean-field limit of McKean–Vlasov and Vlasov equations in the late seventies and eighties of the last century. We refer to the classical references [5, 29, 63–65] proving these large particle limits by means of the bounded Lipschitz distance and the coupling method. See the recent results and surveys in [3, 12, 13, 35, 36]. The optimal transport viewpoint for one-dimensional models was developed using pseudoinverse distributions for nonlinear aggregation and diffusion equations in [15, 19, 20, 61, 66] and the references therein, showing the contractivity of the Wasserstein distance in one dimension without the heavy machinery of optimal transport developed for general gradient flows in [1]. More recently, these metrics have been used with success to show uniqueness past the blow-up time for multidimensional aggregation equations [14] using gradient flow solutions. It is also interesting to point out that gradient flow solutions of the aggregation equation in one dimension with particular potentials are equivalent to entropy solutions of the Burgers

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equation as proven in [4]. Another strategy using unbalanced optimal transport tools has been recently analyzed in [34] with the objective of understanding the relation between the incompressible Euler equations and the CH equations.

Finally, it is worth mentioning that there have been several works [2, 17, 18, 30, 37, 38, 59, 62, 68] making use of this change of variables to produce numerical schemes capable of going over blow-up of solutions to nonlinear aggregations and being able to capture the blow-up of solutions of aggregation–diffusion models in one dimension such as toy versions of the Keller–Segel model for chemotaxis. It is a nice avenue of research to use this approach to produce numerical schemes for conservative solutions of the CH equation; see [23] for related particle methods.

In the present work, we will adapt this strategy of defining suitable distances between measures to the present problem of finding good metrics for solutions of the CH equation. Let $(u(t, \cdot), \mu(t, \cdot))$ be a weak conservative solution to the CH equation with total energy $\mu(t, \mathbb{R}) = C > 0$ (which for simplicity here is assumed to be smooth). Let

$$F(t, x) = \mu(t, (-\infty, x)) = \int_{-\infty}^{x} d\mu(t)$$

due to the smoothness, and introduce the basic quantity

$$G(t, x) = \int_{-\infty}^{x} (2p - u^2)(t, y)\, dy + F(t, x) = 2p_x(t, x) + 2F(t, x).$$

The key function here is the (spatial) inverse of the strictly increasing function G for fixed time t. To that end, we define

$$\mathcal{Y}(t, \eta) = \sup\{x \mid G(t, x) < \eta\}.$$

Formally we have that $G(t, \mathcal{Y}(t, \eta)) = \eta$ for all $\eta \in (0, 2C)$ and $\mathcal{Y}(t, G(t, x)) = x$ for all $x \in \mathbb{R}$. Here it is important to note that the domain of $\mathcal{Y}$ depends on the total energy C. Next, we want to determine the time evolution of $\mathcal{Y}$. Direct formal calculations yield that

$$\mathcal{Y}_t(t, G(t, x)) + \mathcal{Y}_\eta(t, G(t, x))G_t(t, x) = 0, \tag{1.4a}$$

$$\mathcal{Y}_\eta(t, G(t, x))G_x(t, x) = 1. \tag{1.4b}$$

Thus we need to compute the time evolution of $G(t, x)$ before being able to compute the time evolution of $\mathcal{Y}(t, \eta)$. To that end, we find, after some computations, that

$$G_t(t, x) + uG_x(t, x) = \tfrac{2}{3}u^3(t, x) + S(t, x), \tag{1.5}$$

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where

$$S(t,x)=\int_{\mathbb{R}} e^{-|x-y|}\Big(\frac{2}{3}u^3-u_xp_x-2pu\Big)(t,y)\,dy. \tag{1.6}$$

Introducing $\eta=G(t,x)$, $\mathcal{S}(t,\eta)=S(t,\mathcal{Y}(t,\eta))$, and

$$\mathcal{U}(t,\eta)=u(t,\mathcal{Y}(t,\eta)), \tag{1.7}$$

we find by combining (1.4) and (1.5) that

$$\mathcal{Y}_t(t,\eta)+(\tfrac{2}{3}\mathcal{U}^3+\mathcal{S})(t,\eta)\mathcal{Y}_\eta(t,\eta)=\mathcal{U}(t,\eta),$$

where we used that $\mathcal{Y}(t,G(t,x))=x$ for all $x\in\mathbb{R}$. As far as the time evolution of $\mathcal{U}(t,\eta)$ is concerned, we find

$$\mathcal{U}_t(t,\eta)=-\mathcal{Q}(t,\eta)-(\tfrac{2}{3}\mathcal{U}^3+\mathcal{S})\mathcal{U}_\eta(t,\eta),$$

where we introduced $\mathcal{Q}(t,\eta)=p_x(t,\mathcal{Y}(t,\eta))$. Thus, formally we end up with the system

$$\begin{aligned}\mathcal{Y}_t(t,\eta)+(\tfrac{2}{3}\mathcal{U}^3+\mathcal{S})\mathcal{Y}_\eta(t,\eta)&=\mathcal{U}(t,\eta),\\ \mathcal{U}_t(t,\eta)+(\tfrac{2}{3}\mathcal{U}^3+\mathcal{S})\mathcal{U}_\eta(t,\eta)&=-\mathcal{Q}(t,\eta).\end{aligned}$$

However, this system is not closed, and we need to introduce the function

$$\mathcal{P}(t,\eta)=p(t,\mathcal{Y}(t,\eta)), \tag{1.8}$$

and determine its time evolution. We find, after some computations, that

$$\mathcal{P}_t(t,\eta)+(\tfrac{2}{3}\mathcal{U}^3+\mathcal{S})\mathcal{P}_\eta(t,\eta)=\mathcal{Q}\mathcal{U}(t,\eta)+\mathcal{R}(t,\eta),$$

where

$$\begin{aligned}\mathcal{R}(t,\eta)=\frac{1}{4}\int_0^{2C}&\operatorname{sign}(\eta-\theta)e^{-|\mathcal{Y}(t,\eta)-\mathcal{Y}(t,\theta)|}\Big(\frac{2}{3}\mathcal{U}^3\mathcal{Y}_\eta+\mathcal{U}\Big)(t,\theta)\,d\theta\\ &-\frac{1}{2}\int_0^{2C}e^{-|\mathcal{Y}(t,\eta)-\mathcal{Y}(t,\theta)|}\mathcal{U}\mathcal{Q}\mathcal{Y}_\eta(t,\theta)\,d\theta.\end{aligned}$$

To summarize, we have established the following system of differential equations

$$\mathcal{Y}_t(t,\eta)+(\tfrac{2}{3}\mathcal{U}^3+\mathcal{S})\mathcal{Y}_\eta(t,\eta)=\mathcal{U}(t,\eta), \tag{1.9a}$$

$$\mathcal{U}_t(t,\eta)+(\tfrac{2}{3}\mathcal{U}^3+\mathcal{S})\mathcal{U}_\eta(t,\eta)=-\mathcal{Q}(t,\eta), \tag{1.9b}$$

$$\mathcal{P}_t(t,\eta)+(\tfrac{2}{3}\mathcal{U}^3+\mathcal{S})\mathcal{P}_\eta(t,\eta)=\mathcal{Q}\mathcal{U}(t,\eta)+\mathcal{R}(t,\eta), \tag{1.9c}$$

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where

$$\mathcal{Q}(t,\eta) = -\frac{1}{4}\int_0^{2C} \operatorname{sign}(\eta-\theta)e^{-|\mathcal{Y}(t,\eta)-\mathcal{Y}(t,\theta)|}(2(\mathcal{U}^2-\mathcal{P})\mathcal{Y}_\eta(t,\theta)+1)\,d\theta, \tag{1.10a}$$

$$\mathcal{S}(t,\eta) = \int_0^{2C} e^{-|\mathcal{Y}(t,\eta)-\mathcal{Y}(t,\theta)|}\left(\frac{2}{3}\mathcal{U}^3\mathcal{Y}_\eta - \mathcal{U}_\eta\mathcal{Q} - 2\mathcal{P}\mathcal{U}\mathcal{Y}_\eta\right)(t,\theta)\,d\theta, \tag{1.10b}$$

$$\mathcal{R}(t,\eta) = \frac{1}{4}\int_0^{2C} \operatorname{sign}(\eta-\theta)e^{-|\mathcal{Y}(t,\eta)-\mathcal{Y}(t,\theta)|}\left(\frac{2}{3}\mathcal{U}^3\mathcal{Y}_\eta + \mathcal{U}\right)(t,\theta)\,d\theta$$
$$- \frac{1}{2}\int_0^{2C} e^{-|\mathcal{Y}(t,\eta)-\mathcal{Y}(t,\theta)|}\mathcal{U}\mathcal{Q}\mathcal{Y}_\eta(t,\theta)\,d\theta. \tag{1.10c}$$

Derived under assumptions of smoothness of the functions involved, the same system is valid also in the general case of weak conservation solutions. However, that requires considerable analysis, and Section 3 is devoted to that. The next step is to estimate the time evolution of these quantities $(\mathcal{Y}, \mathcal{U}, \mathcal{P})$. It turns out that the natural functional space is the space of square integrable functions for the unknowns $(\mathcal{Y}, \mathcal{U}, \mathcal{P}^{1/2})$. For this reason, we prefer to work with $\mathcal{P}^{1/2}$ rather than $\mathcal{P}$. Section 3 focuses on the first qualitative properties of the time evolution of the solutions of (1.9) for weak conservative solutions of the CH equation (1.1) as well as the propagation in time of the L^2-norm of the unknowns.

The main aim of our work is to identify the right distance between two general conservative solutions of the CH equation (1.1), or equivalently, between two general L^2 solutions $(\mathcal{Y}, \mathcal{U}, \mathcal{P}^{1/2})$ of system (1.9) with possibly different energies. In order to compare solutions with different energies, we need to rescale the solutions of (1.9) in such a way that they are defined on the same interval. Since the natural functional space for our unknowns $(\mathcal{Y}, \mathcal{U}, \mathcal{P}^{1/2})$ was identified as the L^2-functional space, it seems natural to do a scaling conserving the L^2-norms of the unknowns $(\mathcal{Y}, \mathcal{U}, \mathcal{P}^{1/2})$, but leading to the domain being independent of the total energy C.

Let us define the scaled unknowns $(\tilde{\mathcal{Y}}, \tilde{\mathcal{U}}, \tilde{\mathcal{P}}^{1/2})$ associated to a conservative solution $(u(t), \mu(t))$ with energy $C = \mu(t, \mathbb{R})$ of the CH equation (1.1) as $\tilde{\mathcal{Y}}(t, \eta) = \sqrt{2C}\,\mathcal{Y}(t, 2C\eta)$, $\tilde{\mathcal{U}}(t,\eta) = \sqrt{2C}\,\mathcal{U}(t, 2C\eta)$, and $\tilde{\mathcal{P}}^{1/2}(t,\eta) = \sqrt{2C}\,\mathcal{P}^{1/2}(t, 2C\eta)$, where $(\mathcal{Y}, \mathcal{U}, \mathcal{P}^{1/2})$ is the solution of (1.9). This scaling allows also for the zero solution to (1.1) to be included in our considerations, as outlined in Section 4. A similar system to (1.9) can be written for $(\tilde{\mathcal{Y}}, \tilde{\mathcal{U}}, \tilde{\mathcal{P}}^{1/2})$, but this is postponed to Section 4. With this new set of unknowns in place, we can now define a metric to compare two general conservative solutions (u_i, μ_i), $i = 1, 2$, of (1.1) with total

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energy $C_i = \mu_i(\mathbb{R})$. We define it as

$$d((u_1,\mu_1),(u_2,\mu_2)) = \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|_{L^2([0,1])} + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|_{L^2([0,1])} + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|_{L^2([0,1])} + |\sqrt{2C_1} - \sqrt{2C_2}|.$$

Our main result reads as follows.

THEOREM 1.1. *Consider initial data* $u_{i,0} \in H^1(\mathbb{R})$, $\mu_{i,0} \in \mathcal{M}_+(\mathbb{R})$ *such that* $d(\mu_{\mathrm{ac}})_{i,0} = (u_i^2 + u_{i,x}^2)\,dx$ *and* $C_i = \mu_i(\mathbb{R})$, *and let* (u_i, μ_i) *for* $i = 1, 2$ *denote the corresponding weak conservative solutions of the CH equation* (1.1). *Then we have that*

$$d((u_1(t),\mu_1(t)),(u_2(t),\mu_2(t))) \leqslant e^{\mathcal{O}(1)t} d((u_{1,0},\mu_{1,0}),(u_{2,0},\mu_{2,0})),$$

where $\mathcal{O}(1)$ *denotes a constant depending only on* $\max_j(C_j)$ *remaining bounded as* $\max_j(C_j) \to 0$.

The main core of this work lies in estimating the Lipschitz property of the right-hand side of the equivalent system to (1.9) in the L^2-sense for the unknowns $(\tilde{\mathcal{Y}}, \tilde{\mathcal{U}}, \tilde{\mathcal{P}}^{1/2})$. This is much easier in case we compare to the zero solution as it coincides with the propagation of the L^2-norms of the unknowns. Due to the intricate nonlinearities of the right-hand sides of (1.9), this leads in the general case to long detailed technical estimates that are displayed in full in Subsections 5.1, 5.2, and 5.3. In the case of peakon–antipeakon solutions, as solution (1.3) is denoted, all quantities described in this paper can be computed explicitly. The details are to be found in Appendix B.

A notational comment is in order. We decided to denote by $\mathcal{O}(1)$ constants depending on $\max_j(C_j)$ that may change from line to line along the proofs, but remain bounded as $\max_j(C_j) \to 0$. Explicit tracking of the constants could be possible but it is highly cumbersome and avoided for the sake of the reader.

2. Formal ideas: transformations with smoothness

Let us start by explaining all the mathematical details for the transformation in the case of smooth solutions as outlined in the introduction. Let $(u(t,\cdot), \mu(t,\cdot))$ be a weak conservative solution to the CH equation with total energy $\mu(t,\mathbb{R}) = C > 0$. We assume that $F(t,x)$, given by

$$F(t,x) = \int_{-\infty}^{x} d\mu(t), \tag{2.1}$$

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is increasing and smooth, and, in particular, that $\mu = \mu_{\text{ac}} = (u^2 + u_x^2)\, dx$ for all t. Introduce the function

$$\begin{aligned} G(t,x) &= \int_{-\infty}^{x} (2p - u^2)(t,y)\, dy + F(t,x) \\ &= 2p_x(t,x) + 2F(t,x), \end{aligned}$$

where we used integration by parts and (1.2). First of all, note that the function $G(t,x)$ satisfies

$$\lim_{x\to-\infty} G(x) = 0 \quad \text{and} \quad \lim_{x\to\infty} G(x) = 2C,$$

since $|p_x(t,x)| \leqslant p(t,x)$ and p is an H^1 function on the line due to (1.2). Moreover, the function $(2p - u^2)(t,x) \geqslant 0$ for all $(t,x) \in \mathbb{R}^2$ as the following computation shows,

$$\begin{aligned} (2p - u^2)(t,x) &= \frac{1}{2}\int_{\mathbb{R}} e^{-|x-y|}(u^2 + F_x)(t,y)\, dy - u^2(t,x) \\ &= \frac{1}{2}\int_{-\infty}^{x} e^{y-x} u^2(t,y)\, dy + \frac{1}{2}\int_{x}^{\infty} e^{x-y} u^2(t,y)\, dy \\ &\quad + \frac{1}{2}\int_{\mathbb{R}} e^{-|x-y|} F_x(t,y)\, dy - u^2(t,x) \\ &= \frac{1}{2} u^2(t,x) - \frac{1}{2}\int_{-\infty}^{x} e^{y-x} 2uu_x(t,y)\, dy \\ &\quad + \frac{1}{2} u^2(t,x) + \frac{1}{2}\int_{x}^{\infty} e^{x-y} 2uu_x(t,y)\, dy \\ &\quad + \frac{1}{2}\int_{\mathbb{R}} e^{-|x-y|} F_x(t,y)\, dy - u^2(t,x) \\ &\geqslant \frac{1}{2}\int_{\mathbb{R}} e^{-|x-y|}(F_x(t,y) - 2|uu_x|(t,y))\, dy \\ &= \frac{1}{2}\int_{\mathbb{R}} e^{-|x-y|}(|u(t,y)| - |u_x(t,y)|)^2\, dy \geqslant 0. \end{aligned} \tag{2.2}$$

Thus the function $G(t,x)$ is nondecreasing and, in our case, since the function $F(t,x)$ is smooth, also $G(t,x)$ is smooth.

REMARK 2.1. Estimate (2.2), that is, $2p - u^2 \geqslant 0$, remains valid also in the case where the functions are nonsmooth.

Last but not least, we want to make sure that $G(t,x)$ is strictly increasing, so that its pseudoinverse will have no jumps. $F(t,x)$ is constant if and only if $d\mu$,

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u, and u_x are equal to zero. Therefore assume that there exists (for fixed t) some interval $[b, c]$ such that $d\mu(t,x) = u(t,x) = u_x(t,x) = 0$ for all $x \in [b, c]$. Then the only term that can save us is $p(t,x)$, which in general satisfies $p(t,x) \geqslant 0$ for all $(t,x) \in \mathbb{R}^2$. However, whenever $\mu(t,\mathbb{R}) \neq 0$, one has by its definition in (1.2) that $p(t,x) > 0$ and the claim follows.

Thus the function $G(t,x)$ is strictly increasing and continuous, and we can consider its pseudoinverse $\mathcal{Y}\colon [0, 2C] \to \mathbb{R}$, which in this case coincides with its inverse and which is given by

$$\mathcal{Y}(t,\eta) = \sup\{x \mid G(t,x) < \eta\}. \tag{2.3}$$

Since $G(t,x)$ is strictly increasing and continuous, we have that $G(t,\mathcal{Y}(t,\eta)) = \eta$ for all $\eta \in (0, 2C)$ and $\mathcal{Y}(t, G(t,x)) = x$ for all $x \in \mathbb{R}$. By the smoothness assumption on F, direct calculations yield that

$$\mathcal{Y}_t(t, G(t,x)) + \mathcal{Y}_\eta(t, G(t,x))G_t(t,x) = 0, \tag{2.4a}$$

$$\mathcal{Y}_\eta(t, G(t,x))G_x(t,x) = 1. \tag{2.4b}$$

Thus we need to compute the time evolution of $G(t,x)$ before being able to compute the time evolution of $\mathcal{Y}(t,\eta)$. The following calculations are only valid in the case of smooth solutions, but we will show in the next section how to overcome this issue for weak conservative solutions. Since $e^{-|x-y|}/2$ is the integral kernel of $(-\partial_x^2 + 1)^{-1}$, we observe from (1.2) that p is the solution to

$$p - p_{xx} = \tfrac{1}{2}u^2 + \tfrac{1}{2}\mu$$

and hence

$$\begin{aligned} p_t - p_{txx} &= uu_t + \tfrac{1}{2}F_{xt} \\ &= -u^2u_x - up_x - \tfrac{1}{2}(uF_x)_x + \tfrac{1}{2}(u^3)_x - (pu)_x \\ &= \tfrac{1}{6}(u^3)_x - (pu)_x - \tfrac{1}{2}(uF_x)_x - up_x, \end{aligned}$$

where we used the abbreviation $\mu = F_x$. Thus we end up with

$$\begin{aligned} p_t(t,x) = &-\frac{1}{2}\int_{\mathbb{R}} \operatorname{sign}(x-y)e^{-|x-y|}\left(\frac{1}{6}u^3 - pu - \frac{1}{2}uF_x\right)(t,y)\,dy \\ &- \frac{1}{2}\int_{\mathbb{R}} e^{-|x-y|}up_x(t,y)\,dy. \end{aligned}$$

Similar calculations yield that

$$p_{xt}(t,x) = -\frac{1}{6}u^3 + pu + \frac{1}{2}uF_x + \frac{1}{2}\int_{\mathbb{R}} e^{-|x-y|}\left(\frac{2}{3}u^3 - u_xp_x - 2pu\right)(t,y)\,dy.$$

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Thus we get for the time evolution of $G(t,x)$ that

$$G_t(t,x)+uG_x(t,x)=\tfrac{2}{3}u^3(t,x)+S(t,x), \tag{2.5}$$

where

$$S(t,x)=\int_{\mathbb{R}} e^{-|x-y|}\left(\frac{2}{3}u^3-u_xp_x-2pu\right)(t,y)\,dy.$$

Combining (2.4) and (2.5), we end up with

$$\mathcal{Y}_t(t,G(t,x))+(\tfrac{2}{3}u^3(t,x)+S(t,x))\mathcal{Y}_\eta(t,G(t,x))=u(t,x).$$

Introducing $\eta=G(t,x)$, we deduce

$$\mathcal{Y}_t(t,\eta)+(\tfrac{2}{3}u^3+S)(t,\mathcal{Y}(t,\eta))\mathcal{Y}_\eta(t,\eta)=u(t,\mathcal{Y}(t,\eta)),$$

where we used that $\mathcal{Y}(t,G(t,x))=x$ for all $x\in\mathbb{R}$. As far as the time evolution of

$$\mathcal{U}(t,\eta)=u(t,\mathcal{Y}(t,\eta)) \tag{2.6}$$

is concerned, we have

$$\begin{aligned}
\mathcal{U}_t(t,\eta)&=u_t(t,\mathcal{Y}(t,\eta))+u_x(t,\mathcal{Y}(t,\eta))\mathcal{Y}_t(t,\eta)\\
&=u_t(t,\mathcal{Y}(t,\eta))+uu_x(t,\mathcal{Y}(t,\eta))-(\tfrac{2}{3}u^3+S)u_x(t,\mathcal{Y}(t,\eta))\mathcal{Y}_n(t,\eta)\\
&=-p_x(t,\mathcal{Y}(t,\eta))-(\tfrac{2}{3}u^3+S)u_x(t,\mathcal{Y}(t,\eta))\mathcal{Y}_n(t,\eta)\\
&=-\mathcal{Q}(t,\eta)-(\tfrac{2}{3}\mathcal{U}^3+\mathcal{S})\mathcal{U}_\eta(t,\eta),
\end{aligned}$$

where we introduced $\mathcal{Q}(t,\eta)=p_x(t,\mathcal{Y}(t,\eta))$ and $\mathcal{S}(t,\eta)=S(t,\mathcal{Y}(t,\eta))$. Thus, formally we end up with the system

$$\mathcal{Y}_t(t,\eta)+(\tfrac{2}{3}\mathcal{U}^3+\mathcal{S})\mathcal{Y}_\eta(t,\eta)=\mathcal{U}(t,\eta), \tag{2.7a}$$

$$\mathcal{U}_t(t,\eta)+(\tfrac{2}{3}\mathcal{U}^3+\mathcal{S})\mathcal{U}_\eta(t,\eta)=-\mathcal{Q}(t,\eta), \tag{2.7b}$$

where $\mathcal{Q}(t,\eta)$ and $\mathcal{S}(t,\eta)$ can be written as

$$\begin{aligned}
&\mathcal{Q}(t,\eta)\\
&=-\frac{1}{4}\int_{\mathbb{R}}\operatorname{sign}(\mathcal{Y}(t,\eta)-y)e^{-|\mathcal{Y}(t,\eta)-y|}(u^2(t,y)+F_x(t,y))\,dy\\
&=-\frac{1}{4}\int_{\mathbb{R}}\operatorname{sign}(\mathcal{Y}(t,\eta)-y)e^{-|\mathcal{Y}(t,\eta)-y|}(2(u^2(t,y)-p(t,y))+G_x(t,y))\,dy\\
&=-\frac{1}{4}\int_0^{2C}\operatorname{sign}(\eta-\theta)e^{-|\mathcal{Y}(t,\eta)-\mathcal{Y}(t,\theta)|}(2(\mathcal{U}^2-\mathcal{P})\mathcal{Y}_\eta(t,\theta)+1)\,d\theta,
\end{aligned}$$

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and

$$
\begin{aligned}
\mathcal{S}(t,\eta) &= \int_{\mathbb{R}} e^{-|\mathcal{Y}(t,\eta)-y|}\Big(\frac{2}{3}u^3 - u_x p_x - 2pu\Big)(t,y)\,dy\\
&= \int_0^{2C} e^{-|\mathcal{Y}(t,\eta)-\mathcal{Y}(t,\theta)|}\Big(\frac{2}{3}\mathcal{U}^3\mathcal{Y}_\eta - \mathcal{U}_\eta\mathcal{Q} - 2\mathcal{P}\mathcal{U}\mathcal{Y}_\eta\Big)(t,\theta)\,d\theta,
\end{aligned}
$$

with

$$
\mathcal{P}(t,\eta) = p(t,\mathcal{Y}(t,\eta)). \tag{2.8}
$$

It is then natural, in order to close system (2.7), that besides the quantities u and μ, p in the new variables must be considered. One main reason is that these three quantities turn up in the definition of G. We already computed before that

$$
\begin{aligned}
p_t(t,x) &= -\frac{1}{2}\int_{\mathbb{R}} \mathrm{sign}(x-y)e^{-|x-y|}\Big(\frac{1}{6}u^3 - pu - \frac{1}{2}uF_x\Big)(t,y)\,dy\\
&\quad - \frac{1}{2}\int_{\mathbb{R}} e^{-|x-y|}up_x(t,y)\,dy\\
&= \frac{1}{2}\int_{\mathbb{R}} \mathrm{sign}(x-y)e^{-|x-y|}\Big(\frac{1}{3}u^3 + \frac{1}{2}uG_x\Big)(t,y)\,dy\\
&\quad - \frac{1}{2}\int_{\mathbb{R}} e^{-|x-y|}up_x(t,y)\,dy,
\end{aligned}
$$

where we used that

$$
G_x(t,x) = 2p(t,x) - u^2(t,x) + F_x(t,x).
$$

Thus direct computations yield the additional equation

$$
\mathcal{P}_t(t,\eta) + \Big(\frac{2}{3}\mathcal{U}^3 + \mathcal{S}\Big)\mathcal{P}_\eta(t,\eta) = \mathcal{Q}\mathcal{U}(t,\eta) + \mathcal{R}(t,\eta),
$$

where

$$
\begin{aligned}
\mathcal{R}(t,\eta) &= \frac{1}{4}\int_{\mathbb{R}} \mathrm{sign}(\mathcal{Y}(t,\eta)-y)e^{-|\mathcal{Y}(t,\eta)-y|}\Big(\frac{2}{3}u^3 + uG_x\Big)(t,y)\,dy\\
&\quad - \frac{1}{2}\int_{\mathbb{R}} e^{-|\mathcal{Y}(t,\eta)-y|}up_x(t,y)\,dy\\
&= \frac{1}{4}\int_0^{2C} \mathrm{sign}(\eta-\theta)e^{-|\mathcal{Y}(t,\eta)-\mathcal{Y}(t,\theta)|}\Big(\frac{2}{3}\mathcal{U}^3\mathcal{Y}_\eta + \mathcal{U}\Big)(t,\theta)\,d\theta\\
&\quad - \frac{1}{2}\int_0^{2C} e^{-|\mathcal{Y}(t,\eta)-\mathcal{Y}(t,\theta)|}\mathcal{U}\mathcal{Q}\mathcal{Y}_\eta(t,\theta)\,d\theta.
\end{aligned}
$$

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We summarize the result in the following proposition.

PROPOSITION 2.2. *Let* (u, μ) *denote a smooth solution of* (1.1). *Define* $\mathcal{Y}$ *by* (2.3), $\mathcal{U}$ *by* (2.6), *and* $\mathcal{P}$ *by* (2.8). *Then the following system of differential equations holds:*

$$\mathcal{Y}_t(t,\eta) + (\tfrac{2}{3}\mathcal{U}^3 + \mathcal{S})\mathcal{Y}_\eta(t,\eta) = \mathcal{U}(t,\eta), \tag{2.9a}$$

$$\mathcal{U}_t(t,\eta) + (\tfrac{2}{3}\mathcal{U}^3 + \mathcal{S})\mathcal{U}_\eta(t,\eta) = -\mathcal{Q}(t,\eta), \tag{2.9b}$$

$$\mathcal{P}_t(t,\eta) + (\tfrac{2}{3}\mathcal{U}^3 + \mathcal{S})\mathcal{P}_\eta(t,\eta) = \mathcal{Q}\mathcal{U}(t,\eta) + \mathcal{R}(t,\eta), \tag{2.9c}$$

where

$$\mathcal{Q}(t,\eta) = -\frac{1}{4}\int_0^{2C} \operatorname{sign}(\eta-\theta)e^{-|\mathcal{Y}(t,\eta)-\mathcal{Y}(t,\theta)|}(2(\mathcal{U}^2-\mathcal{P})\mathcal{Y}_\eta(t,\theta)+1)\,d\theta, \tag{2.10a}$$

$$\mathcal{S}(t,\eta) = \int_0^{2C} e^{-|\mathcal{Y}(t,\eta)-\mathcal{Y}(t,\theta)|}\left(\frac{2}{3}\mathcal{U}^3\mathcal{Y}_\eta - \mathcal{U}_\eta\mathcal{Q} - 2\mathcal{P}\mathcal{U}\mathcal{Y}_\eta\right)(t,\theta)\,d\theta, \tag{2.10b}$$

$$\begin{aligned}\mathcal{R}(t,\eta) = {}& \frac{1}{4}\int_0^{2C} \operatorname{sign}(\eta-\theta)e^{-|\mathcal{Y}(t,\eta)-\mathcal{Y}(t,\theta)|}\left(\frac{2}{3}\mathcal{U}^3\mathcal{Y}_\eta + \mathcal{U}\right)(t,\theta)\,d\theta \\ & - \frac{1}{2}\int_0^{2C} e^{-|\mathcal{Y}(t,\eta)-\mathcal{Y}(t,\theta)|}\mathcal{U}\mathcal{Q}\mathcal{Y}_\eta(t,\theta)\,d\theta.\end{aligned} \tag{2.10c}$$

In the next section we will derive this system of equations also in the general case without assuming smoothness of the quantities involved; see (3.28).

Let us finish this section by checking some properties of system (2.9), which will also hold in the case of weak conservative solutions as we will see in the next section.

The quantity $\frac{2}{3}\mathcal{U}^3 + \mathcal{S}$ is the velocity field of the three equations in (2.9). Instead of applying a characteristic method to estimate the solutions to this system, we will perform integration by parts by which the η-derivative of this quantity will naturally appear.

LEMMA 2.3. *Given* (u, μ) *a smooth solution of* (1.1), *then the solution to* (2.9) *satisfies*

$$|(\tfrac{2}{3}\mathcal{U}^3 + \mathcal{S})_\eta| \leqslant \mathcal{O}(1).$$

Proof. Since $G(t, \mathcal{Y}(t,\eta)) = 2p_x(t, \mathcal{Y}(t,\eta)) + 2F(t, \mathcal{Y}(t,\eta)) = \eta$, we have due to the smoothness that

$$G_x(t, \mathcal{Y}(t,\eta))\mathcal{Y}_\eta(t,\eta) = (2p - u^2 + F_x)(t, \mathcal{Y}(t,\eta))\mathcal{Y}_\eta(t,\eta) = 1.$$

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Since $(2p - u^2)(t,x) \geqslant 0$ due to (2.2) and $(F_x - u^2)(t,x) \geqslant 0$ due to (2.1), we have that

$$\begin{aligned}
2\mathcal{P}\mathcal{Y}_\eta(t,\eta) &\leqslant (2p - u^2 + F_x)(t,\mathcal{Y})\mathcal{Y}_\eta(t,\eta) = 1,\\
\mathcal{U}^2\mathcal{Y}_\eta(t,\eta) &\leqslant 2\mathcal{P}\mathcal{Y}_\eta(t,\eta) \leqslant 1,\\
2|\mathcal{U}\mathcal{U}_\eta(t,\eta)| &= 2|uu_x(t,\mathcal{Y})\mathcal{Y}_\eta(t,\eta)| \leqslant (u^2 + u_x^2)(t,\mathcal{Y})\mathcal{Y}_\eta(t,\eta)\\
&= F_x(t,\mathcal{Y})\mathcal{Y}_\eta(t,\eta) \leqslant (2p - u^2 + F_x)(t,\mathcal{Y})\mathcal{Y}_\eta(t,\eta) = 1.
\end{aligned}$$

We conclude that

$$\mathcal{P}\mathcal{Y}_\eta(t,\eta) \leqslant \tfrac{1}{2}, \quad |\mathcal{U}\mathcal{U}_\eta(t,\eta)| \leqslant \tfrac{1}{2}, \quad \text{and} \quad \mathcal{U}^2\mathcal{Y}_\eta(t,\eta) \leqslant 1. \tag{2.11}$$

From the fact that the energy is conserved, it follows that $u(t) \in H^1(\mathbb{R})$ for all $t \geqslant 0$ and

$$\|u(t,\cdot)\|_{L^\infty(\mathbb{R})} \leqslant \sqrt{C}.$$

Therefore, the term $\mathcal{U}^2\mathcal{U}_\eta$ is bounded by $\mathcal{O}(1)$.

Furthermore, $\mathcal{S}_\eta(t,\eta)$ is bounded. In particular, one can establish that

$$\mathcal{S}_\eta(t,\eta) \leqslant \mathcal{O}(1)\mathcal{P}\mathcal{Y}_\eta(t,\eta),$$

which is going to play a key role. Indeed, by definition one has

$$S(t,x) = \int_{\mathbb{R}} e^{-|x-y|}\left(\frac{2}{3}u^3 - u_x p_x - 2pu\right)(t,y)\,dy,$$

and hence

$$S_x(t,x) = -\int_{\mathbb{R}} \operatorname{sign}(x-y)e^{-|x-y|}\left(\frac{2}{3}u^3 - u_x p_x - 2pu\right)(t,y)\,dy.$$

Our aim is to show that

$$S_x(t,x) \leqslant \mathcal{O}(1)p(t,x).$$

First of all, note that we have

$$\begin{aligned}
\left|\int_{\mathbb{R}} \operatorname{sign}(x-y)e^{-|x-y|}\frac{2}{3}u^3(t,y)\,dy\right| &\leqslant \frac{8}{3}\|u(t,\,\cdot)\|_{L^\infty(\mathbb{R})}\frac{1}{4}\int_{\mathbb{R}} e^{-|x-y|}u^2(t,y)\,dy\\
&\leqslant \mathcal{O}(1)p(t,x).
\end{aligned}$$

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Moreover,

$$\begin{aligned}&\left|\int_{\mathbb{R}} \operatorname{sign}(x-y)e^{-|x-y|}(u_x p_x + 2pu)(t,y)\,dy\right|\\&\quad\leqslant \frac{1}{2}\int_{\mathbb{R}} e^{-|x-y|}(2u^2+u_x^2+3p^2)(t,y)\,dy\\&\quad\leqslant 2p(t,x)+\frac{3}{2}\int_{\mathbb{R}} e^{-|x-y|}p^2(t,y)\,dy,\end{aligned}$$

since $|p_x(t,x)| \leqslant p(t,x)$. Thus it remains to show that the last term can be bounded by a multiple of $p(t,x)$. Our reasoning will be based on integration by parts and the fact that

$$p(t,x)-p_{xx}(t,x)=\tfrac{1}{2}u^2(t,x)+\tfrac{1}{2}F_x(t,x).$$

Indeed, first direct computations yield

$$\begin{aligned}&\int_{\mathbb{R}} e^{-|x-y|}p^2(t,y)\,dy\\&\quad=\int_{\mathbb{R}} e^{-|x-y|}p\left(\frac{1}{2}u^2+\frac{1}{2}F_x\right)(t,y)\,dy+\int_{\mathbb{R}} e^{-|x-y|}pp_{xx}(t,y)\,dy\\&\quad=I_1(t,x)+I_2(t,x).\end{aligned}$$

Since $p(t,x)\leqslant \frac{1}{2}\int_{\mathbb{R}} F_x(t,y)\,dy \leqslant \frac{1}{2}C$, we have that

$$I_1(t,x)\leqslant \|p(t,\cdot)\|_{L^\infty(\mathbb{R})}\frac{1}{2}\int_{\mathbb{R}} e^{-|x-y|}(u^2+F_x)(t,y)\,dy \leqslant 2\|p(t,x)\|_{L^\infty(\mathbb{R})}p(t,x).$$

As far as I_2 is concerned, we have

$$\begin{aligned}&\int_{-\infty}^{x} e^{y-x}pp_{xx}(t,y)\,dy\\&\quad=pp_x(t,x)-\int_{-\infty}^{x} e^{y-x}(pp_x+p_x^2)(t,y)\,dy\\&\quad=pp_x(t,x)-\frac{1}{2}p^2(t,x)+\int_{-\infty}^{x} e^{y-x}\left(\frac{1}{2}p^2-p_x^2\right)(t,y)\,dy,\end{aligned}$$

and

$$\begin{aligned}&\int_{x}^{\infty} e^{x-y}pp_{xx}(t,y)\,dy\\&\quad=-pp_x(t,x)-\int_{x}^{\infty} e^{x-y}(p_x^2-pp_x)(t,y)\,dy\\&\quad=-pp_x(t,x)-\frac{1}{2}p^2(t,x)+\int_{x}^{\infty} e^{x-y}\left(\frac{1}{2}p^2-p_x^2\right)(t,y)\,dy.\end{aligned}$$

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Thus

$$I_2(t,x) = -p^2(t,x) + \int_{\mathbb{R}} e^{-|x-y|}\left(\frac{1}{2}p^2 - p_x^2\right)(t,y)\,dy$$

and subsequently

$$\begin{aligned}
&\int_{\mathbb{R}} e^{-|x-y|} p^2(t,y)\,dy \\
&\quad \leqslant 2\|p(t,\cdot)\|_{L^\infty(\mathbb{R})} p(t,x) - p^2(t,x) + \int_{\mathbb{R}} e^{-|x-y|}\left(\frac{1}{2}p^2 - p_x^2\right)(t,y)\,dy \\
&\quad \leqslant 2\|p(t,\cdot)\|_{L^\infty(\mathbb{R})} p(t,x) + \int_{\mathbb{R}} e^{-|x-y|}\left(\frac{1}{2}p^2 - p_x^2\right)(t,y)\,dy.
\end{aligned}$$

Reshuffling the terms, we end up with

$$\begin{aligned}
\frac{1}{2}\int_{\mathbb{R}} e^{-|x-y|} p^2(t,y)\,dy &\leqslant \frac{1}{2}\int_{\mathbb{R}} e^{-|x-y|}(p^2 + p_x^2)(t,y)\,dy \\
&\leqslant 2\|p(t,\cdot)\|_{L^\infty(\mathbb{R})} p(t,x) \leqslant Cp(t,x),
\end{aligned} \tag{2.12}$$

showing the desired estimate. □

Next, we show that all properties seen in this section for smooth solutions remain true for weak conservative solutions to (1.1).

3. Rigorous transformation: weak conservative solutions

To accommodate for the wave breaking of the solutions, it has turned out to be advantageous to rewrite the CH equation from the original Eulerian variables into Lagrangian variables; see [6, 54]. We will show that the system of equations obtained in Proposition 2.2 holds for the weak conservative solutions introduced in [54]. With this aim in mind let us start by summarizing their approach, which uses the different adopted convention followed in this work for cumulative distribution functions to be continuous from the left and with limit from the right (caglad) at all points $x \in \mathbb{R}$. Therefore, both $F(t,\cdot)$ and $G(t,\cdot)$ are nondecreasing and caglad functions.

Given some initial data (u_0, μ_0), the corresponding initial data in Lagrangian coordinates is then given by

$$y(0,\xi) = \sup\{x \mid x + F_0(x) < \xi\}, \tag{3.1a}$$

$$H(0,\xi) = \xi - y(0,\xi), \tag{3.1b}$$

$$U(0,\xi) = u(0, y(0,\xi)), \tag{3.1c}$$

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and $(y(t,\ \cdot), U(t,\ \cdot), H(t,\ \cdot))$ are the solutions of

$$y_t(t,\xi) = U(t,\xi), \tag{3.2a}$$
$$U_t(t,\xi) = -Q(t,\xi), \tag{3.2b}$$
$$H_t(t,\xi) = (U^3 - 2PU)(t,\xi), \tag{3.2c}$$

where

$$P(t,\xi) = \frac{1}{4}\int_{\mathbb{R}} e^{-|y(t,\xi)-y(t,\sigma)|}(U^2 y_\xi + H_\xi)(t,\sigma)\,d\sigma, \tag{3.3a}$$
$$Q(t,\xi) = -\frac{1}{4}\int_{\mathbb{R}} \operatorname{sign}(\xi-\sigma)e^{-|y(t,\xi)-y(t,\sigma)|}(U^2 y_\xi + H_\xi)(t,\sigma)\,d\sigma. \tag{3.3b}$$

Moreover, the relation between P and Q is given by

$$P_\xi(t,\xi) = Q(t,\xi)y_\xi(t,\xi), \tag{3.4a}$$
$$Q_\xi(t,\xi) = (P - \tfrac{1}{2}U^2)y_\xi(t,\xi) - \tfrac{1}{2}H_\xi(t,\xi). \tag{3.4b}$$

Introduce the function

$$\begin{aligned} I(t,\xi) &= \int_{-\infty}^{\xi}(2P - U^2)y_\xi(t,\sigma)\,d\sigma = \int_{-\infty}^{\xi}(2Q_\xi + H_\xi)(t,\sigma)\,d\sigma \\ &= 2Q(t,\xi) + H(t,\xi), \end{aligned}$$

where we used that

$$\lim_{\xi\to-\infty} Q(t,\xi) = 0 = \lim_{\xi\to-\infty} H(t,\xi),$$

which follows from the definition of $H(t,\xi)$. The relation between H and F is given by

$$F(t, y(t,\xi)) \leqslant H(t,\xi) \leqslant F(t, y(t,\xi)+).$$

Here we have introduced the common notation

$$\Phi(x\pm) = \lim_{\epsilon\downarrow 0}\Phi(x\pm\epsilon). \tag{3.5}$$

Notice that $I(t,\xi)+H(t,\xi)$ is the Lagrangian counterpart to the function $G(t,x)$. To convince oneself that this is really the case, one should take a quick look back first. The function $G(t,x)$ was defined as

$$G(t,x) = \int_{-\infty}^{x}(2p - u^2)(t,y)\,dy + F(t,x).$$

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Thus, whenever $F(t,x)$ has a jump of height α at a point $\bar{x}$, that is, $\mu(t,\{\bar{x}\})=\alpha$, then also $G(t,x)$ has a jump of height α at $\bar{x}$, since the function $2p-u^2$ is continuous. Furthermore, the point $\bar{x}$ in Eulerian coordinates is mapped to some maximal interval $[\xi_l,\xi_r]$ in Lagrangian coordinates, on which $y_\xi(t,\xi)=0$ and $H_\xi(t,\xi)=1$ for a specific choice of a relabeling function. In fact, $H_\xi(t,\xi)>0$ for all $\xi\in[\xi_l,\xi_r]$ as proven in [54, Theorem 4.2] and [54, Definition 2.6]. Thus a close look at $Q(t,\xi)$ reveals that $Q(t,\xi)=Q(t,\xi_l)-\frac{1}{2}\int_{\xi_l}^{\xi}H_\xi(t,\sigma)\,d\sigma$ for all $\xi\in[\xi_l,\xi_r]$ and hence

$$\begin{aligned}I(t,\xi)&=2Q(t,\xi)+H(t,\xi)\\&=2Q(t,\xi_l)-\int_{\xi_l}^{\xi}H_\xi(t,\sigma)\,d\sigma+H(t,\xi_l)+\int_{\xi_l}^{\xi}H_\xi(t,\sigma)\,d\sigma\\&=2Q(t,\xi_l)+H(t,\xi_l)=I(t,\xi_l).\end{aligned}$$

In short, we have that $I(t,\xi)=I(t,\xi_l)$ for all $\xi\in[\xi_l,\xi_r]$. This allows us now to follow a similar approach as for the Hunter–Saxton (HS) equation in [9, 16]. Therefore introduce

$$J(t,\xi)=I(t,\xi)+H(t,\xi)=\int_{-\infty}^{\xi}(2P-U^2)y_\xi(t,\sigma)\,d\sigma+H(t,\xi), \tag{3.6}$$

and observe that for all solutions except the zero solution $J(t,\xi)$ is strictly increasing and continuous. In more detail, one has for all solutions except the zero solution that $P(t,\xi)\neq 0$ for all $\xi\in\mathbb{R}$, since $y_\xi+H_\xi>0$ almost everywhere due to [54, Definition 2.6]. Moreover, if $H_\xi(t,\bar{\xi})=0$ for some $\bar{\xi}$ one has that $U(t,\bar{\xi})=0$ and $y_\xi(t,\bar{\xi})\neq 0$, since $y_\xi H_\xi(t,\bar{\xi})=U^2y_\xi(t,\bar{\xi})+U_\xi^2(t,\bar{\xi})$ almost everywhere due to [54, Definition 2.6], and hence the ξ-derivative of $J(t,\xi)$ is strictly positive at the point $\bar{\xi}$.

Lemma 3.1. *Given $\mathcal{Y}(t,\eta)=\sup\{x\mid G(t,x)<\eta\}$, then*

$$\mathcal{Y}(t,\eta)=y(t,l(t,\eta)), \tag{3.7}$$

where we have introduced $l(t,\cdot)\colon[0,2C]\to\mathbb{R}$ by

$$l(t,\eta)=\sup\{\xi\mid J(t,\xi)<\eta\}. \tag{3.8}$$

Proof. For each time t we have

$$y(t,\xi)=\sup\{x\mid x+F(t,x)<y(t,\xi)+H(t,\xi)\},$$

which implies that

$$y(t,\xi)+F(t,y(t,\xi))\leqslant y(t,\xi)+H(t,\xi)\leqslant y(t,\xi)+F(t,y(t,\xi)+).$$

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Moreover, one has that $G(t,x) - F(t,x)$ is continuous, and, in particular,

$$G(t, y(t,\xi)) - F(t, y(t,\xi)) = \int_{-\infty}^{y(t,\xi)} (2p - u^2)(t,y)\,dy$$
$$= \int_{-\infty}^{\xi} (2P - U^2)y_\xi(t,\sigma)\,d\sigma = I(t,\xi).$$

Thus one has

$$y(t,\xi) + G(t, y(t,\xi)) \leqslant y(t,\xi) + J(t,\xi) \leqslant y(t,\xi) + G(t, y(t,\xi)+).$$

Subtracting $y(t,\xi)$ in the above inequality, we end up with

$$G(t, y(t,\xi)) \leqslant J(t,\xi) \leqslant G(t, y(t,\xi)+) \quad \text{for all } \xi \in \mathbb{R}.$$

Comparing the last equation and (3.8), we have

$$G(t, y(t, l(t,\eta))) \leqslant J(t, l(t,\eta)) = \eta \leqslant G(t, y(t, l(t,\eta))+).$$

Since $y(t,\cdot)$ is surjective and nondecreasing, we end up with

$$\mathcal{Y}(t,\eta) = \sup\{x \mid G(t,x) < \eta\} = y(t, l(t,\eta)), \tag{3.9}$$

thereby proving (3.7). □

In the next step we want to establish rigorously the corresponding system of differential equations. Hence we first have to establish that the function $l(t,\eta)$ is differentiable, both with respect to time and space.

3.1. The differentiability of $Q(t,\xi)$. The differentiability of $Q(t,\xi)$ with respect to ξ has been proven in [54] and $Q_\xi(t,\xi)$ is given by (3.4b). Thus it is left to establish the differentiability with respect to time of $Q(t,\xi)$. To be more precise, we are going to establish the Lipschitz continuity of $Q(t,\xi)$ with respect to time.

Let us recall that since $U(t,\cdot) \in H^1(\mathbb{R})$, one has in particular that $U(t,\cdot) \in L^\infty(\mathbb{R})$. Moreover, direct calculations yield

$$U^2(t,\xi) = U^2(t,\xi) - U^2(t,-\infty) = \int_{-\infty}^{\xi} 2UU_\xi(t,\sigma)\,d\sigma$$
$$\leqslant \int_{-\infty}^{\xi} H_\xi(t,\sigma)\,d\sigma \leqslant H(t,\infty) = C, \tag{3.10}$$

and we end up with

$$\|U(t,\cdot)\|_{L^\infty(\mathbb{R})} \leqslant \sqrt{C} \quad \text{for all } t \in \mathbb{R}.$$

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Here we used *Xavier's relation* from [54] which asserts that

$$U^2 y_\xi^2(t,\xi) + U_\xi^2(t,\xi) = y_\xi H_\xi(t,\xi), \tag{3.11}$$

and hence

$$|U_\xi(t,\xi)| \leqslant \sqrt{y_\xi H_\xi}(t,\xi) \leqslant \tfrac{1}{2}(y_\xi + H_\xi)(t,\xi), \tag{3.12a}$$
$$|Uy_\xi(t,\xi)| \leqslant \sqrt{y_\xi H_\xi}(t,\xi) \leqslant \tfrac{1}{2}(y_\xi + H_\xi)(t,\xi), \tag{3.12b}$$
$$|UU_\xi(t,\xi)| \leqslant \tfrac{1}{2}H_\xi(t,\xi). \tag{3.12c}$$

Similar considerations apply for $P(t,\xi)$. Indeed one has

$$\begin{aligned} 0 \leqslant P(t,\xi) &= \frac{1}{4}\int_{\mathbb{R}} e^{-|y(t,\sigma)-y(t,\xi)|}(U^2 y_\xi + H_\xi)(t,\sigma)\,d\sigma \\ &\leqslant \frac{1}{4}\int_{\mathbb{R}} 2H_\xi(t,\sigma)\,d\sigma = \frac{1}{2}C. \end{aligned} \tag{3.13}$$

Since $|Q(t,\xi)| \leqslant P(t,\xi)$, we end up with

$$\|Q(t,\,\cdot\,)\|_{L^\infty(\mathbb{R})} \leqslant \tfrac{1}{2}C, \quad \|P(t,\,\cdot\,)\|_{L^\infty(\mathbb{R})} \leqslant \tfrac{1}{2}C \quad \text{for all } t \in \mathbb{R}. \tag{3.14}$$

Direct calculations, using (3.11) and (3.12), yield

$$\begin{aligned} &-\mathcal{O}(1)(y_\xi(t,\xi) + H_\xi(t,\xi)) \\ &\quad\leqslant (y_\xi(t,\xi) + H_\xi(t,\xi))_t \\ &\quad= U_\xi(t,\xi) + 3U^2U_\xi(t,\xi) - 2QUy_\xi(t,\xi) - 2PU_\xi(t,\xi) \\ &\quad\leqslant \mathcal{O}(1)(y_\xi(t,\xi) + H_\xi(t,\xi)), \end{aligned}$$

since $U(t,\xi)$, $P(t,\xi)$, and $Q(t,\xi)$ are uniformly bounded with respect to both space and time due to (3.10), (3.13), and (3.14). Thus, we have for $s < t$ that

$$\begin{aligned} &(y_\xi(s,\xi) + H_\xi(s,\xi))e^{-\mathcal{O}(1)(t-s)} \\ &\quad\leqslant y_\xi(t,\xi) + H_\xi(t,\xi) \leqslant (y_\xi(s,\xi) + H_\xi(s,\xi))e^{\mathcal{O}(1)(t-s)} \end{aligned}$$

or equivalently

$$\begin{aligned} (y_\xi(t,\xi) + H_\xi(t,\xi))e^{-\mathcal{O}(1)(t-s)} &\leqslant (y_\xi(s,\xi) + H_\xi(s,\xi)) \\ &\leqslant (y_\xi(t,\xi) + H_\xi(t,\xi))e^{\mathcal{O}(1)(t-s)}. \end{aligned} \tag{3.15}$$

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Recall that

$$\begin{aligned}Q(t,\xi) &= -\frac{1}{4}\int_{-\infty}^{\xi} e^{y(t,\sigma)-y(t,\xi)}(U^2 y_\xi + H_\xi)(t,\sigma)\,d\sigma\\ &\quad + \frac{1}{4}\int_{\xi}^{\infty} e^{y(t,\xi)-y(t,\sigma)}(U^2 y_\xi + H_\xi)(t,\sigma)\,d\sigma\\ &=: \frac{1}{4}Q_1(t,\xi) + \frac{1}{4}Q_2(t,\xi).\end{aligned}$$

We are only establishing the Lipschitz continuity with respect to time for $Q_1(t,\xi)$, since the argument for $Q_2(t,\xi)$ follows the same lines. Let $t < \hat{t}$. Then one has

$$\begin{aligned}&|Q_1(t,\xi) - Q_1(\hat{t},\xi)|\\ &\leqslant \int_{-\infty}^{\xi} |e^{y(t,\sigma)-y(t,\xi)} - e^{y(\hat{t},\sigma)-y(\hat{t},\xi)}|(U^2 y_\xi + H_\xi)(t,\sigma)\,d\sigma\\ &\quad + \int_{-\infty}^{\xi} e^{y(\hat{t},\sigma)-y(\hat{t},\xi)}|(U^2 y_\xi + H_\xi)(t,\sigma) - (U^2 y_\xi + H_\xi)(\hat{t},\sigma)|\,d\sigma\\ &=: I_1 + I_2.\end{aligned}$$

As far as I_1 is concerned, we observe, using Lemma A.1(i), that

$$\begin{aligned}|e^{y(t,\sigma)-y(t,\xi)} - e^{y(\hat{t},\sigma)-y(\hat{t},\xi)}| &\leqslant |y(t,\sigma) - y(\hat{t},\sigma)| + |y(t,\xi) - y(\hat{t},\xi)|\\ &\leqslant \int_t^{\hat{t}} (|U(s,\sigma)| + |U(s,\xi)|)\,ds \leqslant \mathcal{O}(1)|t-\hat{t}|,\end{aligned}$$

where we used that both $e^{y(\hat{t},\sigma)-y(\hat{t},\xi)}$ and $e^{y(t,\sigma)-y(t,\xi)}$ are bounded from above by one, and that U can be uniformly bounded both with respect to space and time due to (3.10). Thus we end up with

$$\begin{aligned}I_1 &= \int_{-\infty}^{\xi} |e^{y(t,\sigma)-y(t,\xi)} - e^{y(\hat{t},\sigma)-y(\hat{t},\xi)}|(U^2 y_\xi + H_\xi)(t,\sigma)\,d\sigma\\ &\leqslant \mathcal{O}(1)|t-\hat{t}| \int_{-\infty}^{\xi} 2H_\xi(t,\sigma)\,d\sigma \leqslant \mathcal{O}(1)|t-\hat{t}|,\end{aligned}$$

where we used that $U^2 y_\xi \leqslant H_\xi$.

Thus it remains to establish a similar estimate for I_2. The idea is to use a similar strategy combined with (3.12a) and (3.15). We have

$$\begin{aligned}I_2 &= \int_{-\infty}^{\xi} e^{y(\hat{t},\sigma)-y(\hat{t},\xi)}|(U^2 y_\xi + H_\xi)(t,\sigma) - (U^2 y_\xi + H_\xi)(\hat{t},\sigma)|\,d\sigma\\ &\leqslant \int_{-\infty}^{\xi} e^{y(\hat{t},\sigma)-y(\hat{t},\xi)} \int_t^{\hat{t}} |4U^2 U_\xi - 4QUy_\xi - 2PU_\xi|(s,\sigma)\,ds\,d\sigma\end{aligned}$$

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$$
\begin{aligned}
&\leqslant \int_{-\infty}^{\xi} e^{y(\hat{t},\sigma)-y(\hat{t},\xi)} \mathcal{O}(1) \int_{t}^{\hat{t}} (y_\xi + H_\xi)(s,\sigma)\,ds\,d\sigma \\
&\leqslant \mathcal{O}(1) \int_{-\infty}^{\xi} e^{y(\hat{t},\sigma)-y(\hat{t},\xi)} \int_{t}^{\hat{t}} e^{\mathcal{O}(1)(\hat{t}-s)} (y_\xi + H_\xi)(\hat{t},\sigma)\,ds\,d\sigma \\
&\leqslant \mathcal{O}(1) \int_{-\infty}^{\xi} e^{y(\hat{t},\sigma)-y(\hat{t},\xi)} (y_\xi + H_\xi)(\hat{t},\sigma) \int_{t}^{\hat{t}} e^{\mathcal{O}(1)(\hat{t}-s)}\,ds\,d\sigma \\
&\leqslant \mathcal{O}(1) e^{\mathcal{O}(1)(\hat{t}-t)} |\hat{t}-t| \leqslant \mathcal{O}(1)|\hat{t}-t|,
\end{aligned}
$$

under the assumption that $|\hat{t}-t| \leqslant 1$ (or in general bounded). Thus we established that

$$|Q_1(t,\xi) - Q_1(\hat{t},\xi)| \leqslant \mathcal{O}(1)|\hat{t}-t|$$

for all pairs t, $\hat{t}$, such that $|\hat{t}-t| \leqslant 1$. Furthermore, one has

$$|Q(t,\xi) - Q(\hat{t},\xi)| \leqslant \mathcal{O}(1)|\hat{t}-t|$$

for all pairs t, $\hat{t}$, such that $|\hat{t}-t| \leqslant 1$. Thus for fixed ξ, the function Q is locally Lipschitz with respect to time, and hence differentiable almost everywhere by Rademacher's theorem.

A close look at the above argument reveals that we cannot only differentiate $Q(t,\xi)$ with respect to time, but also that we can apply the dominated convergence theorem, which yields

$$
\begin{aligned}
Q_t(t,\xi) &= -\frac{1}{4} \int_{\mathbb{R}} \frac{d}{dt} (\operatorname{sign}(\xi-\sigma) e^{-|y(t,\xi)-y(t,\sigma)|} (U^2 y_\xi + H_\xi)(t,\sigma))\,d\sigma \\
&= -\frac{2}{3} U^3(t,\xi) + 2PU(t,\xi) \\
&\quad + \frac{1}{2} \int_{\mathbb{R}} e^{-|y(t,\xi)-y(t,\sigma)|} \left(\frac{2}{3} U^3 y_\xi - 2PU y_\xi - QU_\xi \right)(t,\sigma)\,d\sigma \qquad (3.16)
\end{aligned}
$$

after some integrations by parts.

Next we want to show that $Q_t(t,\xi)$ can be uniformly bounded by a constant $\mathcal{O}(1)$. Observe that (3.10) and (3.13) imply

$$
\begin{aligned}
&\left| \int_{\mathbb{R}} e^{-|y(t,\sigma)-y(t,\xi)|} \left(\frac{2}{3} U^3 y_\xi - 2PU y_\xi - QU y_\xi \right)(t,\sigma)\,d\sigma \right| \\
&\qquad \leqslant \mathcal{O}(1) \left(\int_{\mathbb{R}} e^{-|y(t,\sigma)-y(t,\xi)|} y_\xi(t,\sigma)\,d\sigma \right) \leqslant \mathcal{O}(1). \qquad (3.17)
\end{aligned}
$$

Thus recalling (3.16) and combining (3.10), (3.13), and (3.17), we finally end up with

$$\|Q_t(t,\,\cdot\,)\|_{L^\infty(\mathbb{R})} \leqslant \mathcal{O}(1) \quad \text{for all } t \in \mathbb{R}. \qquad (3.18)$$

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This completes the argument regarding the differentiability of Q. Notice that a direct application of the dominated convergence theorem using formula (3.16) shows that Q_t is a continuous function in time, under the assumption that P_t exists and satisfies an estimate similar to (3.18).

Next we focus on the differentiability with respect to time of P. An analogous argument to the one for Q leads to the differentiability in time of P. Let us show that the derivative of P with respect to time exists by applying the dominated convergence theorem to

$$\lim_{s\to t}\frac{P(t,\xi)-P(s,\xi)}{t-s}, \tag{3.19}$$

where ξ is chosen such that y is differentiable with respect to time. Since

$$\frac{d}{dt}(e^{-|y(t,\xi)-y(t,\sigma)|}(U^2y_\xi+H_\xi)(t,\sigma))$$

exists almost everywhere, it is left to show that we can find a function in $L^1(\mathbb{R})$, which bounds the integrand of (3.19) uniformly in s. Therefore observe that we can write, using (3.2), for $s<t$, that

$$\begin{aligned}
&P(t,\xi)-P(s,\xi)\\
&=\frac{1}{4}\int_{-\infty}^{\xi}\int_s^t e^{y(l,\sigma)-y(l,\xi)}(U(l,\sigma)-U(l,\xi))(U^2y_\xi+H_\xi)(l,\sigma)\,dl\,d\sigma\\
&\quad+\frac{1}{4}\int_{\xi}^{\infty}\int_s^t e^{y(l,\xi)-y(l,\sigma)}(U(l,\xi)-U(l,\sigma))(U^2y_\xi+H_\xi)(l,\sigma)\,dl\,d\sigma\\
&\quad+\frac{1}{4}\int_{\mathbb{R}}\int_s^t e^{-|y(l,\sigma)-y(l,\xi)|}(4U^2U_\xi-4UQy_\xi-2PU_\xi)(l,\sigma)\,dl\,d\sigma\\
&=I_1+I_2+I_3.
\end{aligned}$$

As far as the first term is concerned, observe that (3.15) implies that

$$\begin{aligned}
&\left|\int_s^t e^{y(l,\sigma)-y(l,\xi)}(U(l,\sigma)-U(l,\xi))(U^2y_\xi+H_\xi)(l,\sigma)\,dl\right|\\
&\qquad\leqslant \mathcal{O}(1)\int_s^t e^{y(l,\sigma)-y(l,\xi)}|(U^2y_\xi+H_\xi)(l,\sigma)|\,dl\\
&\qquad\leqslant \mathcal{O}(1)e^{\mathcal{O}(1)|t-s|}e^{y(t,\sigma)-y(t,\xi)}(y_\xi+H_\xi)(t,\sigma)|t-s|\\
&\qquad\leqslant \mathcal{O}(1)e^{y(t,\sigma)-y(t,\xi)}(y_\xi+H_\xi)(t,\sigma)|t-s|,
\end{aligned}$$

if we assume that $|t-s|\leqslant 1$ or any other fixed constant. Moreover, the function in the last line belongs to $L^1(\mathbb{R})$. The remaining two terms can be bounded by a function, which is of the same form and belongs to $L^1(\mathbb{R})$ uniformly in s.

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Thus all assumptions of the dominated convergence theorem are fulfilled, and we end up with

$$\begin{aligned}
P_t(t,\xi) &= UQ(t,\xi) + \frac{1}{4}\int_{\mathbb{R}} \operatorname{sign}(\xi-\sigma)e^{-|y(t,\xi)-y(t,\sigma)|}U(U^2y_\xi + H_\xi)(t,\sigma)\,d\sigma \\
&\quad + \frac{1}{4}\int_{\mathbb{R}} e^{-|y(t,\xi)-y(t,\sigma)|}(4U^2U_\xi - 4UQy_\xi - 2PU_\xi)(t,\sigma)\,d\sigma \\
&= UQ(t,\xi) + \frac{1}{2}\int_{\mathbb{R}} \operatorname{sign}(\xi-\sigma)e^{-|y(t,\xi)-y(t,\sigma)|} \\
&\quad \times U\left(\frac{1}{3}U^2y_\xi + Q_\xi + H_\xi\right)(t,\sigma)\,d\sigma \\
&\quad - \frac{1}{2}\int_{\mathbb{R}} e^{-|y(t,\xi)-y(t,\sigma)|}UQy_\xi(t,\sigma)\,d\sigma, \qquad (3.20)
\end{aligned}$$

where we used integration by parts together with (3.4b) in the last step. Moreover, we have that

$$\begin{aligned}
|P_t(t,\xi)| &\leqslant |UQ(t,\xi)| + \frac{1}{4}\int_{\mathbb{R}} e^{-|y(t,\xi)-y(t,\sigma)|}|U|(U^2y_\xi + H_\xi)(t,\sigma)\,d\sigma \\
&\quad + \frac{1}{4}\int_{\mathbb{R}} e^{-|y(t,\xi)-y(t,\sigma)|}(4U^2|U_\xi| + 4|UQ|y_\xi + 2P|U_\xi|)(t,\sigma)\,d\sigma \\
&\leqslant \|U(t,\cdot)\|_{L^\infty}P(t,\xi) \\
&\quad + \frac{1}{4}\|U(t,\cdot)\|_{L^\infty}\int_{\mathbb{R}} e^{-|y(t,\xi)-y(t,\sigma)|}(U^2y_\xi + H_\xi)(t,\sigma)\,d\sigma \\
&\quad + \frac{1}{4}\int_{\mathbb{R}} e^{-|y(t,\xi)-y(t,\sigma)|}(2U^4y_\xi + 3H_\xi + 2U^2y_\xi + 3P^2y_\xi)(t,\sigma)\,d\sigma \\
&\leqslant \mathcal{O}(1)P(t,\xi) + \frac{3}{4}\int_{\mathbb{R}} e^{-|y(t,\xi)-y(t,\sigma)|}P^2y_\xi(t,\sigma)\,d\sigma \\
&\quad + \frac{1}{4}\int_{\mathbb{R}} e^{-|y(t,\xi)-y(t,\sigma)|}(2\|U(t,\cdot)\|^2_{L^\infty}U^2y_\xi + 3H_\xi + 2U^2y_\xi)(t,\sigma)\,d\sigma \\
&\leqslant \mathcal{O}(1)P(t,\xi) + \frac{3}{4}\int_{\mathbb{R}} e^{-|y(t,\xi)-y(t,\sigma)|}P^2y_\xi(t,\sigma)\,d\sigma \\
&\leqslant \mathcal{O}(1)\left(P(t,\xi) + \int_{\mathbb{R}} e^{-|y(t,\xi)-y(t,\sigma)|}P^2y_\xi(t,\sigma)\,d\sigma\right)
\end{aligned}$$

due to (3.10). It remains to show that

$$\int_{\mathbb{R}} e^{-|y(t,\xi)-y(t,\sigma)|}P^2y_\xi(t,\sigma)\,d\eta \leqslant \mathcal{O}(1)P(t,\xi).$$

The proof follows the same lines as the one of (2.12) in Eulerian coordinates.

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3.2. The differentiability of $l(t, \eta)$. Finally, we can start thinking about the differentiability with respect to space and time of $l(t, \eta)$. Therefore, recall that the function $l(t, \eta)$ is defined as

$$l(t, \eta) = \sup\{\xi \in \mathbb{R} \mid J(t, \xi) < \eta\}.$$

Differentiability with respect to η: For every $t \in \mathbb{R}$, the function $J(t, \cdot) = 2Q(t, \cdot) + 2H(t, \cdot)$ is strictly increasing and continuous, and hence differentiable almost everywhere with respect to ξ. Its inverse $J^{-1}(t, \cdot)\colon [0, 2C] \to \mathbb{R}$ is therefore also strictly increasing and continuous, and hence differentiable almost everywhere. Since we have by definition that $J(t, l(t, \eta)) = \eta$ for all η, it follows immediately that $J(t, l(t, \cdot))$ is Lipschitz continuous with Lipschitz constant one and hence differentiable almost everywhere with respect to η. Since

$$J^{-1}(t, J(t, l(t, \eta))) = l(t, \eta) \quad \text{for all } \eta,$$

we can finally conclude that $l(t, \eta)$ is differentiable almost everywhere with respect to η. In particular, one has (cf. (3.6)) that

$$J_\xi(t, l(t, \eta)) l_\eta(t, \eta) = 2\mathcal{P}\mathcal{Y}_\eta(t, \eta) - \mathcal{U}^2\mathcal{Y}_\eta(t, \eta) + \mathcal{H}_\eta(t, \eta) = 1, \tag{3.21}$$

with

$$\mathcal{H}(t, \eta) = H(t, l(t, \eta)), \quad \mathcal{P}(t, \eta) = P(t, l(t, \eta)),$$
$$\text{and} \quad \mathcal{U}(t, \eta) = U(t, l(t, \eta)).$$

These identities follow from equality (3.9) since $\mathcal{P}(t, \eta) = p(t, \mathcal{Y}(t, \eta))$ and $P(t, \xi) = p(t, y(t, \xi))$, and analogously $\mathcal{U}(t, \eta) = u(t, \mathcal{Y}(t, \eta))$ and $U(t, \xi) = u(t, y(t, \xi))$.

Following the same argument as in the smooth case (see (2.11)), we end up with

$$0 \leqslant 2\mathcal{P}\mathcal{Y}_\eta(t, \eta) \leqslant 1, \quad 2|\mathcal{U}\mathcal{U}_\eta(t, \eta)| \leqslant 1, \quad \text{and} \quad 0 \leqslant \mathcal{U}^2\mathcal{Y}_\eta(t, \eta) \leqslant 1. \tag{3.22}$$

Differentiability with respect to t: As far as the differentiability of $l(t, \eta)$ with respect to time is concerned, the argument is a bit more involved. First of all, note that for any time t we have $J(t, l(t, \eta)) = \eta$ and hence

$$J(t, l(t, \eta)) = J(\tilde{t}, l(\tilde{t}, \eta)) \quad \text{for all } \eta \text{ and } \tilde{t}.$$

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In particular, we can conclude that

$$\begin{aligned} J(t, l(t, \eta)) - J(t, l(\tilde{t}, \eta)) &= J(\tilde{t}, l(\tilde{t}, \eta)) - J(t, l(\tilde{t}, \eta)) \\ &= \int_t^{\tilde{t}} J_t(s, l(\tilde{t}, \eta))\, ds \\ &= 2 \int_t^{\tilde{t}} (Q_t(s, l(\tilde{t}, \eta)) + H_t(s, l(\tilde{t}, \eta)))\, ds. \end{aligned}$$

We already showed that $Q_t(t, \xi)$ exists and is given by (3.16). However, a close look reveals that $Q_t(t, \xi)$ is uniformly bounded both with respect to space and time and $Q_t(t, \xi)$ is continuous with respect to space for fixed t. The same holds for

$$H_t(t, \xi) = U^3(t, \xi) - 2PU(t, \xi).$$

Thus we have that

$$\lim_{\tilde{t} \to t} \frac{J(t, l(t, \eta)) - J(t, l(\tilde{t}, \eta))}{t - \tilde{t}} = \lim_{\tilde{t} \to t} 2 \int_t^{\tilde{t}} \frac{Q_t(s, l(\tilde{t}, \eta)) + H_t(s, l(\tilde{t}, \eta))}{t - \tilde{t}}\, ds \tag{3.23}$$

from a formal point of view. Thus it is left to establish that the limit exists and is finite.

Since $J_t(t, \xi) = 2Q_t(t, \xi) + 2H_t(t, \xi)$ and both $Q_t(t, \xi)$ and $H_t(t, \xi)$ can be bounded uniformly both in space and time by a constant only dependent on the total energy C, it follows that there exists a constant $\mathcal{O}(1)$ such that

$$|J_t(t, \xi)| \leqslant \mathcal{O}(1) \quad \text{for all } t \text{ and } \xi.$$

Thus it follows that for all $s \in [t - |t - \tilde{t}|, t + |t - \tilde{t}|]$, we have

$$J(s, \xi) \in [J(t, \xi) - \mathcal{O}(1)|\tilde{t} - t|, J(t, \xi) + \mathcal{O}(1)|\tilde{t} - t|].$$

Accordingly we can conclude, since $J(t, l(t, \eta)) = \eta$ for all η, that

$$J(s, l(t, \eta)) \in [\eta - \mathcal{O}(1)|\tilde{t} - t|, \eta + \mathcal{O}(1)|\tilde{t} - t|],$$

and hence

$$l(t, \eta) = l(s, \eta(s)) \quad \text{for some } \eta(s) \in [\eta - \mathcal{O}(1)|\tilde{t} - t|, \eta + \mathcal{O}(1)|\tilde{t} - t|], \tag{3.24}$$

and, in particular,

$$|\eta - \eta(s)| \leqslant \mathcal{O}(1)|\tilde{t} - t| \quad \text{for all } s \in [t - |t - \tilde{t}|, t + |t - \tilde{t}|].$$

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Similar considerations yield that for all $s \in [\tilde{t} - |t - \tilde{t}|, \tilde{t} + |t - \tilde{t}|]$, we can find $\tilde{\eta}(s)$ such that

$$l(\tilde{t}, \eta) = l(s, \tilde{\eta}(s)) \quad \text{and} \quad |\eta - \tilde{\eta}(s)| \leqslant \mathcal{O}(1)|t - \tilde{t}|.$$

Let us return to the integral we are actually interested in. Namely,

$$\begin{aligned}
\int_t^{\tilde{t}} J_t(s, l(\tilde{t}, \eta))\, ds &= 2\int_t^{\tilde{t}} (Q_t(s, l(\tilde{t}, \eta)) + H_t(s, l(\tilde{t}, \eta)))\, ds \\
&= 2\int_t^{\tilde{t}} (Q_t(s, l(t, \eta)) + H_t(s, l(t, \eta)))\, ds \\
&\quad + 2\int_t^{\tilde{t}} ((Q_t(s, l(\tilde{t}, \eta)) + H_t(s, l(\tilde{t}, \eta))) \\
&\quad - (Q_t(s, l(t, \eta)) + H_t(s, l(t, \eta))))\, ds \\
&= \tilde{I}_1 + \tilde{I}_2.
\end{aligned}$$

What we are hoping for is that the second term $\tilde{I}_2$ is of order $o(|t - \tilde{t}|)$, and hence plays no role when computing (3.23). Therefore observe that

$$\begin{aligned}
H_{t\xi}(t, \xi) &= 3U^2 U_\xi(t, \xi) - 2QUy_\xi(t, \xi) - 2PU_\xi(t, \xi), \\
Q_{t\xi}(t, \xi) &= -2U^2 U_\xi(t, \xi) + 2QUy_\xi(t, \xi) + 2PU_\xi(t, \xi) \\
&\quad - \frac{1}{2}\int_{\mathbb{R}} \operatorname{sign}(\xi - \sigma) e^{-|y(t,\xi) - y(t,\sigma)|} \\
&\quad \times \left(\frac{2}{3}U^3 y_\xi - 2PUy_\xi - QU_\xi\right)(t, \sigma)\, d\sigma\, y_\xi(t, \xi).
\end{aligned}$$

Notice that the second derivative $Q_{t\xi}$ is well defined by a dominated convergence argument as before for Q_t. Recalling (3.24), we have

$$\begin{aligned}
\tilde{I}_2 &= \int_t^{\tilde{t}} J_t(s, l(\tilde{t}, \eta)) - J_t(s, l(t, \eta))\, ds \\
&= 2\int_t^{\tilde{t}} ((Q_t(s, l(\tilde{t}, \eta)) + H_t(s, l(\tilde{t}, \eta))) - (Q_t(s, l(t, \eta)) + H_t(s, l(t, \eta))))\, ds \\
&= 2\int_t^{\tilde{t}} \int_{l(t,\eta)}^{l(\tilde{t},\eta)} (Q_{t,\xi} + H_{t,\xi})(s, z)\, dz\, ds \\
&= 2\int_t^{\tilde{t}} \int_{l(t,\eta)}^{l(\tilde{t},\eta)} \Big[U^2 U_\xi(s, z) \\
&\quad - \frac{1}{2}\int_{\mathbb{R}} \operatorname{sign}(z - \sigma) e^{-|y(s,z) - y(s,\sigma)|}
\end{aligned}$$

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$$\times \left(\frac{2}{3}U^3 y_\xi - 2PUy_\xi - QU_\xi\right)(s,\sigma)d\sigma y_\xi(s,z)\Bigg] dz\,ds$$

$$= 2\int_t^{\tilde t}\int_{l(s,\eta(s))}^{l(s,\tilde\eta(s))}\Bigg[U^2U_\xi(s,z)$$

$$-\frac{1}{2}\int_{\mathbb{R}} \operatorname{sign}(z-\sigma)e^{-|y(s,z)-y(s,\sigma)|}$$

$$\times \left(\frac{2}{3}U^3 y_\xi - 2PUy_\xi - QU_\xi\right)(s,\sigma)\,d\sigma\, y_\xi(s,z)\Bigg] dz\,ds$$

$$= 2\int_t^{\tilde t}\int_{\eta(s)}^{\tilde\eta(s)} \mathcal{U}^2\mathcal{U}_\eta(s,m)\,dm\,ds$$

$$-\int_t^{\tilde t}\int_{\eta(s)}^{\tilde\eta(s)}\int_0^{2C} \operatorname{sign}(m-n)e^{-|\mathcal{Y}(s,m)-\mathcal{Y}(s,n)|}$$

$$\times \left(\frac{2}{3}\mathcal{U}^3\mathcal{Y}_\eta - 2\mathcal{P}\mathcal{U}\mathcal{Y}_\eta - \mathcal{Q}\mathcal{U}_\eta\right)(s,n)dn\mathcal{Y}_\eta(s,m)\,dm\,ds,$$

where η, $\eta(s)$, and $\tilde\eta(s)$ satisfy

$$l(t,\eta) = l(s,\eta(s)) \quad \text{and} \quad l(\tilde t,\eta) = l(s,\tilde\eta(s)),$$

and

$$|\eta - \eta(s)| \leqslant \mathcal{O}(1)|t-\tilde t| \quad \text{and} \quad |\eta - \tilde\eta(s)| \leqslant \mathcal{O}(1)|t-\tilde t|. \tag{3.25}$$

As far as the first term is concerned, we can apply (3.10), (3.22) and (3.25) as follows:

$$\left|2\int_t^{\tilde t}\int_{\eta(s)}^{\tilde\eta(s)} \mathcal{U}^2\mathcal{U}_\eta(s,m)\,dm\,ds\right|$$
$$\leqslant \int_t^{\tilde t} \|\mathcal{U}(s,\cdot)\|_{L^\infty([0,2C])}|\eta(s)-\tilde\eta(s)|\,ds \leqslant \mathcal{O}(1)|t-\tilde t|^2. \tag{3.26}$$

As far as the second and last term is concerned, we want to apply the Cauchy–Schwarz inequality to the integral term at first. Indeed,

$$\Bigg|\int_{\eta(s)}^{\tilde\eta(s)}\int_0^{2C} \operatorname{sign}(m-n)e^{-|\mathcal{Y}(s,m)-\mathcal{Y}(s,n)|}$$
$$\times \left(\frac{2}{3}\mathcal{U}^3\mathcal{Y}_\eta - 2\mathcal{P}\mathcal{U}\mathcal{Y}_\eta - \mathcal{Q}\mathcal{U}_\eta\right)(s,n)dn\mathcal{Y}_\eta(s,m)\,dm\Bigg|$$
$$\leqslant \Bigg|\int_{\eta(s)}^{\tilde\eta(s)}\int_0^{2C} e^{-|\mathcal{Y}(s,m)-\mathcal{Y}(s,n)|}\Bigg[\frac{2}{3}\mathcal{U}^2\sqrt{\mathcal{Y}_\eta}(\mathcal{U}^2\mathcal{Y}_\eta)^{1/2}$$

$$
\begin{aligned}
&+2\mathcal{P}\sqrt{\mathcal{Y}_\eta}(\mathcal{U}^2\mathcal{Y}_\eta)^{1/2}+\mathcal{P}\sqrt{\mathcal{Y}_\eta}\sqrt{\mathcal{H}_\eta}\Big](s,n)\,dn\,\mathcal{Y}_\eta(s,m)\,dm\Big|\\
&\leqslant \left|\int_{\eta(s)}^{\tilde\eta(s)}\int_0^{2C} e^{-|\mathcal{Y}(s,m)-\mathcal{Y}(s,n)|}\left(\frac{2}{3}\mathcal{U}^2\sqrt{\mathcal{Y}_\eta}+3\mathcal{P}\sqrt{\mathcal{Y}_\eta}\right)\sqrt{\mathcal{H}_\eta}(s,n)dn\,\mathcal{Y}_\eta(s,m)\,dm\right|\\
&\leqslant \int_{\eta(s)}^{\tilde\eta(s)}\left(\int_0^{2C} e^{-|\mathcal{Y}(s,m)-\mathcal{Y}(s,n)|}\left(\frac{2}{3}\mathcal{U}^2+3\mathcal{P}\right)^2\mathcal{Y}_\eta(s,n)\,dn\right)^{1/2}\\
&\quad\times\left(\int_0^{2C} e^{-|\mathcal{Y}(s,m)-\mathcal{Y}(s,n)|}\mathcal{H}_\eta(s,n)\,dn\right)^{1/2}\mathcal{Y}_\eta(s,m)\,dm\\
&\leqslant \int_{\eta(s)}^{\tilde\eta(s)}\left(\int_0^{2C} e^{-|\mathcal{Y}(s,m)-\mathcal{Y}(s,n)|}\left(\frac{8}{9}\mathcal{U}^4+18\mathcal{P}^2\right)\mathcal{Y}_\eta(s,n)\,dn\right)^{1/2}\\
&\quad\times\left(\int_0^{2C} e^{-|\mathcal{Y}(s,m)-\mathcal{Y}(s,n)|}\mathcal{H}_\eta(s,n)\,dn\right)^{1/2}\mathcal{Y}_\eta(s,m)\,dm\\
&\leqslant \mathcal{O}(1)\left(\int_{\eta(s)}^{\tilde\eta(s)}\int_0^{2C} e^{-|\mathcal{Y}(s,m)-\mathcal{Y}(s,n)|}(\mathcal{U}^2+\mathcal{P})\mathcal{Y}_\eta(s,n)\,dn\mathcal{Y}_\eta(s,m)\,dm\right)^{1/2}\\
&\quad\times\left(\int_{\eta(s)}^{\tilde\eta(s)}\int_0^{2C} e^{-|\mathcal{Y}(s,m)-\mathcal{Y}(s,n)|}\mathcal{H}_\eta(s,n)dn\mathcal{Y}_\eta(s,m)\,dm\right)^{1/2}\\
&\leqslant \mathcal{O}(1)\left(\int_{\eta(s)}^{\tilde\eta(s)}\int_0^{2C} e^{-|\mathcal{Y}(s,m)-\mathcal{Y}(s,n)|}(\mathcal{U}^2+\mathcal{P})\mathcal{Y}_\eta(s,n)\,dn\mathcal{Y}_\eta(s,m)\,dm\right)^{1/2}\\
&\quad\times\left(\int_{\eta(s)}^{\tilde\eta(s)} 4\mathcal{P}\mathcal{Y}_\eta(s,m)\,dm\right)^{1/2}\\
&\leqslant \mathcal{O}(1)\left(\int_{\eta(s)}^{\tilde\eta(s)}\int_0^{2C} e^{-|\mathcal{Y}(s,m)-\mathcal{Y}(s,n)|}(\mathcal{U}^2+\mathcal{P})\mathcal{Y}_\eta(s,n)\,dn\mathcal{Y}_\eta(s,m)\,dm\right)^{1/2}\\
&\quad\times\sqrt{|\eta(s)-\tilde\eta(s)|}\\
&\leqslant \mathcal{O}(1)\left(|t-\tilde t|\int_{\eta(s)}^{\tilde\eta(s)}\int_0^{2C} e^{-|\mathcal{Y}(s,m)-\mathcal{Y}(s,n)|}(\mathcal{U}^2+\mathcal{P})\mathcal{Y}_\eta(s,n)\,dn\mathcal{Y}_\eta(s,m)\,dm\right)^{1/2}.
\end{aligned}
$$

Thus it is left to show that

$$
\left|\int_{\eta(s)}^{\tilde\eta(s)}\int_0^{2C} e^{-|\mathcal{Y}(s,m)-\mathcal{Y}(s,n)|}(\mathcal{U}^2+\mathcal{P})\mathcal{Y}_\eta(s,n)\,dn\,\mathcal{Y}_\eta(s,m)\,dm\right|
$$

is bounded uniformly with respect to both space and time. Therefore observe that, since all the involved terms are positive, we have

$$
\begin{aligned}
&\left|\int_{\eta(s)}^{\tilde\eta(s)}\int_0^{2C} e^{-|\mathcal{Y}(s,m)-\mathcal{Y}(s,n)|}(\mathcal{U}^2+\mathcal{P})\mathcal{Y}_\eta(s,n)dn\,\mathcal{Y}_\eta(s,m)\,dm\right|\\
&\quad\leqslant \int_0^{2C}\int_0^{2C} e^{-|\mathcal{Y}(s,m)-\mathcal{Y}(s,n)|}(\mathcal{U}^2+\mathcal{P})\mathcal{Y}_\eta(s,n)\,dn\,\mathcal{Y}_\eta(s,m)\,dm
\end{aligned}
$$

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$$
\begin{aligned}
&= \int_0^{2C} \int_0^{2C} e^{-|\mathcal{Y}(s,m)-\mathcal{Y}(s,n)|} \mathcal{Y}_\eta(s,m)\, dm\, (\mathcal{U}^2 + \mathcal{P})\mathcal{Y}_\eta(s,n)\, dn \\
&\leqslant 2 \int_0^{2C} (\mathcal{U}^2 + \mathcal{P})\mathcal{Y}_\eta(s,n)\, dn \leqslant 6C,
\end{aligned}
$$

where we used (3.22) in the last step. Thus we obtain

$$
\begin{aligned}
&\bigg| \int_{\eta(s)}^{\tilde\eta(s)} \int_0^{2C} \operatorname{sign}(m-n) e^{-|\mathcal{Y}(s,m)-\mathcal{Y}(s,n)|} \\
&\qquad \times \left(\frac{2}{3} \mathcal{U}^3 \mathcal{Y}_\eta - 2\mathcal{P}\mathcal{U}\mathcal{Y}_\eta - \mathcal{Q}\mathcal{U}_\eta \right)(s,n)\, dn\, \mathcal{Y}_\eta(s,m)\, dm \bigg| \\
&\quad \leqslant \mathcal{O}(1) \left(|t-\tilde t| \int_{\eta(s)}^{\tilde\eta(s)} \int_0^{2C} e^{-|\mathcal{Y}(s,m)-\mathcal{Y}(s,n)|} (\mathcal{U}^2 + \mathcal{P})\mathcal{Y}_\eta(s,n)\, dn\, \mathcal{Y}_\eta(s,m)\, dm \right)^{1/2} \\
&\quad \leqslant \mathcal{O}(1)|t-\tilde t|^{1/2},
\end{aligned}
$$

and

$$
\begin{aligned}
&\bigg| \int_t^{\tilde t} \int_{\eta(s)}^{\tilde\eta(s)} \int_0^{2C} \operatorname{sign}(m-n) e^{-|\mathcal{Y}(s,m)-\mathcal{Y}(s,n)|} \\
&\qquad \times \left(\frac{2}{3} \mathcal{U}^3 \mathcal{Y}_\eta - 2\mathcal{P}\mathcal{U}\mathcal{Y}_\eta - \mathcal{Q}\mathcal{U}_\eta \right)(s,n)\, dn\, \mathcal{Y}_\eta(s,m)\, dm\, ds \bigg| \\
&\quad \leqslant \mathcal{O}(1)|t-\tilde t|^{3/2}.
\end{aligned} \tag{3.27}
$$

Combining (3.26) and (3.27), we end up with

$$
\begin{aligned}
|\tilde I_2| &= \left| 2 \int_t^{\tilde t} (Q_t(s, l(\tilde t, \eta)) + H_t(s, l(\tilde t, \eta))) - (Q_t(s, l(t,\eta)) + H_t(s, l(t,\eta)))\, ds \right| \\
&\leqslant \mathcal{O}(1)(|t-\tilde t|^{3/2} + |t-\tilde t|^2).
\end{aligned}
$$

As far as $\tilde I_1$ is concerned, we would like to apply the mean-value theorem. Note therefore that we showed before that the function $(Q_t + H_t)(t, \xi)$ is continuous with respect to time, and hence the function $(Q_t + H_t)(s, l(t, \eta))$, considered as a function of s, is continuous with respect to s. Thus we end up with

$$
\tilde I_1 = 2 \int_t^{\tilde t} (Q_t + H_t)(s, l(t,\eta))\, ds = 2(\tilde t - t)(Q_t + H_t)(\tilde s, l(t,\eta))
$$

for some $\tilde s$ between t and $\tilde t$.

Last but not least, we therefore have

$$\begin{aligned}2\int_t^{\tilde t}(Q_t+H_t)(s,l(\tilde t,\eta))\,ds &= \tilde I_1+\tilde I_2\\ &= 2(\tilde t-t)(Q_t+H_t)(\tilde s,l(t,\eta))+\mathcal{O}(1)|\tilde t-t|^{3/2}\\ &\quad+\mathcal{O}(1)|\tilde t-t|^2,\end{aligned}$$

for some $\tilde s$ between t and $\tilde t$, which implies (cf. (3.23)) that

$$\begin{aligned}\lim_{\tilde t\to t}\frac{J(t,l(t,\eta))-J(t,l(\tilde t,\eta))}{t-\tilde t} &= \lim_{\tilde t\to t}2\int_t^{\tilde t}\frac{(Q_t+H_t)(s,l(\tilde t,\eta))}{t-\tilde t}\,ds\\ &= 2(Q_t+H_t)(t,l(t,\eta)).\end{aligned}$$

Recalling that $J(t,l(t,\eta))=\eta$ for all t and η, we have that

$$J^{-1}(t,J(t,l(t,\eta)))=l(t,\eta),$$

if we, as before, denote the inverse to $J(t,\,\cdot)$ by $J^{-1}(t,\,\cdot)$. Thus we have

$$\begin{aligned}l_t(t,\eta) &= \lim_{\tilde t\to t}\frac{l(t,\eta)-l(\tilde t,\eta)}{t-\tilde t}\\ &= \lim_{\tilde t\to t}\frac{J^{-1}(t,J(t,l(t,\eta)))-J^{-1}(t,J(t,l(\tilde t,\eta)))}{t-\tilde t}\\ &= \lim_{\tilde t\to t}\frac{J^{-1}(t,J(t,l(t,\eta)))-J^{-1}(t,J(t,l(\tilde t,\eta))}{J(t,l(t,\eta))-J(t,l(\tilde t,\eta))}\\ &\quad\times\lim_{\tilde t\to t}\frac{J(t,l(t,\eta))-J(t,l(\tilde t,\eta))}{t-\tilde t}.\end{aligned}$$

We have established that the last limit on the right-hand side exists and the existence of the first one follows from the fact that $J^{-1}(t,\,\cdot)$ is differentiable almost everywhere.

REMARK 3.2. The above argument relies heavily on the fact that the function $l(t,\cdot)\colon[0,2C]\to\mathbb{R}$ is continuous and strictly increasing, the reason being that $J(t,\,\cdot)$ is continuous and strictly increasing. This is in contrast to the HS equation in [16], where the function $H(t,\,\cdot)$ is increasing, but not necessarily strictly increasing and its inverse may have jumps. In the HS equations, if $H(t,\,\cdot)$ is constant on some interval I, then $H(s,\,\cdot)$ will be constant on I for any s, since H is independent of time. Furthermore, this means that the jumps in the new coordinates might change in height with respect to time, but never change their position. This is the essential difference to the CH equation where the jumps would turn up and disappear again immediately, which motivates the choice of $G(t,x)$ and $J(t,\xi)$.

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3.3. New system: the right quantities. Since we have shown that $l(t,\eta)$ is differentiable almost everywhere with respect to both space and time, we can now establish rigorously the system of differential equations in our new coordinates. Recall that $J(t,l(t,\eta))=\eta$ and hence direct calculations yield

$$\begin{aligned} J_t(t,l(t,\eta))+J_\xi(t,l(t,\eta))l_t(t,\eta)&=0,\\ J_\xi(t,l(t,\eta))l_\eta(t,\eta)&=1. \end{aligned}$$

According to the time evolution in Lagrangian coordinates, we thus end up with

$$\begin{aligned} l_\eta(t,\eta)&=\frac{1}{J_\xi(t,l(t,\eta))},\\ l_t(t,\eta)&=-\frac{J_t(t,l(t,\eta))}{J_\xi(t,l(t,\eta))}=-J_t(t,l(t,\eta))l_\eta(t,\eta). \end{aligned}$$

As far as the new coordinates are concerned, we recall that

$$\begin{aligned} \mathcal{Y}(t,\eta)=y(t,l(t,\eta)),\quad \mathcal{U}(t,\eta)=U(t,l(t,\eta)),\\ \text{and}\quad \mathcal{P}(t,\eta)=P(t,l(t,\eta)), \end{aligned}$$

and direct calculations using the differentiabilities proved for Q, P, and l in Subsections 3.1 and 3.2 yield the following differential equations for the unknowns $(\mathcal{Y},\mathcal{U},\mathcal{P})$.

THEOREM 3.3. *Let (u,μ) denote a weak conservative solution of (1.1). Define $\mathcal{Y}$ by (2.3), $\mathcal{U}$ by (2.6), and $\mathcal{P}$ by (2.8). Then the following system of differential equations holds:*

$$\mathcal{Y}_t(t,\eta)+(\tfrac{2}{3}\mathcal{U}^3+\mathcal{S})\mathcal{Y}_\eta(t,\eta)=\mathcal{U}(t,\eta), \tag{3.28a}$$

$$\mathcal{U}_t(t,\eta)+(\tfrac{2}{3}\mathcal{U}^3+\mathcal{S})\mathcal{U}_\eta(t,\eta)=-\mathcal{Q}(t,\eta), \tag{3.28b}$$

$$\mathcal{P}_t(t,\eta)+(\tfrac{2}{3}\mathcal{U}^3+\mathcal{S})\mathcal{P}_\eta(t,\eta)=\mathcal{Q}\mathcal{U}(t,\eta)+\mathcal{R}(t,\eta), \tag{3.28c}$$

where

$$\mathcal{Q}(t,\eta)=-\frac{1}{4}\int_0^{2C}\operatorname{sign}(\eta-\theta)e^{-|\mathcal{Y}(t,\eta)-\mathcal{Y}(t,\theta)|}(2(\mathcal{U}^2-\mathcal{P})\mathcal{Y}_\eta(t,\theta)+1)\,d\theta, \tag{3.29a}$$

$$\mathcal{S}(t,\eta)=\int_0^{2C}e^{-|\mathcal{Y}(t,\eta)-\mathcal{Y}(t,\theta)|}\left(\frac{2}{3}\mathcal{U}^3\mathcal{Y}_\eta-\mathcal{U}_\eta\mathcal{Q}-2\mathcal{P}\mathcal{U}\mathcal{Y}_\eta\right)(t,\theta)\,d\theta, \tag{3.29b}$$

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$$\mathcal{R}(t,\eta) = \frac{1}{4}\int_0^{2C} \operatorname{sign}(\eta-\theta)e^{-|\mathcal{Y}(t,\eta)-\mathcal{Y}(t,\theta)|}\left(\frac{2}{3}\mathcal{U}^3\mathcal{Y}_\eta + \mathcal{U}\right)(t,\theta)\,d\theta$$
$$- \frac{1}{2}\int_0^{2C} e^{-|\mathcal{Y}(t,\eta)-\mathcal{Y}(t,\theta)|}\mathcal{U}\mathcal{Q}\mathcal{Y}_\eta(t,\theta)\,d\theta. \tag{3.29c}$$

REMARK 3.4. This system is identical to the system derived in the smooth case (cf. (2.9)) as expected. We also remind the reader that $\mathcal{Q}(t,\eta) = Q(t, l(t,\eta))$, $\mathcal{S}(t,\eta) = J_t(t, l(t,\eta)) - \frac{2}{3}\mathcal{U}^3(t,\eta)$, and $\mathcal{R}(t,\eta) = P_t(t, l(t,\eta)) - \mathcal{Q}\mathcal{U}(t,\eta)$.

3.4. Functional setting: consistency of the new coordinates.

The next main question is which functional space we should work in such that the right-hand side of system (3.28) can be regarded as a Lipschitz function of the chosen unknowns. For instance, it is difficult or rather impossible to establish the Lipschitz continuity of $\mathcal{Q}$ with respect to $\mathcal{P}, \mathcal{U}$, and $\mathcal{Y}$. However, we will see that one can establish that $\mathcal{Q}$ is Lipschitz continuous with respect to $\mathcal{P}^{1/2}, \mathcal{U}$, and $\mathcal{Y}$. At first sight this seems to be surprising, but on the other hand we established in Eulerian coordinates that $2p(t,x) \geqslant u^2(t,x)$, which rewrites in our new coordinates as

$$2\mathcal{P}(t,\eta) \geqslant \mathcal{U}^2(t,\eta). \tag{3.30}$$

Thus it seems somehow natural that $\mathcal{P}(t,\eta)^{1/2}$ and $|\mathcal{U}(t,\eta)|$ will behave similarly.

The aim of this subsection is to derive the system of differential equations for $\mathcal{Y}$, $\mathcal{U}$, and $\mathcal{P}^{1/2}$ and subsequently to establish that all terms turning up in this system are well defined by assuming that each of the new variables $(\mathcal{Y}, \mathcal{U}, \mathcal{P}^{1/2})$ is in $L^2([0, 2C])$.

Replacing the equation for $\mathcal{P}$ with the corresponding one for $\mathcal{P}^{1/2}$, we find that system (3.28) reads as

$$\mathcal{Y}_t + (\tfrac{2}{3}\mathcal{U}^3 + \mathcal{S})\mathcal{Y}_\eta = \mathcal{U}, \tag{3.31a}$$
$$\mathcal{U}_t + (\tfrac{2}{3}\mathcal{U}^3 + \mathcal{S})\mathcal{U}_\eta = -\mathcal{Q}, \tag{3.31b}$$
$$(\mathcal{P}^{1/2})_t + \left(\frac{2}{3}\mathcal{U}^3 + \mathcal{S}\right)(\mathcal{P}^{1/2})_\eta = \frac{\mathcal{Q}\mathcal{U}}{2\mathcal{P}^{1/2}} + \frac{\mathcal{R}}{2\mathcal{P}^{1/2}}, \tag{3.31c}$$

where $\mathcal{Q}$, $\mathcal{S}$, and $\mathcal{R}$ are given by (3.29).

Assume for the moment that $\mathcal{P}^{1/2}$, $\mathcal{U}$, and $\mathcal{Y}$ belong to $L^2([0, 2C])$. Then we want to show that all terms appearing in the above system also belong to $L^2([0, 2C])$. Therefore it is important to keep in mind, in addition to (3.30), that

$$2\mathcal{P}\mathcal{Y}_\eta(t,\eta) - \mathcal{U}^2\mathcal{Y}_\eta(t,\eta) + \mathcal{H}_\eta(t,\eta) = 1 \tag{3.32}$$

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and, in particular (cf. (3.12), (3.21), and (3.22)),

$$2|\mathcal{U}\mathcal{U}_\eta(t,\eta)| \leqslant \mathcal{H}_\eta(t,\eta) \leqslant 1, \tag{3.33a}$$
$$\mathcal{U}^2\mathcal{Y}_\eta(t,\eta) \leqslant \mathcal{H}_\eta(t,\eta) \leqslant 1, \tag{3.33b}$$
$$2|\mathcal{Q}|\mathcal{Y}_\eta(t,\eta) \leqslant 2\mathcal{P}\mathcal{Y}_\eta(t,\eta) \leqslant 1. \tag{3.33c}$$

As an immediate consequence we obtain that

$$\mathcal{U}^3\mathcal{Y}_\eta(t,\eta) \quad \text{and} \quad \mathcal{U}^3\mathcal{U}_\eta(t,\eta)$$

are uniformly bounded. The last term of this form can be estimated as follows:

$$|\mathcal{U}^3(\mathcal{P}^{1/2})_\eta(t,\eta)| = \left|\frac{\mathcal{U}^3\mathcal{Q}\mathcal{Y}_\eta}{2\mathcal{P}^{1/2}}(t,\eta)\right| \leqslant \left|\frac{\mathcal{U}^3}{2\mathcal{P}^{1/2}}(t,\eta)\right| \leqslant \mathcal{U}^2(t,\eta),$$

and is therefore uniformly bounded. Here we used (3.30).

As far as $\mathcal{S}$ is concerned, we want to establish that

$$|\mathcal{S}(t,\eta)| \leqslant \mathcal{O}(1)\mathcal{P}(t,\eta),$$

implying that

$$\begin{aligned} |\mathcal{S}\mathcal{Y}_\eta(t,\eta)| &\leqslant \mathcal{O}(1)\mathcal{P}\mathcal{Y}_\eta(t,\eta) \leqslant \mathcal{O}(1),\\ |\mathcal{S}\mathcal{U}_\eta(t,\eta)| &\leqslant \mathcal{S}^2\mathcal{Y}_\eta(t,\eta) + \mathcal{H}_\eta(t,\eta) \leqslant \mathcal{O}(1),\\ |\mathcal{S}(\mathcal{P}^{1/2})_\eta(t,\eta)| &\leqslant \mathcal{O}(1)\mathcal{P}^{3/2}\mathcal{Y}_\eta(t,\eta) \leqslant \mathcal{O}(1). \end{aligned}$$

Indeed, by the definition of $\mathcal{S}$ and since $|\mathcal{Q}| \leqslant \mathcal{P}$ and (cf. (3.12a))

$$\mathcal{U}_\eta^2(t,\eta) \leqslant \mathcal{H}_\eta\mathcal{Y}_\eta(t,\eta),$$

we have

$$\begin{aligned} |\mathcal{S}(t,\eta)| &= \left|\int_0^{2C} e^{-|\mathcal{Y}(t,\eta)-\mathcal{Y}(t,\theta)|}\left(\frac{2}{3}\mathcal{U}^3\mathcal{Y}_\eta - \mathcal{U}_\eta\mathcal{Q} - 2\mathcal{P}\mathcal{U}\mathcal{Y}_\eta\right)(t,\theta)\,d\theta\right|\\ &\leqslant \|\mathcal{U}(t,\cdot)\|_{L^\infty([0,2C])}\int_0^{2C} e^{-|\mathcal{Y}(t,\eta)-\mathcal{Y}(t,\theta)|}\mathcal{U}^2\mathcal{Y}_\eta(t,\theta)\,d\theta\\ &\quad + 2\left(\int_0^{2C} e^{-|\mathcal{Y}(t,\eta)-\mathcal{Y}(t,\theta)|}\mathcal{P}^2\mathcal{Y}_\eta(t,\theta)\,d\theta\right)^{1/2}\\ &\quad \times \left(\int_0^{2C} e^{-|\mathcal{Y}(t,\eta)-\mathcal{Y}(t,\theta)|}\mathcal{U}^2\mathcal{Y}_\eta(t,\theta)\,d\theta\right)^{1/2} \end{aligned}$$

$$
\begin{aligned}
&+\left(\int_0^{2C} e^{-|\mathcal{Y}(t,\eta)-\mathcal{Y}(t,\theta)|}\mathcal{P}^2\mathcal{Y}_\eta(t,\theta)\,d\theta\right)^{1/2}\\
&\times\left(\int_0^{2C} e^{-|\mathcal{Y}(t,\eta)-\mathcal{Y}(t,\theta)|}\mathcal{H}_\eta(t,\theta)\,d\theta\right)^{1/2}\\
&\leqslant 4\sqrt{C}\,\mathcal{P}(t,\eta)+6\left(\int_0^{2C} e^{-|\mathcal{Y}(t,\eta)-\mathcal{Y}(t,\theta)|}\mathcal{P}^2\mathcal{Y}_\eta(t,\theta)\,d\theta\right)^{1/2}\mathcal{P}^{1/2}(t,\eta).
\end{aligned}
$$

Thus it is left to show that

$$
\int_0^{2C} e^{-|\mathcal{Y}(t,\eta)-\mathcal{Y}(t,\theta)|}\mathcal{P}^2\mathcal{Y}_\eta(t,\theta)\,d\theta \leqslant \mathcal{O}(1)\mathcal{P}(t,\eta). \tag{3.34}
$$

As several times before, the main tool will be integration by parts together with

$$
\mathcal{P}(t,\eta)=\frac{1}{4}\int_0^{2C} e^{-|\mathcal{Y}(t,\eta)-\mathcal{Y}(t,\theta)|}(2(\mathcal{U}^2-\mathcal{P})\mathcal{Y}_\eta(t,\theta)+1)\,d\theta.
$$

Direct computations yield

$$
\begin{aligned}
&\int_0^{\eta} e^{\mathcal{Y}(t,\theta)-\mathcal{Y}(t,\eta)}\mathcal{P}^2\mathcal{Y}_\eta(t,\theta)\,d\theta\\
&\quad=\mathcal{P}^2(t,\eta)-\int_0^{\eta} e^{\mathcal{Y}(t,\theta)-\mathcal{Y}(t,\eta)}2\mathcal{P}\mathcal{Q}\mathcal{Y}_\eta(t,\theta)\,d\theta\\
&\quad=\mathcal{P}^2(t,\eta)-2\mathcal{P}\mathcal{Q}(t,\eta)\\
&\qquad+\int_0^{\eta} e^{\mathcal{Y}(t,\theta)-\mathcal{Y}(t,\eta)}(2\mathcal{Q}^2\mathcal{Y}_\eta+2\mathcal{P}\mathcal{Q}_\eta)(t,\theta)\,d\theta\\
&\quad=\mathcal{P}^2(t,\eta)-2\mathcal{P}\mathcal{Q}(t,\eta)\\
&\qquad+\int_0^{\eta} e^{\mathcal{Y}(t,\theta)-\mathcal{Y}(t,\eta)}2(\mathcal{P}^2+\mathcal{Q}^2)\mathcal{Y}_\eta(t,\theta)\,d\theta\\
&\qquad+\int_0^{\eta} e^{\mathcal{Y}(t,\theta)-\mathcal{Y}(t,\eta)}(2(\mathcal{P}-\mathcal{U}^2)\mathcal{Y}_\eta(t,\theta)-1)\mathcal{P}(t,\theta)\,d\theta.
\end{aligned} \tag{3.35}
$$

Here we have used (3.4b) and (3.32). Thus, rearranging the terms, we end up with

$$
\begin{aligned}
&\int_0^{\eta} e^{\mathcal{Y}(t,\theta)-\mathcal{Y}(t,\eta)}\mathcal{P}^2\mathcal{Y}_\eta(t,\theta)\,d\theta\\
&\quad\leqslant\int_0^{\eta} e^{\mathcal{Y}(t,\theta)-\mathcal{Y}(t,\eta)}(\mathcal{P}^2+2\mathcal{Q}^2)\mathcal{Y}_\eta(t,\theta)\,d\theta\\
&\quad=2\mathcal{P}\mathcal{Q}(t,\eta)-\mathcal{P}^2(t,\eta)
\end{aligned}
$$

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$$+\int_0^\eta e^{\mathcal{Y}(t,\theta)-\mathcal{Y}(t,\eta)}(2(\mathcal{U}^2-\mathcal{P})\mathcal{Y}_\eta(t,\theta)+1)\mathcal{P}(t,\theta)\,d\theta$$
$$\leqslant 2\mathcal{P}^2(t,\eta)+4\|\mathcal{P}(t,\cdot)\|_{L^\infty([0,2C])}\mathcal{P}(t,\eta)$$
$$\leqslant 6\|\mathcal{P}(t,\cdot)\|_{L^\infty([0,2C])}\mathcal{P}(t,\eta). \tag{3.36}$$

Similar computations yield that

$$\begin{aligned}\int_\eta^{2C} e^{\mathcal{Y}(t,\eta)-\mathcal{Y}(t,\theta)}\mathcal{P}^2\mathcal{Y}_\eta(t,\theta)d\theta &\leqslant \int_\eta^{2C} e^{\mathcal{Y}(t,\eta)-\mathcal{Y}(t,\theta)}(\mathcal{P}^2+2\mathcal{Q}^2)\mathcal{Y}_\eta(t,\theta)\,d\theta\\ &= -\mathcal{P}^2(t,\eta)-2\mathcal{P}\mathcal{Q}(t,\eta)\\ &\quad+\int_\eta^{2C} e^{\mathcal{Y}(t,\eta)-\mathcal{Y}(t,\theta)}\\ &\quad\times(2(\mathcal{U}^2-\mathcal{P})\mathcal{Y}_\eta(t,\theta)+1)\mathcal{P}(t,\theta)\,d\theta\\ &\leqslant 2\mathcal{P}^2(t,\eta)+4\|\mathcal{P}(t,\cdot)\|_{L^\infty([0,2C])}\mathcal{P}(t,\eta)\\ &\leqslant 6\|\mathcal{P}(t,\cdot)\|_{L^\infty([0,2C])}\mathcal{P}(t,\eta).\end{aligned}$$

This proves (3.34) due to (3.13).

There are two terms left to investigate. Namely, the term $\mathcal{Q}\mathcal{U}/\mathcal{P}^{1/2}$, and we easily find that

$$\left|\frac{\mathcal{Q}\mathcal{U}}{\sqrt{2}\mathcal{P}^{1/2}}(t,\eta)\right|\leqslant \mathcal{P}(t,\eta),$$

and hence it is uniformly bounded. The last term $\mathcal{R}/\mathcal{P}^{1/2}$ is a bit more involved. We have

$$\begin{aligned}&|\mathcal{R}(t,\eta)|\\ &\leqslant \left|\frac{1}{4}\int_0^{2C}\operatorname{sign}(\eta-\theta)e^{-|\mathcal{Y}(t,\eta)-\mathcal{Y}(t,\theta)|}\left(\frac{2}{3}\mathcal{U}^3\mathcal{Y}_\eta+\mathcal{U}\right)(t,\theta)\,d\theta\right|\\ &\quad+\left|\frac{1}{2}\int_0^{2C} e^{-|\mathcal{Y}(t,\eta)-\mathcal{Y}(t,\theta)|}\mathcal{U}\mathcal{Q}\mathcal{Y}_\eta(t,\theta)\,d\theta\right|\\ &\leqslant \frac{1}{4}\int_0^{2C} e^{-|\mathcal{Y}(t,\eta)-\mathcal{Y}(t,\theta)|}(2(\mathcal{U}^2-\mathcal{P})\mathcal{Y}_\eta(t,\theta)+1)|\mathcal{U}|(t,\theta)\,d\theta\\ &\quad+\frac{1}{4}\int_0^{2C} e^{-|\mathcal{Y}(t,\eta)-\mathcal{Y}(t,\theta)|}2\mathcal{P}|\mathcal{U}|\mathcal{Y}_\eta(t,\theta)\,d\theta\\ &\quad+\frac{1}{2}\int_0^{2C} e^{-|\mathcal{Y}(t,\eta)-\mathcal{Y}(t,\theta)|}|\mathcal{U}\mathcal{Q}|\mathcal{Y}_\eta(t,\theta)\,d\theta\\ &\leqslant \|\mathcal{U}(t,\cdot)\|_{L^\infty([0,2C])}\mathcal{P}(t,\eta)+\frac{1}{2}\int_0^{2C} e^{-|\mathcal{Y}(t,\eta)-\mathcal{Y}(t,\theta)|}(\mathcal{P}^2+\mathcal{U}^2)\mathcal{Y}_\eta(t,\theta)\,d\theta\\ &\leqslant \mathcal{O}(1)\mathcal{P}(t,\eta).\end{aligned}\tag{3.37}$$

Thus

$$\left|\frac{\mathcal{R}}{\mathcal{P}^{1/2}(t,\eta)}\right| \leqslant \mathcal{O}(1)\mathcal{P}^{1/2}(t,\eta),$$

and hence belongs to $L^2([0,2C])$.

Later on, we will need in some of our estimates (cf. (3.3a)) that

$$\mathcal{P}(t,\eta) = \frac{1}{4}\int_0^{2C} e^{-|\mathcal{Y}(t,\eta)-\mathcal{Y}(t,\theta)|}(\mathcal{U}^2\mathcal{Y}_\eta + \mathcal{H}_\eta)(t,\theta)\,d\theta,$$

which implies the estimate

$$\int_0^{\eta} e^{-(\mathcal{Y}(t,\eta)-\mathcal{Y}(t,\theta))}(\mathcal{U}^2\mathcal{Y}_\eta + \mathcal{H}_\eta)(t,\theta)\,d\theta \leqslant 4\mathcal{P}(t,\eta). \tag{3.38}$$

3.5. The choice of the distance. In order to motivate the choice of the distance between two solutions with the same energy and to outline the strategy for proving the Lipschitz continuity, let us come back to system (3.31). We observe that the unknowns $(\mathcal{Y}, \mathcal{U}, \mathcal{P}^{1/2})$ are advected by the same velocity field in the η variable. Moreover, since all terms make sense under the assumption that the unknowns belong to $L^2([0,2C])$, as checked in the previous subsection, we consider a toy problem of the form

$$f_t + \mathcal{A}(f)f_\eta = \mathcal{B}(f),$$

where f might be a vector valued function in $L^2([0,2C])$. This toy problem resembles our situation with the complicated operator dependencies on the unknowns. Within this functional framework, it is natural to look at the evolution of the L^2-norm of the unknown f. Given two solutions f_1 and f_2 to $f_t+\mathcal{A}(f)f_\eta = \mathcal{B}(f)$, direct computations yield

$$\begin{aligned}
\frac{d}{dt}\|f_1 - f_2\|^2_{L^2([0,2C])} &= 2\int_0^{2C}(f_1-f_2)(f_{1,t}-f_{2,t})(t,\theta)\,d\theta \\
&= -2\int_0^{2C}(\mathcal{A}(f_1)f_{1,\eta} - \mathcal{A}(f_2)f_{2,\eta})(f_1-f_2)(t,\theta)\,d\theta \\
&\quad + 2\int_0^{2C}(\mathcal{B}(f_1)-\mathcal{B}(f_2))(f_1-f_2)(t,\theta)\,d\theta \\
&= -2\int_0^{2C}\mathcal{A}(f_1)(f_{1,\eta}-f_{2,\eta})(f_1-f_2)(t,\theta)\,d\theta \\
&\quad - 2\int_0^{2C} f_{2,\eta}(\mathcal{A}(f_1)-\mathcal{A}(f_2))(f_1-f_2)(t,\theta)\,d\theta
\end{aligned}$$

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$$
\begin{aligned}
&+ 2\int_0^{2C} (\mathcal{B}(f_1) - \mathcal{B}(f_2))(f_1 - f_2)(t,\theta)\,d\theta \\
&= \mathcal{A}(f_1)(f_1 - f_2)^2(t,0) - \mathcal{A}(f_1)(f_1 - f_2)^2(t,2C) \\
&+ \int_0^{2C} \mathcal{A}'(f_1) f_{1,\eta}(f_1 - f_2)^2(t,\theta)\,d\theta \\
&- 2\int_0^{2C} f_{2,\eta}(\mathcal{A}(f_1) - \mathcal{A}(f_2))(f_1 - f_2)(t,\theta)\,d\theta \\
&+ 2\int_0^{2C} (\mathcal{B}(f_1) - \mathcal{B}(f_2))(f_1 - f_2)(t,\theta)\,d\theta,
\end{aligned}
$$

where the first two terms in the last line are going to vanish if we impose the correct boundary conditions or rather if we have a good behavior of the solutions at both boundaries. Under this assumption, we can then use norm estimates if we know that $\mathcal{A}'(f_1) f_{1,\eta}$ and $f_{2,\eta}$ are uniformly bounded by a constant $\mathcal{O}(1)$ and that $\mathcal{A}$ and $\mathcal{B}$ are Lipschitz continuous with Lipschitz constant $\mathcal{O}(1)$ with respect to f to conclude that

$$
\frac{d}{dt}\|f_1 - f_2\|^2_{L^2([0,2C])} \leqslant \mathcal{O}(1)\|f_1 - f_2\|^2_{L^2([0,2C])},
$$

and Gronwall's lemma then implies

$$
\|(f_1 - f_2)(t)\|_{L^2([0,2C])} \leqslant \|(f_1 - f_2)(0)\|_{L^2([0,2C])} e^{\mathcal{O}(1)t}.
$$

Hence the strategy consists in proving first a propagation in time for the L^2-norm of f in order to get bounds on the unknowns and check the validity of the approach. Then the second, and the most important and technical point, is to establish the Lipschitz estimates. We will come back to this point in Section 4, where we will also fix the following problem in the definition of our metric: the domain of definition of our unknowns depends on the total energy. This is unsatisfactory, if we want to compare solutions with different total energies.

3.6. Propagation of L^2 bounds: moment conditions. Consider (u,μ) a weak conservative solution of (1.1) and $f = (\mathcal{Y}, \mathcal{U}, \mathcal{P}^{1/2})$, defined by (2.3), (2.6), and (2.8), the solution to system (3.31). We want to show that the L^2-norm of f is propagated in time. Since both $\mathcal{U}$ and $\mathcal{P}$ are bounded functions due to (3.10) and (3.13), we are reduced to show the propagation of the L^2-norm of $\mathcal{Y}$. First of all, note that we have

$$
\mathcal{Y}(t, G(t,x)) = x \quad \text{for all } x \in \mathbb{R},
$$

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where $G(t,x)$ is given by

$$G(t,x) = \int_{-\infty}^{x} (2p - u^2)(t,y)dy + F(t,x).$$

Notice that the x-distributional derivative of G is a measure that we denoted by $\nu(t,\cdot) \in \mathcal{M}_+(\mathbb{R})$ given by

$$d\nu(t,x) = (2p - u^2)(t,x) + d\mu(t,x).$$

Since $\mathcal{Y}(t,\eta)$ is the pseudoinverse to $G(t,x)$, then $\mathcal{Y}$ pushes forward the uniform distribution on $[0,2C]$ to ν (see [67]) and then

$$\int_{\mathbb{R}} x^2 d\nu = \int_0^{2C} \mathcal{Y}^2(t,\eta)\, d\eta.$$

Thus $\mathcal{Y} \in L^2([0,2C])$ is equivalent to $\nu(t,\cdot)$ having a finite second moment.

PROPOSITION 3.5. *Let (u,μ) denote a weak conservative solution of (1.1), and by $f = (\mathcal{Y},\mathcal{U},\mathcal{P}^{1/2})$ the solution to system (3.31); then*

$$\int_0^{2C} \mathcal{Y}^2(t,\eta)\, d\eta \leqslant e^{\mathcal{O}(1)t}\left(1 + \int_0^{2C} \mathcal{Y}^2(0,\eta)\, d\eta\right).$$

Proof. We show the propagation of the second moment in time using the Lagrangian coordinates. Note first that

$$\begin{aligned}
\int_{\mathbb{R}} x^2\, d\mu(t,x) &= \int_{\mathbb{R}} y^2 H_\xi(t,\xi)\, d\xi,\\
\int_{\mathbb{R}} x^2 u^2(t,x)\, dx &= \int_{\mathbb{R}} y^2 U^2 y_\xi(t,\xi)\, d\xi,\\
\int_{\mathbb{R}} x^2 p(t,x)\, dx &= \int_{\mathbb{R}} y^2 P y_\xi(t,\xi)\, d\xi.
\end{aligned}$$

We start by computing the derivatives of $y^2 H_\xi$, $y^2 U^2 y_\xi$, and $y^2 P y_\xi$ with respect to time. Direct computations using the formulas in Section 3.1 then yield

$$\begin{aligned}
|(y^2 H_\xi)_t| &= |2yUH_\xi + y^2(3U^2U_\xi - 2PU_\xi - 2QUy_\xi)|\\
&\leqslant y^2 H_\xi + U^2 H_\xi + 3y^2|U|H_\xi + 2y^2P^2y_\xi + 2y^2H_\xi\\
&\leqslant \mathcal{O}(1)(y^2 H_\xi + y^2 P y_\xi + H_\xi)
\end{aligned}$$

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and

$$\begin{aligned}|(y^2 P y_\xi)_t| &= |2yUPy_\xi + y^2 P_t y_\xi + y^2 P U_\xi| \\ &\leqslant y^2 P^2 y_\xi + U^2 y_\xi + y^2 |P_t| y_\xi + y^2 P(y_\xi + H_\xi) \\ &\leqslant \mathcal{O}(1)(y^2 H_\xi + y^2 P y_\xi + H_\xi),\end{aligned}$$

where we used (3.20). Thus

$$\frac{d}{dt}(\|y^2 P y_\xi\|_{L^1} + \|y^2 H_\xi\|_{L^1}) \leqslant \mathcal{O}(1)(\|y^2 P y_\xi\|_{L^1} + \|y^2 H_\xi\|_{L^1} + C)$$

and

$$\int_{\mathbb{R}} (y^2 P y_\xi + y^2 H_\xi)(t,\xi)\, d\xi \leqslant e^{\mathcal{O}(1)t}\left(\int_{\mathbb{R}} (y^2 P y_\xi + y^2 H_\xi)(0,\xi)\, d\xi + C\right).$$

Since $y^2 U^2 y_\xi(t,\xi) \leqslant y^2 H_\xi(t,\xi)$, it follows that also

$$\int_{\mathbb{R}} y^2 U^2 y_\xi(t,\xi) d\xi$$

remains finite for all times, which proves the desired estimate since

$$\int_{\mathbb{R}} x^2\, d\nu(t,x) = \int_{\mathbb{R}} x^2 (2p - u^2)(t,x)\, dx + \int_{\mathbb{R}} x^2\, d\mu(t,x). \qquad \square$$

The propagation of moments not only implies the feasibility of the strategy illustrated in Section 3.5 but also gives us a control on the 'boundary terms'. In fact, since $(u^2 + u_x^2)(t,x) \leqslant d\mu(t,x)$ and $(2p - u^2)(t,x) \geqslant 0$, the previous result implies that

$$\int_{\mathbb{R}} x^2 p(t,x)\, dx, \quad \int_{\mathbb{R}} x^2 u^2(t,x)\, dx, \quad \int_{\mathbb{R}} x^2 u_x^2(t,x)\, dx, \quad \text{and} \quad \int_{\mathbb{R}} x^2\, d\mu(t,x) \tag{3.39}$$

are all finite. We show next that (3.39) implies that

$$xu(t,x) \to 0 \quad \text{and} \quad x^2 p(t,x) \to 0 \quad \text{as } x \to \pm\infty. \tag{3.40}$$

Indeed, (3.39) and $u \in H^1(\mathbb{R})$ imply that

$$xu(t,x) \in L^2(\mathbb{R}) \quad \text{and} \quad (xu(t,x))_x = u(t,x) + xu_x(t,x) \in L^2(\mathbb{R}).$$

Hence $xu(t,x)$ belongs to $H^1(\mathbb{R})$ for any fixed t and in particular $xu(t,x) \to 0$ as $x \to \pm\infty$ for any fixed t.

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The argument for $xp(t,x)$ is a bit more involved, but follows the same lines. To be more precise, $p(t,x) \in L^2(\mathbb{R})$, $|p_x(t,x)| \leqslant p(t,x)$, and

$$\int_{\mathbb{R}} x^2 p^2(t,x)\,dx \leqslant \mathcal{O}(1) \int_{\mathbb{R}} x^2 p(t,x)\,dx < \infty,$$

imply that

$$xp(t,x) \in L^2(\mathbb{R}) \quad \text{and} \quad (xp(t,x))_x = p(t,x) + xp_x(t,x) \in L^2(\mathbb{R}).$$

Hence $xp(t,x)$ belongs to $H^1(\mathbb{R})$ for any fixed t and in particular $xp(t,x) \to 0$ as $x \to \pm\infty$.

Finally, it is left to show that $x\sqrt{p(t,x)} \to 0$ as $x \to \pm\infty$. Therefore note that $p(t,x) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and

$$|(x\sqrt{p(t,x)})_x| = \left|\sqrt{p(t,x)} + x\frac{p_x(t,x)}{2\sqrt{p(t,x)}}\right| \leqslant \sqrt{p(t,x)} + |x|\sqrt{p(t,x)},$$

which together with (3.39) implies that

$$x\sqrt{p(t,x)} \in L^2(\mathbb{R}) \quad \text{and} \quad (x\sqrt{p(t,x)})_x \in L^2(\mathbb{R}).$$

Hence $x\sqrt{p(t,x)}$ belongs to $H^1(\mathbb{R})$ for any fixed t and in particular $x^2 p(t,x) \to 0$ as $x \to \pm\infty$. These properties will allow us to carry out several integrations by parts when deriving the anticipated Lipschitz estimate.

4. The right metric: solutions with different energies

For the remaining part of this paper, we will consider two distinct solutions (u_j, μ_j) for $j = 1, 2$ of the CH equation (1.1) and estimate their difference using a carefully selected norm based on our new variables $(\mathcal{Y}_j, \mathcal{U}_j, \mathcal{P}_j^{1/2})$ for $j = 1$, 2 that yields a Lipschitz metric. The main idea remains to a large extent the one from the HS equation in [16], where a Lipschitz metric for measuring the distance between solutions with nonzero energy has been constructed.

We want to define a Lipschitz metric, which can measure the distance between solutions with different total energy based on our new coordinates. At first sight this seems to be impossible for two reasons:

1. The support of the new variables depends on the total energy since

$$\mathcal{Y}, \mathcal{U}, \mathcal{P}^{1/2} : [0, 2C] \to \mathbb{R}.$$

2. In the case of the zero solution, that is, $(u, \mu) = (0, 0)$, the function $\mathcal{Y}$ is not defined and the same applies to the new variables $(\mathcal{Y}, \mathcal{U}, \mathcal{P}^{1/2})$.

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The key idea is to use a rescaling; more precisely, given $(\mathcal{Y}, \mathcal{U}, \mathcal{P}^{1/2})$, where $\mathcal{Y} : [0, 2C] \to \mathbb{R}$ with $C \neq 0$. Then we can introduce

$$\tilde{\mathcal{Y}}(t, \eta) = A\mathcal{Y}(t, A^2\eta), \quad \tilde{\mathcal{U}}(t, \eta) = A\mathcal{U}(t, A^2\eta),$$
$$\tilde{\mathcal{P}}^{1/2}(t, \eta) = A\mathcal{P}^{1/2}(t, A^2\eta), \quad \tilde{\mathcal{H}}(t, \eta) = A^3\mathcal{H}(t, A^2\eta), \tag{4.1}$$

with

$$A = \sqrt{2C} \tag{4.2}$$

for notational purposes.

This rescaling has three main properties that motivate our choice:

1. The support of the variables $(\tilde{\mathcal{Y}}, \tilde{\mathcal{U}}, \tilde{\mathcal{P}}^{1/2})$ is independent of the energy, which allows us to compare arbitrary solutions with nonzero energy.

2. In the last section, we discussed that the natural solution space for the unknowns $(\mathcal{Y}, \mathcal{U}, \mathcal{P}^{1/2})$ is $L^2([0, 2C])$. Since our rescaling preserves the L^2-norm, the natural solution space for the unknowns $(\tilde{\mathcal{Y}}, \tilde{\mathcal{U}}, \tilde{\mathcal{P}}^{1/2})$ is $L^2([0, 1])$.

3. The most important property is that this rescaling allows us to set

$$(\tilde{\mathcal{Y}}, \tilde{\mathcal{U}}, \tilde{\mathcal{P}}^{1/2}) = (0, 0, 0) \quad \text{for } (u, \mu) = (0, 0).$$

A close look at the definition of $\tilde{\mathcal{Y}}(t, \eta)$ reveals that $\tilde{\mathcal{Y}}(t, \eta)$ is the pseudoinverse to $\tilde{G}(t, x)$, which is given by

$$\tilde{G}(t, x) = \frac{1}{A^2} G\left(t, \frac{x}{A}\right) \quad \text{for } A \neq 0.$$

If one compares $\tilde{G}(t, x)$ with $G(t, x)$ for small A, then the graph of $G(t, x)$ is squeezed in the x direction, but stretched in the y direction (cf. Figure 1). Taking a sequence of functions $G_n(t, x)$ with A_n tending to zero, then the corresponding sequence of functions $\tilde{G}_n(t, x)$ tends to the Heaviside function, which has as pseudoinverse $\tilde{\mathcal{Y}}(t, \eta) = 0$ (cf. Figure 2).

Direct calculation yields the following theorem.

THEOREM 4.1. *Let (u, μ) denote a weak conservative solution of (1.1). Define $\tilde{\mathcal{Y}}, \tilde{\mathcal{U}}$ and $\tilde{\mathcal{P}}^{1/2}$ by (4.1). Then the following system of differential equations holds:*

$$\tilde{\mathcal{Y}}_t + \left(\frac{2}{3}\frac{1}{A^5}\tilde{\mathcal{U}}^3 + \frac{1}{A^6}\tilde{\mathcal{S}}\right)\tilde{\mathcal{Y}}_\eta = \tilde{\mathcal{U}}, \tag{4.3a}$$

$$\tilde{\mathcal{U}}_t + \left(\frac{2}{3}\frac{1}{A^5}\tilde{\mathcal{U}}^3 + \frac{1}{A^6}\tilde{\mathcal{S}}\right)\tilde{\mathcal{U}}_\eta = -\frac{1}{A^2}\tilde{\mathcal{Q}}, \tag{4.3b}$$

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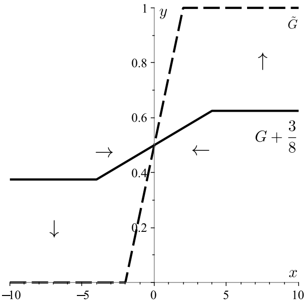

Figure 1. An example of how the rescaling changes the graph of a function $G : \mathbb{R} \to \frac{1}{4}$. The picture shows the function $G + \frac{3}{8}$ (solid) and its rescaled version $\tilde{G}$ (dash).

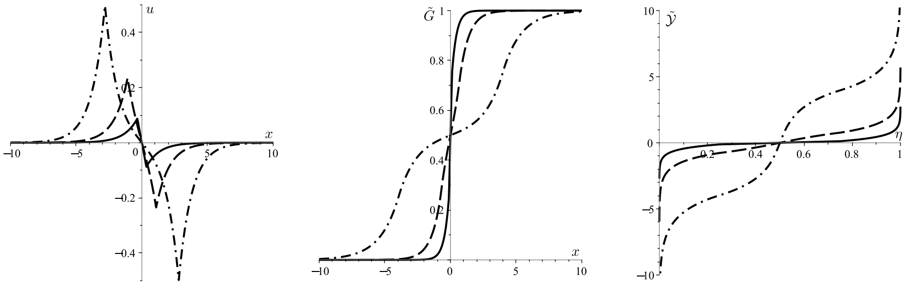

Figure 2. Three different peakon–antipeakon solutions u (left) with total energy 1 (dash–dot), 0.5 (dash), and 0.25 (solid) and the corresponding functions $\tilde{G}$ (middle) and $\tilde{\mathcal{Y}}$ (right).

$$(\tilde{\mathcal{P}}^{1/2})_t + \left(\frac{2}{3}\frac{1}{A^5}\tilde{\mathcal{U}}^3 + \frac{1}{A^6}\tilde{\mathcal{S}}\right)(\tilde{\mathcal{P}}^{1/2})_\eta = \frac{1}{2A^2}\frac{\tilde{\mathcal{Q}}\tilde{\mathcal{U}}}{\tilde{\mathcal{P}}^{1/2}} + \frac{1}{2A^3}\frac{\tilde{\mathcal{R}}}{\tilde{\mathcal{P}}^{1/2}}, \tag{4.3c}$$

where

$$\begin{aligned}\tilde{\mathcal{Q}}(t,\eta) &= A^3\mathcal{Q}(t, A^2\eta)\\ &= -\frac{1}{4}\int_0^1 \operatorname{sign}(\eta-\theta)e^{-\frac{1}{A}|\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta)|}(2(\tilde{\mathcal{U}}^2-\tilde{\mathcal{P}})\tilde{\mathcal{Y}}_\eta(t,\theta)+A^5)\,d\theta,\end{aligned} \tag{4.4a}$$

$$\begin{aligned}\tilde{\mathcal{S}}(t,\eta) &= A^4\mathcal{S}(t, A^2\eta)\\ &= \int_0^1 e^{-\frac{1}{A}|\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta)|}\left(\frac{2}{3}\tilde{\mathcal{U}}^3\tilde{\mathcal{Y}}_\eta - \tilde{\mathcal{U}}_\eta\tilde{\mathcal{Q}} - 2\tilde{\mathcal{P}}\tilde{\mathcal{U}}\tilde{\mathcal{Y}}_\eta\right)(t,\theta)\,d\theta,\end{aligned} \tag{4.4b}$$

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$$\begin{aligned}\tilde{\mathcal{R}}(t,\eta) &= A^5\mathcal{R}(t,A^2\eta)\\ &= \frac{1}{4}\int_0^1 \operatorname{sign}(\eta-\theta)e^{-\frac{1}{A}|\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta)|}\left(\frac{2}{3}A\tilde{\mathcal{U}}^3\tilde{\mathcal{Y}}_\eta + A^6\tilde{\mathcal{U}}\right)(t,\theta)\\ &\quad -\frac{1}{2}\int_0^1 e^{-\frac{1}{A}|\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta)|}\tilde{\mathcal{U}}\tilde{\mathcal{Q}}\tilde{\mathcal{Y}}_\eta(t,\theta)\,d\theta. \end{aligned} \tag{4.4c}$$

Now, we can define the metric between two general conservative solutions to (1.1). To simplify the notation we will from now on assume that all norms are on $L^2([0,1])$, unless otherwise indicated, and write

$$\|\Psi\| = \|\Psi\|_{L^2([0,1])},$$

for any function Ψ. We will not always explicitly indicate the time dependence in all quantities and write $\|\Psi\|$ rather than $\|\Psi(t)\|$.

DEFINITION 4.2. Let (u_i,μ_i) for $i=1,2$ denote two conservative solutions of (1.1). We define the distance between them as

$$\begin{aligned} d((u_1,\mu_1),(u_2,\mu_2)) = \|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\| + \|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2\|\\ + \|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\| + |A_1-A_2|,\end{aligned}$$

where $A_i=\sqrt{2C_i}$ and $C_i=\mu_i(\mathbb{R})$, $i=1,2$, and $(\tilde{\mathcal{Y}}_i,\tilde{\mathcal{U}}_i,\tilde{\mathcal{P}}_i^{1/2})$ are given by (4.1).

REMARK 4.3. Notice that in the case of one of the two solutions being the trivial solution, the distance reduces to the L^2-norm of the solution in the right functional space, that is,

$$d((u,\mu),(0,0)) = \|\tilde{\mathcal{Y}}\| + \|\tilde{\mathcal{U}}\| + \|\tilde{\mathcal{P}}^{1/2}\| + \sqrt{2C},$$

where $C=\mu(\mathbb{R})$, that we already estimated in Section 3.6. It remains to show that the right-hand side of system (4.3) is Lipschitz continuous with respect to the new unknowns $(\tilde{\mathcal{Y}},\tilde{\mathcal{U}},\tilde{\mathcal{P}}^{1/2})$.

The main result of this work can finally be stated. The proof consists in showing the corresponding Lipschitz estimates in each of the components of the distance in Definition 4.2. This is done in the next section in Lemmas 5.3–5.5. Collecting these results leads to our main theorem due to Gronwall's lemma.

THEOREM 4.4. *Consider initial data $u_{i,0}\in H^1(\mathbb{R})$, $\mu_{i,0}\in\mathcal{M}_+(\mathbb{R})$ such that $d(\mu_{\mathrm{ac}})_{i,0}=(u_i^2+u_{i,x}^2)\,dx$ and $C_i=\mu_i(\mathbb{R})$, and let (u_i,μ_i) for $i=1,2$ denote*

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the corresponding conservative solutions of the CH equation (1.1). Then we have that

$$d((u_1(t),\mu_1(t)),(u_2(t),\mu_2(t))) \leqslant e^{\mathcal{O}(1)t} d((u_{1,0},\mu_{1,0}),(u_{2,0},\mu_{2,0})),$$

where $\mathcal{O}(1)$ *denotes a constant depending only on* $\max_j(C_j)$ *remaining bounded as* $\max_j(C_j)\to 0$.

Now, let us start to do some preparatory work for the next section by collecting the main estimates we need in the new variables. Introduce

$$\begin{aligned}\tilde{\mathcal{D}}(t,\eta) &= \int_0^\eta e^{\frac{1}{A}(\tilde{\mathcal{Y}}(t,\theta)-\tilde{\mathcal{Y}}(t,\eta))}\Big((\tilde{\mathcal{U}}^2-\tilde{\mathcal{P}})\tilde{\mathcal{Y}}_\eta(t,\theta)+\frac{1}{2}A^5\Big)\,d\theta\\ &= \frac{1}{2}\int_0^\eta e^{\frac{1}{A}(\tilde{\mathcal{Y}}(t,\theta)-\tilde{\mathcal{Y}}(t,\eta))}(\tilde{\mathcal{U}}^2\tilde{\mathcal{Y}}_\eta+\tilde{\mathcal{H}}_\eta)(t,\theta)\,d\theta, \qquad (4.5a)\\ \tilde{\mathcal{E}}(t,\eta) &= \int_\eta^1 e^{\frac{1}{A}(\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta))}\Big((\tilde{\mathcal{U}}^2-\tilde{\mathcal{P}})\tilde{\mathcal{Y}}_\eta(t,\theta)+\frac{1}{2}A^5\Big)\,d\theta\\ &= \frac{1}{2}\int_\eta^1 e^{\frac{1}{A}(\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta))}(\tilde{\mathcal{U}}^2\tilde{\mathcal{Y}}_\eta+\tilde{\mathcal{H}}_\eta)(t,\theta)\,d\theta, \qquad (4.5b)\end{aligned}$$

so that we can write

$$\tilde{\mathcal{P}}(t,\eta)=\frac{1}{2A}(\tilde{\mathcal{D}}(t,\eta)+\tilde{\mathcal{E}}(t,\eta)) \quad \text{and} \quad \tilde{\mathcal{Q}}(t,\eta)=\frac{1}{2}(-\tilde{\mathcal{D}}(t,\eta)+\tilde{\mathcal{E}}(t,\eta)) \quad (4.6)$$

and

$$\tilde{\mathcal{Q}}(t,\eta)=-\tilde{\mathcal{D}}(t,\eta)+A\tilde{\mathcal{P}}(t,\eta). \qquad (4.7)$$

Note that both $\tilde{\mathcal{D}}(t,\eta)$ and $\tilde{\mathcal{E}}(t,\eta)$ have some very nice properties. Namely,

$$0\leqslant \tilde{\mathcal{D}}(t,\eta)\leqslant 2A\tilde{\mathcal{P}}(t,\eta) \quad \text{and} \quad 0\leqslant \tilde{\mathcal{E}}(t,\eta)\leqslant 2A\tilde{\mathcal{P}}(t,\eta), \qquad (4.8)$$

and

$$\begin{aligned}\left|\frac{d}{d\eta}\tilde{\mathcal{D}}(t,\eta)\right| &= \left|(\tilde{\mathcal{U}}^2-\tilde{\mathcal{P}})\tilde{\mathcal{Y}}_\eta(t,\eta)+\frac{1}{2}A^5-\frac{1}{A}\tilde{\mathcal{D}}\tilde{\mathcal{Y}}_\eta(t,\eta)\right| \leqslant \mathcal{O}(1)A^5, \qquad (4.9)\\ \left|\frac{d}{d\eta}\tilde{\mathcal{E}}(t,\eta)\right| &= \left|-(\tilde{\mathcal{U}}^2-\tilde{\mathcal{P}})\tilde{\mathcal{Y}}_\eta(t,\eta)-\frac{1}{2}A^5+\frac{1}{A}\tilde{\mathcal{E}}\tilde{\mathcal{Y}}_\eta(t,\eta)\right| \leqslant \mathcal{O}(1)A^5. \qquad (4.10)\end{aligned}$$

The scaling leads to some changes in the standard estimates. Some of the problematic terms depending on $\frac{1}{A}$ in system (4.3) can be controlled since

$$\begin{aligned}2\tilde{\mathcal{P}}\tilde{\mathcal{Y}}_\eta(t,\eta)\leqslant A^5, \quad 2|\tilde{\mathcal{U}}\tilde{\mathcal{U}}_\eta(t,\eta)|\leqslant A^4, \quad \tilde{\mathcal{U}}^2\tilde{\mathcal{Y}}_\eta(t,\eta)\leqslant A^5,\\ \text{and} \quad \tilde{\mathcal{H}}_\eta(t,\eta)\leqslant A^5,\end{aligned} \qquad (4.11)$$

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from (3.33) and (4.1), as well as

$$A^2\tilde{\mathcal{U}}_\eta^2(t,\eta) \leqslant \tilde{\mathcal{H}}_\eta\tilde{\mathcal{Y}}_\eta(t,\eta) \leqslant A^5\tilde{\mathcal{Y}}_\eta(t,\eta). \tag{4.12}$$

In addition, equation (3.32) becomes

$$2\tilde{\mathcal{P}}\tilde{\mathcal{Y}}_\eta(t,\eta) - \tilde{\mathcal{U}}^2\tilde{\mathcal{Y}}_\eta(t,\eta) + \tilde{\mathcal{H}}_\eta(t,\eta) = A^5. \tag{4.13}$$

Moreover, we can show that

$$\begin{aligned} 4\|\tilde{\mathcal{P}}(t,\,\cdot\,)\|_{L^\infty} &= 4\|A^2\mathcal{P}(t,\,\cdot\,)\|_{L^\infty} \leqslant A^4 \\ 2\|\tilde{\mathcal{U}}^2(t,\,\cdot\,)\|_{L^\infty} &= 2\|A^2\mathcal{U}^2(t,\,\cdot\,)\|_{L^\infty} \leqslant A^4. \end{aligned} \tag{4.14}$$

We now collect all the estimates we will need on our solutions in the new variables with the explicit dependence on A. This will be repeatedly used in the next section.

$$0 \leqslant 4\tilde{\mathcal{P}} \leqslant A^4, \quad \text{[from (4.14) and (3.13)]}, \tag{4.15a}$$

$$\sqrt{2}|\tilde{\mathcal{U}}| \leqslant A^2, \quad \text{[from (4.14) and (3.10)]}, \tag{4.15b}$$

$$|\tilde{\mathcal{Q}}| \leqslant A\tilde{\mathcal{P}}, \quad \text{[from common sense]}, \tag{4.15c}$$

$$\tilde{\mathcal{U}}^2 \leqslant 2\tilde{\mathcal{P}}, \quad \text{[from (3.30)]}, \tag{4.15d}$$

$$2\tilde{\mathcal{P}}\tilde{\mathcal{Y}}_\eta \leqslant A^5, \quad \text{[from (4.11)]}, \tag{4.15e}$$

$$2|\tilde{\mathcal{U}}\tilde{\mathcal{U}}_\eta| \leqslant A^4, \quad \text{[from (4.11)]}, \tag{4.15f}$$

$$0 \leqslant \tilde{\mathcal{U}}^2\tilde{\mathcal{Y}}_\eta \leqslant A^5, \quad \text{[from (4.11)]}, \tag{4.15g}$$

$$\tilde{\mathcal{U}}_\eta^2 \leqslant A^3\tilde{\mathcal{Y}}_\eta, \quad \text{[from (4.12) and (3.12a)]}, \tag{4.15h}$$

$$\sqrt{2}|\tilde{\mathcal{U}}|\tilde{\mathcal{P}}^{1/2}\tilde{\mathcal{Y}}_\eta \leqslant A^5, \quad \text{[from (4.15e) and (4.15g)]}, \tag{4.15i}$$

$$|\tilde{\mathcal{R}}| \leqslant \mathcal{O}(1)A^3\tilde{\mathcal{P}}, \quad \text{[from (3.37)]}, \tag{4.15j}$$

$$0 \leqslant \tilde{\mathcal{H}}_\eta \leqslant A^5, \quad \text{[from (4.11)]}, \tag{4.15k}$$

$$0 \leqslant \tilde{\mathcal{Y}}_\eta, \quad \text{[from the definition]}, \tag{4.15l}$$

$$A^2\tilde{\mathcal{U}}_\eta^2 \leqslant \tilde{\mathcal{H}}_\eta\tilde{\mathcal{Y}}_\eta, \quad \text{[from (4.12) and (3.12a)]}, \tag{4.15m}$$

$$0 \leqslant \tilde{\mathcal{D}} \leqslant 2A\tilde{\mathcal{P}}, \quad \text{[from (4.8)]}, \tag{4.15n}$$

$$0 \leqslant \tilde{\mathcal{E}} \leqslant 2A\tilde{\mathcal{P}}, \quad \text{[from (4.8)]}, \tag{4.15o}$$

$$2\sqrt{2}\tilde{\mathcal{P}}\tilde{\mathcal{U}}_\eta \leqslant A^6, \quad \text{[from (4.15a), (4.15e), and (4.15h)]}, \tag{4.15p}$$

$$2\tilde{\mathcal{P}}\tilde{\mathcal{U}}_\eta^2 \leqslant A^8, \quad \text{[from (4.15e) and (4.15h)]}, \tag{4.15q}$$

$$4\tilde{\mathcal{P}}\tilde{\mathcal{U}}_\eta^2 \leqslant A^7\tilde{\mathcal{Y}}_\eta, \quad \text{[from (4.15a) and (4.15h)]}. \tag{4.15r}$$

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In addition, we have several integrals that appear frequently:

$$\int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta))}\tilde{\mathcal{U}}^2(t,\theta)\,d\theta \leqslant 6\tilde{\mathcal{P}}(t,\eta), \quad \text{[from (4.16g)]}, \tag{4.16a}$$

$$\int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta))}\tilde{\mathcal{P}}^2\tilde{\mathcal{Y}}_\eta(t,\theta)\,d\theta \leqslant \frac{3}{2}A^5\tilde{\mathcal{P}}(t,\eta), \quad \text{[from (3.36) and (4.15a)]}, \tag{4.16b}$$

$$\int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta))}\tilde{\mathcal{U}}^2\tilde{\mathcal{Y}}_\eta(t,\theta)\,d\theta \leqslant 4A\tilde{\mathcal{P}}(t,\eta), \quad \text{[from (3.38)]}, \tag{4.16c}$$

$$\int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta))}\tilde{\mathcal{H}}_\eta(t,\theta)\,d\theta \leqslant 4A\tilde{\mathcal{P}}(t,\eta), \quad \text{[from (3.38)]}, \tag{4.16d}$$

$$\int_0^\eta e^{-\frac{3}{2A}(\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta))}\tilde{\mathcal{P}}\tilde{\mathcal{Y}}_\eta(t,\theta)\,d\theta \leqslant 2A\tilde{\mathcal{P}}(t,\eta), \quad \text{[from Lemma A.10]}, \tag{4.16e}$$

$$\int_0^\eta e^{-\frac{3}{2A}(\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta))}\tilde{\mathcal{H}}_\eta(t,\theta)\,d\theta \leqslant 4A\tilde{\mathcal{P}}(t,\eta), \quad \text{[from Lemma A.10]}, \tag{4.16f}$$

$$\int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta))}\tilde{\mathcal{U}}^2(t,\theta)\,d\theta \leqslant 6\tilde{\mathcal{P}}(t,\eta), \quad \text{[from Lemma A.10]}, \tag{4.16g}$$

$$\int_0^\eta e^{-\frac{5}{4A}(\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta))}\tilde{\mathcal{P}}\tilde{\mathcal{Y}}_\eta(t,\theta)\,d\theta \leqslant 4A\tilde{\mathcal{P}}(t,\eta), \quad \text{[from Lemma A.10]}, \tag{4.16h}$$

$$\int_0^\eta e^{-\frac{3}{2A}(\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta))}\tilde{\mathcal{P}}(t,\theta)\,d\theta \leqslant 7\tilde{\mathcal{P}}(t,\eta), \quad \text{[from Lemma A.10]}, \tag{4.16i}$$

$$\int_0^\eta e^{-\frac{1}{2A}(\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta))}\tilde{\mathcal{P}}^2\tilde{\mathcal{Y}}_\eta(t,\theta)\,d\theta \leqslant A\mathcal{O}(1)\tilde{\mathcal{P}}^{1/2}(t,\eta), \quad \text{[from Lemma A.10]}, \tag{4.16j}$$

$$\int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta))}\tilde{\mathcal{P}}^{1+\beta}\tilde{\mathcal{Y}}_\eta(t,\theta)\,d\theta \leqslant 3\frac{1+\beta}{\beta}\frac{A^{1+4\beta}}{4^\beta}\tilde{\mathcal{P}}(t,\eta), \quad \beta > 0, \quad \text{[from Lemma A.10]}. \tag{4.16k}$$

As for derivatives of the quantity $\tilde{\mathcal{P}}^{1/2}$ we have

$$|(\tilde{\mathcal{P}}^{1/2})_\eta| \leqslant \frac{1}{2A}\tilde{\mathcal{P}}^{1/2}\tilde{\mathcal{Y}}_\eta, \tag{4.17a}$$

$$|\tilde{\mathcal{U}}(\tilde{\mathcal{P}}^{1/2})_\eta| \leqslant \tfrac{3}{8}A^4, \tag{4.17b}$$

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$$((\tilde{\mathcal{P}}^{1/2})_\eta)^2 \leqslant \tfrac{1}{8} A^3 \tilde{\mathcal{Y}}_\eta, \tag{4.17c}$$
$$((\tilde{\mathcal{P}}^{1/2})_\eta)^2 \leqslant \tfrac{1}{16} A^2 \tilde{\mathcal{Y}}_\eta^2. \tag{4.17d}$$

5. Lipschitz estimates in the new metric

We will repeatedly use the following elementary identity and estimate.

LEMMA 5.1. *Let $a_j, b_j \in \mathbb{R}$ for $j = 1, 2$. Then we have*

$$a_2 b_2 - a_1 b_1 = (b_2 - b_1)(a_2 \mathbb{1}_{b_1 < b_2} + a_1 \mathbb{1}_{b_1 \geqslant b_2}) + \min(b_1, b_2)(a_2 - a_1).$$

Here

$$\mathbb{1}_{\mathcal{K}} = \begin{cases} 1, & \mathcal{K} \text{ is true}, \\ 0, & \mathcal{K} \text{ is false}. \end{cases}$$

We also use $\mathbb{1}_{\mathcal{K}}$ to denote the characteristic (indicator) function of a set $\mathcal{K}$. Furthermore, we have the estimate

$$|\min(a_1, b_1) - \min(a_2, b_2)| \leqslant \max(|a_1 - a_2|, |b_1 - b_2|).$$

LEMMA 5.2. *Let $f, g: \mathbb{R} \to \mathbb{R}$ be two Lipschitz continuous functions with Lipschitz constants c_f and c_g, respectively. Then the function $h: \mathbb{R} \to \mathbb{R}$ defined by $h = \min(f, g)$ is a Lipschitz continuous function with Lipschitz constant bounded by* $\max(c_f, c_g)$.

Proof. Use the previous lemma. □

Introduce for any function Φ the functions

$$\Phi^- = \min(0, \Phi), \quad \Phi^+ = \max(0, \Phi). \tag{5.1}$$

We then have

$$\Phi^- \leqslant 0 \leqslant \Phi^+, \quad \Phi^- \Phi^+ = 0, \quad \Phi = \Phi^+ + \Phi^-, \quad |\Phi| = \Phi^+ - \Phi^-. \tag{5.2}$$

For two functions Φ, Ψ we have

$$|\Phi^\pm - \Psi^\pm| \leqslant |\Phi - \Psi|. \tag{5.3}$$

Frequently, we will have to estimate quantities like

$$\int_0^1 e^{-\frac{1}{A}|\tilde{\mathcal{Y}}(t,\eta) - \tilde{\mathcal{Y}}(t,\theta)|} (\cdots)(t, \theta)\, d\theta.$$

We will rewrite it as follows:

$$\begin{aligned}\int_0^1 & e^{-\frac{1}{A}|\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta)|}(\cdots)(t,\theta)\,d\theta\\ &= \left(\int_0^\eta + \int_\eta^1\right) e^{-\frac{1}{A}|\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta)|}(\cdots)(t,\theta)\,d\theta\\ &= \int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta))}(\cdots)(t,\theta)\,d\theta\\ &\quad + \int_\eta^1 e^{\frac{1}{A}(\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta))}(\cdots)(t,\theta)\,d\theta. \end{aligned} \tag{5.4}$$

Both integrals can be estimated in the same manner, using that the argument in the exponential is negative in both cases. Eliminating the absolute value will allow us to perform integration by parts.

5.1. Lipschitz estimates for $\tilde{\mathcal{Y}}$. From the system of differential equations, we have

$$\tilde{\mathcal{Y}}_{i,t} + \left(\frac{2}{3}\frac{1}{A_i^5}\tilde{\mathcal{U}}_i^3 + \frac{1}{A_i^6}\tilde{\mathcal{S}}_i\right)\tilde{\mathcal{Y}}_{i,\eta} = \tilde{\mathcal{U}}_i, \tag{5.5}$$

where

$$\begin{aligned}\tilde{\mathcal{P}}_i(t,\eta) &= \frac{1}{4A_i}\int_0^1 e^{-\frac{1}{A_i}|\tilde{\mathcal{Y}}_i(t,\eta)-\tilde{\mathcal{Y}}_i(t,\theta)|}(2(\tilde{\mathcal{U}}_i^2-\tilde{\mathcal{P}}_i)\tilde{\mathcal{Y}}_{i,\eta}(t,\theta)+A_i^5)\,d\theta,\\ \tilde{\mathcal{Q}}_i(t,\eta) &= -\frac{1}{4}\int_0^1 \operatorname{sign}(\eta-\theta)e^{-\frac{1}{A_i}|\tilde{\mathcal{Y}}_i(t,\eta)-\tilde{\mathcal{Y}}_i(t,\theta)|}(2(\tilde{\mathcal{U}}_i^2-\tilde{\mathcal{P}}_i)\tilde{\mathcal{Y}}_{i,\eta}(t,\theta)+A_i^5)\,d\theta,\\ \tilde{\mathcal{S}}_i(t,\eta) &= \int_0^1 e^{-\frac{1}{A_i}|\tilde{\mathcal{Y}}_i(t,\eta)-\tilde{\mathcal{Y}}_i(t,\theta)|}\left(\frac{2}{3}\tilde{\mathcal{U}}_i^3\tilde{\mathcal{Y}}_{i,\eta}-\tilde{\mathcal{Q}}_i\tilde{\mathcal{U}}_{i,\eta}-2\tilde{\mathcal{P}}_i\tilde{\mathcal{U}}_i\tilde{\mathcal{Y}}_{i,\eta}\right)(t,\theta)\,d\theta.\end{aligned}$$

Thus we have

$$\begin{aligned}\frac{d}{dt}&\int_0^1(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,\eta)\,d\eta\\ &= 2\int_0^1(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)(\tilde{\mathcal{Y}}_{1,t}-\tilde{\mathcal{Y}}_{2,t})(t,\eta)\,d\eta\\ &= 2\int_0^1(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)(t,\eta)\,d\eta\\ &\quad+\frac{4}{3}\int_0^1(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)\left(\frac{1}{A_2^5}\tilde{\mathcal{U}}_2^3\tilde{\mathcal{Y}}_{2,\eta}-\frac{1}{A_1^5}\tilde{\mathcal{U}}_1^3\tilde{\mathcal{Y}}_{1,\eta}\right)(t,\eta)\,d\eta\end{aligned}$$

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$$
\begin{aligned}
&+2\int_0^1 (\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)\Big(\frac{1}{A_2^6}\tilde{\mathcal{S}}_2\tilde{\mathcal{Y}}_{2,\eta}-\frac{1}{A_1^6}\tilde{\mathcal{S}}_1\tilde{\mathcal{Y}}_{1,\eta}\Big)(t,\eta)\,d\eta\\
&= I_1+I_2+I_3.
\end{aligned}\tag{5.6}
$$

The strategy is to use integration by parts for the last two integrals I_2 and I_3, while we want to use straightforward estimates for I_1, which will finally yield that

$$
\begin{aligned}
&\frac{d}{dt}\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2\\
&\quad\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2+\|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2\|^2+\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+|A_1-A_2|^2),
\end{aligned}
$$

where $\mathcal{O}(1)$ denotes some constant which depends on $A=\max_j(A_j)$ and which remains bounded as $A\to 0$.

The first integral I_1: Note that

$$
\begin{aligned}
|I_1| &= \Big|2\int_0^1 (\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)(t,\eta)\,d\eta\Big|\\
&\leqslant \int_0^1 ((\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2+(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)^2)(t,\eta)\,d\eta\\
&= \|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2+\|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2\|^2.
\end{aligned}
$$

The second integral I_2: Note that

$$
\begin{aligned}
\frac{3}{4}|I_2| &= \Big|\int_0^1 (\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)\Big(\frac{1}{A_2^5}\tilde{\mathcal{U}}_2^3\tilde{\mathcal{Y}}_{2,\eta}-\frac{1}{A_1^5}\tilde{\mathcal{U}}_1^3\tilde{\mathcal{Y}}_{1,\eta}\Big)(t,\eta)\,d\eta\Big|\\
&\leqslant \frac{1}{(\max_j(A_j))^5}\Big|\int_0^1 (\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)(\tilde{\mathcal{U}}_2^3\tilde{\mathcal{Y}}_{2,\eta}-\tilde{\mathcal{U}}_1^3\tilde{\mathcal{Y}}_{1,\eta})(t,\eta)\,d\eta\Big|\\
&\quad+\frac{|A_1^5-A_2^5|}{A_1^5A_2^5}\Big|\int_0^1 (\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)(\tilde{\mathcal{U}}_1^3\tilde{\mathcal{Y}}_{1,\eta}\mathbb{1}_{A_1\leqslant A_2}+\tilde{\mathcal{U}}_2^3\tilde{\mathcal{Y}}_{2,\eta}\mathbb{1}_{A_2<A_1})(t,\eta)\,d\eta\Big|\\
&\leqslant \frac{1}{A^5}\Big|\int_0^1 (\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)(\tilde{\mathcal{U}}_2-\tilde{\mathcal{U}}_1)\tilde{\mathcal{U}}_2^2\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\,d\eta\Big|\\
&\quad+\frac{1}{A^5}\Big|\int_0^1 (\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)[\tilde{\mathcal{U}}_1\tilde{\mathcal{Y}}_{1,\eta}\mathbb{1}_{\tilde{\mathcal{U}}_2^2\leqslant\tilde{\mathcal{U}}_1^2}+\tilde{\mathcal{U}}_1\tilde{\mathcal{Y}}_{2,\eta}\mathbb{1}_{\tilde{\mathcal{U}}_1^2<\tilde{\mathcal{U}}_2^2}]\\
&\quad\times(\tilde{\mathcal{U}}_2+\tilde{\mathcal{U}}_1)(\tilde{\mathcal{U}}_2-\tilde{\mathcal{U}}_1)(t,\eta)\,d\eta\Big|\\
&\quad+\frac{1}{A^5}\Big|\int_0^1 (\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)\tilde{\mathcal{U}}_1\min_j(\tilde{\mathcal{U}}_j^2)(\tilde{\mathcal{Y}}_{2,\eta}-\tilde{\mathcal{Y}}_{1,\eta})(t,\eta)\,d\eta\Big|
\end{aligned}
$$

$$+\frac{|A_1^5 - A_2^5|}{A_1^5 A_2^5}\left|\int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(\tilde{\mathcal{U}}_1^3\tilde{\mathcal{Y}}_{1,\eta}\mathbb{1}_{A_1\leqslant A_2} + \tilde{\mathcal{U}}_2^3\tilde{\mathcal{Y}}_{2,\eta}\mathbb{1}_{A_2<A_1})(t,\eta)\,d\eta\right|$$
$$= J_1 + J_2 + J_3 + J_4.$$

We will write

$$A = \max_j(A_j), \quad a = \min_j(A_j). \tag{5.7}$$

We commence with J_1: Since $\tilde{\mathcal{U}}_i^2\tilde{\mathcal{Y}}_{i,\eta}(t,\eta) \leqslant A_i^5 \leqslant A^5$ for all t and η, we have

$$J_1 = \frac{1}{A^5}\left|\int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1)\tilde{\mathcal{U}}_2^2\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\,d\eta\right|$$
$$\leqslant \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2.$$

Next term is J_2: Using once more that $\tilde{\mathcal{U}}_i^2\tilde{\mathcal{Y}}_{i,\eta}(t,\eta) \leqslant A^5$ for all t and η, we get

$$J_2 = \frac{1}{A^5}\left|\int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)[\tilde{\mathcal{U}}_1\tilde{\mathcal{Y}}_{1,\eta}\mathbb{1}_{\tilde{\mathcal{U}}_2^2\leqslant\tilde{\mathcal{U}}_1^2} + \tilde{\mathcal{U}}_1\tilde{\mathcal{Y}}_{2,\eta}\mathbb{1}_{\tilde{\mathcal{U}}_1^2<\tilde{\mathcal{U}}_2^2}]\right.$$
$$\left.\times\ (\tilde{\mathcal{U}}_2 + \tilde{\mathcal{U}}_1)(\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1)(t,\eta)\,d\eta\right|$$
$$\leqslant \frac{1}{A^5}\int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2||\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2|(2\tilde{\mathcal{U}}_1^2\tilde{\mathcal{Y}}_{1,\eta}\mathbb{1}_{\tilde{\mathcal{U}}_2^2\leqslant\tilde{\mathcal{U}}_1^2} + 2\tilde{\mathcal{U}}_2^2\tilde{\mathcal{Y}}_{2,\eta}\mathbb{1}_{\tilde{\mathcal{U}}_1^2<\tilde{\mathcal{U}}_2^2})(t,\eta)\,d\eta$$
$$\leqslant 2(\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2).$$

Next, it is J_3: Here integration by parts plays an essential role. Thus we have to determine first whether or not the function $\eta \mapsto \tilde{\mathcal{U}}_1 \min_j(\tilde{\mathcal{U}}_j^2)(t,\eta)$ is differentiable, and, if so, if its derivative is bounded. Recall from Lemma A.2 (ii) that the function $\eta \mapsto \min_j(\tilde{\mathcal{U}}_j^2)(t,\eta)$ is Lipschitz continuous with Lipschitz constant at most A^4. Thus

$$\left|\frac{d}{d\eta}\left(\tilde{\mathcal{U}}_1(t,\eta)\min_j(\tilde{\mathcal{U}}_j^2(t,\eta))\right)\right| \leqslant \frac{3}{2}A^4\|\tilde{\mathcal{U}}_1\|_{L^\infty} \leqslant \mathcal{O}(1)A^6, \tag{5.8}$$

and integration by parts together with (3.40) yields

$$J_3 = \frac{1}{A^5}\left|\int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)\tilde{\mathcal{U}}_1\min_j(\tilde{\mathcal{U}}_j^2)(\tilde{\mathcal{Y}}_{2,\eta} - \tilde{\mathcal{Y}}_{1,\eta})(t,\eta)\,d\eta\right|$$
$$= \left|-\frac{1}{2A^5}(\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2\tilde{\mathcal{U}}_1\min_j(\tilde{\mathcal{U}}_j^2)(t,\eta)\right|_{\eta=0}^1$$

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$$+\frac{1}{2A^5}\int_0^1(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2\frac{d}{d\eta}(\tilde{\mathcal{U}}_1\min_j(\tilde{\mathcal{U}}_j^2))(t,\eta)\,d\eta\bigg|$$
$$=\frac{1}{2A^5}\bigg|\int_0^1(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2\frac{d}{d\eta}(\tilde{\mathcal{U}}_1\min_j(\tilde{\mathcal{U}}_j^2))(t,\eta)\,d\eta\bigg|$$
$$\leqslant\frac{3}{4A}\|\tilde{\mathcal{U}}_1(t,\cdot)\|_{L^\infty}\int_0^1(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,\eta)\,d\eta$$
$$\leqslant\mathcal{O}(1)\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2.$$

Finally, we consider J_4: Direct calculations yield

$$J_4\leqslant\frac{|A_1^5-A_2^5|}{A_1^5A_2^5}\bigg(\mathbb{1}_{A_1\leqslant A_2}\bigg|\int_0^1(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)\tilde{\mathcal{U}}_1^3\tilde{\mathcal{Y}}_{1,\eta}(t,\eta)\,d\eta\bigg|$$
$$+\mathbb{1}_{A_2<A_1}\bigg|\int_0^1(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)\tilde{\mathcal{U}}_2^3\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\,d\eta\bigg|\bigg)$$
$$\leqslant\frac{|A_1^5-A_2^5|}{A_1^5A_2^5}(\mathbb{1}_{A_1\leqslant A_2}A_1^7+\mathbb{1}_{A_2<A_1}A_2^7)\int_0^1|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|(t,\eta)\,d\eta$$
$$\leqslant\frac{|A_1^5-A_2^5|}{A^3}\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|$$
$$\leqslant 5A|A_1-A_2|\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|$$
$$\leqslant\mathcal{O}(1)(\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2+|A_1-A_2|^2),$$

where we again used that $\tilde{\mathcal{U}}_i^2\tilde{\mathcal{Y}}_{i,\eta}\leqslant A_i^5$ and $\|\tilde{\mathcal{U}}_i\|_{L^\infty}\leqslant A_i^2$.

The third integral I_3: We will consider several smaller parts of I_3 by inserting the definition of $\tilde{\mathcal{S}}_i$, and combine them in the end. We write

$$I_3=\frac{4}{3}I_{31}-4I_{32}-2I_{33},\tag{5.9}$$

where

$$I_{31}=\int_0^1(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)(t,\eta)$$
$$\times\bigg(\frac{1}{A_2^6}\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\int_0^1 e^{-\frac{1}{A_2}|\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta)|}\tilde{\mathcal{U}}_2^3\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta$$
$$-\frac{1}{A_1^6}\tilde{\mathcal{Y}}_{1,\eta}(t,\eta)\int_0^1 e^{-\frac{1}{A_1}|\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta)|}\tilde{\mathcal{U}}_1^3\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\bigg)\,d\eta,$$

$$
\begin{aligned}
I_{32} = \int_0^1 & (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t,\eta) \\
& \times \Bigg(\frac{1}{A_2^6} \tilde{\mathcal{Y}}_{2,\eta}(t,\eta) \int_0^1 e^{-\frac{1}{A_2}|\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta)|} \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\, d\theta \\
& - \frac{1}{A_1^6} \tilde{\mathcal{Y}}_{1,\eta}(t,\eta) \int_0^1 e^{-\frac{1}{A_1}|\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta)|} \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1 \tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\, d\theta \Bigg) d\eta, \\
I_{33} = \int_0^1 & (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t,\eta) \\
& \times \Bigg(\frac{1}{A_2^6} \tilde{\mathcal{Y}}_{2,\eta}(t,\eta) \int_0^1 e^{-\frac{1}{A_2}|\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta)|} \tilde{\mathcal{Q}}_2 \tilde{\mathcal{U}}_{2,\eta}(t,\theta)\, d\theta \\
& - \frac{1}{A_1^6} \tilde{\mathcal{Y}}_{1,\eta}(t,\eta) \int_0^1 e^{-\frac{1}{A_1}|\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta)|} \tilde{\mathcal{Q}}_1 \tilde{\mathcal{U}}_{1,\eta}(t,\theta)\, d\theta \Bigg) d\eta.
\end{aligned}
$$

Recall definitions (5.1). Then we have

$$
\begin{aligned}
I_{31} = \int_0^1 & (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t,\eta) \\
& \times \Bigg(\frac{1}{A_2^6} \tilde{\mathcal{Y}}_{2,\eta}(t,\eta) \int_0^1 e^{-\frac{1}{A_2}|\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta)|} \tilde{\mathcal{U}}_2^3 \tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\, d\theta \\
& - \frac{1}{A_1^6} \tilde{\mathcal{Y}}_{1,\eta}(t,\eta) \int_0^1 e^{-\frac{1}{A_1}|\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta)|} \tilde{\mathcal{U}}_1^3 \tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\, d\theta \Bigg) d\eta \\
= \int_0^1 & (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t,\eta) \\
& \times \Bigg(\frac{1}{A_2^6} \tilde{\mathcal{Y}}_{2,\eta}(t,\eta) \int_0^1 e^{-\frac{1}{A_2}|\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta)|} (\tilde{\mathcal{U}}_2^-)^3 \tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\, d\theta \\
& - \frac{1}{A_1^6} \tilde{\mathcal{Y}}_{1,\eta}(t,\eta) \int_0^1 e^{-\frac{1}{A_1}|\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta)|} (\tilde{\mathcal{U}}_1^-)^3 \tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\, d\theta \Bigg) d\eta \\
& + \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t,\eta) \\
& \times \Bigg(\frac{1}{A_2^6} \tilde{\mathcal{Y}}_{2,\eta}(t,\eta) \int_0^1 e^{-\frac{1}{A_2}|\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta)|} (\tilde{\mathcal{U}}_2^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\, d\theta \\
& - \frac{1}{A_1^6} \tilde{\mathcal{Y}}_{1,\eta}(t,\eta) \int_0^1 e^{-\frac{1}{A_1}|\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta)|} (\tilde{\mathcal{U}}_1^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\, d\theta \Bigg) d\eta \\
= \int_0^1 & (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(\tilde{K}^- + \tilde{K}^+)(t,\eta)\, d\eta.
\end{aligned}
$$

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Since both integrals have the same structure, it suffices to consider the second integral. Furthermore, applying the device described in (5.4), it suffices to study the terms, denoted by $K^{\pm}$, with the upper limit of the inner integral replaced by η.

Therefore observe that we can write

$$
\begin{aligned}
&K^+(t,\eta)\\
&= \frac{1}{A_2^6}\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{U}}_2^+)^3\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\\
&\quad -\frac{1}{A_1^6}\tilde{\mathcal{Y}}_{1,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}(\tilde{\mathcal{U}}_1^+)^3\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\\
&= \frac{1}{(\max_j(A_j))^6}\Big(\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{U}}_2^+)^3\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\\
&\quad -\tilde{\mathcal{Y}}_{1,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}(\tilde{\mathcal{U}}_1^+)^3\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\Big)\\
&\quad +\frac{A_1^6-A_2^6}{A_1^6A_2^6}\mathbb{1}_{A_1\leqslant A_2}\tilde{\mathcal{Y}}_{1,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}(\tilde{\mathcal{U}}_1^+)^3\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\\
&\quad +\frac{A_1^6-A_2^6}{A_1^6A_2^6}\mathbb{1}_{A_2<A_1}\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{U}}_2^+)^3\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\\
&= \frac{1}{A^6}\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\\
&\quad\times\Big(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}((\tilde{\mathcal{U}}_2^+)^3-(\tilde{\mathcal{U}}_1^+)^3)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\mathbb{1}_{\tilde{\mathcal{U}}_1^+\leqslant\tilde{\mathcal{U}}_2^+}(t,\theta)\,d\theta\Big)\\
&\quad +\frac{1}{A^6}\tilde{\mathcal{Y}}_{1,\eta}(t,\eta)\\
&\quad\times\Big(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}((\tilde{\mathcal{U}}_2^+)^3-(\tilde{\mathcal{U}}_1^+)^3)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\mathbb{1}_{\tilde{\mathcal{U}}_2^+<\tilde{\mathcal{U}}_1^+}(t,\theta)\,d\theta\Big)\\
&\quad +\frac{1}{A^6}\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\Big(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\min_j(\tilde{\mathcal{U}}_j^+)^3\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\Big)\\
&\quad -\frac{1}{A^6}\tilde{\mathcal{Y}}_{1,\eta}(t,\eta)\Big(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\min_j(\tilde{\mathcal{U}}_j^+)^3\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\Big)\\
&\quad +\frac{A_1^6-A_2^6}{A_1^6A_2^6}\mathbb{1}_{A_1\leqslant A_2}\tilde{\mathcal{Y}}_{1,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}(\tilde{\mathcal{U}}_1^+)^3\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\\
&\quad +\frac{A_1^6-A_2^6}{A_1^6A_2^6}\mathbb{1}_{A_2<A_1}\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{U}}_2^+)^3\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\\
&= \frac{1}{A^6}\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\Big(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}
\end{aligned}
$$

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$$
\begin{aligned}
&\times ((\tilde{\mathcal{U}}_2^+)^3 - (\tilde{\mathcal{U}}_1^+)^3)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\mathbb{1}_{\tilde{\mathcal{U}}_1^+ \leqslant \tilde{\mathcal{U}}_2^+}(t,\theta)\,d\theta\Big) \\
&+ \frac{1}{A^6}\tilde{\mathcal{Y}}_{1,\eta}(t,\eta)\Big(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))} \\
&\times ((\tilde{\mathcal{U}}_2^+)^3 - (\tilde{\mathcal{U}}_1^+)^3)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\mathbb{1}_{\tilde{\mathcal{U}}_2^+ < \tilde{\mathcal{U}}_1^+}(t,\theta)\,d\theta\Big) \\
&+ \frac{1}{A^6}\mathbb{1}_{A_1 \leqslant A_2}\tilde{\mathcal{Y}}_{2,\eta}(t,\eta) \\
&\times \Big(\int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))})\min_j(\tilde{\mathcal{U}}_j^+)^3\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\Big) \\
&+ \frac{1}{A^6}\mathbb{1}_{A_2 < A_1}\tilde{\mathcal{Y}}_{1,\eta}(t,\eta) \\
&\times \Big(\int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))})\min_j(\tilde{\mathcal{U}}_j^+)^3\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\Big) \\
&+ \frac{1}{A^6}\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\Big(\int_0^\eta (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}) \\
&\times \min_j(\tilde{\mathcal{U}}_j^+)^3\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\mathbb{1}_{B(\eta)}(t,\theta)\,d\theta\Big) \\
&- \frac{1}{A^6}\tilde{\mathcal{Y}}_{1,\eta}(t,\eta)\Big(\int_0^\eta (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}) \\
&\times \min_j(\tilde{\mathcal{U}}_j^+)^3\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\mathbb{1}_{B^c(\eta)}(t,\theta)\,d\theta\Big) \\
+ &\frac{1}{A^6}\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\Big(\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}) \\
&\times \min_j(\tilde{\mathcal{U}}_j^+)^3\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\Big) \\
&- \frac{1}{A^6}\tilde{\mathcal{Y}}_{1,\eta}(t,\eta)\Big(\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}) \\
&\times \min_j(\tilde{\mathcal{U}}_j^+)^3\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\Big) \\
&+ \frac{A_1^6 - A_2^6}{A_1^6A_2^6}\mathbb{1}_{A_1 \leqslant A_2}\tilde{\mathcal{Y}}_{1,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}(\tilde{\mathcal{U}}_1^+)^3\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta \\
&+ \frac{A_1^6 - A_2^6}{A_1^6A_2^6}\mathbb{1}_{A_2 < A_1}\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{U}}_2^+)^3\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta \\
= &(J_1 + J_2 + J_3 + J_4 + J_5 + J_6 + J_7 + J_8 + J_9 + J_{10})(t,\eta). \qquad (5.10)
\end{aligned}
$$

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Here

$$B(\eta) = \{(t,\theta) \mid e^{\tilde{\mathcal{Y}}_1(t,\theta)-\tilde{\mathcal{Y}}_1(t,\eta)} \leqslant e^{\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,\eta)}\} \tag{5.11}$$

which means especially that $B(\eta)$ depends heavily on η! Observe that the set $B(\eta)$ is scale invariant in the sense that $(\tilde{\mathcal{Y}}_1, \tilde{\mathcal{Y}}_2)$ and $(c\tilde{\mathcal{Y}}_1, c\tilde{\mathcal{Y}}_2)$ define the same set $B(\eta)$ for any constant c.

As far as the first two terms J_1 and J_2 are concerned, they have the same structure, and hence we only consider the term J_1, that is,

$$\begin{aligned}
&\int_0^1 J_1(\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t,\eta)\,d\eta \\
&\quad = \frac{1}{A^6}\int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)\tilde{\mathcal{Y}}_{2,\eta}(t,\eta) \\
&\qquad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}((\tilde{\mathcal{U}}_2^+)^3 - (\tilde{\mathcal{U}}_1^+)^3)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\mathbb{1}_{\tilde{\mathcal{U}}_1^+\leqslant\tilde{\mathcal{U}}_2^+}(t,\theta)\,d\theta\right)d\eta.
\end{aligned}$$

The main ingredients are the following observations:

$$|(\tilde{\mathcal{U}}_1^+)^3(t,\eta) - (\tilde{\mathcal{U}}_2^+)^3(t,\eta)|\mathbb{1}_{\tilde{\mathcal{U}}_1^+\leqslant\tilde{\mathcal{U}}_2^+} \leqslant 3\tilde{\mathcal{U}}_2^2(t,\eta)|\tilde{\mathcal{U}}_1(t,\eta) - \tilde{\mathcal{U}}_2(t,\eta)|,$$

and (4.16c).

Thus we have

$$\begin{aligned}
&\left|\int_0^1 J_1(\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t,\eta)\,d\eta\right| \\
&\quad = \frac{1}{A^6}\left|\int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\right. \\
&\qquad \left.\times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}((\tilde{\mathcal{U}}_2^+)^3 - (\tilde{\mathcal{U}}_1^+)^3)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\mathbb{1}_{\tilde{\mathcal{U}}_1^+\leqslant\tilde{\mathcal{U}}_2^+}(t,\theta)\,d\theta\right)d\eta\right| \\
&\quad \leqslant \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{1}{A^{12}}\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta) \\
&\qquad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}((\tilde{\mathcal{U}}_2^+)^3 - (\tilde{\mathcal{U}}_1^+)^3)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\mathbb{1}_{\tilde{\mathcal{U}}_1^+\leqslant\tilde{\mathcal{U}}_2^+}(t,\theta)\,d\theta\right)^2 d\eta \\
&\quad \leqslant \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
&\qquad + \frac{9}{A^{12}}\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right) \\
&\qquad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^2\tilde{\mathcal{Y}}_{2,\eta}(\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2(t,\theta)\,d\theta\right)d\eta \\
&\quad \leqslant \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2
\end{aligned}$$

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$$
\begin{aligned}
&+\frac{36}{A^{11}}\int_0^1 \tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^2\tilde{\mathcal{Y}}_{2,\eta}(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)^2(t,\theta)\,d\theta\bigg)\,d\eta\\
&\leqslant \|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2\\
&\quad+\frac{18}{A^6}\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t,\eta)e^{-\frac{1}{A_2}\tilde{\mathcal{Y}}_2(t,\eta)}\int_0^\eta e^{\frac{1}{A_2}\tilde{\mathcal{Y}}_2(t,\theta)}\tilde{\mathcal{U}}_2^2\tilde{\mathcal{Y}}_{2,\eta}(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)^2(t,\theta)\,d\theta\,d\eta\\
&=\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2\\
&\quad-\frac{18A_2}{A^6}\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^2\tilde{\mathcal{Y}}_{2,\eta}(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)^2(t,\theta)\,d\theta\,\bigg|_{\eta=0}^1\\
&\quad+\frac{18A_2}{A^6}\int_0^1 \tilde{\mathcal{U}}_2^2\tilde{\mathcal{Y}}_{2,\eta}(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)^2(t,\eta)\,d\eta\\
&\leqslant \|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2+18\|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2\|^2,
\end{aligned}
$$

where in the last step we used (4.15b) and (4.16c), which imply that

$$
\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^2\tilde{\mathcal{Y}}_{2,\eta}(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)^2(t,\theta)\,d\theta\leqslant 8A^5\tilde{\mathcal{P}}_2(t,\eta), \tag{5.12}
$$

and hence the left-hand side of (5.12) tends to 0 as η to 0 or 1 according to (3.40). Thus the second term above vanishes.

As far as the third and fourth terms J_3 and J_4 are concerned, they again have the same structure, and hence we only consider the integral corresponding to J_3, that is,

$$
\begin{aligned}
&\int_0^1 J_3(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)(t,\eta)\,d\eta\\
&=\frac{1}{A^6}\mathbb{1}_{A_1\leqslant A_2}\int_0^1(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\\
&\quad\times\bigg(\int_0^\eta(e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}-e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))})\min_j(\tilde{\mathcal{U}}_j^+)^3\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\bigg)\,d\eta.
\end{aligned}
$$

Recall Lemma A.1(ii). Direct computations yield

$$
\begin{aligned}
&\bigg|\int_0^1 J_3(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)(t,\eta)\,d\eta\bigg|\\
&=\frac{1}{A^6}\mathbb{1}_{A_1\leqslant A_2}\bigg|\int_0^1(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\\
&\quad\times\bigg(\int_0^\eta(e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}-e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))})\min_j(\tilde{\mathcal{U}}_j^+)^3\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\bigg)\,d\eta\bigg|\\
&\leqslant\frac{4|A_1-A_2|}{aA^6e}\mathbb{1}_{A_1\leqslant A_2}\int_0^1|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)
\end{aligned}
$$

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$$
\begin{aligned}
&\times \left(\int_0^\eta e^{-\frac{3}{4A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\, d\theta\right) d\eta \\
&\leqslant \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{16|A_1 - A_2|^2}{a^2 A^{12} e^2} \mathbb{1}_{A_1 \leqslant A_2} \\
&\times \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta) \left(\int_0^\eta e^{-\frac{3}{4A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} |\tilde{\mathcal{U}}_1| \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\, d\theta\right)^2 d\eta \\
&\leqslant \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
&+ \frac{16a^4|A_1 - A_2|^2}{a^2 A^{12} e^2} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\, d\theta\right) \\
&\times \left(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\, d\theta\right) d\eta \\
&\leqslant \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
&+ \frac{64|A_1 - A_2|^2}{A^9 e^2} \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta) \left(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\, d\theta\right) d\eta \\
&\leqslant \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
&+ \frac{32|A_1 - A_2|^2}{A^4 e^2} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t,\eta) e^{-\frac{1}{2A_2}\tilde{\mathcal{Y}}_2(t,\eta)} \int_0^\eta e^{\frac{1}{2A_2}\tilde{\mathcal{Y}}_2(t,\theta)} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\, d\theta\, d\eta \\
&\leqslant \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{64A_2|A_1 - A_2|^2}{A^4 e^2} \\
&\times \left(-\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\, d\theta\Big|_{\eta=0}^1 + \int_0^1 \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\, d\eta\right) \\
&\leqslant \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{128A^2}{e^2}|A_1 - A_2|^2 \\
&\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + |A_1 - A_2|^2).
\end{aligned}
$$

As far as the terms J_5 and J_6 are concerned, they again have the same structure, and hence we only consider the integral corresponding to J_5, that is,

$$
\begin{aligned}
\int_0^1 J_5(\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t,\eta)\, d\eta = \frac{1}{A^6} &\int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t,\eta) \\
&\times \left(\int_0^\eta (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))})\right. \\
&\times \left. \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t,\theta) \mathbb{1}_{B(\eta)}(t,\theta)\, d\theta\right) d\eta.
\end{aligned}
$$

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The main ingredient is the estimate

$$\begin{aligned}
&|e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}| \\
&\quad \leqslant \frac{1}{a} e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(|\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_1(t,\theta)| + |\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_1(t,\eta)|) \\
&\qquad \text{for all } (t,\theta) \in B(\eta),
\end{aligned} \tag{5.13}$$

which follows from Lemma A.1 (i).

Direct computations yield

$$\begin{aligned}
&\left| \int_0^1 J_5(\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t,\eta)\, d\eta \right| \\
&= \frac{1}{A^6} \Bigg| \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)\tilde{\mathcal{Y}}_{2,\eta}(t,\eta) \Bigg(\int_0^\eta (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}) \\
&\quad \times \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t,\theta) \mathbb{1}_{B(\eta)}(t,\theta)\, d\theta \Bigg) d\eta \Bigg| \\
&\leqslant \frac{1}{aA^6} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \tilde{\mathcal{Y}}_{2,\eta}(t,\eta) \\
&\quad \times \Bigg(\int_0^\eta (|\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_1(t,\eta)| + |\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_1(t,\theta)|) \\
&\quad \times e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\, d\theta \Bigg) d\eta \\
&\leqslant \frac{1}{aA^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \tilde{\mathcal{Y}}_{2,\eta}(t,\eta) \\
&\quad \times \Bigg(\int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\, d\theta \Bigg) d\eta \\
&\quad + \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{1}{a^2A^{12}} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta) \\
&\quad \times \Bigg(\int_0^\eta |\tilde{\mathcal{Y}}_1(t,\theta) - \tilde{\mathcal{Y}}_2(t,\theta)| e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\, d\theta \Bigg)^2 d\eta \\
&\leqslant \frac{4}{A^4} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\, d\eta + \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
&\quad + \frac{1}{a^2A^{12}} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta) \Bigg(\int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\, d\theta \Bigg) \\
&\quad \times \Bigg(\int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t,\theta) e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} \min_j (\tilde{\mathcal{U}}_j^4) \tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\, d\theta \Bigg) d\eta
\end{aligned}$$

$$
\begin{aligned}
&\leqslant (2A+1)\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2+\frac{4}{a^2A^{11}}\int_0^1 \tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\\
&\quad\times\left(\int_0^\eta(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,\theta)e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\min_j(\tilde{\mathcal{U}}_j^4)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)d\eta\\
&\leqslant \mathcal{O}(1)\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2+\frac{2}{a^2A^6}\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\\
&\quad\times\left(\int_0^\eta(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,\theta)e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\min_j(\tilde{\mathcal{U}}_j^4)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)d\eta\\
&= \mathcal{O}(1)\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2+\frac{2}{a^2A^6}\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t,\eta)e^{-\frac{1}{a}\tilde{\mathcal{Y}}_2(t,\eta)}\\
&\quad\times\int_0^\eta(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,\theta)e^{\frac{1}{a}\tilde{\mathcal{Y}}_2(t,\theta)}\min_j(\tilde{\mathcal{U}}_j^4)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\,d\eta\\
&= \mathcal{O}(1)\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2\\
&\quad-\frac{2}{aA^6}\int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2\min_j(\tilde{\mathcal{U}}_j^4)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\,\Big|_{\eta=0}^1\\
&\quad+\frac{2}{aA^6}\int_0^1(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2\min_j(\tilde{\mathcal{U}}_j^4)\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\,d\eta\\
&\leqslant \mathcal{O}(1)\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2,
\end{aligned}
$$

where $\mathcal{O}(1)$ denotes some constant which only depends on A and which remains bounded as $A\to 0$. We used (cf. (4.16c)) that

$$
\begin{aligned}
\int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta &\leqslant \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\\
&\leqslant 4A_2\tilde{\mathcal{P}}_2(t,\eta).
\end{aligned}
$$

In the last step we used (4.15b), (4.15g), and finally that

$$
\begin{aligned}
&\left|\int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2\min_j(\tilde{\mathcal{U}}_j^4)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right|\\
&\qquad\leqslant (\|\tilde{\mathcal{Y}}_1\tilde{\mathcal{U}}_1\|_{L^\infty}^2+\|\tilde{\mathcal{Y}}_2\tilde{\mathcal{U}}_2\|_{L^\infty}^2)8A\tilde{\mathcal{P}}_2(t,\eta).
\end{aligned}
$$

Since $\tilde{\mathcal{P}}_i(t,\eta)$ tends to 0 as η tends to 0 and 1, the term on the left-hand side tends to zero as η tends to 0 and 1, respectively (cf. (3.40)).

Consider next the terms J_7 and J_8, that is,

$$
\begin{aligned}
(J_7+J_8)(t,\eta) = {} & \frac{1}{A^6}\tilde{\mathcal{Y}}_{2,\eta}(t,\eta) \\
& \times \left(\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{U}}_j^+)^3\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right) \\
& - \frac{1}{A^6}\tilde{\mathcal{Y}}_{1,\eta}(t,\eta) \\
& \times \left(\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{U}}_j^+)^3\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\right).
\end{aligned}
$$

Here we have to be a bit more careful. Introducing

$$
E = \left\{(t,\eta) \;\middle|\; \int_0^\eta \Omega(t,\eta,\theta)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta \leqslant \int_0^\eta \Omega(t,\eta,\theta)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\right\}, \tag{5.14}
$$

with

$$
\Omega(t,\eta,\theta) = \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{U}}_j^+)^3(t,\theta),
$$

we can write

$$
\begin{aligned}
&(J_7+J_8)(t,\eta) \\
&\quad = \frac{1}{A^6}\tilde{\mathcal{Y}}_{2,\eta}(t,\eta) \\
&\qquad \times \left(\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{U}}_j^+)^3\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right) \\
&\qquad - \frac{1}{A^6}\tilde{\mathcal{Y}}_{1,\eta}(t,\eta) \\
&\qquad \times \left(\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{U}}_j^+)^3\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\right) \\
&\quad = \frac{1}{A^6}(\tilde{\mathcal{Y}}_{2,\eta}-\tilde{\mathcal{Y}}_{1,\eta})(t,\eta) \\
&\qquad \times \min_k\left[\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{U}}_j^+)^3\tilde{\mathcal{Y}}_{k,\eta}(t,\theta)\,d\theta\right] \\
&\qquad + \frac{1}{A^6}\tilde{\mathcal{Y}}_{1,\eta}(t,\eta)\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}) \\
&\qquad \times \min_j(\tilde{\mathcal{U}}_j^+)^3(\tilde{\mathcal{Y}}_{2,\eta}-\tilde{\mathcal{Y}}_{1,\eta})(t,\theta)\,d\theta\,\mathbb{1}_E(t,\eta) \\
&\qquad + \frac{1}{A^6}\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})
\end{aligned}
$$

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$$\times \min_j (\tilde{\mathcal{U}}_j^+)^3(\tilde{\mathcal{Y}}_{2,\eta} - \tilde{\mathcal{Y}}_{1,\eta})(t,\theta)\, d\theta \mathbb{1}_{E^c}(t,\eta)$$
$$= (L_1 + L_2 + L_3)(t,\eta).$$

As far as the first term L_1 is concerned, the corresponding integral can be estimated as follows (we use (3.40)):

$$\begin{aligned}
&\left| \int_0^1 L_1(\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t,\eta)\, d\eta \right| \\
&= \frac{1}{A^6} \left| \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(\tilde{\mathcal{Y}}_{2,\eta} - \tilde{\mathcal{Y}}_{1,\eta})(t,\eta) \right. \\
&\quad \left. \times \min_k \left[\int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{k,\eta}(t,\theta)\, d\theta \right] d\eta \right| \\
&= \left| -\frac{1}{2A^6} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t,\eta) \right. \\
&\quad \times \min_k \left[\int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{k,\eta}(t,\theta)\, d\theta \right] \Bigg|_{\eta=0}^1 \\
&\quad + \frac{1}{2A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t,\eta) \\
&\quad \left. \times \frac{d}{d\eta} \min_k \left[\int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{k,\eta}(t,\theta)\, d\theta \right] d\eta \right| \\
&\leqslant \mathcal{O}(1) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2,
\end{aligned}$$

where $\mathcal{O}(1)$ denotes some constant only depending on A, which remains bounded as $A \to 0$, since the derivative

$$\frac{d}{d\eta} \min_k \left[\int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{k,\eta}(t,\theta)\, d\theta \right]$$

exists and is uniformly bounded; see Lemma A.8.

As far as the last term L_3 (a similar argument works for L_2) is concerned, the corresponding integral can be estimated as follows:

$$\begin{aligned}
&\int_0^1 L_3(\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t,\eta)\, d\eta \\
&\quad = \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)\tilde{\mathcal{Y}}_{2,\eta}(t,\eta) \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \\
&\quad\quad \times \min_j (\tilde{\mathcal{U}}_j^+)^3 (\tilde{\mathcal{Y}}_{2,\eta} - \tilde{\mathcal{Y}}_{1,\eta})(t,\theta) d\theta \mathbb{1}_{E^c}(t,\eta)\, d\eta
\end{aligned}$$

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$$
\begin{aligned}
&= \frac{1}{A^6}\int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\mathbb{1}_{E^c}(t,\eta) \\
&\quad \times \bigg[(\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)(t,\theta)\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{U}}_j^+)^3(t,\theta)\bigg|_{\theta=0}^{\eta} \\
&\quad - \int_0^\eta (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)(t,\theta) \\
&\quad \times \bigg[\bigg(\frac{d}{d\theta}\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\bigg)\min_j(\tilde{\mathcal{U}}_j^+)^3(t,\theta) \\
&\quad + \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\bigg(\frac{d}{d\theta}\min_j(\tilde{\mathcal{U}}_j^+)^3\bigg)(t,\theta)\bigg]d\theta\bigg]d\eta \\
&= -\frac{1}{A^6}\int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2\min_j(\tilde{\mathcal{U}}_j^+)^3\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\mathbb{1}_{E^c}(t,\eta)\,d\eta \\
&\quad + \frac{1}{A^6}\int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\mathbb{1}_{E^c}(t,\eta) \\
&\quad \times \int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t,\theta) \\
&\times \bigg[\bigg(\frac{d}{d\theta}\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\bigg)\min_j(\tilde{\mathcal{U}}_j^+)^3(t,\theta) \\
&\quad + \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\bigg(\frac{d}{d\theta}\min_j(\tilde{\mathcal{U}}_j^+)^3\bigg)(t,\theta)\bigg]d\theta\,d\eta \\
&= L_{31} + L_{32}.
\end{aligned}
$$

As far as the first term L_{31} is concerned, we have, since

$$
\min_j(\tilde{\mathcal{U}}_j^+)^3\tilde{\mathcal{Y}}_{i,\eta}(t,\eta) \leqslant A^5\min_j(\tilde{\mathcal{U}}_j^+) \leqslant \frac{1}{\sqrt{2}}A^7,
$$

that

$$
\begin{aligned}
|L_{31}| &\leqslant \bigg|\frac{1}{A^6}\int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2\min_j(\tilde{\mathcal{U}}_j^+)^3\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\mathbb{1}_{E^c}(t,\eta)\,d\eta\bigg| \\
&\leqslant \frac{A}{\sqrt{2}}\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 = \mathcal{O}(1)\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2,
\end{aligned}
$$

where $\mathcal{O}(1)$ denotes a constant, which only depends on A and which remains bounded as $A \to 0$.

The second term L_{32}, on the other hand, is a bit more demanding. We start by considering the first part of L_{32}. From Lemma A.2 we have that

$$
\bigg|\frac{d}{d\theta}\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\bigg| \leqslant \frac{1}{a}\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\max_j(\tilde{\mathcal{Y}}_{j,\eta}(t,\theta)).
$$

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This implies that

$$
\begin{aligned}
&\frac{1}{A^6}\bigg|\int_0^1 (\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\mathbb{1}_{E^c}(t,\eta)\int_0^\eta (\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)(t,\theta)\\
&\quad\times\Big(\frac{d}{d\theta}\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{U}}_j^+)^3(t,\theta)\,d\theta\,d\eta\bigg|\\
&\quad\leqslant \|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2+\frac{1}{A^{12}}\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\Big(\int_0^\eta (\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)(t,\theta)\\
&\quad\times\Big(\frac{d}{d\theta}\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{U}}_j^+)^3(t,\theta)\,d\theta\Big)^2 d\eta\\
&\quad\leqslant \|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2+\frac{1}{a^2A^{12}}\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\\
&\quad\times\Big(\int_0^\eta (\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)(t,\theta)\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\\
&\quad\times\min_j(\tilde{\mathcal{U}}_j^+)^3\max_j(\tilde{\mathcal{Y}}_{j,\eta})(t,\theta)\,d\theta\Big)^2 d\eta\\
&\quad\leqslant \|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2+\frac{1}{a^2A^2}\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\\
&\quad\times\Big(\int_0^\eta (\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)(t,\theta)\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{U}}_j^+)(t,\theta)\,d\theta\Big)^2 d\eta\\
&\quad\leqslant \|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2+\frac{a^2}{2A^2}\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\Big(\int_0^\eta (\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)(t,\theta)\\
&\quad\times\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\,d\theta\Big)^2 d\eta\\
&\quad\leqslant \|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2\\
&\quad+\frac{1}{2}\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\Big(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,\theta)\,d\theta\Big)\\
&\quad\times\Big(\int_0^\eta e^{-\frac{3}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\,d\theta\Big)d\eta\\
&\quad\leqslant \|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2\\
&\quad+\frac{1}{2A^5}\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\Big(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,\theta)\,d\theta\Big)\\
&\quad\times\Big(\int_0^\eta e^{-\frac{3}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(2\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}+\tilde{\mathcal{H}}_{2,\eta})(t,\theta)\,d\theta\Big)d\eta
\end{aligned}
$$

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$$
\begin{aligned}
&\leqslant \|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2 \\
&\quad + \frac{1}{2A^5}\int_0^1 8A\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\left(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,\theta)\,d\theta\right)d\eta \\
&\leqslant \|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2 \\
&\quad + 2A\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t,\eta)e^{-\frac{1}{2A_2}\tilde{\mathcal{Y}}_2(t,\eta)}\int_0^\eta e^{\frac{1}{2A_2}\tilde{\mathcal{Y}}_2(t,\theta)}(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,\theta)\,d\theta\,d\eta \\
&\leqslant \|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2 + 4AA_2\left(-\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,\theta)\,d\theta\,\Big|_{\eta=0}^1\right. \\
&\quad \left. + \int_0^1 (\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,\eta)\,d\eta\right) \\
&\leqslant \mathcal{O}(1)\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2.
\end{aligned}
$$

Here we used (4.13) and (4.16e). Furthermore, we use (4.16f).

Next, we turn to the second half of L_{32}. Recall first (4.16g).

From Lemma A.2 we have that

$$
\left|\frac{d}{d\theta}\min_j(\tilde{\mathcal{U}}_j^+)^3(t,\theta)\right| \leqslant 2A^4\min_j(\tilde{\mathcal{U}}_j^+)(t,\theta).
$$

Thus we can conclude as before

$$
\begin{aligned}
&\frac{1}{A^6}\left|\int_0^1(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\mathbb{1}_{E^c}(t,\eta)\int_0^\eta(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)(t,\theta)\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\right. \\
&\qquad \left.\times\left(\frac{d}{d\theta}\min_j(\tilde{\mathcal{U}}_j^+)^3\right)(t,\theta)\,d\theta\,d\eta\right| \\
&\leqslant \|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2 + \frac{1}{A^{12}}\int_0^1\tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\left(\int_0^\eta(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)(t,\theta)\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\right. \\
&\qquad \left.\times\left(\frac{d}{d\theta}\min_j(\tilde{\mathcal{U}}_j^+)^3\right)(t,\theta)\,d\theta\right)^2 d\eta \\
&\leqslant \|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2 + \frac{4}{A^4}\int_0^1\tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\left(\int_0^\eta|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|(t,\theta)e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\right. \\
&\qquad \left.\times\min_j(\tilde{\mathcal{U}}_j^+)(t,\theta)\,d\theta\right)^2 d\eta \\
&\leqslant \|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2 \\
&\qquad + \frac{4}{A^4}\int_0^1\tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\left(\int_0^\eta(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,\theta)e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\,d\theta\right) \\
&\qquad \times\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^2(t,\theta)\,d\theta\right)d\eta
\end{aligned}
$$

$$\begin{aligned}
&\leqslant \|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2\\
&\quad+\frac{24}{A^4}\int_0^1 \tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\bigg(\int_0^\eta (\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,\theta)e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\,d\theta\bigg)\,d\eta\\
&\leqslant \|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2\\
&\quad+12A\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t,\eta)e^{-\frac{1}{a}\tilde{\mathcal{Y}}_2(t,\eta)}\bigg(\int_0^\eta (\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,\theta)e^{\frac{1}{a}\tilde{\mathcal{Y}}_2(t,\theta)}\,d\theta\bigg)\,d\eta\\
&=\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2+12aA\bigg(-\int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,\theta)\,d\theta\,\bigg|_{\eta=0}^1\\
&\quad+\int_0^1(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,\eta)\,d\eta\bigg)\\
&\leqslant \mathcal{O}(1)\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2.
\end{aligned}$$

We conclude that

$$|L_{32}|\leqslant \mathcal{O}(1)\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2.$$

Finally, we have a look at J_9 (the argument for J_{10} follows the same lines). We have

$$\begin{aligned}
&\bigg|\int_0^1 J_9(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)(t,\eta)\,d\eta\bigg|\\
&\quad=\bigg|\frac{A_1^6-A_2^6}{A_1^6A_2^6}\mathbb{1}_{A_1\leqslant A_2}\int_0^1(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)\tilde{\mathcal{Y}}_{1,\eta}(t,\eta)\\
&\qquad\times\bigg(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}(\tilde{\mathcal{U}}_1^+)^3\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\bigg)\,d\eta\bigg|\\
&\qquad\leqslant \frac{A_2^6-A_1^6}{\sqrt{2}A_1^6A_2^6}A_1^2\mathbb{1}_{A_1\leqslant A_2}\\
&\qquad\times\int_0^1|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|\tilde{\mathcal{Y}}_{1,\eta}(t,\eta)\bigg(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\tilde{\mathcal{U}}_1^2\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\bigg)\,d\eta\\
&\quad\leqslant 4\frac{A_2^6-A_1^6}{\sqrt{2}A_1^6A_2^6}A_1^3\mathbb{1}_{A_1\leqslant A_2}\int_0^1|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|\tilde{\mathcal{P}}_1\tilde{\mathcal{Y}}_{1,\eta}(t,\eta)\,d\eta\\
&\quad\leqslant 2\frac{A_2^6-A_1^6}{\sqrt{2}A_1^6A_2^6}A_1^8\mathbb{1}_{A_1\leqslant A_2}\int_0^1|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|(t,\eta)\,d\eta\\
&\quad\leqslant 6\sqrt{2}A(A_2-A_1)\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|\\
&\quad\leqslant 6A(\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2+|A_1-A_2|^2).
\end{aligned}$$

We now turn our attention to I_{32}, that is,

$$I_{32} = \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t,\eta)\Big(\frac{1}{A_2^6}\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\int_0^1 e^{-\frac{1}{A_2}|\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta)|}\tilde{\mathcal{P}}_2\tilde{\mathcal{U}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta$$
$$- \frac{1}{A_1^6}\tilde{\mathcal{Y}}_{1,\eta}(t,\eta)\int_0^1 e^{-\frac{1}{A_1}|\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta)|}\tilde{\mathcal{P}}_1\tilde{\mathcal{U}}_1\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\Big)\,d\eta.$$

As before, we are only going to establish the estimates for one part of it, since the other parts can be treated similarly. Let

$$\begin{aligned}
\tilde{I}_{32} &= \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t,\eta)\Big(\frac{1}{A_2^6}\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta \\
&\quad - \frac{1}{A_1^6}\tilde{\mathcal{Y}}_{1,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\tilde{\mathcal{P}}_1\tilde{\mathcal{U}}_1^+\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\Big)\,d\eta \\
&= \frac{1}{A^6}\int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t,\eta)\Big(\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta \\
&\quad - \tilde{\mathcal{Y}}_{1,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\tilde{\mathcal{P}}_1\tilde{\mathcal{U}}_1^+\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\Big)\,d\eta \\
&\quad + \Big(\frac{1}{A_2^6} - \frac{1}{A_1^6}\Big)\mathbb{1}_{A_1\leqslant A_2}\int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)\tilde{\mathcal{Y}}_{1,\eta}(t,\eta) \\
&\quad \times \Big(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\tilde{\mathcal{P}}_1\tilde{\mathcal{U}}_1^+\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\Big)\,d\eta \\
&\quad + \Big(\frac{1}{A_2^6} - \frac{1}{A_1^6}\Big)\mathbb{1}_{A_2< A_1}\int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)\tilde{\mathcal{Y}}_{2,\eta}(t,\eta) \\
&\quad \times \Big(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\Big)\,d\eta \\
&= \tilde{K}_1 + \tilde{K}_2 + \tilde{K}_3.
\end{aligned} \tag{5.15}$$

Direct calculations yield for $\tilde{K}_2$ (and similarly for $\tilde{K}_3$) that

$$\begin{aligned}
|\tilde{K}_2| &\leqslant \frac{A_2^6 - A_1^6}{A_1^6 A_2^6}\mathbb{1}_{A_1\leqslant A_2} \\
&\qquad \times \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|\tilde{\mathcal{Y}}_{1,\eta}(t,\eta)\Big(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\tilde{\mathcal{P}}_1\tilde{\mathcal{U}}_1^+\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\Big)\,d\eta \\
&\leqslant \frac{A_2^6 - A_1^6}{A_1^6 A_2^6}\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| \\
&\qquad \times \Big(\int_0^1 \tilde{\mathcal{Y}}_{1,\eta}^2(t,\eta)\Big(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\tilde{\mathcal{P}}_1\tilde{\mathcal{U}}_1^+\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\Big)^2\,d\eta\Big)^{1/2}
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \frac{A_2^6 - A_1^6}{A_1^6 A_2^6} \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| \bigg(\int_0^1 \tilde{\mathcal{Y}}_{1,\eta}^2(t,\eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{P}}_1^2 \tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\, d\theta \\
&\quad \times \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{U}}_1^2 \tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\, d\theta\, d\eta \bigg)^{1/2} \\
&\leqslant \sqrt{6} \frac{A_2^6 - A_1^6}{A_1^3 A_2^6} \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| \bigg(\int_0^1 \tilde{\mathcal{P}}_1^2 \tilde{\mathcal{Y}}_{1,\eta}^2(t,\eta)\, d\eta \bigg)^{1/2} \\
&\leqslant \frac{\sqrt{6}}{2} \frac{A_2^6 - A_1^6}{A^4} \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| \\
&\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + |A_2 - A_1|^2),
\end{aligned}
$$

where we used (4.16b), (4.16c), and $A_1 \leqslant A_2$.

On the other hand, the term $\tilde{K}_1$ needs to be rewritten a bit more. Namely,

$$
\begin{aligned}
\tilde{K}_1 &= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t,\eta) \bigg(\tilde{\mathcal{Y}}_{2,\eta}(t,\eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\, d\theta \\
&\quad - \tilde{\mathcal{Y}}_{1,\eta}(t,\eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\, d\theta \bigg)\, d\eta \\
&= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t,\eta) \\
&\quad \times \bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{P}}_2 - \tilde{\mathcal{P}}_1) \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t,\theta) \mathbb{1}_{\tilde{\mathcal{P}}_1 \leqslant \tilde{\mathcal{P}}_2}(t,\theta)\, d\theta \bigg)\, d\eta \\
&\quad + \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{1,\eta}(t,\eta) \\
&\quad \times \bigg(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))} (\tilde{\mathcal{P}}_2 - \tilde{\mathcal{P}}_1) \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1,\eta}(t,\theta) \mathbb{1}_{\tilde{\mathcal{P}}_2 < \tilde{\mathcal{P}}_1}(t,\theta)\, d\theta \bigg)\, d\eta \\
&\quad + \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t,\eta) \\
&\quad \times \bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} \min_j(\tilde{\mathcal{P}}_j) (\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+) \tilde{\mathcal{Y}}_{2,\eta}(t,\theta) \mathbb{1}_{\tilde{\mathcal{U}}_1^+ \leqslant \tilde{\mathcal{U}}_2^+}(t,\theta)\, d\theta \bigg)\, d\eta \\
&\quad + \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{1,\eta}(t,\eta) \\
&\quad \times \bigg(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))} \min_j(\tilde{\mathcal{P}}_j) (\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+) \tilde{\mathcal{Y}}_{1,\eta}(t,\theta) \mathbb{1}_{\tilde{\mathcal{U}}_2^+ < \tilde{\mathcal{U}}_1^+}(t,\theta)\, d\theta \bigg)\, d\eta \\
&\quad + \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t,\eta) \\
&\quad \times \bigg(\tilde{\mathcal{Y}}_{2,\eta}(t,\eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\, d\theta
\end{aligned}
$$

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$$- \tilde{\mathcal{Y}}_{1,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\Big)\,d\eta$$
$$= \tilde{J}_1+\tilde{J}_2+\tilde{J}_3+\tilde{J}_4+\tilde{J}_5.$$

We start by having a close look at $\tilde{J}_1$ ($\tilde{J}_2$ can be handled similarly). One has

$$\begin{aligned}
|\tilde{J}_1| &\leqslant \frac{1}{A^6}\bigg|\int_0^1(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\\
&\quad\times\bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{P}}_2-\tilde{\mathcal{P}}_1)\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\mathbb{1}_{\tilde{\mathcal{P}}_1\leqslant\tilde{\mathcal{P}}_2}(t,\theta)\,d\theta\bigg)\,d\eta\bigg|\\
&\leqslant \|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2+\frac{1}{A^{12}}\int_0^1\tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\\
&\quad\times(\tilde{\mathcal{P}}_2^{1/2}-\tilde{\mathcal{P}}_1^{1/2})(\tilde{\mathcal{P}}_2^{1/2}+\tilde{\mathcal{P}}_1^{1/2})\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\mathbb{1}_{\tilde{\mathcal{P}}_1\leqslant\tilde{\mathcal{P}}_2}(t,\theta)\,d\theta\bigg)^2\,d\eta\\
&\leqslant \|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2+\frac{1}{A^{12}}\int_0^1\tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\\
&\quad\times\bigg(2\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{P}}_2^{1/2}-\tilde{\mathcal{P}}_1^{1/2})\tilde{\mathcal{P}}_2^{1/2}\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\bigg)^2\,d\eta\\
&\leqslant \|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2+\frac{4}{A^{12}}\int_0^1\tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\bigg)\\
&\quad\times\bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{P}}_2^{1/2}-\tilde{\mathcal{P}}_1^{1/2})^2\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\bigg)\,d\eta\\
&\leqslant \|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2+\frac{16}{A^{11}}\int_0^1\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\\
&\quad\times\bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{P}}_2^{1/2}-\tilde{\mathcal{P}}_1^{1/2})^2\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\bigg)\,d\eta\\
&\leqslant \|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2+\frac{8}{A^6}\int_0^1\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\\
&\quad\times\bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{P}}_2^{1/2}-\tilde{\mathcal{P}}_1^{1/2})^2\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\bigg)\,d\eta\\
&\leqslant \|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2+\frac{8}{A^6}\int_0^1\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)e^{-\frac{1}{A_2}\tilde{\mathcal{Y}}_2(t,\eta)}\\
&\quad\times\bigg(\int_0^\eta e^{\frac{1}{A_2}\tilde{\mathcal{Y}}_2(t,\theta)}(\tilde{\mathcal{P}}_2^{1/2}-\tilde{\mathcal{P}}_1^{1/2})^2\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\bigg)\,d\eta\\
&\leqslant \|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2
\end{aligned}$$

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$$
\begin{aligned}
&+\frac{8A_2}{A^6}\Big(-\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{P}}_2^{1/2}-\tilde{\mathcal{P}}_1^{1/2})^2\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\,\Big|_{\eta=0}^{1}\\
&+\int_0^1(\tilde{\mathcal{P}}_2^{1/2}-\tilde{\mathcal{P}}_1^{1/2})^2\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\,d\eta\Big)\\
\leqslant{}&\mathcal{O}(1)(\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2+\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2),
\end{aligned}
$$

where we used that

$$
\begin{aligned}
0&\leqslant\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{P}}_2^{1/2}-\tilde{\mathcal{P}}_1^{1/2})^2\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\,d\eta\\
&\leqslant A^4\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\\
&\leqslant A^4\Big(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\Big)^{1/2}\\
&\quad\times\Big(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2^2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\Big)^{1/2}\\
&\leqslant A^5\mathcal{O}(1)\tilde{\mathcal{P}}_2^{1/2}(t,\eta),
\end{aligned}
$$

where the very last term tends to 0 as $\eta\to 0, 1$. In the last step we used (4.16b).

Next, we investigate $\tilde{J}_3$ ($\tilde{J}_4$ can be handled in a similar way). The argument is a bit more involved and hence we start with some preliminary estimates. One has

$$
\begin{aligned}
&\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{U}}_2^+-\tilde{\mathcal{U}}_1^+)^2\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\\
&\quad\leqslant 2A^4\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\\
&\quad\leqslant 2A^4\Big(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\Big)^{1/2}\\
&\qquad\times\Big(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2^2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\Big)^{1/2}\\
&\quad\leqslant A^5\mathcal{O}(1)\tilde{\mathcal{P}}_2^{1/4}(t,\eta),
\end{aligned}
$$

where the term on the right-hand side tends to 0 for $\eta\to 0, 1$. In the last step we used (4.16j); cf. (3.35) (and the derivative $\tilde{\mathcal{Q}}_{2,\eta}$ can be found from (4.7) and (4.9)).

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With these estimates in mind, we end up with, recalling (4.16e),

$$
\begin{aligned}
|\tilde J_3| &= \frac{1}{A^6}\bigg|\int_0^1 (\tilde{\mathcal Y}_1-\tilde{\mathcal Y}_2)\tilde{\mathcal Y}_{2,\eta}(t,\eta)\\
&\quad\times\bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal Y}_2(t,\eta)-\tilde{\mathcal Y}_2(t,\theta))}\min_j(\tilde{\mathcal P}_j)(\tilde{\mathcal U}_2^+-\tilde{\mathcal U}_1^+)\tilde{\mathcal Y}_{2,\eta}(t,\theta)\mathbb{1}_{\tilde{\mathcal U}_1^+\leqslant\tilde{\mathcal U}_2^+}(t,\theta)\,d\theta\bigg)\,d\eta\bigg|\\
&\leqslant \|\tilde{\mathcal Y}_1-\tilde{\mathcal Y}_2\|^2\\
&\quad+\frac{1}{A^{12}}\int_0^1 \tilde{\mathcal Y}_{2,\eta}^2(t,\eta)\\
&\quad\times\bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal Y}_2(t,\eta)-\tilde{\mathcal Y}_2(t,\theta))}\min_j(\tilde{\mathcal P}_j)(\tilde{\mathcal U}_2^+-\tilde{\mathcal U}_1^+)\tilde{\mathcal Y}_{2,\eta}(t,\theta)\mathbb{1}_{\tilde{\mathcal U}_1^+\leqslant\tilde{\mathcal U}_2^+}(t,\theta)\,d\theta\bigg)^2\,d\eta\\
&\leqslant \|\tilde{\mathcal Y}_1-\tilde{\mathcal Y}_2\|^2\\
&\quad+\frac{1}{A^{12}}\int_0^1 \tilde{\mathcal Y}_{2,\eta}^2(t,\eta)\bigg(\int_0^\eta e^{-\frac{3}{2A_2}(\tilde{\mathcal Y}_2(t,\eta)-\tilde{\mathcal Y}_2(t,\theta))}\tilde{\mathcal P}_2\tilde{\mathcal Y}_{2,\eta}(t,\theta)\,d\theta\bigg)\\
&\quad\times\bigg(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal Y}_2(t,\eta)-\tilde{\mathcal Y}_2(t,\theta))}(\tilde{\mathcal U}_2^+-\tilde{\mathcal U}_1^+)^2\tilde{\mathcal P}_2\tilde{\mathcal Y}_{2,\eta}(t,\theta)\,d\theta\bigg)\,d\eta\\
&\leqslant \|\tilde{\mathcal Y}_1-\tilde{\mathcal Y}_2\|^2+\frac{2}{A^{11}}\int_0^1 \tilde{\mathcal P}_2\tilde{\mathcal Y}_{2,\eta}^2(t,\eta)\\
&\quad\times\bigg(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal Y}_2(t,\eta)-\tilde{\mathcal Y}_2(t,\theta))}(\tilde{\mathcal U}_2^+-\tilde{\mathcal U}_1^+)^2\tilde{\mathcal P}_2\tilde{\mathcal Y}_{2,\eta}(t,\theta)\,d\theta\bigg)\,d\eta\\
&\leqslant \|\tilde{\mathcal Y}_1-\tilde{\mathcal Y}_2\|^2\\
&\quad+\frac{1}{A^6}\int_0^1 \tilde{\mathcal Y}_{2,\eta}(t,\eta)e^{-\frac{1}{2A_2}\tilde{\mathcal Y}_2(t,\eta)}\bigg(\int_0^\eta e^{\frac{1}{2A_2}\tilde{\mathcal Y}_2(t,\theta)}(\tilde{\mathcal U}_2^+-\tilde{\mathcal U}_1^+)^2\tilde{\mathcal P}_2\tilde{\mathcal Y}_{2,\eta}(t,\theta)\,d\theta\bigg)\,d\eta\\
&= \|\tilde{\mathcal Y}_1-\tilde{\mathcal Y}_2\|^2\\
&\quad+\frac{1}{A^6}\bigg(-2A_2\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal Y}_2(t,\eta)-\tilde{\mathcal Y}_2(t,\theta))}(\tilde{\mathcal U}_2^+-\tilde{\mathcal U}_1^+)^2\tilde{\mathcal P}_2\tilde{\mathcal Y}_{2,\eta}(t,\theta)\,d\theta\Big|_{\eta=0}^1\\
&\quad+2A_2\int_0^1(\tilde{\mathcal U}_2^+-\tilde{\mathcal U}_1^+)^2\tilde{\mathcal P}_2\tilde{\mathcal Y}_{2,\eta}(t,\eta)\,d\eta\bigg)\\
&\leqslant \mathcal O(1)(\|\tilde{\mathcal Y}_1-\tilde{\mathcal Y}_2\|^2+\|\tilde{\mathcal U}_2-\tilde{\mathcal U}_1\|^2).
\end{aligned}
$$

As far as $\tilde J_5$ is concerned, we have to rewrite it a bit more. Namely,

$$
\begin{aligned}
\tilde J_5 &= \frac{1}{A^6}\int_0^1(\tilde{\mathcal Y}_1-\tilde{\mathcal Y}_2)(t,\eta)\\
&\quad\times\bigg(\tilde{\mathcal Y}_{2,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal Y}_2(t,\eta)-\tilde{\mathcal Y}_2(t,\theta))}\min_j(\tilde{\mathcal P}_j)\min_j(\tilde{\mathcal U}_j^+)\tilde{\mathcal Y}_{2,\eta}(t,\theta)\,d\theta
\end{aligned}
$$

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$$
\begin{aligned}
&- \tilde{\mathcal{Y}}_{1,\eta}(t,\eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))} \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\, d\theta \Big)\, d\eta \\
= &\frac{1}{A^6} \mathbb{1}_{A_1 \leqslant A_2} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)\tilde{\mathcal{Y}}_{2,\eta}(t,\eta) \\
&\times \Big(\int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}) \\
&\times \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\, d\theta \Big)\, d\eta \\
&+ \frac{1}{A^6} \mathbb{1}_{A_2 < A_1} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)\tilde{\mathcal{Y}}_{1,\eta}(t,\eta) \\
&\times \Big(\int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}) \\
&\times \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\, d\theta \Big)\, d\eta \\
&+ \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)\tilde{\mathcal{Y}}_{2,\eta}(t,\eta) \\
&\times \Big(\int_0^\eta (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}) \min_j(\tilde{\mathcal{P}}_j) \\
&\times \min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta) \mathbb{1}_{B(\eta)}(t,\theta)\, d\theta \Big)\, d\eta \\
&+ \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)\tilde{\mathcal{Y}}_{1,\eta}(t,\eta) \\
&\times \Big(\int_0^\eta (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}) \min_j(\tilde{\mathcal{P}}_j) \\
&\times \min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta) \mathbb{1}_{B^c(\eta)}(t,\theta)\, d\theta \Big)\, d\eta \\
&+ \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t,\eta) \\
&\times \Big(\tilde{\mathcal{Y}}_{2,\eta}(t,\eta) \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{P}}_j) \\
&\times \min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\, d\theta \\
&- \tilde{\mathcal{Y}}_{1,\eta}(t,\eta) \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{P}}_j)
\end{aligned}
$$

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$$\times \min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\Big)\,d\eta$$
$$= \tilde{L}_1 + \tilde{L}_2 + \tilde{L}_3 + \tilde{L}_4 + \tilde{L}_5,$$

where $B(\eta)$ is given by (5.11).

Both $\tilde{L}_1$ and $\tilde{L}_2$ can be handled in much the same way, and therefore we only consider $\tilde{L}_1$. One has

$$
\begin{aligned}
|\tilde{L}_1| &= \frac{1}{A^6}\mathbb{1}_{A_1\leqslant A_2}\Big|\int_0^1(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\\
&\quad\times\Big(\int_0^\eta(e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}-e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))})\\
&\quad\times\min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\Big)\,d\eta\Big|\\
&\leqslant \frac{4}{aA^6e}\mathbb{1}_{A_1\leqslant A_2}\int_0^1|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\\
&\quad\times\Big(\int_0^\eta e^{-\frac{3}{4A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\Big)\,d\eta|A_1-A_2|\\
&\leqslant \|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2+\frac{16}{a^2A^{12}e^2}\mathbb{1}_{A_1\leqslant A_2}\int_0^1\tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\\
&\quad\times\Big(\int_0^\eta e^{-\frac{3}{4A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}|\tilde{\mathcal{U}}_1|\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\Big)^2\,d\eta|A_1-A_2|^2\\
&\leqslant \|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2+\frac{8}{A^{10}e^2}\int_0^1\tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\\
&\quad\times\Big(\int_0^\eta e^{-\frac{3}{4A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\Big)^2\,d\eta|A_1-A_2|^2\\
&\leqslant \|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2+\frac{8}{A^{10}e^2}\int_0^1\tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\Big(\int_0^\eta e^{-\frac{5}{4A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\Big)\\
&\quad\times\Big(\int_0^\eta e^{-\frac{1}{4A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\Big)\,d\eta|A_1-A_2|^2\\
&\leqslant \|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2+\frac{32}{A^9e^2}\int_0^1\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\\
&\quad\times\Big(\int_0^\eta e^{-\frac{1}{4A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\Big)\,d\eta|A_1-A_2|^2\\
&\leqslant \|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2+\frac{16}{A^4e^2}\int_0^1\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)e^{-\frac{1}{4A_2}\tilde{\mathcal{Y}}_2(t,\eta)}\\
&\quad\times\int_0^\eta e^{\frac{1}{4A_2}\tilde{\mathcal{Y}}_2(t,\theta)}\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\,d\eta|A_1-A_2|^2
\end{aligned}
$$

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$$
\begin{aligned}
&\leqslant \|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2+\frac{64}{A^3e^2}\bigg(-\int_0^\eta e^{-\frac{1}{4A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\,\bigg|_{\eta=0}^{1}\\
&\quad+\int_0^1\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\,d\eta\bigg)|A_1-A_2|^2\\
&\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2+|A_1-A_2|^2),
\end{aligned}
$$

where we used (4.16h).

Next we turn our attention to $\tilde{L}_3$ and $\tilde{L}_4$, which can be handled in much the same way, and therefore we only consider $\tilde{L}_3$. One has

$$
\begin{aligned}
|\tilde{L}_3|&=\frac{1}{A^6}\bigg|\int_0^1(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\\
&\quad\times\bigg(\int_0^\eta(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}-e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))})\min_j(\tilde{\mathcal{P}}_j)\\
&\quad\times\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\mathbb{1}_{B(\eta)}(t,\theta)\,d\theta\bigg)\,d\eta\bigg|\\
&\leqslant\frac{1}{aA^6}\int_0^1|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\\
&\quad\times\bigg(\int_0^\eta(|\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_2(t,\eta)|+|\tilde{\mathcal{Y}}_1(t,\theta)-\tilde{\mathcal{Y}}_2(t,\theta)|)\\
&\quad\times e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\bigg)\,d\eta\\
&=\frac{1}{aA^6}\int_0^1(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\\
&\quad\times\bigg(\int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\bigg)\,d\eta\\
&\quad+\frac{1}{aA^6}\int_0^1|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\bigg(\int_0^\eta|\tilde{\mathcal{Y}}_1(t,\theta)-\tilde{\mathcal{Y}}_2(t,\theta)|\\
&\quad\times e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\bigg)\,d\eta\\
&\leqslant\frac{1}{\sqrt{2}A^6}\int_0^1(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\\
&\quad\times\bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2^{3/4}\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\bigg)\,d\eta\\
&\quad+\frac{1}{\sqrt{2}A^6}\int_0^1|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\\
&\quad\times\bigg(\int_0^\eta|\tilde{\mathcal{Y}}_1(t,\theta)-\tilde{\mathcal{Y}}_2(t,\theta)|e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2^{3/4}\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\bigg)\,d\eta
\end{aligned}
$$

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$$
\begin{aligned}
&\leqslant \frac{1}{\sqrt{2}A^6}\int_0^1 (\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2^{3/2}\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)^{1/2}\\
&\quad\times\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)^{1/2}d\eta\\
&\quad+\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2+\frac{1}{2A^{12}}\int_0^1\tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\\
&\quad\times\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|\tilde{\mathcal{P}}_2^{3/4}\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)^2d\eta\\
&\leqslant\frac{3}{A^4}\int_0^1(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\,d\eta+\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2\\
&\quad+\frac{1}{2A^{12}}\int_0^1\tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)\\
&\quad\times\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2\tilde{\mathcal{P}}_2^{3/2}\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)d\eta\\
&\leqslant\mathcal{O}(1)\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2\\
&\quad+\frac{1}{A^6}\int_0^1\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)e^{-\frac{1}{A_2}\tilde{\mathcal{Y}}_2(t,\eta)}\left(\int_0^\eta e^{\frac{1}{A_2}\tilde{\mathcal{Y}}_2(t,\theta)}(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2\tilde{\mathcal{P}}_2^{3/2}\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)d\eta\\
&\leqslant\mathcal{O}(1)\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2\\
&\quad+\frac{1}{A^5}\Bigg(-\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2\tilde{\mathcal{P}}_2^{3/2}\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\,\Bigg|_{\eta=0}^1\\
&\quad+\int_0^1(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2\tilde{\mathcal{P}}_2^{3/2}\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\,d\eta\Bigg)\\
&\leqslant\mathcal{O}(1)\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2,
\end{aligned}
$$

where we used (4.16k) for $\beta=\frac{1}{2}$.

Finally, we can turn our attention to $\tilde{L}_5$, which we need to split into several parts. We write

$$
\begin{aligned}
\tilde{L}_5=\frac{1}{A^6}\int_0^1&(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)(t,\eta)\\
&\times\Bigg(\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\int_0^\eta\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{P}}_j)\\
&\times\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\\
&-\tilde{\mathcal{Y}}_{1,\eta}(t,\eta)\int_0^\eta\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{P}}_j)
\end{aligned}
$$

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$$
\begin{aligned}
&\times \min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\bigg)\,d\eta\\
&= \frac{1}{A^6}\int_0^1 (\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)(\tilde{\mathcal{Y}}_{2,\eta}-\tilde{\mathcal{Y}}_{1,\eta})(t,\eta)\\
&\quad\times \min_k\bigg(\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{P}}_j)\\
&\quad\times \min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{k,\eta}(t,\theta)\,d\theta\bigg)\,d\eta\\
&\quad+\frac{1}{A^6}\int_0^1 (\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\mathbb{1}_{\tilde{E}^c}(t,\eta)\\
&\quad\times \bigg(\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{P}}_j)\\
&\quad\times \min_j(\tilde{\mathcal{U}}_j^+)(\tilde{\mathcal{Y}}_{2,\eta}-\tilde{\mathcal{Y}}_{1,\eta})(t,\theta)\,d\theta\bigg)\,d\eta\\
&\quad+\frac{1}{A^6}\int_0^1 (\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)\tilde{\mathcal{Y}}_{1,\eta}(t,\eta)\mathbb{1}_{\tilde{E}}(t,\eta)\\
&\quad\times \bigg(\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{P}}_j)\\
&\quad\times \min_j(\tilde{\mathcal{U}}_j^+)(\tilde{\mathcal{Y}}_{2,\eta}-\tilde{\mathcal{Y}}_{1,\eta})(t,\theta)\,d\theta\bigg)\,d\eta\\
&= \tilde{L}_{51}+\tilde{L}_{52}+\tilde{L}_{53},
\end{aligned}
$$

where $\tilde{E}$ is given by

$$
\begin{aligned}
\tilde{E} = \bigg\{(t,\eta)\ \bigg|\ &\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\\
&\leqslant \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\bigg\}. \qquad (5.16)
\end{aligned}
$$

As far as the first term $\tilde{L}_{51}$ is concerned, it can be estimated as follows (we use (3.40)):

$$
\begin{aligned}
|\tilde{L}_{51}| = \frac{1}{A^6}\bigg|&\int_0^1 (\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)(\tilde{\mathcal{Y}}_{2,\eta}-\tilde{\mathcal{Y}}_{1,\eta})(t,\eta)\\
&\times \min_k\bigg(\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{P}}_j)\\
&\times \min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{k,\eta}(t,\theta)\,d\theta\bigg)\,d\eta\bigg|
\end{aligned}
$$

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$$
\begin{aligned}
&= \frac{1}{2A^6}\Bigg| -(\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t,\eta) \\
&\quad \times \min_k \bigg(\int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{P}}_j) \\
&\quad \times \min_j (\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{k,\eta}(t,\theta)\, d\theta \bigg) \bigg|_{\eta=0}^1 \\
&\quad + \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t,\eta) \\
&\quad \times \frac{d}{d\eta} \min_k \bigg(\int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{P}}_j) \\
&\quad \times \min_j (\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{k,\eta}(t,\theta)\, d\theta \bigg)\, d\eta \Bigg| \\
&\leqslant \mathcal{O}(1) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2,
\end{aligned} \tag{5.17}
$$

where $\mathcal{O}(1)$ denotes some constant only depending on A, which remains bounded as $A \to 0$, provided we can show that the derivative in the latter integral exists and is uniformly bounded. In fact, from Lemma A.6 we have that

$$
\frac{d}{d\eta} \min_k \bigg(\int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{P}}_j) \min_j (\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{k,\eta}(t,\theta)\, d\theta \bigg)
$$

exists and is uniformly bounded.

As far as the term $\tilde{L}_{52}$ (a similar argument works for $\tilde{L}_{53}$) is concerned, the integral can be estimated as follows:

$$
\begin{aligned}
\tilde{L}_{52} &= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t,\eta) \mathbb{1}_{\tilde{E}^c}(t,\eta) \\
&\quad \times \bigg(\int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{P}}_j) \\
&\quad \times \min_j (\tilde{\mathcal{U}}_j^+)(\tilde{\mathcal{Y}}_{2,\eta} - \tilde{\mathcal{Y}}_{1,\eta})(t,\theta)\, d\theta \bigg)\, d\eta \\
&= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t,\eta) \mathbb{1}_{\tilde{E}^c}(t,\eta) \\
&\quad \times \bigg[(\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)(t,\theta) \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{P}}_j) \\
&\quad \times \min_j (\tilde{\mathcal{U}}_j^+)(t,\theta) \bigg|_{\theta=0}^\eta
\end{aligned}
$$

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$$
\begin{aligned}
&- \int_0^\eta (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)(t,\theta) \\
&\times \bigg[\bigg(\frac{d}{d\theta} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \bigg) \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+)(t,\theta) \\
&+ \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \bigg(\frac{d}{d\theta} \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \bigg)(t,\theta)\, d\theta \bigg] d\eta \bigg] \\
= &- \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t,\eta) \mathbb{1}_{\tilde{E}^c}(t,\eta)\, d\eta \\
&+ \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t,\eta) \mathbb{1}_{\tilde{E}^c}(t,\eta) \\
&\times \int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t,\theta) \\
&\times \bigg[\bigg(\frac{d}{d\theta} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \bigg) \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+)(t,\theta) \\
&+ \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \bigg(\frac{d}{d\theta} \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \bigg)(t,\theta)\, d\theta \bigg] d\eta \\
= &\, M_1 + M_2.
\end{aligned}
$$

As far as the first term M_1 is concerned, we have since

$$
2 \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{i,\eta}(t,\eta) \leqslant A^5 \min_j(\tilde{\mathcal{U}}_j^+)(t,\eta) \leqslant \frac{A^7}{\sqrt{2}},
$$

that

$$
\begin{aligned}
|M_1| &\leqslant \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\, d\eta \\
&\leqslant \frac{A}{2\sqrt{2}} \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 = \mathcal{O}(1) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2,
\end{aligned}
$$

where $\mathcal{O}(1)$ denotes some constant, which only depends on A and which remains bounded as $A \to 0$.

The second term M_2, on the other hand, is a bit more demanding. (i): First of all, recall (A.4), that is,

$$
\left| \frac{d}{d\theta} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \right| \leqslant \frac{1}{a} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \max_j(\tilde{\mathcal{Y}}_{j,\eta})(t,\theta),
$$

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which implies that

$$
\begin{aligned}
&\frac{1}{A^6}\bigg|\int_0^1 (\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\mathbb{1}_{\tilde{E}^c}(t,\eta)\int_0^\eta (\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)(t,\theta)\\
&\qquad\times\left(\frac{d}{d\theta}\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\right)\min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)(t,\theta)\,d\theta\,d\eta\\
&\leqslant \|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2+\frac{1}{A^{12}}\int_0^1\tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\mathbb{1}_{\tilde{E}^c}(t,\eta)\bigg(\int_0^\eta(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)(t,\theta)\\
&\qquad\times\left(\frac{d}{d\theta}\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\right)\min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)(t,\theta)\,d\theta\bigg)^2\,d\eta\\
&\leqslant \|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2+\frac{1}{4a^2A^2}\int_0^1\tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\\
&\qquad\times\left(\int_0^\eta|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|(t,\theta)e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\min_j(\tilde{\mathcal{U}}_j^+)(t,\theta)\,d\theta\right)^2 d\eta\\
&\leqslant \|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2+\frac{a^2}{8A^2A_2^5}\int_0^1\tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\\
&\qquad\times\left(\int_0^\eta|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|(t,\theta)e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(2\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}+\tilde{\mathcal{H}}_{2,\eta})^{1/2}(t,\theta)\,d\theta\right)^2 d\eta\\
&\leqslant \|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2\\
&\qquad+\frac{1}{8A_2^5}\int_0^1\tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\left(\int_0^\eta(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,\theta)e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\,d\theta\right)\\
&\qquad\times\left(\int_0^\eta e^{-\frac{3}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(2\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}+\tilde{\mathcal{H}}_{2,\eta})(t,\theta)\,d\theta\right)d\eta\\
&\leqslant \|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2\\
&\qquad+\frac{1}{A_2^4}\int_0^1\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\left(\int_0^\eta(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,\theta)e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\,d\theta\right)d\eta\\
&\leqslant \|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2\\
&\qquad+\frac{A}{2}\int_0^1\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)e^{-\frac{1}{2A_2}\tilde{\mathcal{Y}}_2(t,\eta)}\left(\int_0^\eta(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,\theta)e^{\frac{1}{2A_2}\tilde{\mathcal{Y}}_2(t,\theta)}\,d\theta\right)d\eta\\
&=\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2-AA_2\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,\theta)\,d\theta\,\bigg|_{\eta=0}^1\\
&\qquad+AA_2\int_0^1(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,\eta)\,d\eta\\
&\leqslant \mathcal{O}(1)\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2,
\end{aligned}
$$

where we used (4.16e) and (4.16f).

(ii): First of all, we have to establish that $\min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)(t,\theta)$ is Lipschitz continuous with a uniformly bounded Lipschitz constant. More precisely, in Lemma A.3 we show that

$$\left|\frac{d}{d\theta}(\min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+))(t,\theta)\right| \leqslant 2A^4(\min_j(\tilde{\mathcal{P}}_j)^{1/2}+|\tilde{\mathcal{U}}_2|)(t,\theta).$$

We are now ready to establish a Lipschitz estimate for the second part of M_2. Indeed,

$$\begin{aligned}
&\frac{1}{A^6}\left|\int_0^1(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\mathbb{1}_{\tilde{E}^c}(t,\eta)\right.\\
&\quad\times\int_0^\eta(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)(t,\theta)\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\\
&\quad\left.\times\left(\frac{d}{d\theta}\min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\right)(t,\theta)\,d\theta\,d\eta\right|\\
&\leqslant\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2+\frac{1}{A^{12}}\int_0^1\tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\mathbb{1}_{\tilde{E}^c}(t,\eta)\\
&\quad\times\left(\int_0^\eta(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)(t,\theta)\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\right.\\
&\quad\left.\times\left(\frac{d}{d\theta}\min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\right)(t,\theta)\,d\theta\right)^2d\eta\\
&\leqslant\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2+\frac{4}{A^4}\int_0^1\tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\\
&\quad\times\left(\int_0^\eta\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\right.\\
&\quad\left.\times|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|(\min_j(\tilde{\mathcal{P}}_j)^{1/2}+|\tilde{\mathcal{U}}_2|)(t,\theta)\,d\theta\right)^2d\eta\\
&\leqslant\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2+\frac{4(1+\sqrt{2})^2}{A^4}\int_0^1\tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\\
&\quad\times\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|\tilde{\mathcal{P}}_2^{1/2}(t,\theta)\,d\theta\right)^2d\eta\\
&\leqslant\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2\\
&\quad+\frac{36}{A^4}\int_0^1\tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\left(\int_0^\eta e^{-\frac{3}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2(t,\theta)\,d\theta\right)\\
&\quad\times\left(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,\theta)\,d\theta\right)d\eta
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2\\
&\quad+\frac{252}{A^4}\int_0^1 \tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\left(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,\theta)\,d\theta\right)d\eta\\
&\leqslant \|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2\\
&\quad+126A\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t,\eta)e^{-\frac{1}{2A_2}\tilde{\mathcal{Y}}_2(t,\eta)}\left(\int_0^\eta e^{\frac{1}{2A_2}\tilde{\mathcal{Y}}_2(t,\theta)}(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,\theta)\,d\theta\right)d\eta\\
&\leqslant \|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2\\
&\quad+126A\left(-2A_2\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,\theta)\,d\theta\,\Big|_{\eta=0}^{1}\right.\\
&\quad\left.+\int_0^1 2A_2(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,\eta)\,d\eta\right)\\
&\leqslant \mathcal{O}(1)\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2,
\end{aligned}
$$

where we used (4.16i). Thus we find that

$$
|\tilde{L}_{52}|+|\tilde{L}_{53}|\leqslant \mathcal{O}(1)\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2.
$$

We now turn our attention to I_{33}, that is,

$$
\begin{aligned}
I_{33}=\int_0^1(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)(t,\eta)&\left(\frac{1}{A_2^6}\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\int_0^1 e^{-\frac{1}{A_2}|\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta)|}\tilde{\mathcal{Q}}_2\tilde{\mathcal{U}}_{2,\eta}(t,\theta)\,d\theta\right.\\
&\left.-\frac{1}{A_1^6}\tilde{\mathcal{Y}}_{1,\eta}(t,\eta)\int_0^1 e^{-\frac{1}{A_1}|\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta)|}\tilde{\mathcal{Q}}_1\tilde{\mathcal{U}}_{1,\eta}(t,\theta)\,d\theta\right)d\eta.
\end{aligned}
$$

As before, we apply (5.4). Thus it suffices to study the following term.

$$
\begin{aligned}
\bar{I}_{33}&=\int_0^1(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)(t,\eta)\left(\frac{1}{A_2^6}\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{Q}}_2\tilde{\mathcal{U}}_{2,\eta}(t,\theta)\,d\theta\right.\\
&\quad\left.-\frac{1}{A_1^6}\tilde{\mathcal{Y}}_{1,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\tilde{\mathcal{Q}}_1\tilde{\mathcal{U}}_{1,\eta}(t,\theta)\,d\theta\right)d\eta\\
&=\frac{1}{A^6}\int_0^1(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)(t,\eta)\left(\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{Q}}_2\tilde{\mathcal{U}}_{2,\eta}(t,\theta)\,d\theta\right.\\
&\quad\left.-\tilde{\mathcal{Y}}_{1,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\tilde{\mathcal{Q}}_1\tilde{\mathcal{U}}_{1,\eta}(t,\theta)\,d\theta\right)d\eta\\
&\quad+\left(\frac{1}{A_2^6}-\frac{1}{A_1^6}\right)\mathbb{1}_{A_1\leqslant A_2}\int_0^1(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)\tilde{\mathcal{Y}}_{1,\eta}(t,\eta)\\
&\quad\times\left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\tilde{\mathcal{Q}}_1\tilde{\mathcal{U}}_{1,\eta}(t,\theta)\,d\theta\right)d\eta
\end{aligned}
$$

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$$
\begin{aligned}
&+\left(\frac{1}{A_2^6}-\frac{1}{A_1^6}\right)\mathbb{1}_{A_2<A_1}\int_0^1(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\\
&\times\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{Q}}_2\tilde{\mathcal{U}}_{2,\eta}(t,\theta)\,d\theta\right)d\eta\\
=\;&\bar{K}_1+\bar{K}_2+\bar{K}_3.
\end{aligned}
$$

Direct calculations yield for $\bar{K}_2$ (and similarly for $\bar{K}_3$) that

$$
\begin{aligned}
|\bar{K}_2|&\leqslant\frac{A_2^6-A_1^6}{A_1^5A_2^6}\int_0^1|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|\tilde{\mathcal{Y}}_{1,\eta}(t,\eta)\left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\tilde{\mathcal{P}}_1|\tilde{\mathcal{U}}_{1,\eta}|(t,\theta)\,d\theta\right)d\eta\\
&\leqslant\frac{A_2^6-A_1^6}{A_1^5A_2^6}\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|\\
&\quad\times\left(\int_0^1\tilde{\mathcal{Y}}_{1,\eta}^2(t,\eta)\left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\tilde{\mathcal{P}}_1|\tilde{\mathcal{U}}_{1,\eta}|(t,\theta)\,d\theta\right)^2d\eta\right)^{1/2}\\
&\leqslant\frac{A_2^6-A_1^6}{A_1^6A_2^6}\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|\left(\int_0^1\tilde{\mathcal{Y}}_{1,\eta}^2(t,\eta)\left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\tilde{\mathcal{P}}_1^2\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\right)\right.\\
&\quad\times\left.\left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\tilde{\mathcal{H}}_{1,\eta}(t,\theta)\,d\theta\right)d\eta\right)^{1/2}\\
&\leqslant\sqrt{6}\frac{A_2^6-A_1^6}{A_1^3A_2^6}\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|\left(\int_0^1\tilde{\mathcal{P}}_1^2\tilde{\mathcal{Y}}_{1,\eta}^2(t,\eta)\,d\eta\right)^{1/2}\\
&\leqslant\frac{\sqrt{6}}{2}\frac{A_2^6-A_1^6}{A^4}\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|\\
&\leqslant\mathcal{O}(1)(\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2+|A_1-A_2|^2).
\end{aligned}
$$

Here we used (4.15m), (4.16b), and (4.16d).

$\bar{K}_1$, on the other hand, needs to be rewritten a bit more. Recall (4.7). Then we can write

$$
\begin{aligned}
\bar{K}_1&=\frac{1}{A^6}\int_0^1(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)(t,\eta)\left(\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{Q}}_2\tilde{\mathcal{U}}_{2,\eta}(t,\theta)\,d\theta\right.\\
&\qquad\left.-\tilde{\mathcal{Y}}_{1,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\tilde{\mathcal{Q}}_1\tilde{\mathcal{U}}_{1,\eta}(t,\theta)\,d\theta\right)d\eta\\
&=\frac{1}{A^6}\int_0^1(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)(t,\eta)\left(A_2\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2\tilde{\mathcal{U}}_{2,\eta}(t,\theta)\,d\theta\right.\\
&\qquad\left.-A_1\tilde{\mathcal{Y}}_{1,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\tilde{\mathcal{P}}_1\tilde{\mathcal{U}}_{1,\eta}(t,\theta)\,d\theta\right)d\eta
\end{aligned}
$$

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$$+\frac{1}{A^6}\int_0^1(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)(t,\eta)\bigg(\tilde{\mathcal{Y}}_{1,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\tilde{\mathcal{D}}_1\tilde{\mathcal{U}}_{1,\eta}(t,\theta)\,d\theta$$
$$-\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{D}}_2\tilde{\mathcal{U}}_{2,\eta}(t,\theta)\,d\theta\bigg)\,d\eta. \tag{5.18}$$

In the next step we are applying integration by parts, so that we get rid of $\tilde{\mathcal{U}}_{j,\eta}(t,\theta)$ in the integrands and hence can therefore have a splitting into positive and negative parts. Therefore note that for $i=1,2$, we have

$$\begin{aligned}
&\int_0^\eta e^{-\frac{1}{A_i}(\tilde{\mathcal{Y}}_i(t,\eta)-\tilde{\mathcal{Y}}_i(t,\theta))}\tilde{\mathcal{P}}_i\tilde{\mathcal{U}}_{i,\eta}(t,\theta)\,d\theta\\
&\quad= e^{-\frac{1}{A_i}(\tilde{\mathcal{Y}}_i(t,\eta)-\tilde{\mathcal{Y}}_i(t,\theta))}\tilde{\mathcal{P}}_i\tilde{\mathcal{U}}_i(t,\theta)\Big|_{\theta=0}^{\eta}\\
&\qquad-\int_0^\eta e^{-\frac{1}{A_i}(\tilde{\mathcal{Y}}_i(t,\eta)-\tilde{\mathcal{Y}}_i(t,\theta))}\bigg(\frac{1}{A_i}\tilde{\mathcal{P}}_i+\frac{1}{A_i^2}\tilde{\mathcal{Q}}_i\bigg)\tilde{\mathcal{U}}_i\tilde{\mathcal{Y}}_{i,\eta}(t,\theta)\,d\theta\\
&\quad=\tilde{\mathcal{P}}_i\tilde{\mathcal{U}}_i(t,\eta)\\
&\qquad-\int_0^\eta e^{-\frac{1}{A_i}(\tilde{\mathcal{Y}}_i(t,\eta)-\tilde{\mathcal{Y}}_i(t,\theta))}\bigg(\frac{2}{A_i}\tilde{\mathcal{P}}_i-\frac{1}{A_i^2}\tilde{\mathcal{D}}_i\bigg)\tilde{\mathcal{U}}_i\tilde{\mathcal{Y}}_{i,\eta}(t,\theta)\,d\theta\\
&\quad=\tilde{\mathcal{P}}_i\tilde{\mathcal{U}}_i(t,\eta)\\
&\qquad-\frac{2}{A_i}\int_0^\eta e^{-\frac{1}{A_i}(\tilde{\mathcal{Y}}_i(t,\eta)-\tilde{\mathcal{Y}}_i(t,\theta))}\tilde{\mathcal{P}}_i\tilde{\mathcal{U}}_i\tilde{\mathcal{Y}}_{i,\eta}(t,\theta)\,d\theta\\
&\qquad+\frac{1}{A_i^2}\int_0^\eta e^{-\frac{1}{A_i}(\tilde{\mathcal{Y}}_i(t,\eta)-\tilde{\mathcal{Y}}_i(t,\theta))}\tilde{\mathcal{D}}_i\tilde{\mathcal{U}}_i\tilde{\mathcal{Y}}_{i,\eta}(t,\theta)\,d\theta,
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^\eta e^{-\frac{1}{A_i}(\tilde{\mathcal{Y}}_i(t,\eta)-\tilde{\mathcal{Y}}_i(t,\theta))}\tilde{\mathcal{D}}_i\tilde{\mathcal{U}}_{i,\eta}(t,\theta)\,d\theta\\
&\quad=\int_0^\eta e^{-\frac{1}{A_i}(\tilde{\mathcal{Y}}_i(t,\eta)-\tilde{\mathcal{Y}}_i(t,\theta))}\\
&\qquad\times\bigg(\int_0^\theta e^{-\frac{1}{A_i}(\tilde{\mathcal{Y}}_i(t,\theta)-\tilde{\mathcal{Y}}_i(t,l))}\Big((\tilde{\mathcal{U}}_i^2-\tilde{\mathcal{P}}_i)\tilde{\mathcal{Y}}_{i,\eta}(t,l)+\frac{1}{2}A_i^5\Big)\,dl\bigg)\tilde{\mathcal{U}}_{i,\eta}(t,\theta)\,d\theta\\
&\quad=\int_0^\eta\bigg(\int_0^\theta e^{-\frac{1}{A_i}(\tilde{\mathcal{Y}}_i(t,\eta)-\tilde{\mathcal{Y}}_i(t,l))}\Big((\tilde{\mathcal{U}}_i^2-\tilde{\mathcal{P}}_i)\tilde{\mathcal{Y}}_{i,\eta}(t,l)+\frac{1}{2}A_i^5\Big)\,dl\bigg)\tilde{\mathcal{U}}_{i,\eta}(t,\theta)\,d\theta\\
&\quad=\bigg(\int_0^\theta e^{-\frac{1}{A_i}(\tilde{\mathcal{Y}}_i(t,\eta)-\tilde{\mathcal{Y}}_i(t,l))}\Big((\tilde{\mathcal{U}}_i^2-\tilde{\mathcal{P}}_i)\tilde{\mathcal{Y}}_{i,\eta}(t,l)+\frac{1}{2}A_i^5\Big)\,dl\bigg)\tilde{\mathcal{U}}_i(t,\theta)\Big|_{\theta=0}^{\eta}\\
&\qquad-\int_0^\eta e^{-\frac{1}{A_i}(\tilde{\mathcal{Y}}_i(t,\eta)-\tilde{\mathcal{Y}}_i(t,\theta))}\Big((\tilde{\mathcal{U}}_i^2-\tilde{\mathcal{P}}_i)\tilde{\mathcal{Y}}_{i,\eta}(t,\theta)+\frac{1}{2}A_i^5\Big)\tilde{\mathcal{U}}_i(t,\theta)\,d\theta
\end{aligned}$$

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$$
\begin{aligned}
&= \tilde{\mathcal{D}}_i \tilde{\mathcal{U}}_i(t,\eta) \\
&\quad - \int_0^\eta e^{-\frac{1}{A_i}(\tilde{\mathcal{Y}}_i(t,\eta)-\tilde{\mathcal{Y}}_i(t,\theta))}\Big((\tilde{\mathcal{U}}_i^2 - \tilde{\mathcal{P}}_i)\tilde{\mathcal{Y}}_{i,\eta}(t,\theta) + \frac{1}{2}A_i^5\Big)\tilde{\mathcal{U}}_i(t,\theta)\,d\theta.
\end{aligned}
$$

Here we used that for $\theta \leqslant \eta$,

$$
\begin{aligned}
0 &\leqslant \int_0^\theta e^{-\frac{1}{A_i}(\tilde{\mathcal{Y}}_i(t,\eta)-\tilde{\mathcal{Y}}_i(t,l))}\Big((\tilde{\mathcal{U}}_i^2 - \tilde{\mathcal{P}}_i)\tilde{\mathcal{Y}}_{i,\eta}(t,l) + \frac{1}{2}A_i^5\Big)\,dl \\
&\leqslant \int_0^\eta e^{-\frac{1}{A_i}(\tilde{\mathcal{Y}}_i(t,\eta)-\tilde{\mathcal{Y}}_i(t,l))}\Big((\tilde{\mathcal{U}}_i^2 - \tilde{\mathcal{P}}_i)\tilde{\mathcal{Y}}_{i,\eta}(t,l) + \frac{1}{2}A_i^5\Big)\,dl \\
&\leqslant \tilde{\mathcal{D}}_i(t,\eta) \leqslant 2A_i\tilde{\mathcal{P}}_i(t,\eta).
\end{aligned}
$$

We finally end up with

$$
\begin{aligned}
\bar{K}_1 &= \frac{1}{A^6}\int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(A_2\tilde{\mathcal{P}}_2\tilde{\mathcal{U}}_2\tilde{\mathcal{Y}}_{2,\eta} - A_1\tilde{\mathcal{P}}_1\tilde{\mathcal{U}}_1\tilde{\mathcal{Y}}_{1,\eta})(t,\eta)\,d\eta \\
&\quad + \frac{1}{A^6}\int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(\tilde{\mathcal{D}}_1\tilde{\mathcal{U}}_1\tilde{\mathcal{Y}}_{1,\eta} - \tilde{\mathcal{D}}_2\tilde{\mathcal{U}}_2\tilde{\mathcal{Y}}_{2,\eta})(t,\eta)\,d\eta \\
&\quad + \frac{1}{A^6}\int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t,\eta) \\
&\quad \times \Big(\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\frac{1}{A_2}\tilde{\mathcal{D}}_2\tilde{\mathcal{U}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta \\
&\quad - \tilde{\mathcal{Y}}_{1,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\frac{1}{A_1}\tilde{\mathcal{D}}_1\tilde{\mathcal{U}}_1\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\Big)\,d\eta \\
&\quad + \frac{3}{A^6}\int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t,\eta)\Big(\tilde{\mathcal{Y}}_{1,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\tilde{\mathcal{P}}_1\tilde{\mathcal{U}}_1\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta \\
&\quad - \tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2\tilde{\mathcal{U}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\Big)\,d\eta \\
&\quad + \frac{1}{A^6}\int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t,\eta)\Big(\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^3\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta \\
&\qquad - \tilde{\mathcal{Y}}_{1,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\tilde{\mathcal{U}}_1^3\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\Big)\,d\eta \\
&\quad + \frac{1}{2A^6}\int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t,\eta)\Big(\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}A_2^5\tilde{\mathcal{U}}_2(t,\theta)\,d\theta \\
&\quad - \tilde{\mathcal{Y}}_{1,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}A_1^5\tilde{\mathcal{U}}_1(t,\theta)\,d\theta\Big)\,d\eta \\
&= \bar{K}_{11} + \bar{K}_{12} + \bar{K}_{13} + \bar{K}_{14} + \bar{K}_{15} + \bar{K}_{16}. \qquad (5.19)
\end{aligned}
$$

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We start by considering $\bar{K}_{11}$, which can be further split into

$$\begin{aligned}
\bar{K}_{11} &= \frac{1}{A^6}\int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(A_2\tilde{\mathcal{P}}_2\tilde{\mathcal{U}}_2\tilde{\mathcal{Y}}_{2,\eta} - A_1\tilde{\mathcal{P}}_1\tilde{\mathcal{U}}_1\tilde{\mathcal{Y}}_{1,\eta})(t,\eta)\,d\eta \\
&= \frac{1}{A^6}\mathbb{1}_{A_1\leqslant A_2}(A_2 - A_1)\int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)\tilde{\mathcal{P}}_2\tilde{\mathcal{U}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\,d\eta \\
&\quad + \frac{1}{A^6}\mathbb{1}_{A_2< A_1}(A_2 - A_1)\int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)\tilde{\mathcal{P}}_1\tilde{\mathcal{U}}_1\tilde{\mathcal{Y}}_{1,\eta}(t,\eta)\,d\eta \\
&\quad + \frac{a}{A^6}\int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(\tilde{\mathcal{P}}_2 - \tilde{\mathcal{P}}_1)\tilde{\mathcal{U}}_2\tilde{\mathcal{Y}}_{2,\eta}\mathbb{1}_{\tilde{\mathcal{P}}_1\leqslant\tilde{\mathcal{P}}_2}(t,\eta)\,d\eta \\
&\quad + \frac{a}{A^6}\int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(\tilde{\mathcal{P}}_2 - \tilde{\mathcal{P}}_1)\tilde{\mathcal{U}}_1\tilde{\mathcal{Y}}_{1,\eta}\mathbb{1}_{\tilde{\mathcal{P}}_2<\tilde{\mathcal{P}}_1}(t,\eta)\,d\eta \\
&\quad + \frac{a}{A^6}\int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(\tilde{\mathcal{Y}}_{2,\eta} - \tilde{\mathcal{Y}}_{1,\eta})\min_j(\tilde{\mathcal{P}}_j)\tilde{\mathcal{U}}_2(t,\eta)\,d\eta \\
&\quad + \frac{a}{A^6}\int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1)\min_j(\tilde{\mathcal{P}}_j)\tilde{\mathcal{Y}}_{1,\eta}(t,\eta)\,d\eta \\
&= \bar{B}_{11} + \bar{B}_{12} + \bar{B}_{13} + \bar{B}_{14} + \bar{B}_{15} + \bar{B}_{16}.
\end{aligned}$$

For $\bar{B}_{11}$ we have (and similarly for B_{12}) that

$$\begin{aligned}
|\bar{B}_{11}| &= \frac{1}{A^6}\mathbb{1}_{A_1\leqslant A_2}(A_2 - A_1)\left|\int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)\tilde{\mathcal{P}}_2\tilde{\mathcal{U}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\,d\eta\right| \\
&\leqslant \frac{A}{2\sqrt{2}}|A_1 - A_2|\int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t,\eta)\,d\eta \\
&\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + |A_1 - A_2|).
\end{aligned}$$

Recalling (4.15i), we have for $\bar{B}_{13}$ (and similarly for $\bar{B}_{14}$) that

$$\begin{aligned}
|\bar{B}_{13}| &= \frac{a}{A^6}\left|\int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(\tilde{\mathcal{P}}_2 - \tilde{\mathcal{P}}_1)\tilde{\mathcal{U}}_2\tilde{\mathcal{Y}}_{2,\eta}\mathbb{1}_{\tilde{\mathcal{P}}_1\leqslant\tilde{\mathcal{P}}_2}(t,\eta)\,d\eta\right| \\
&\leqslant \frac{2}{A^5}\int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|\,|\tilde{\mathcal{P}}_2^{1/2} - \tilde{\mathcal{P}}_1^{1/2}|\tilde{\mathcal{P}}_2^{1/2}|\tilde{\mathcal{U}}_2|\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\,d\eta \\
&\leqslant \sqrt{2}\int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|\,|\tilde{\mathcal{P}}_2^{1/2} - \tilde{\mathcal{P}}_1^{1/2}|(t,\eta)\,d\eta \\
&\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \|\tilde{\mathcal{P}}_2^{1/2} - \tilde{\mathcal{P}}_1^{1/2}\|^2).
\end{aligned}$$

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As far as $\bar{B}_{15}$ is concerned, we want to use integration by parts. Therefore it is important to recall (3.40) and Lemma A.3 (ii), which imply that

$$\begin{aligned}
|\bar{B}_{15}| &= \frac{a}{A^6}\left|\int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(\tilde{\mathcal{Y}}_{1,\eta} - \tilde{\mathcal{Y}}_{2,\eta}) \min_j(\tilde{\mathcal{P}}_j)\tilde{\mathcal{U}}_2(t,\eta)\, d\eta\right| \\
&= \frac{a}{2A^6}\left|(\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \min_j(\tilde{\mathcal{P}}_j)\tilde{\mathcal{U}}_2(t,\eta)\,\right|_{\eta=0}^{1} \\
&\quad - \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \frac{d}{d\eta}(\min_j(\tilde{\mathcal{P}}_j)\tilde{\mathcal{U}}_2)(t,\eta)\, d\eta\Bigg| \\
&= \frac{a}{2A^6}\left|\int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \frac{d}{d\eta}(\min_j(\tilde{\mathcal{P}}_j)\tilde{\mathcal{U}}_2)(t,\eta)\, d\eta\right| \\
&\leqslant \frac{A}{2\sqrt{2}}\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2.
\end{aligned} \tag{5.20}$$

$\bar{B}_{16}$ is straightforward. Indeed, one has using (4.15e) that

$$\begin{aligned}
|\bar{B}_{16}| &= \frac{a}{A^6}\left|\int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \min_j(\tilde{\mathcal{P}}_j)\tilde{\mathcal{Y}}_{1,\eta}(t,\eta)\, d\eta\right| \\
&\leqslant \frac{1}{2}\int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2||\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2|(t,\eta)\, d\eta \\
&\leqslant \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|.
\end{aligned}$$

We continue with $\bar{K}_{12}$. We can split $\bar{K}_{12}$ as follows:

$$\begin{aligned}
\bar{K}_{12} &= \frac{1}{A^6}\int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(\tilde{\mathcal{D}}_1\tilde{\mathcal{U}}_1\tilde{\mathcal{Y}}_{1,\eta} - \tilde{\mathcal{D}}_2\tilde{\mathcal{U}}_2\tilde{\mathcal{Y}}_{2,\eta})(t,\eta)\, d\eta \\
&= \frac{1}{A^6}\int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(\tilde{\mathcal{D}}_1 - \tilde{\mathcal{D}}_2)\tilde{\mathcal{U}}_1\tilde{\mathcal{Y}}_{1,\eta}\mathbb{1}_{\tilde{\mathcal{D}}_2 \leqslant \tilde{\mathcal{D}}_1}(t,\eta)\, d\eta \\
&\quad + \frac{1}{A^6}\int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(\tilde{\mathcal{D}}_1 - \tilde{\mathcal{D}}_2)\tilde{\mathcal{U}}_2\tilde{\mathcal{Y}}_{2,\eta}\mathbb{1}_{\tilde{\mathcal{D}}_1 < \tilde{\mathcal{D}}_2}(t,\eta)\, d\eta \\
&\quad + \frac{1}{A^6}\int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \min_j(\tilde{\mathcal{D}}_j)\tilde{\mathcal{Y}}_{1,\eta}(t,\eta)\, d\eta \\
&\quad + \frac{1}{A^6}\int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(\tilde{\mathcal{Y}}_{1,\eta} - \tilde{\mathcal{Y}}_{2,\eta}) \min_j(\tilde{\mathcal{D}}_j)\tilde{\mathcal{U}}_2(t,\eta)\, d\eta \\
&= \bar{B}_{21} + \bar{B}_{22} + \bar{B}_{23} + \bar{B}_{24}.
\end{aligned}$$

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As the first step we need to establish an estimate for $(\tilde{\mathcal{D}}_1 - \tilde{\mathcal{D}}_2)(t,\eta)$. Note that $|\tilde{\mathcal{U}}_i \tilde{\mathcal{Y}}_{i,\eta}(t,\eta)| \leqslant A^{5/2}\sqrt{\tilde{\mathcal{Y}}_{i,\eta}}(t,\eta)$, and hence it is in general unbounded. Thus our estimate for $(\tilde{\mathcal{D}}_1 - \tilde{\mathcal{D}}_2)(t,\eta)$ must take care of this problem.

Applying Lemma A.9 finally yields

$$
\begin{aligned}
|\bar{B}_{21}| &\leqslant \frac{1}{A^6}\int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|\,|\tilde{\mathcal{D}}_1 - \tilde{\mathcal{D}}_2|\,|\tilde{\mathcal{U}}_1|\tilde{\mathcal{Y}}_{1,\eta}(t,\eta)\mathbb{1}_{\tilde{\mathcal{D}}_2\leqslant\tilde{\mathcal{D}}_1}(t,\eta)\,d\eta\\
&\leqslant \frac{2}{A^{9/2}}\int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2\tilde{\mathcal{D}}_1^{1/2}|\tilde{\mathcal{U}}_1|\tilde{\mathcal{Y}}_{1,\eta}(t,\eta)\,d\eta\\
&\quad + \frac{1}{A^6}\int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(\tilde{\mathcal{U}}_1^2 + \tilde{\mathcal{P}}_1)|\tilde{\mathcal{U}}_1|\tilde{\mathcal{Y}}_{1,\eta}(t,\eta)\,d\eta\\
&\quad + \frac{2\sqrt{2}}{A^{9/2}}\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|\int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|\tilde{\mathcal{D}}_1^{1/2}|\tilde{\mathcal{U}}_1|\tilde{\mathcal{Y}}_{1,\eta}(t,\eta)\,d\eta\\
&\quad + \frac{4}{A^3}\int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t,\eta)\\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}(\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2(t,\theta)\,d\theta\right)^{1/2}|\tilde{\mathcal{U}}_1|\tilde{\mathcal{Y}}_{1,\eta}(t,\eta)\,d\eta\\
&\quad + \frac{2\sqrt{2}}{A^3}\int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t,\eta)\\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}(\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2(t,\theta)\,d\theta\right)^{1/2}|\tilde{\mathcal{U}}_1|\tilde{\mathcal{Y}}_{1,\eta}(t,\eta)\,d\eta\\
&\quad + \frac{3}{\sqrt{2}A^2}\int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t,\eta)\\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}(\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t,\theta)\,d\theta\right)^{1/2}|\tilde{\mathcal{U}}_1|\tilde{\mathcal{Y}}_{1,\eta}(t,\eta)\,d\eta\\
&\quad + \frac{12\sqrt{2}}{\sqrt{3e}A^2}|A_1 - A_2|\int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t,\eta)\\
&\quad \times \left(\int_0^\eta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\,d\theta\right)^{1/2}|\tilde{\mathcal{U}}_1|\tilde{\mathcal{Y}}_{1,\eta}(t,\eta)\,d\eta\\
&\quad + \frac{3}{2A^2}\int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t,\eta)\\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t,\theta)\,d\theta\right)|\tilde{\mathcal{U}}_1|\tilde{\mathcal{Y}}_{1,\eta}(t,\eta)\,d\eta\\
&\quad + \frac{6}{A^2}|A_1 - A_2|\int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t,\eta)
\end{aligned}
$$

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$$
\begin{aligned}
&\times \left(\int_0^\eta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\,d\theta\right)|\tilde{\mathcal{U}}_1|\tilde{\mathcal{Y}}_{1,\eta}(t,\eta)\,d\eta\\
&\leqslant \mathcal{O}(1)\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2+\frac{16}{A}\int_0^1 \tilde{\mathcal{Y}}_{1,\eta}(t,\eta)e^{-\frac{1}{A}\tilde{\mathcal{Y}}_1(t,\eta)}\\
&\times \left(\int_0^\eta e^{\frac{1}{A}\tilde{\mathcal{Y}}_1(t,\theta)}((\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)^2+(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})^2+A(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2)(t,\theta)\,d\theta\right)d\eta\\
&+12A^{1/2}|A_1-A_2|\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|\\
&\times \left(\int_0^1 \tilde{\mathcal{Y}}_{1,\eta}(t,\eta)e^{-\frac{3}{4A}\tilde{\mathcal{Y}}_1(t,\eta)}\left(\int_0^\eta e^{\frac{3}{4A}\tilde{\mathcal{Y}}_1(t,\theta)}\,d\theta\right)d\eta\right)^{1/2}\\
&+\frac{3A^{1/2}}{2}\left(\int_0^1 \tilde{\mathcal{Y}}_{1,\eta}(t,\eta)e^{-\frac{2}{a}\tilde{\mathcal{Y}}_1(t,\eta)}\left(\int_0^\eta e^{\frac{2}{a}\tilde{\mathcal{Y}}_1(t,\theta)}\,d\theta\right)d\eta\right)^{1/2}\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2\\
&+6A^{1/2}|A_1-A_2|\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|\\
&\times \left(\int_0^1 \tilde{\mathcal{Y}}_{1,\eta}(t,\eta)e^{-\frac{3}{4A}\tilde{\mathcal{Y}}_1(t,\eta)}\left(\int_0^\eta e^{\frac{3}{4A}\tilde{\mathcal{Y}}_1(t,\theta)}\,d\theta\right)d\eta\right)^{1/2}\\
&\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2+\|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2\|^2+\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+|A_1-A_2|^2).
\end{aligned}
$$

Following the same lines, one obtains

$$
|\bar{B}_{22}|\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2+\|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2\|^2+\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+|A_1-A_2|^2).
$$

The estimate for $\bar{B}_{23}$ is straightforward. Namely,

$$
\begin{aligned}
|\bar{B}_{23}| &\leqslant \frac{2}{A^5}\int_0^1 |\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2||\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2|\tilde{\mathcal{P}}_1\tilde{\mathcal{Y}}_{1,\eta}(t,\eta)\,d\eta\\
&\leqslant \|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2+\|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2\|^2.
\end{aligned}
$$

As far as $\bar{B}_{24}$ is concerned, recall from Lemma A.4 (ii) that

$$
\left|\frac{d}{d\eta}\big(\min_j(\tilde{\mathcal{D}}_j)\tilde{\mathcal{U}}_2(t,\eta)\big)\right|\leqslant \mathcal{O}(1)A^7,
$$

which yields, together with (3.40), that

$$
\begin{aligned}
|\bar{B}_{24}| &= \frac{1}{A^6}\left|\int_0^1 (\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)(\tilde{\mathcal{Y}}_{1,\eta}-\tilde{\mathcal{Y}}_{2,\eta})\min_j(\tilde{\mathcal{D}}_j)\tilde{\mathcal{U}}_2(t,\eta)\,d\eta\right|\\
&= \frac{1}{2A^6}\left|(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2\min_j(\tilde{\mathcal{D}}_j)\tilde{\mathcal{U}}_2(t,\eta)\,\Big|_{\eta=0}^1\right.\\
&\quad \left.-\int_0^1 (\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,\eta)\frac{d}{d\eta}(\min_j(\tilde{\mathcal{D}}_j)\tilde{\mathcal{U}}_2)(t,\eta)\,d\eta\right|
\end{aligned}
$$

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$$= \frac{1}{2A^6}\left|\int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t,\eta)\frac{d}{d\eta}(\min_j(\tilde{\mathcal{D}}_j)\tilde{\mathcal{U}}_2)(t,\eta)\,d\eta\right|$$
$$\leqslant \mathcal{O}(1)\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2.$$

Next, we have a look at $\bar{K}_{13}$, which can be rewritten as follows:

$$\begin{aligned}
\bar{K}_{13} &= \frac{1}{A^6}\int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t,\eta) \\
&\quad \times \bigg(\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\frac{1}{A_2}\tilde{\mathcal{D}}_2\tilde{\mathcal{U}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta \\
&\quad - \tilde{\mathcal{Y}}_{1,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\frac{1}{A_1}\tilde{\mathcal{D}}_1\tilde{\mathcal{U}}_1\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\bigg)\,d\eta \\
&= \frac{1}{A^6}\int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t,\eta) \\
&\quad \times \bigg(\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\frac{1}{A_2}\tilde{\mathcal{D}}_2\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta \\
&\quad - \tilde{\mathcal{Y}}_{1,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\frac{1}{A_1}\tilde{\mathcal{D}}_1\tilde{\mathcal{U}}_1^+\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\bigg)\,d\eta \\
&\quad + \frac{1}{A^6}\int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t,\eta) \\
&\quad \times \bigg(\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\frac{1}{A_2}\tilde{\mathcal{D}}_2\tilde{\mathcal{U}}_2^-\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta \\
&\quad - \tilde{\mathcal{Y}}_{1,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\frac{1}{A_1}\tilde{\mathcal{D}}_1\tilde{\mathcal{U}}_1^-\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\bigg)\,d\eta \\
&= \bar{K}_{13}^+ + \bar{K}_{13}^-.
\end{aligned}$$

Note that both $\bar{K}_{13}^+$ and $\bar{K}_{13}^-$ have the same structure and hence we are only going to present the details for $\bar{K}_{13}^+$, which needs to be rewritten a bit more. Namely,

$$\begin{aligned}
\bar{K}_{13}^+ &= \frac{1}{A^6}\int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t,\eta) \\
&\quad \times \bigg(\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\frac{1}{A_2}\tilde{\mathcal{D}}_2\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta \\
&\quad - \tilde{\mathcal{Y}}_{1,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\frac{1}{A_1}\tilde{\mathcal{D}}_1\tilde{\mathcal{U}}_1^+\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\bigg)\,d\eta \\
&= \mathbb{1}_{A_1\leqslant A_2}\frac{1}{A^6}\left(\frac{1}{A_2} - \frac{1}{A_1}\right)\int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)\tilde{\mathcal{Y}}_{1,\eta}(t,\eta)
\end{aligned}$$

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$$\times \left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{D}}_1 \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\, d\theta \right) d\eta$$

$$+ \mathbb{1}_{A_2<A_1} \frac{1}{A^6} \left(\frac{1}{A_2} - \frac{1}{A_1} \right) \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t,\eta)$$

$$\times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{D}}_2 \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\, d\theta \right) d\eta$$

$$+ \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t,\eta)$$

$$\times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{D}}_2 - \tilde{\mathcal{D}}_1) \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{\tilde{\mathcal{D}}_1 \leqslant \tilde{\mathcal{D}}_2}(t,\theta)\, d\theta \right) d\eta$$

$$+ \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{1,\eta}(t,\eta)$$

$$\times \left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))} (\tilde{\mathcal{D}}_2 - \tilde{\mathcal{D}}_1) \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1,\eta} \mathbb{1}_{\tilde{\mathcal{D}}_2 < \tilde{\mathcal{D}}_1}(t,\theta)\, d\theta \right) d\eta$$

$$+ \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t,\eta)$$

$$\times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} \min_j(\tilde{\mathcal{D}}_j)(\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+) \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{\tilde{\mathcal{U}}_1^+ \leqslant \tilde{\mathcal{U}}_2^+}(t,\theta)\, d\theta \right) d\eta$$

$$+ \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{1,\eta}(t,\eta)$$

$$\times \left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))} \min_j(\tilde{\mathcal{D}}_j)(\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+) \tilde{\mathcal{Y}}_{1,\eta} \mathbb{1}_{\tilde{\mathcal{U}}_2^+ < \tilde{\mathcal{U}}_1^+}(t,\theta)\, d\theta \right) d\eta$$

$$+ \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t,\eta)$$

$$\times \left(\int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}) \right.$$

$$\left. \times \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{B(\eta)}(t,\theta)\, d\theta \right) d\eta$$

$$+ \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{1,\eta}(t,\eta)$$

$$\times \left(\int_0^\eta (e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}) \right.$$

$$\left. \times \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta} \mathbb{1}_{B(\eta)^c}(t,\theta)\, d\theta \right) d\eta$$

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$$
\begin{aligned}
&+ \mathbb{1}_{A_1 \leqslant A_2} \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)\tilde{\mathcal{Y}}_{2,\eta}(t,\eta) \\
&\times \bigg(\int_0^\eta (\min_j (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) - \min_j (e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))})) \\
&\times \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\, d\theta \bigg) \, d\eta \\
&+ \mathbb{1}_{A_2 < A_1} \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)\tilde{\mathcal{Y}}_{1,\eta}(t,\eta) \\
&\times \bigg(\int_0^\eta (\min_j (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) - \min_j (e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))})) \\
&\times \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\, d\theta \bigg) \, d\eta \\
&+ \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(\tilde{\mathcal{Y}}_{2,\eta} - \tilde{\mathcal{Y}}_{1,\eta})(t,\eta) \\
&\times \min_k \bigg(\int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{k,\eta}(t,\theta)\, d\theta \bigg) \, d\eta \\
&+ \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)\tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{D^c}(t,\eta) \\
&\times \bigg(\int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \\
&\times \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) (\tilde{\mathcal{Y}}_{2,\eta} - \tilde{\mathcal{Y}}_{1,\eta})(t,\theta)\, d\theta \bigg) \, d\eta \\
&+ \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)\tilde{\mathcal{Y}}_{1,\eta} \mathbb{1}_{D}(t,\eta) \\
&\times \bigg(\int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \\
&\times \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) (\tilde{\mathcal{Y}}_{2,\eta} - \tilde{\mathcal{Y}}_{1,\eta})(t,\theta)\, d\theta \bigg) \, d\eta \\
= {}& \hat{B}_{31} + \hat{B}_{32} + \bar{B}_{31} + \bar{B}_{32} + \bar{B}_{33} + \bar{B}_{34} \\
&+ \bar{B}_{35} + \bar{B}_{36} + \hat{B}_{35} + \hat{B}_{36} + \bar{B}_{37} + \bar{B}_{38} + \bar{B}_{39},
\end{aligned}
$$

where $B(\eta)$ by (5.11) and

$$
D = \left\{ (t,\eta) \,\middle|\, \int_0^\eta \Upsilon(t,\eta,\theta)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\, d\theta \leqslant \int_0^\eta \Upsilon(t,\eta,\theta)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\, d\theta \right\}, \tag{5.21}
$$

where

$$\Upsilon(t,\eta,\theta)=\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+).$$

We start by estimating $\hat{B}_{31}$ (a similar argument works for $\hat{B}_{32}$). Direct calculations yield

$$\begin{aligned}
|\hat{B}_{31}| &= \mathbb{1}_{A_1\leqslant A_2}\frac{1}{aA^7}|A_2-A_1|\int_0^1|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|\tilde{\mathcal{Y}}_{1,\eta}(t,\eta)\\
&\quad\times\left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\tilde{\mathcal{D}}_1\tilde{\mathcal{U}}_1^+\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\right)d\eta\\
&\leqslant \frac{2}{A^7}|A_1-A_2|\int_0^1|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|\tilde{\mathcal{Y}}_{1,\eta}(t,\eta)\\
&\quad\times\left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\tilde{\mathcal{P}}_1\tilde{\mathcal{U}}_1^+\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\right)d\eta\\
&\leqslant \frac{2}{A^7}|A_1-A_2|\int_0^1|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|\tilde{\mathcal{Y}}_{1,\eta}(t,\eta)\\
&\quad\times\left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\tilde{\mathcal{P}}_1^2\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\right)^{1/2}\\
&\quad\times\left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\tilde{\mathcal{U}}_1^2\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\right)^{1/2}d\eta\\
&\leqslant \frac{2\sqrt{6}}{A^4}|A_1-A_2|\int_0^1|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|\tilde{\mathcal{P}}_1\tilde{\mathcal{Y}}_{1,\eta}(t,\eta)\,d\eta\\
&\leqslant \sqrt{6}A|A_1-A_2|\int_0^1|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|(t,\eta)\,d\eta\\
&\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2+|A_1-A_2|^2).
\end{aligned}$$

For $\bar{B}_{31}$ (a similar argument works for $\bar{B}_{32}$) we would like to apply some of the estimates established when investigating $\bar{K}_1$. Thus we split $\bar{B}_{31}$ into even smaller parts, that is,

$$\begin{aligned}
|\bar{B}_{31}| &\leqslant \frac{1}{A^7}\int_0^1|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\\
&\quad\times\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\bar{d}_{11}+\bar{d}_{12})\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}\mathbb{1}_{\tilde{\mathcal{D}}_1\leqslant\tilde{\mathcal{D}}_2}(t,\theta)\,d\theta\right)d\eta\\
&\quad+\frac{1}{A^7}\int_0^1|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)
\end{aligned}$$

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$$
\begin{aligned}
&\times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\bar{T}_1\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}\mathbb{1}_{\tilde{\mathcal{D}}_1\leqslant\tilde{\mathcal{D}}_2}(t,\theta)\,d\theta\right)d\eta\\
&+\frac{1}{A^7}\int_0^1|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\\
&\times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\bar{T}_2\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}\mathbb{1}_{\tilde{\mathcal{D}}_1\leqslant\tilde{\mathcal{D}}_2}(t,\theta)\,d\theta\right)d\eta\\
&+\frac{1}{A^7}\int_0^1|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\\
&\times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\bar{T}_3\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}\mathbb{1}_{\tilde{\mathcal{D}}_1\leqslant\tilde{\mathcal{D}}_2}(t,\theta)\,d\theta\right)d\eta\\
={}&B_{311}+B_{312}+B_{313}+B_{314}.
\end{aligned}
$$

By (A.12) and (A.13), we have that

$$
(\bar{d}_{11}+\bar{d}_{12})(t,\theta)\leqslant 2A^{3/2}\tilde{\mathcal{D}}_2^{1/2}|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|(t,\theta)+2\sqrt{2}A^{3/2}\tilde{\mathcal{D}}_2^{1/2}(t,\theta)\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|,
$$

and hence

$$
\begin{aligned}
B_{311}\leqslant{}&\frac{2}{A^{11/2}}\int_0^1|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\\
&\times\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|\tilde{\mathcal{D}}_2^{1/2}\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)d\eta\\
&+\frac{2\sqrt{2}}{A^{11/2}}\int_0^1|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\\
&\times\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{D}}_2^{1/2}\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)d\eta\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|\\
\leqslant{}&\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2\\
&+\frac{4}{A^{11}}\int_0^1\tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\\
&\times\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|\tilde{\mathcal{D}}_2^{1/2}\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)^2d\eta\\
&+\frac{2\sqrt{2}}{A^{11/2}}\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2\\
&\times\left(\int_0^1\tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{D}}_2^{1/2}\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)^2d\eta\right)^{1/2}\\
\leqslant{}&\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2\\
&+\frac{4}{A^{11}}\int_0^1\tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)
\end{aligned}
$$

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$$
\begin{aligned}
&\times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2\tilde{\mathcal{D}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)d\eta\\
&+\frac{2\sqrt{2}}{A^{11/2}}\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2\left(\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)\right.\\
&\left.\times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{D}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)d\eta\right)^{1/2}\\
\leqslant{}& \|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2\\
&+\frac{8}{A^5}\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t,\eta)e^{-\frac{1}{A_2}\tilde{\mathcal{Y}}_2(t,\eta)}\left(\int_0^\eta e^{\frac{1}{A_2}\tilde{\mathcal{Y}}_2(t,\theta)}(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2\tilde{\mathcal{D}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)d\eta\\
&+\frac{4}{A^{5/2}}\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2\\
&\times\left(\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{D}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)d\eta\right)^{1/2}\\
\leqslant{}& \|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2\\
&+\frac{8A_2}{A^5}\left[-\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2\tilde{\mathcal{D}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\,\Big|_{\eta=0}^1\right.\\
&\left.+\int_0^1(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2\tilde{\mathcal{D}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\,d\eta\right]\\
&+4A^{1/2}\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2\left(\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t,\eta)e^{-\frac{1}{A_2}\tilde{\mathcal{Y}}_2(t,\eta)}\left(\int_0^\eta e^{\frac{1}{A_2}\tilde{\mathcal{Y}}_2(t,\theta)}\,d\theta\right)d\eta\right)^{1/2}\\
\leqslant{}& \mathcal{O}(1)\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2.
\end{aligned}
$$

Recalling the estimate for $\bar{T}_1$, we have that

$$
\begin{aligned}
B_{312}\leqslant{}& \frac{1}{A^7}\int_0^1 |\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\\
&\times\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2||\tilde{\mathcal{U}}_2|^3\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)d\eta\\
&+\frac{4}{A^4}\int_0^1 |\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\right.\\
&\left.\times\left(\int_0^\theta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,l))}(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)^2(t,l)\,dl\right)^{1/2}\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)d\eta\\
&+\frac{\sqrt{2}}{A^3}\int_0^1 |\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\right.
\end{aligned}
$$

$$\times\left(\int_0^\theta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,l))}(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,l)\,dl\right)^{1/2}\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\Bigg)\,d\eta$$

$$+\frac{1}{A^3}\int_0^1|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}$$

$$\times\left(\int_0^\theta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,l))}|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|(t,l)\,dl\right)\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\Bigg)\,d\eta$$

$$+\frac{8\sqrt{2}}{\sqrt{3}eA^3}\int_0^1|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}$$

$$\times\left(\int_0^\theta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,l))}\,dl\right)^{1/2}\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\Bigg)\,d\eta|A_1-A_2|$$

$$\leqslant 5\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2+\frac{1}{A^{14}}\int_0^1\tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)$$

$$\times\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|\,|\tilde{\mathcal{U}}_2|^3\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)^2 d\eta$$

$$+\frac{16}{A^8}\int_0^1\tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}$$

$$\times\left(\int_0^\theta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,l))}(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)^2(t,l)\,dl\right)^{1/2}\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\Bigg)^2 d\eta$$

$$+\frac{2}{A^6}\int_0^1\tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}$$

$$\times\left(\int_0^\theta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,l))}(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,l)\,dl\right)^{1/2}\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\Bigg)^2 d\eta$$

$$+\frac{1}{A^6}\int_0^1\tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}$$

$$\times\left(\int_0^\theta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,l))}|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|(t,l)\,dl\right)\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\Bigg)^2 d\eta$$

$$+\frac{128}{3e^2A^6}\int_0^1\tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}$$

$$\times\left(\int_0^\theta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,l))}\,dl\right)^{1/2}\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\Bigg)^2 d\eta|A_1-A_2|^2$$

$$\leqslant 5\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2+\frac{1}{A^{14}}\int_0^1\tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^4\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)$$

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$$
\begin{aligned}
&\times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2\tilde{\mathcal{U}}_2^2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)d\eta\\
&+\frac{16}{A^8}\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)\\
&\times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\right.\\
&\times \left.\left(\int_0^\theta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,l))}(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)^2(t,l)\,dl\right)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)d\eta\\
&+\frac{2}{A^6}\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)\\
&\times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\right.\\
&\times \left.\left(\int_0^\theta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,l))}(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,l)\,dl\right)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)d\eta\\
&+\frac{1}{A^6}\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)\\
&\times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\right.\\
&\times \left.\left(\int_0^\theta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,l))}|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|(t,l)\,dl\right)^2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)d\eta\\
&+\frac{128}{3e^2A^6}\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)\\
&\times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\right.\\
&\times \left.\left(\int_0^\theta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,l)}\,dl\right)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)d\eta|A_1-A_2|^2\\
\leqslant{}& 5\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2\\
&+\frac{2}{A^4}\int_0^1 \tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,\theta)\,d\theta\right)d\eta\\
&+\frac{64}{A^7}\int_0^1 \tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\left(\int_0^\eta e^{-\frac{1}{2A}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\right.
\end{aligned}
$$

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$$\times \left(\int_0^\theta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,l))}(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)^2(t,l)\,dl\right)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)d\eta$$

$$+\frac{8}{A^5}\int_0^1 \tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\left(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\right.$$

$$\times \left(\int_0^\theta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,l))}(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,l)\,dl\right)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)d\eta$$

$$+\frac{4}{A^5}\int_0^1 \tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\left(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\left(\int_0^\theta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,l))}\,dl\right)\right.$$

$$\times \left(\int_0^\theta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,l))}(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,l)\,dl\right)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)d\eta$$

$$+\frac{512}{3e^2A^5}\int_0^1 \tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\left(\int_0^\eta e^{-\frac{1}{2A}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\right.$$

$$\times \left(\int_0^\theta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,l))}\,dl\right)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)d\eta|A_1-A_2|^2$$

$$\leqslant 5\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2$$

$$+A\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t,\eta)e^{-\frac{1}{A_2}\tilde{\mathcal{Y}}_2(t,\eta)}\left(\int_0^\eta e^{\frac{1}{A_2}\tilde{\mathcal{Y}}_2(t,\theta)}(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,\theta)\,d\theta\right)d\eta$$

$$+\frac{32}{A^2}\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\left(\int_0^\eta \tilde{\mathcal{Y}}_{2,\eta}(t,\theta)e^{-\frac{1}{2A}\tilde{\mathcal{Y}}_2(t,\theta)}\right.$$

$$\times \left(\int_0^\theta e^{-(\frac{1}{2A}\tilde{\mathcal{Y}}_2(t,\eta)-\frac{1}{A}\tilde{\mathcal{Y}}_2(t,l))}(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)^2(t,l)\,dl\right)d\theta\right)d\eta$$

$$+4\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\left(\int_0^\eta \tilde{\mathcal{Y}}_{2,\eta}(t,\theta)e^{-\frac{1}{2A_2}\tilde{\mathcal{Y}}_2(t,\theta)}\right.$$

$$\times \left(\int_0^\theta e^{-(\frac{1}{2A_2}\tilde{\mathcal{Y}}_2(t,\eta)-\frac{1}{A_2}\tilde{\mathcal{Y}}_2(t,l))}(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,l)\,dl\right)d\theta\right)d\eta$$

$$+2\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\left(\int_0^\eta \tilde{\mathcal{Y}}_{2,\eta}(t,\theta)e^{-\frac{1}{2A_2}\tilde{\mathcal{Y}}_2(t,\theta)}\right.$$

$$\times \left(\int_0^\theta e^{-(\frac{1}{2A_2}\tilde{\mathcal{Y}}_2(t,\eta)-\frac{1}{A_2}\tilde{\mathcal{Y}}_2(t,l))}\,dl\right)d\theta\right)d\eta\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2$$

$$+\frac{256}{3e^2}\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\left(\int_0^\eta \tilde{\mathcal{Y}}_{2,\eta}(t,\theta)e^{-\frac{1}{4A}\tilde{\mathcal{Y}}_2(t,\theta)}\right.$$

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$$
\begin{aligned}
&\times \left(\int_0^\theta e^{-(\frac{1}{2A}\tilde{\mathcal{Y}}_2(t,\eta)-\frac{3}{4A}\tilde{\mathcal{Y}}_2(t,l))}\,dl\right)d\theta\bigg)\,d\eta|A_1-A_2|^2\\
&\leqslant 5\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2 + A\bigg[-A_2\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,\theta)\,d\theta\,\bigg|_{\eta=0}^1\\
&\quad+\int_0^1(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,\eta)\,d\eta\bigg]\\
&\quad+\frac{32}{A}\int_0^1\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\bigg[-2\int_0^\theta e^{\frac{1}{A}\tilde{\mathcal{Y}}_2(t,l)-\frac{1}{2A}(\tilde{\mathcal{Y}}_2(t,\eta)+\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)^2(t,l)\,dl\,\bigg|_{\theta=0}^\eta\\
&\quad+2\int_0^\eta e^{\frac{1}{2A}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,\eta))}(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)^2(t,\theta)\,d\theta\bigg]\,d\eta\\
&\quad+4A_2\int_0^1\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\\
&\quad\times\bigg[-2\int_0^\theta e^{\frac{1}{A_2}\tilde{\mathcal{Y}}_2(t,l)-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)+\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,l)\,dl\,\bigg|_{\theta=0}^\eta\\
&\quad+2\int_0^\eta e^{\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,\eta))}(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,\theta)\,d\theta\bigg]\,d\eta\\
&\quad+2A_2\int_0^1\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\bigg[-2\int_0^\theta e^{\frac{1}{A_2}\tilde{\mathcal{Y}}_2(t,l)-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\theta)+\tilde{\mathcal{Y}}_2(t,\eta))}\,dl\,\bigg|_{\theta=0}^\eta\\
&\quad+2\int_0^\eta e^{\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,\eta))}\,d\theta\bigg]\,d\eta\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2\\
&\quad+\frac{256A}{3e^2}\int_0^1\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\bigg[-4\int_0^\theta e^{\frac{3}{4A}\tilde{\mathcal{Y}}_2(t,l)-(\frac{1}{4A}\tilde{\mathcal{Y}}_2(t,\theta)+\frac{1}{2A}\tilde{\mathcal{Y}}_2(t,\eta))}\,\bigg|_{\theta=0}^\eta\\
&\quad+4\int_0^\eta e^{\frac{1}{2A}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,\eta))}\,d\theta\bigg]\,d\eta|A_1-A_2|^2\\
&\leqslant \mathcal{O}(1)\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2\\
&\quad+\frac{64}{A}\int_0^1\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)e^{-\frac{1}{2A}\tilde{\mathcal{Y}}_2(t,\eta)}\left(\int_0^\eta e^{\frac{1}{2A}\tilde{\mathcal{Y}}_2(t,\theta)}(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)^2(t,\theta)\,d\theta\right)d\eta\\
&\quad+8A\int_0^1\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)e^{-\frac{1}{2A_2}\tilde{\mathcal{Y}}_2(t,\eta)}\left(\int_0^\eta e^{\frac{1}{2A_2}\tilde{\mathcal{Y}}_2(t,\theta)}(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,\theta)\,d\theta\right)d\eta\\
&\quad+4A\int_0^1\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)e^{-\frac{1}{2A_2}\tilde{\mathcal{Y}}_2(t,\eta)}\left(\int_0^\eta e^{\frac{1}{2A_2}\tilde{\mathcal{Y}}_2(t,\theta)}\,d\theta\right)d\eta\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1024A}{3e^2}\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t,\eta)e^{-\frac{1}{2A}\tilde{\mathcal{Y}}_2(t,\eta)}\left(\int_0^\eta e^{\frac{1}{2A}\tilde{\mathcal{Y}}_2(t,\theta)}\,d\theta\right)d\eta|A_1-A_2|^2\\
&\leqslant \mathcal{O}(1)\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2\\
&+64\bigg[-2\int_0^\eta e^{\frac{1}{2A}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,\eta))}(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)^2(t,\theta)\,d\theta\,\bigg|_{\eta=0}^1\\
&+2\int_0^1(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)^2(t,\eta)\,d\eta\bigg]\\
&+8AA_2\bigg[-2\int_0^\eta e^{\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,\eta))}(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,\theta)\,d\theta\,\bigg|_{\eta=0}^1\\
&+2\int_0^1(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,\eta)\,d\eta\bigg]\\
&+4AA_2\bigg[-2\int_0^\eta e^{\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,\eta))}\,d\theta\,\bigg|_{\eta=0}^1+2\int_0^1 d\eta\bigg]\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2\\
&+\frac{1024A^2}{3e^2}\bigg[-2\int_0^\eta e^{\frac{1}{2A}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,\eta))}\,d\theta\,\bigg|_{\eta=0}^1+2\int_0^1 d\eta\bigg]|A_1-A_2|^2\\
&\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2+\|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2\|^2+|A_1-A_2|^2).
\end{aligned}
$$

Similar calculations yield for B_{313},

$$
\begin{aligned}
B_{313}\leqslant{}&\frac{1}{A^7}\int_0^1|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\\
&\times\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|\tilde{\mathcal{P}}_2|\tilde{\mathcal{U}}_2|\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)d\eta\\
&+\frac{2\sqrt{2}}{A^4}\int_0^1|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\\
&\times\left(\int_0^\theta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,l))}(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})^2(t,l)\,dl\right)^{1/2}\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\bigg)\,d\eta\\
&+\frac{1}{\sqrt{2}A^3}\int_0^1|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\\
&\times\left(\int_0^\theta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,l))}(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,l)\,dl\right)^{1/2}\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\bigg)\,d\eta\\
&+\frac{1}{2A^3}\int_0^1|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}
\end{aligned}
$$

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$$\times \left(\int_0^\theta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,l))}|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t,l)\,dl\right)\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\Bigg)\,d\eta$$

$$+ \frac{4\sqrt{2}}{\sqrt{3}eA^3}\int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\Bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}$$

$$\times \left(\int_0^\theta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,l))}\,dl\right)^{1/2}\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\Bigg)\,d\eta|A_1 - A_2|$$

$$\leqslant 5\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{1}{A^{14}}\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)$$

$$\times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|\,\tilde{\mathcal{P}}_2|\tilde{\mathcal{U}}_2|\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)^2 d\eta$$

$$+ \frac{8}{A^8}\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\Bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}$$

$$\times \left(\int_0^\theta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,l))}(\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2(t,l)\,dl\right)^{1/2}\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\Bigg)^2 d\eta$$

$$+ \frac{1}{2A^6}\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\Bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}$$

$$\times \left(\int_0^\theta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,l))}(\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t,l)\,dl\right)^{1/2}\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\Bigg)^2 d\eta$$

$$+ \frac{1}{4A^6}\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\Bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}$$

$$\times \left(\int_0^\theta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,l))}|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t,l)\,dl\right)\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\Bigg)^2 d\eta$$

$$+ \frac{32}{3e^2A^6}\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\Bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}$$

$$\times \left(\int_0^\theta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,l))}\,dl\right)^{1/2}\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\Bigg)^2 d\eta|A_1 - A_2|^2$$

$$\leqslant 5\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2$$

$$+ \frac{1}{A^{14}}\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)$$

$$\times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2\tilde{\mathcal{P}}_2^2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)d\eta$$

$$+ \frac{8}{A^8}\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)$$

$$
\begin{aligned}
&\times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} \right.\\
&\times \left. \left(\int_0^\theta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,l))} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2(t,l)\, dl \right) \tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\, d\theta \right) d\eta \\
&+ \frac{1}{2A^6} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\, d\theta \right) \\
&\times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} \right.\\
&\times \left. \left(\int_0^\theta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,l))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t,l)\, dl \right) \tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\, d\theta \right) d\eta \\
&+ \frac{1}{4A^6} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\, d\theta \right) \\
&\times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} \right.\\
&\times \left. \left(\int_0^\theta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,l))} |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t,l)\, dl \right)^2 \tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\, d\theta \right) d\eta \\
&+ \frac{32}{3e^2 A^6} \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\, d\theta \right) \\
&\times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} \right.\\
&\times \left. \left(\int_0^\theta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,l))}\, dl \right) \tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\, d\theta \right) d\eta |A_1 - A_2|^2 \\
\leqslant{}& 5\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
&+ \frac{1}{2A^4} \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t,\theta)\, d\theta \right) d\eta \\
&+ \frac{32}{A^7} \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta) \left(\int_0^\eta e^{-\frac{1}{2A}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} \right.\\
&\times \left. \left(\int_0^\theta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,l))} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2(t,l)\, dl \right) \tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\, d\theta \right) d\eta \\
&+ \frac{2}{A^5} \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta) \left(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} \right.\\
&\times \left. \left(\int_0^\theta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,l))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t,l)\, dl \right) \tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\, d\theta \right) d\eta
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{A^5}\int_0^1 \tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\bigg(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\bigg(\int_0^\theta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,l))}\,dl\bigg)\\
&\times\bigg(\int_0^\theta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,l))}(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,l)\,dl\bigg)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\bigg)\,d\eta\\
&+\frac{128}{3e^2A^5}\int_0^1 \tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\bigg(\int_0^\eta e^{-\frac{1}{2A}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\\
&\times\bigg(\int_0^\theta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,l))}\,dl\bigg)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\bigg)\,d\eta|A_1-A_2|^2\\
\leqslant{}& 5\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2\\
&+\frac{A}{4}\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t,\eta)e^{-\frac{1}{A_2}\tilde{\mathcal{Y}}_2(t,\eta)}\bigg(\int_0^\eta e^{\frac{1}{A_2}\tilde{\mathcal{Y}}_2(t,\theta)}(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,\theta)\,d\theta\bigg)\,d\eta\\
&+\frac{16}{A^2}\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\bigg(\int_0^\eta \tilde{\mathcal{Y}}_{2,\eta}(t,\theta)e^{-\frac{1}{2A}\tilde{\mathcal{Y}}_2(t,\theta)}\\
&\times\bigg(\int_0^\theta e^{-(\frac{1}{2A}\tilde{\mathcal{Y}}_2(t,\eta)-\frac{1}{A}\tilde{\mathcal{Y}}_2(t,l))}(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})^2(t,l)\,dl\bigg)\,d\theta\bigg)\,d\eta\\
&+\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\bigg(\int_0^\eta \tilde{\mathcal{Y}}_{2,\eta}(t,\theta)e^{-\frac{1}{2A_2}\tilde{\mathcal{Y}}_2(t,\theta)}\\
&\times\bigg(\int_0^\theta e^{-(\frac{1}{2A_2}\tilde{\mathcal{Y}}_2(t,\eta)-\frac{1}{A_2}\tilde{\mathcal{Y}}_2(t,l))}(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,l)\,dl\bigg)\,d\theta\bigg)\,d\eta\\
&+\frac{1}{2}\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\bigg(\int_0^\eta \tilde{\mathcal{Y}}_{2,\eta}(t,\theta)e^{-\frac{1}{2A_2}\tilde{\mathcal{Y}}_2(t,\theta)}\\
&\times\bigg(\int_0^\theta e^{-(\frac{1}{2A_2}\tilde{\mathcal{Y}}_2(t,\eta)-\frac{1}{A_2}\tilde{\mathcal{Y}}_2(t,l))}\,dl\bigg)\,d\theta\bigg)\,d\eta\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2\\
&+\frac{64}{3e^2}\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\bigg(\int_0^\eta \tilde{\mathcal{Y}}_{2,\eta}(t,\theta)e^{-\frac{1}{4A}\tilde{\mathcal{Y}}_2(t,\theta)}\\
&\times\bigg(\int_0^\theta e^{-(\frac{1}{2A}\tilde{\mathcal{Y}}_2(t,\eta)-\frac{3}{4A}\tilde{\mathcal{Y}}_2(t,l))}\,dl\bigg)\,d\theta\bigg)\,d\eta|A_1-A_2|^2\\
\leqslant{}& 5\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2+\frac{A}{4}\bigg[-A_2\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,\theta)\,d\theta\,\bigg|_{\eta=0}^1\\
&+\int_0^1(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,\eta)\,d\eta\bigg]\\
&+\frac{16}{A}\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t,\eta)
\end{aligned}
$$

$$
\begin{aligned}
&\times \left[-2\int_0^{\theta} e^{\frac{1}{A}\tilde{\mathcal{Y}}_2(t,l)-\frac{1}{2A}(\tilde{\mathcal{Y}}_2(t,\eta)+\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})^2(t,l)\,dl\,\Big|_{\theta=0}^{\eta}\right.\\
&\left.+\,2\int_0^{\eta} e^{\frac{1}{2A}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,\eta))}(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})^2(t,\theta)\,d\theta\right]d\eta\\
&+\,A_2\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\\
&\times \left[-2\int_0^{\theta} e^{\frac{1}{A_2}\tilde{\mathcal{Y}}_2(t,l)-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)+\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,l)\,dl\,\Big|_{\theta=0}^{\eta}\right.\\
&\left.+\,2\int_0^{\eta} e^{\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,\eta))}(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,\theta)\,d\theta\right]d\eta\\
&+\,\frac{A_2}{2}\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\left[-2\int_0^{\theta} e^{\frac{1}{A_2}\tilde{\mathcal{Y}}_2(t,l)-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\theta)+\tilde{\mathcal{Y}}_2(t,\eta))}\,dl\,\Big|_{\theta=0}^{\eta}\right.\\
&\left.+\,2\int_0^{\eta} e^{\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,\eta))}\,d\theta\right]d\eta\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2\\
&+\,\frac{64A}{3e^2}\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\left[-4\int_0^{\theta} e^{\frac{3}{4A}\tilde{\mathcal{Y}}_2(t,l)-(\frac{1}{4A}\tilde{\mathcal{Y}}_2(t,\theta)+\frac{1}{2A}\tilde{\mathcal{Y}}_2(t,\eta))}\,\Big|_{\theta=0}^{\eta}\right.\\
&\left.+\,4\int_0^{\eta} e^{\frac{1}{2A}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,\eta))}\,d\theta\right]d\eta|A_1-A_2|^2\\
\leqslant\;&\mathcal{O}(1)\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2\\
&+\,\frac{32}{A}\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t,\eta)e^{-\frac{1}{2A}\tilde{\mathcal{Y}}_2(t,\eta)}\left(\int_0^{\eta} e^{\frac{1}{2A}\tilde{\mathcal{Y}}_2(t,\theta)}(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})^2(t,\theta)\,d\theta\right)d\eta\\
&+\,2A\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t,\eta)e^{-\frac{1}{2A_2}\tilde{\mathcal{Y}}_2(t,\eta)}\left(\int_0^{\eta} e^{\frac{1}{2A_2}\tilde{\mathcal{Y}}_2(t,\theta)}(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,\theta)\,d\theta\right)d\eta\\
&+\,A\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t,\eta)e^{-\frac{1}{2A_2}\tilde{\mathcal{Y}}_2(t,\eta)}\left(\int_0^{\eta} e^{\frac{1}{2A_2}\tilde{\mathcal{Y}}_2(t,\theta)}\,d\theta\right)d\eta\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2\\
&+\,\frac{256A}{3e^2}\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t,\eta)e^{-\frac{1}{2A}\tilde{\mathcal{Y}}_2(t,\eta)}\left(\int_0^{\eta} e^{\frac{1}{2A}\tilde{\mathcal{Y}}_2(t,\theta)}\,d\theta\right)d\eta|A_1-A_2|^2\\
\leqslant\;&\mathcal{O}(1)\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2\\
&+\,32\left[-2\int_0^{\eta} e^{\frac{1}{2A}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,\eta))}(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})^2(t,\theta)\,d\theta\,\Big|_{\eta=0}^{1}\right.\\
&\left.+\,2\int_0^1 (\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})^2(t,\eta)\,d\eta\right]
\end{aligned}
$$

$$
\begin{aligned}
&+ 2AA_2\Big[-2\int_0^\eta e^{\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,\eta))}(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,\theta)\,d\theta\,\Big|_{\eta=0}^1\\
&+ 2\int_0^1 (\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,\eta)\,d\eta\Big]\\
&+ AA_2\Big[-2\int_0^\eta e^{\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,\eta))}\,d\theta\,\Big|_{\eta=0}^1 +2\int_0^1 d\eta\Big]\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2\\
&+ \frac{256A^2}{3e^2}\Big[-2\int_0^\eta e^{\frac{1}{2A}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,\eta))}\,d\theta\,\Big|_{\eta=0}^1 +2\int_0^1 d\eta\Big]|A_1-A_2|^2\\
&\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2+\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+|A_1-A_2|^2).
\end{aligned}
$$

Last but not least, direct calculations for B_{314} yield

$$
\begin{aligned}
B_{314} &\leqslant \frac{6}{A^3}|A_1-A_2|\int_0^1 |\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\\
&\quad\times\Big(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\Big(\int_0^\theta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,l))}\,dl\Big)\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\Big)\,d\eta\\
&\leqslant \frac{6}{A^3}|A_1-A_2|\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|\\
&\quad\times\Big(\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\Big(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\\
&\quad\times\Big(\int_0^\theta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,l))}\,dl\Big)\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\Big)^2\,d\eta\Big)^{1/2}\\
&\leqslant \frac{6}{A^3}|A_1-A_2|\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|\\
&\quad\times\Big(\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\Big(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\Big)\\
&\quad\times\Big(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\Big(\int_0^\theta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,l))}\,dl\Big)^2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\Big)\,d\eta\Big)^{1/2}\\
&\leqslant \frac{12}{A^{5/2}}|A_1-A_2|\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|\\
&\quad\times\Big(\int_0^1 \tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\Big(\int_0^\eta e^{-\frac{1}{2A}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\\
&\quad\times\Big(\int_0^\theta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,l))}\,dl\Big)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\Big)\,d\eta\Big)^{1/2}\\
&\leqslant 6\sqrt{2}|A_1-A_2|\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|
\end{aligned}
$$

$$
\begin{aligned}
&\times \bigg(\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\bigg(\int_0^\eta \tilde{\mathcal{Y}}_{2,\eta}(t,\theta)e^{-\frac{1}{4A}\tilde{\mathcal{Y}}_2(t,\theta)} \\
&\times \bigg(\int_0^\theta e^{\frac{3}{4A}\tilde{\mathcal{Y}}_2(t,l)-\frac{1}{2A}\tilde{\mathcal{Y}}_2(t,\eta)}\,dl\bigg)\,d\theta\bigg)\,d\eta\bigg)^{1/2} \\
&\leqslant \mathcal{O}(1)|A_1-A_2|\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|.
\end{aligned}
$$

Collecting the estimates we have obtained, we conclude

$$
|\bar{B}_{31}| \leqslant \mathcal{O}(1)(\|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2\|^2+\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2+\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+|A_1-A_2|^2).
$$

Recalling (4.16e), a direct computation yields for $\bar{B}_{33}$ (and in much the same way for $\bar{B}_{34}$) that

$$
\begin{aligned}
|\bar{B}_{33}| &\leqslant \frac{1}{A^7}\int_0^1 |\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|\tilde{\mathcal{Y}}_{2,\eta}(t,\eta) \\
&\quad\times \bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\min_j(\tilde{\mathcal{D}}_j)|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2|\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\bigg)\,d\eta \\
&\leqslant \|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2+\frac{1}{A^{14}}\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta) \\
&\quad\times \bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\min_j(\tilde{\mathcal{D}}_j)\tilde{\mathcal{Y}}_{2,\eta}|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2|(t,\theta)\,d\theta\bigg)^2\,d\eta \\
&\leqslant \|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2 \\
&\quad+\frac{4}{A^{12}}\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\bigg(\int_0^\eta e^{-\frac{3}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\bigg) \\
&\quad\times \bigg(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)^2(t,\theta)\,d\theta\bigg)\,d\eta \\
&\leqslant \|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2 \\
&\quad+\frac{4}{A^6}\int_0^1 \tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\bigg(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)^2(t,\theta)\,d\theta\bigg)\,d\eta \\
&\leqslant \|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2 \\
&\quad+\frac{2}{A}\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t,\eta)e^{-\frac{1}{2A_2}\tilde{\mathcal{Y}}_2(t,\eta)}\bigg(\int_0^\eta e^{\frac{1}{2A_2}\tilde{\mathcal{Y}}_2(t,\theta)}(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)^2(t,\theta)\,d\theta\bigg)\,d\eta \\
&\leqslant \|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2 \\
&\quad+\frac{2A_2}{A}\bigg[-2\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)^2(t,\theta)\,d\theta\,\bigg|_{\eta=0}^1
\end{aligned}
$$

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$$
\begin{aligned}
&\quad + 2\int_0^1 (\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)^2(t,\eta)\,d\eta\Big]\\
&\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2+\|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2\|^2).
\end{aligned}
$$

Direct calculations yield for $\bar{B}_{35}$ (and in much the same way for $\bar{B}_{36}$) that

$$
\begin{aligned}
|\bar{B}_{35}| &\leqslant \frac{1}{A^7}\int_0^1 |\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\\
&\quad \times \bigg(\int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}-e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))})\\
&\quad \times \min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{2,\eta}\mathbb{1}_{B(\eta)}(t,\theta)\,d\theta\bigg)\,d\eta\\
&\leqslant \frac{1}{A^7A_2}\int_0^1 |\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\bigg(\int_0^\eta (|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|(t,\eta)+|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|(t,\theta))\\
&\quad \times e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{D}}_2\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\bigg)\,d\eta\\
&\leqslant \frac{1}{A^7A_2}\int_0^1 (\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\\
&\quad \times \bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{D}}_2\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\bigg)\,d\eta\\
&\quad + \frac{1}{A^7A_2}\int_0^1 |\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\\
&\quad \times \bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{D}}_2\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|(t,\theta)\,d\theta\bigg)\,d\eta\\
&\leqslant \frac{2}{A^7}\int_0^1 (\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2^2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\bigg)^{1/2}\\
&\quad \times \bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\bigg)^{1/2}\,d\eta\\
&\quad + \|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2\\
&\quad + \frac{4}{A^{14}}\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\bigg)\\
&\quad \times \bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2^2\tilde{\mathcal{Y}}_{2,\eta}(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,\theta)\,d\theta\bigg)\,d\eta
\end{aligned}
$$

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$$
\begin{aligned}
&\leqslant \|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2+\frac{2\sqrt{6}}{A^4}\int_0^1(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\,d\eta\\
&\quad+\frac{16}{A^{13}}\int_0^1\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\\
&\quad\times\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2^2\tilde{\mathcal{Y}}_{2,\eta}(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,\theta)\,d\theta\right)d\eta\\
&\leqslant(1+\sqrt{6}A)\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2\\
&\quad+A\int_0^1\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)e^{-\frac{1}{A_2}\tilde{\mathcal{Y}}_2(t,\eta)}\left(\int_0^\eta e^{\frac{1}{A_2}\tilde{\mathcal{Y}}_2(t,\theta)}(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,\theta)\,d\theta\right)d\eta\\
&=(1+\sqrt{6}A)\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2\\
&\quad+AA_2\Bigg[-\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,\theta)\,d\theta\,\Bigg|_{\eta=0}^1\\
&\quad+\int_0^1(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,\eta)\,d\eta\Bigg]\\
&\leqslant\mathcal{O}(1)\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2. \qquad (5.22)
\end{aligned}
$$

Direct calculations yield for $\hat{B}_{35}$ (and in much the same way for $\hat{B}_{36}$) that

$$
\begin{aligned}
|\hat{B}_{35}|&\leqslant\mathbb{1}_{A_1\leqslant A_2}\frac{4|A_1-A_2|}{aA^7e}\int_0^1|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\\
&\quad\times\left(\int_0^\eta\min_j(e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)d\eta\\
&\leqslant\frac{8}{aA^6e}\int_0^1|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\\
&\quad\times\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)^{1/2}\\
&\quad\times\left(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_1\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)^{1/2}d\eta|A_1-A_2|\\
&\leqslant\frac{16}{aA^{11/2}e}\int_0^1|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|\tilde{\mathcal{P}}_2^{1/2}\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\\
&\quad\times\left(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_1\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)^{1/2}d\eta|A_1-A_2|\\
&\leqslant\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2
\end{aligned}
$$

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$$
\begin{aligned}
&+ \frac{256}{a^2A^{11}e^2}\int_0^1 \tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\\
&\times \left(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_1\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)d\eta|A_1-A_2|^2\\
\leqslant{}& \|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2\\
&+\frac{16A}{e^2}\int_0^1\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)e^{-\frac{1}{2A_2}\tilde{\mathcal{Y}}_2(t,\eta)}\left(\int_0^\eta e^{\frac{1}{2A_2}\tilde{\mathcal{Y}}_2(t,\theta)}\,d\theta\right)d\eta|A_1-A_2|^2\\
\leqslant{}& \mathcal{O}(1)(\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2+|A_1-A_2|^2).
\end{aligned}
$$

As far as the term $\bar{B}_{37}$ is concerned, it can be estimated as follows (we use (3.40)):

$$
\begin{aligned}
|\bar{B}_{37}| ={}& \frac{1}{A^7}\left|\int_0^1(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)(\tilde{\mathcal{Y}}_{1,\eta}-\tilde{\mathcal{Y}}_{2,\eta})(t,\eta)\right.\\
&\times \min_k\left(\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\right.\\
&\left.\left.\times \min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{k,\eta}(t,\theta)\,d\theta\right)d\eta\right|\\
={}& \frac{1}{2A^7}\left|(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,\eta)\min_k\left(\int_0^\eta\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\right.\right.\\
&\left.\times\min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{k,\eta}(t,\theta)\,d\theta\right)\Bigg|_{\eta=0}^1\\
&-\int_0^1(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,\eta)\frac{d}{d\eta}\min_k\left(\int_0^\eta\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\right.\\
&\left.\left.\times\min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{k,\eta}(t,\theta)\,d\theta\right)d\eta\right|\\
={}& \frac{1}{2A^7}\left|\int_0^1(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,\eta)\frac{d}{d\eta}\min_k\left(\int_0^\eta\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\right.\right.\\
&\left.\left.\times\min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{k,\eta}(t,\theta)\,d\theta\right)d\eta\right|\\
\leqslant{}& \mathcal{O}(1)\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2,
\end{aligned}
\tag{5.23}
$$

where $\mathcal{O}(1)$ denotes some constant only depending on A, which remains bounded as $A\to 0$, provided we can show that the derivative in the latter integral exists and is uniformly bounded; see Lemma A.7.

As far as the term $\bar{B}_{38}$ (a similar argument works for $\bar{B}_{39}$) is concerned, the integral can be estimated as follows:

$$\begin{aligned}
\bar{B}_{38} &= \frac{1}{A^7}\int_0^1 (\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)\tilde{\mathcal{Y}}_{2,\eta}\mathbb{1}_{D^c}(t,\eta)\Bigg(\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}) \\
&\quad\times \min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)(\tilde{\mathcal{Y}}_{2,\eta}-\tilde{\mathcal{Y}}_{1,\eta})(t,\theta)\,d\theta\Bigg)\,d\eta \\
&= \frac{1}{A^7}\int_0^1 (\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)\tilde{\mathcal{Y}}_{2,\eta}\mathbb{1}_{D^c}(t,\eta)\Bigg[(\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1)(t,\theta)\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}) \\
&\quad\times \min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)(t,\theta)\Bigg|_{\theta=0}^{\eta} \\
&\quad-\int_0^\eta (\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1)(t,\theta)\Bigg[\Bigg(\frac{d}{d\theta}\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\Bigg) \\
&\quad\times \min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)(t,\theta) \\
&\quad+\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\Bigg(\frac{d}{d\theta}\min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\Bigg)(t,\theta)\,d\theta\Bigg]\,d\eta\Bigg] \\
&= -\frac{1}{A^7}\int_0^1 (\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2\min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{2,\eta}\mathbb{1}_{D^c}(t,\eta)\,d\eta \\
&\quad+\frac{1}{A^7}\int_0^1 (\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\mathbb{1}_{D^c}(t,\eta) \\
&\quad\times\int_0^\eta (\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)(t,\theta)\Bigg[\Bigg(\frac{d}{d\theta}\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\Bigg) \\
&\quad\times \min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)(t,\theta) \\
&\quad+\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\Bigg(\frac{d}{d\theta}\min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\Bigg)(t,\theta)\,d\theta\Bigg]\,d\eta \\
&= \bar{M}_1+\bar{M}_2.
\end{aligned}$$

As far as the first term $\bar{M}_1$ is concerned, we have since

$$\begin{aligned}
\min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{i,\eta}(t,\eta) &\leqslant 2\min_j(A_j\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{i,\eta}(t,\eta) \\
&\leqslant A^6\min_j(\tilde{\mathcal{U}}_j^+)(t,\eta)\leqslant\frac{A^8}{\sqrt{2}},
\end{aligned}$$

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that

$$|\bar{M}_1| \leqslant \frac{1}{A^7}\int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2 \min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\,d\eta$$
$$\leqslant \frac{A}{\sqrt{2}}\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 = \mathcal{O}(1)\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2,$$

where $\mathcal{O}(1)$ denotes some constant, which only depends on A and which remains bounded as $A \to 0$.

The second term $\bar{M}_2$, on the other hand, is a bit more demanding.

(i): First of all, recall (A.4), that is,

$$\left|\frac{d}{d\theta}\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\right| \leqslant \frac{1}{a}\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\max_j(\tilde{\mathcal{Y}}_{j,\eta})(t,\theta),$$

which implies that

$$\frac{1}{A^7}\bigg|\int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\mathbb{1}_{D^c}(t,\eta)\int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t,\theta)$$
$$\times\left(\frac{d}{d\theta}\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\right)\min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)(t,\theta)\,d\theta\,d\eta\bigg|$$
$$\leqslant \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{1}{A^{14}}\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\mathbb{1}_{D^c}(t,\eta)$$
$$\times\bigg(\int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t,\theta)\left(\frac{d}{d\theta}\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\right)$$
$$\times\min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)(t,\theta)\,d\theta\bigg)^2 d\eta$$
$$\leqslant \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \frac{1}{a^2A^2}\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)$$
$$\times\left(\int_0^\eta |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t,\theta)e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\min_j(\tilde{\mathcal{U}}_j^+)(t,\theta)\,d\theta\right)^2 d\eta$$
$$\leqslant \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2$$
$$+\frac{a^2}{2A^2}\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\left(\int_0^\eta |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t,\theta)e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\,d\theta\right)^2 d\eta$$
$$\leqslant \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2$$
$$+\frac{1}{2}\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\left(\int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t,\theta)e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\,d\theta\right)$$

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$$
\begin{aligned}
&\times\left(\int_0^\eta e^{-\frac{3}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\,d\theta\right)d\eta\\
&\leqslant \|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2\\
&\quad+\frac{1}{A_2^5}\int_0^1\tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\left(\int_0^\eta(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,\theta)e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\,d\theta\right)\\
&\quad\times\left(\int_0^\eta e^{-\frac{3}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}+\tilde{\mathcal{H}}_{2,\eta})(t,\theta)\,d\theta\right)d\eta\\
&\leqslant \|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2\\
&\quad+\frac{6}{A_2^4}\int_0^1\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\left(\int_0^\eta(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,\theta)e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\,d\theta\right)d\eta\\
&\leqslant \|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2\\
&\quad+3A\int_0^1\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)e^{-\frac{1}{2A_2}\tilde{\mathcal{Y}}_2(t,\eta)}\left(\int_0^\eta(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,\theta)e^{\frac{1}{2A_2}\tilde{\mathcal{Y}}_2(t,\theta)}\,d\theta\right)d\eta\\
&=\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2-6AA_2\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,\theta)\,d\theta\,\Big|_{\eta=0}^1\\
&\quad+6AA_2\int_0^1(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,\eta)\,d\eta\\
&\leqslant \mathcal{O}(1)\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2,
\end{aligned}
$$

where we used (4.13), (4.16e), and (4.16f). Again, $\mathcal{O}(1)$ denotes some constant only depending on A, which remains bounded as $A\to 0$.

(ii): First of all, we have to establish that $\min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)(t,\theta)$ is Lipschitz continuous with a uniformly bounded Lipschitz constant. More precisely, in Lemma A.4 we show that

$$
\left|\frac{d}{d\theta}(\min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+))(t,\theta)\right|\leqslant \mathcal{O}(1)\sqrt{A}A^4(\min_j(\tilde{\mathcal{D}}_j)^{1/2}+|\tilde{\mathcal{U}}_2|)(t,\theta).
$$

We are now ready to establish a Lipschitz estimate for the second part of $\bar{M}_2$. Indeed,

$$
\begin{aligned}
&\frac{1}{A^7}\left|\int_0^1(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)\tilde{\mathcal{Y}}_{2,\eta}\mathbb{1}_{D^c}(t,\eta)\int_0^\eta(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)(t,\theta)\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\right.\\
&\qquad\left.\times\left(\frac{d}{d\theta}\min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\right)(t,\theta)\,d\theta\,d\eta\right|
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant \|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2+\frac{1}{A^{14}}\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2\mathbb{1}_{D^c}(t,\eta)\\
&\quad\times\bigg(\int_0^\eta(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)(t,\theta)\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\\
&\quad\times\bigg(\frac{d}{d\theta}\min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\bigg)(t,\theta)\,d\theta\bigg)^2\,d\eta\\
&\leqslant \|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2\\
&\quad+\mathcal{O}(1)\frac{1}{A^5}\int_0^1\tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\bigg(\int_0^\eta\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|\\
&\quad\times\big(\min_j(\tilde{\mathcal{D}}_j)^{1/2}+|\tilde{\mathcal{U}}_2|\big)(t,\theta)\,d\theta\bigg)^2\,d\eta\\
&\leqslant \|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2\\
&\quad+\mathcal{O}(1)\frac{1}{A^5}\int_0^1\tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|\tilde{\mathcal{P}}_2^{1/2}(t,\theta)\,d\theta\bigg)^2\,d\eta\\
&\leqslant \|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2\\
&\quad+\mathcal{O}(1)\frac{1}{A^5}\int_0^1\tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\bigg(\int_0^\eta e^{-\frac{3}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2(t,\theta)\,d\theta\bigg)\\
&\quad\times\bigg(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,\theta)\,d\theta\bigg)\,d\eta\\
&\leqslant \|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2\\
&\quad+\mathcal{O}(1)\frac{1}{A^5}\int_0^1\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\bigg(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,\theta)\,d\theta\bigg)\,d\eta\\
&\leqslant \|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2\\
&\quad+\mathcal{O}(1)\int_0^1\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)e^{-\frac{1}{2A_2}\tilde{\mathcal{Y}}_2(t,\eta)}\bigg(\int_0^\eta e^{\frac{1}{2A_2}\tilde{\mathcal{Y}}_2(t,\theta)}(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,\theta)\,d\theta\bigg)\,d\eta\\
&\leqslant \|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2\\
&\quad+\mathcal{O}(1)A_2\bigg(-2\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,\theta)\,d\theta\,\bigg|_{\eta=0}^1\\
&\quad+\int_0^1 2(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,\eta)\,d\eta\bigg)\\
&\leqslant \mathcal{O}(1)\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2,
\end{aligned}
$$

where we used (4.16i). Thus we conclude that

$$|\bar{B}_{38}| + |\bar{B}_{39}| \leqslant \mathcal{O}(1)\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2.$$

Next we have a look at $\bar{K}_{14}$, which can be rewritten as follows:

$$\begin{aligned}
\bar{K}_{14} &= \frac{3}{A^6}\int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t,\eta)\Big(\tilde{\mathcal{Y}}_{1,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\tilde{\mathcal{P}}_1\tilde{\mathcal{U}}_1\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta \\
&\quad - \tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2\tilde{\mathcal{U}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\Big)\,d\eta \\
&= \frac{3}{A^6}\int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t,\eta)\Big(\tilde{\mathcal{Y}}_{1,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\tilde{\mathcal{P}}_1\tilde{\mathcal{U}}_1^+\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta \\
&\quad - \tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\Big)\,d\eta \\
&\quad + \frac{3}{A^6}\int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t,\eta)\Big(\tilde{\mathcal{Y}}_{1,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\tilde{\mathcal{P}}_1\tilde{\mathcal{U}}_1^-\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta \\
&\quad - \tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2\tilde{\mathcal{U}}_2^-\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\Big)\,d\eta \\
&= \bar{K}_{14}^+ + \bar{K}_{14}^-.
\end{aligned}$$

Note that both $\bar{K}_{14}^+$ and $\bar{K}_{14}^-$ have the same structure. Moreover, having a close look at $\bar{K}_{14}^+$ one has

$$\bar{K}_{14}^+ = -3\tilde{K}_1,$$

where $\tilde{K}_1$ is defined in (5.15). Thus we can immediately conclude that

$$|\bar{K}_{14}^+| \leqslant \mathcal{O}(1)(\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + |A_1 - A_2|).$$

Next, we have a look at $\bar{K}_{15}$, which can be rewritten as follows:

$$\begin{aligned}
\bar{K}_{15} &= \frac{1}{A^6}\int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t,\eta)\Big(\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^3\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta \\
&\quad - \tilde{\mathcal{Y}}_{1,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\tilde{\mathcal{U}}_1^3\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\Big)\,d\eta \\
&= \frac{1}{A^6}\int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t,\eta)\Big(\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{U}}_2^+)^3\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta \\
&\quad - \tilde{\mathcal{Y}}_{1,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}(\tilde{\mathcal{U}}_1^+)^3\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\Big)\,d\eta \\
&\quad + \frac{1}{A^6}\int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t,\eta)\Big(\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{U}}_2^-)^3\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta
\end{aligned}$$

$$
\begin{aligned}
&- \tilde{\mathcal{Y}}_{1,\eta}(t,\eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))} (\tilde{\mathcal{U}}_1^-)^3 \tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\, d\theta\Big)\, d\eta \\
&= \bar{K}_{15}^+ + \bar{K}_{15}^-.
\end{aligned}
$$

Note that both $\bar{K}_{15}^+$ and $\bar{K}_{15}^-$ have the same structure. Having a close look at $\bar{K}_{15}^+$ one has

$$
\bar{K}_{15}^+ = \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(J_1 + J_2 + J_3 + J_4 + J_5 + J_6 + J_7 + J_8)(t,\eta)\, d\eta,
$$

where $J_1, \ldots, J_8$ are defined in (5.10). Thus we can conclude immediately that

$$
\begin{aligned}
|\bar{K}_{15}^+| \leqslant \mathcal{O}(1)(&\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 \\
&+ \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + |A_1 - A_2|^2).
\end{aligned}
$$

Finally, we have a look at $\bar{K}_{16}$, which can be rewritten as follows:

$$
\begin{aligned}
\bar{K}_{16} = \frac{1}{2A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t,\eta) \Big(&\tilde{\mathcal{Y}}_{2,\eta}(t,\eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} A_2^5 \tilde{\mathcal{U}}_2(t,\theta)\, d\theta \\
&- \tilde{\mathcal{Y}}_{1,\eta}(t,\eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))} A_1^5 \tilde{\mathcal{U}}_1(t,\theta)\, d\theta \Big)\, d\eta \\
= \frac{A_2^5 - A_1^5}{2A^6} \mathbb{1}_{A_1 \leqslant A_2} &\int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t,\eta) \\
&\times \Big(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2(t,\theta)\, d\theta \Big)\, d\eta \\
&+ \frac{A_2^5 - A_1^5}{2A^6} \mathbb{1}_{A_2 < A_1} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{1,\eta}(t,\eta) \\
&\times \Big(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{U}}_1(t,\theta)\, d\theta \Big)\, d\eta \\
&+ \frac{a^5}{2A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{2,\eta}(t,\eta) \\
&\times \Big(\int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}) \tilde{\mathcal{U}}_2 \mathbb{1}_{B(\eta)}(t,\theta)\, d\theta \Big)\, d\eta \\
&+ \frac{a^5}{2A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \tilde{\mathcal{Y}}_{1,\eta}(t,\eta) \\
&\times \Big(\int_0^\eta (e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}) \tilde{\mathcal{U}}_1 \mathbb{1}_{B(\eta)^c}(t,\theta)\, d\theta \Big)\, d\eta
\end{aligned}
$$

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$$
\begin{aligned}
&+ \mathbb{1}_{A_1 \leqslant A_2} \frac{a^5}{2A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\bigg(\int_0^\eta \big(\min_j(e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\\
&\quad - \min_j(e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\big)\tilde{\mathcal{U}}_2(t,\theta)\,d\theta\bigg)\,d\eta\\
&+ \mathbb{1}_{A_2 < A_1} \frac{a^5}{2A^6} \int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)\tilde{\mathcal{Y}}_{1,\eta}(t,\eta)\bigg(\int_0^\eta \big(\min_j(e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\\
&\quad - \min_j(e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\big)\tilde{\mathcal{U}}_1(t,\theta)\,d\theta\bigg)\,d\eta\\
&+ \frac{a^5}{2A^6}\int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\\
&\times \bigg(\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})(\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+)\mathbb{1}_{\tilde{\mathcal{U}}_1^+ \leqslant \tilde{\mathcal{U}}_2^+}(t,\theta)\,d\theta\bigg)\,d\eta\\
&+ \frac{a^5}{2A^6}\int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\\
&\times \bigg(\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})(\tilde{\mathcal{U}}_2^- - \tilde{\mathcal{U}}_1^-)\mathbb{1}_{\tilde{\mathcal{U}}_2^- \leqslant \tilde{\mathcal{U}}_1^-}(t,\theta)\,d\theta\bigg)\,d\eta\\
&+ \frac{a^5}{2A^6}\int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)\tilde{\mathcal{Y}}_{1,\eta}(t,\eta)\\
&\times \bigg(\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})(\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+)\mathbb{1}_{\tilde{\mathcal{U}}_2^+ \leqslant \tilde{\mathcal{U}}_1^+}(t,\theta)\,d\theta\bigg)\,d\eta\\
&+ \frac{a^5}{2A^6}\int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)\tilde{\mathcal{Y}}_{1,\eta}(t,\eta)\\
&\times \bigg(\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})(\tilde{\mathcal{U}}_2^- - \tilde{\mathcal{U}}_1^-)\mathbb{1}_{\tilde{\mathcal{U}}_1^- \leqslant \tilde{\mathcal{U}}_2^-}(t,\theta)\,d\theta\bigg)\,d\eta\\
&+ \frac{a^5}{2A^6}\int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(\tilde{\mathcal{Y}}_{2,\eta} - \tilde{\mathcal{Y}}_{1,\eta})(t,\eta)\\
&\times \bigg(\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{U}}_j^+)(t,\theta)\,d\theta\bigg)\,d\eta\\
&+ \frac{a^5}{2A^6}\int_0^1 (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(\tilde{\mathcal{Y}}_{2,\eta} - \tilde{\mathcal{Y}}_{1,\eta})(t,\eta)\\
&\times \bigg(\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\max_j(\tilde{\mathcal{U}}_j^-)(t,\theta)\,d\theta\bigg)\,d\eta\\
&= \bar{B}_{61} + \bar{B}_{62} + \bar{B}_{63} + \bar{B}_{64} + \bar{B}_{65} + \bar{B}_{66}\\
&\quad + \bar{B}_{67}^+ + \bar{B}_{67}^- + \bar{B}_{68}^+ + \bar{B}_{68}^- + \bar{B}_{69}^+ + \bar{B}_{69}^-.
\end{aligned}
$$

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The key observation, which rescues the whole paper, is again

$$A_i^5 \leqslant 2(\tilde{\mathcal{P}}_i\tilde{\mathcal{Y}}_{i,\eta} + \tilde{\mathcal{H}}_{i,\eta})(t,\eta),$$

which yields for $\bar{B}_{61}$ (and similarly for $\bar{B}_{62}$) that

$$\begin{aligned}
|\bar{B}_{61}| &\leqslant \frac{A_2^5 - A_1^5}{2A^{11}} \int_0^1 |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|\tilde{\mathcal{Y}}_{2,\eta}(t,\eta) \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} A_2^5 |\tilde{\mathcal{U}}_2|(t,\theta)\, d\theta \right) d\eta \\
&\leqslant \frac{5}{2A^7}|A_2 - A_1| \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| \\
&\quad \times \left(\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} A_2^5 |\tilde{\mathcal{U}}_2|(t,\theta) \right)^2 d\eta \right)^{1/2} \\
&\leqslant \frac{10}{A^7}|A_2 - A_1| \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| \\
&\quad \times \left(\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta) \left[\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 |\tilde{\mathcal{U}}_2| \tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\, d\theta \right)^2 \right. \right. \\
&\quad \left. \left. + \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} |\tilde{\mathcal{U}}_2| \tilde{\mathcal{H}}_{2,\eta}(t,\theta)\, d\theta \right)^2 \right] d\eta \right)^{1/2} \\
&\leqslant \frac{10}{A^7}|A_2 - A_1| \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| \left(\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta) \right. \\
&\quad \times \left[\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\, d\theta \right) \right. \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\, d\theta \right) \\
&\quad \left. \left. + \|\tilde{\mathcal{U}}_2(t,\cdot)\|_{L^\infty}^2 \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{H}}_{2,\eta}(t,\theta)\, d\theta \right)^2 \right] d\eta \right)^{1/2} \\
&\leqslant \frac{10}{A^7}|A_2 - A_1| \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| \\
&\quad \times \left(\int_0^1 (6A^6 + 16A^2 \|\tilde{\mathcal{U}}_2(t,\cdot)\|_{L^\infty}^2) \tilde{\mathcal{P}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\, d\eta \right)^{1/2} \\
&\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + |A_2 - A_1|^2).
\end{aligned}$$

Next, we have a close look at $\bar{B}_{63}$ (and similarly for $\bar{B}_{64}$). Recalling the definition of $B(\eta)$ (5.11), we have

$$
\begin{aligned}
|\bar{B}_{63}| &\leqslant \frac{a^5}{2A^6A_2}\int_0^1 |\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\bigg(\int_0^\eta (|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|(t,\eta)+|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|(t,\theta))\\
&\quad\times e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}|\tilde{\mathcal{U}}_2|(t,\theta)\,d\theta\bigg)\,d\eta\\
&\leqslant \frac{a^5}{2A^6A_2}\int_0^1 (\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}|\tilde{\mathcal{U}}_2|(t,\theta)\,d\theta\bigg)\,d\eta\\
&\quad+\frac{a^5}{2A^6A_2}\int_0^1 |\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\\
&\quad\times\bigg(\int_0^\eta |\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|(t,\theta)e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}|\tilde{\mathcal{U}}_2|(t,\theta)\,d\theta\bigg)\,d\eta\\
&\leqslant \frac{a^5}{A^6A_2^6}\int_0^1 (\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\\
&\quad\times\bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}+\tilde{\mathcal{H}}_{2,\eta})|\tilde{\mathcal{U}}_2|(t,\theta)\,d\theta\bigg)\,d\eta\\
&\quad+\frac{a^5}{A^6A_2^6}\int_0^1 |\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\\
&\quad\times\bigg(\int_0^\eta |\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|(t,\theta)e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}+\tilde{\mathcal{H}}_{2,\eta})|\tilde{\mathcal{U}}_2|(t,\theta)\,d\theta\bigg)\,d\eta\\
&\leqslant \frac{1}{A^6A_2}\int_0^1 (\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\\
&\quad\times\bigg(\frac{1}{A_2}\tilde{\mathcal{P}}_2^2\tilde{\mathcal{Y}}_{2,\eta}+A_2\tilde{\mathcal{U}}_2^2\tilde{\mathcal{Y}}_{2,\eta}+\|\tilde{\mathcal{U}}_2(t,\,\cdot\,)\|_{L^\infty}\tilde{\mathcal{H}}_{2,\eta}\bigg)(t,\theta)\,d\theta\bigg)\,d\eta\\
&\quad+\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2\\
&\quad+\frac{1}{A^{12}A_2^2}\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\bigg(\int_0^\eta |\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|(t,\theta)e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\\
&\quad\times(\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}+\tilde{\mathcal{H}}_{2,\eta})|\tilde{\mathcal{U}}_2|(t,\theta)\,d\theta\bigg)^2\,d\eta\\
&\leqslant \|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2+\mathcal{O}(1)\frac{1}{A^5}\int_0^1 (\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\,d\eta\\
&\quad+\frac{2}{A^{12}A_2^2}\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\\
&\quad\times\bigg[\bigg(\int_0^\eta |\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|(t,\theta)e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2|\tilde{\mathcal{U}}_2|\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\bigg)^2
\end{aligned}
$$

$$
\begin{aligned}
&+\bigg(\int_0^\eta |\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|(t,\theta)e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}|\tilde{\mathcal{U}}_2|\tilde{\mathcal{H}}_{2,\eta}(t,\theta)\,d\theta\bigg)^2\bigg]\,d\eta\\
&\leqslant \mathcal{O}(1)\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2+\frac{2}{A^{12}A_2^2}\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\\
&\quad\times\bigg[\bigg(\int_0^\eta(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,\theta)e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2^2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\bigg)\\
&\quad\times\bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\bigg)\\
&\quad+\bigg(\int_0^\eta(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,\theta)e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^2\tilde{\mathcal{H}}_{2,\eta}(t,\theta)\,d\theta\bigg)\\
&\quad\times\bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{H}}_{2,\eta}(t,\theta)\,d\theta\bigg)\bigg]\,d\eta\\
&\leqslant \mathcal{O}(1)\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2\\
&\quad+\frac{8}{A^{12}A_2}\int_0^1 \tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\\
&\quad\times\bigg(\int_0^\eta(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,\theta)e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{P}}_2^2\tilde{\mathcal{Y}}_{2,\eta}+\tilde{\mathcal{U}}_2^2\tilde{\mathcal{H}}_{2,\eta})(t,\theta)\,d\theta\bigg)\,d\eta\\
&\leqslant \mathcal{O}(1)\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2+\frac{4}{A^8}\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t,\eta)e^{-\frac{1}{A_2}\tilde{\mathcal{Y}}_2(t,\eta)}\\
&\quad\times\bigg(\int_0^\eta(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2e^{\frac{1}{A_2}\tilde{\mathcal{Y}}_2(t,\theta)}(\tilde{\mathcal{P}}_2^2\tilde{\mathcal{Y}}_{2,\eta}+\tilde{\mathcal{U}}_2^2\tilde{\mathcal{H}}_{2,\eta})(t,\theta)\,d\theta\bigg)\,d\eta\\
&=\mathcal{O}(1)\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2+\frac{4A_2}{A^8}\bigg(-\int_0^\eta(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,\theta)e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\\
&\quad\times(\tilde{\mathcal{P}}_2^2\tilde{\mathcal{Y}}_{2,\eta}+\tilde{\mathcal{U}}_2^2\tilde{\mathcal{H}}_{2,\eta})(t,\theta)d\theta\,\bigg|_{\eta=0}^1\\
&\quad+\int_0^1(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(\tilde{\mathcal{P}}_2^2\tilde{\mathcal{Y}}_{2,\eta}+\tilde{\mathcal{U}}_2^2\tilde{\mathcal{H}}_{2,\eta})(t,\eta)\,d\eta\bigg)\\
&\leqslant \mathcal{O}(1)\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2.
\end{aligned}
$$

Next, we have a look at $\bar{B}_{65}$ (a similar argument works for $\bar{B}_{66}$). Direct calculations yield

$$
\begin{aligned}
|\bar{B}_{65}|&\leqslant \frac{a^5}{2A^6}\int_0^1|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\\
&\quad\times\bigg(\frac{4}{ae}\int_0^\eta\min_j(e^{-\frac{3}{4A_2}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})|\tilde{\mathcal{U}}_2|(t,\theta)\,d\theta\bigg)\,d\eta|A_1-A_2|
\end{aligned}
$$

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$$
\begin{aligned}
&\leqslant \frac{2}{A^2 e}\int_0^1 |\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\\
&\quad\times\left(\int_0^\eta e^{-\frac{3}{4A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}|\tilde{\mathcal{U}}_2|(t,\theta)\,d\theta\right)d\eta|A_1-A_2|\\
&\leqslant \frac{4}{A^2 A_2^5 e}\int_0^1 |\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\\
&\quad\times\left(\int_0^\eta e^{-\frac{3}{4A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}+\tilde{\mathcal{H}}_{2,\eta})|\tilde{\mathcal{U}}_2|(t,\theta)\,d\theta\right)d\eta|A_1-A_2|\\
&\leqslant |A_1-A_2|^2+\frac{32}{A^4A_2^{10}e^2}\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2\int_0^1\tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\\
&\quad\times\left[\left(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2^2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)\right.\\
&\quad\times\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)\\
&\quad+\left(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^2\tilde{\mathcal{H}}_{2,\eta}(t,\theta)\,d\theta\right)\\
&\quad\times\left.\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{H}}_{2,\eta}(t,\theta)\,d\theta\right)\right]d\eta\\
&\leqslant |A_1-A_2|^2+\frac{128}{A^4A_2^{9}e^2}\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2\int_0^1\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\\
&\quad\times\left(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{P}}_2^2\tilde{\mathcal{Y}}_{2,\eta}+\tilde{\mathcal{U}}_2^2\tilde{\mathcal{H}}_{2,\eta})(t,\theta)\,d\theta\right)d\eta\\
&\leqslant |A_1-A_2|^2+\frac{64}{A^4A_2^{4}e^2}\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2\\
&\quad\times\int_0^1\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)e^{-\frac{1}{2A_2}\tilde{\mathcal{Y}}_2(t,\eta)}\left(\int_0^\eta e^{\frac{1}{2A_2}\tilde{\mathcal{Y}}_2(t,\theta)}(\tilde{\mathcal{P}}_2^2\tilde{\mathcal{Y}}_{2,\eta}+\tilde{\mathcal{U}}_2^2\tilde{\mathcal{H}}_{2,\eta})(t,\theta)\,d\theta\right)d\eta\\
&\leqslant |A_1-A_2|^2+\frac{128}{A^4A_2^{3}e^2}\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2\\
&\quad\times\left[-\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{P}}_2^2\tilde{\mathcal{Y}}_{2,\eta}+\tilde{\mathcal{U}}_2^2\tilde{\mathcal{H}}_{2,\eta})(t,\theta)\,d\theta\,\Bigg|_{\eta=0}^1\right.\\
&\quad\left.+\int_0^1(\tilde{\mathcal{P}}_2^2\tilde{\mathcal{Y}}_{2,\eta}+\tilde{\mathcal{U}}_2^2\tilde{\mathcal{H}}_{2,\eta})(t,\eta)\,d\eta\right]\\
&\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2+|A_1-A_2|^2).
\end{aligned}
$$

Next, we have a look at $\bar{B}_{67}^{+}$ (a similar argument works for $\bar{B}_{67}^{-}$ and $\bar{B}_{68}^{\pm}$). Direct calculations yield

$$
\begin{aligned}
|\bar{B}_{67}^{+}| &\leqslant \frac{a^5}{2A^6}\int_0^1 |\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}|\tilde{\mathcal{U}}_2^{+}-\tilde{\mathcal{U}}_1^{+}|(t,\theta)\,d\theta\bigg)\,d\eta \\
&\leqslant \|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2 \\
&\quad+\frac{A_2^5}{4A^7}\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}|\tilde{\mathcal{U}}_2-\tilde{\mathcal{U}}_1|(t,\theta)\,d\theta\bigg)^2\,d\eta \\
&\leqslant \|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2+\frac{1}{2A^7}\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta) \\
&\quad\times\bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}+\tilde{\mathcal{H}}_{2,\eta})|\tilde{\mathcal{U}}_2-\tilde{\mathcal{U}}_1|(t,\theta)\,d\theta\bigg)^2\,d\eta \\
&\leqslant \|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2 \\
&\quad+\frac{1}{A^7}\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\bigg[\bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}|\tilde{\mathcal{U}}_2-\tilde{\mathcal{U}}_1|(t,\theta)\,d\theta\bigg)^2 \\
&\quad+\bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{H}}_{2,\eta}|\tilde{\mathcal{U}}_2-\tilde{\mathcal{U}}_1|(t,\theta)\,d\theta\bigg)^2\bigg]\,d\eta \\
&\leqslant \|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2+\frac{1}{A^7}\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\bigg(\int_0^\eta e^{-\frac{3}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\bigg) \\
&\quad\times\bigg(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{U}}_2-\tilde{\mathcal{U}}_1)^2\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\bigg)\,d\eta \\
&\quad+\frac{1}{A^7}\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{H}}_{2,\eta}(t,\theta)\,d\theta\bigg) \\
&\quad\times\bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)^2\tilde{\mathcal{H}}_{2,\eta}(t,\theta)\,d\theta\bigg)\,d\eta \\
&\leqslant \|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2 \\
&\quad+\frac{2}{A^6}\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t,\eta)e^{-\frac{1}{2A_2}\tilde{\mathcal{Y}}_2(t,\eta)}\bigg(\int_0^\eta e^{\frac{1}{2A_2}\tilde{\mathcal{Y}}_2(t,\theta)}(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)^2\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2\eta}(t,\theta)\,d\theta\bigg)\,d\eta \\
&\quad+\frac{4}{A^6}\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t,\eta)e^{-\frac{1}{A_2}\tilde{\mathcal{Y}}_2(t,\eta)}\bigg(\int_0^\eta e^{\frac{1}{A_2}\tilde{\mathcal{Y}}_2(t,\theta)}(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)^2\tilde{\mathcal{H}}_{2,\eta}(t,\theta)\,d\theta\bigg)\,d\eta \\
&\leqslant \|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2 \\
&\quad+\frac{2A_2}{A^6}\bigg(-2\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)^2\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\,\bigg|_{\eta=0}^{1} \\
&\quad+2\int_0^1 (\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)^2\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\,d\eta\bigg)
\end{aligned}
$$

$$+ \frac{4A_2}{A^6}\Big(-\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)^2\tilde{\mathcal{H}}_{2,\eta}(t,\theta)\,d\theta\,\Big|_{\eta=0}^{1}$$

$$+ \int_0^1 (\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)^2\tilde{\mathcal{H}}_{2,\eta}(t,\eta)\,d\eta\Big)$$

$$\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2+\|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2\|^2). \tag{5.24}$$

Finally, we consider $\bar{B}_{69}^{+}$ (a similar argument works for $\bar{B}_{69}^{-}$). Here integration by parts will play the main role. Indeed, we have

$$\begin{aligned}
|\bar{B}_{69}^{+}| &= \frac{a^5}{2A^6}\Big|\int_0^1 (\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)(\tilde{\mathcal{Y}}_{1,\eta}-\tilde{\mathcal{Y}}_{2,\eta})(t,\eta)\\
&\quad\times\Big(\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{U}}_j^{+})(t,\theta)\,d\theta\Big)\,d\eta\Big|\\
&= \frac{a^5}{4A^6}\Big|(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,\eta)\\
&\quad\times\Big(\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{U}}_j^{+})(t,\theta)\,d\theta\Big)\,\Big|_{\eta=0}^{1}\\
&\quad-\int_0^1 (\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,\eta)\\
&\quad\times\frac{d}{d\eta}\Big(\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{U}}_j^{+})(t,\theta)\,d\theta\Big)\,d\eta\Big|\\
&\leqslant \frac{a^5}{4A^6}\Big|\int_0^1 (\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,\eta)\\
&\quad\times\frac{d}{d\eta}\Big(\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{U}}_j^{+})(t,\theta)\,d\theta\Big)\,d\eta\Big|\\
&\leqslant \mathcal{O}(1)\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|,
\end{aligned} \tag{5.25}$$

where $\mathcal{O}(1)$ denotes some constant depending on A, which remains bounded as $A \to 0$, provided that we can show that both

$$a^5\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{U}}_j^{+})(t,\theta)d\theta \leqslant \mathcal{O}(1)A^2\min_j(\tilde{\mathcal{P}}_j)(t,\eta)$$

and the derivative

$$a\frac{d}{d\eta}\Big(\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{U}}_j^{+})(t,\theta)\,d\theta\Big) \tag{5.26}$$

exist and are uniformly bounded.

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Direct computations yield

$$
\begin{aligned}
a^5 &\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{U}}_j^+)(t,\theta)\,d\theta \\
&\leqslant \int_0^\eta e^{-\frac{1}{A_i}(\tilde{\mathcal{Y}}_i(t,\eta)-\tilde{\mathcal{Y}}_i(t,\theta))} A_i^5 |\tilde{\mathcal{U}}_i|(t,\theta)\,d\theta \\
&\leqslant 2\int_0^\eta e^{-\frac{1}{A_i}(\tilde{\mathcal{Y}}_i(t,\eta)-\tilde{\mathcal{Y}}_i(t,\theta))} (\tilde{\mathcal{P}}_i\tilde{\mathcal{Y}}_{i,\eta} + \tilde{\mathcal{H}}_{i,\eta})|\tilde{\mathcal{U}}_i|(t,\theta)\,d\theta \\
&\leqslant 2\int_0^\eta e^{-\frac{1}{A_i}(\tilde{\mathcal{Y}}_i(t,\eta)-\tilde{\mathcal{Y}}_i(t,\theta))} \left(\frac{1}{A_i}\tilde{\mathcal{P}}_i^2\tilde{\mathcal{Y}}_{i,\eta} + A_i\tilde{\mathcal{U}}_i^2\tilde{\mathcal{Y}}_{i,\eta} + \|\tilde{\mathcal{U}}_i(t,\,\cdot\,)\|_{L^\infty}\tilde{\mathcal{H}}_{i,\eta}\right)(t,\theta)\,d\theta \\
&\leqslant \mathcal{O}(1)A_i^2\tilde{\mathcal{P}}_i(t,\eta).
\end{aligned}
$$

The result for (5.26) is contained in Lemma A.5.

LEMMA 5.3. *Let $\tilde{\mathcal{Y}}_i$ denote two solutions of* (5.5). *Then we have*

$$
\begin{aligned}
\frac{d}{dt}\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \leqslant \mathcal{O}(1)(&\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 \\
&+ \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + |A_1 - A_2|^2), \qquad (5.27)
\end{aligned}
$$

where $\mathcal{O}(1)$ denotes some constant which depends on $A = \max_j(A_j)$, which remains bounded as $A \to 0$.

5.2. Lipschitz estimates for $\tilde{\mathcal{U}}$.

From the system of differential equations, we have

$$
\tilde{\mathcal{U}}_{i,t} + \left(\frac{2}{3}\frac{1}{A_i^5}\tilde{\mathcal{U}}_i^3 + \frac{1}{A_i^6}\tilde{\mathcal{S}}_i\right)\tilde{\mathcal{U}}_{i,\eta} = -\frac{1}{A_i^2}\tilde{\mathcal{Q}}_i, \qquad (5.28)
$$

where

$$
\begin{aligned}
\tilde{\mathcal{Q}}_i(t,\eta) &= -\frac{1}{4}\int_0^1 \operatorname{sign}(\eta-\theta)e^{-\frac{1}{A_i}|\tilde{\mathcal{Y}}_i(t,\eta)-\tilde{\mathcal{Y}}_i(t,\theta)|}(2(\tilde{\mathcal{U}}_i^2 - \tilde{\mathcal{P}}_i)\tilde{\mathcal{Y}}_{i,\eta}(t,\theta) + A_i^5)\,d\theta, \\
\tilde{\mathcal{S}}_i(t,\eta) &= \int_0^1 e^{-\frac{1}{A_i}|\tilde{\mathcal{Y}}_i(t,\eta)-\tilde{\mathcal{Y}}_i(t,\theta)|}\left(\frac{2}{3}\tilde{\mathcal{U}}_i^3\tilde{\mathcal{Y}}_{i,\eta} - \tilde{\mathcal{Q}}_i\tilde{\mathcal{U}}_{i,\eta} - 2\tilde{\mathcal{P}}_i\tilde{\mathcal{U}}_i\tilde{\mathcal{Y}}_{i,\eta}\right)(t,\theta)\,d\theta.
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
\frac{d}{dt}&\int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2(t,\eta)\,d\eta \\
&= 2\int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(\tilde{\mathcal{U}}_{1,t} - \tilde{\mathcal{U}}_{2,t})(t,\eta)\,d\eta
\end{aligned}
$$

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$$= 2\int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)\Big(\frac{1}{A_2^2}\tilde{\mathcal{Q}}_2 - \frac{1}{A_1^2}\tilde{\mathcal{Q}}_1\Big)(t,\eta)\,d\eta$$
$$+ \frac{4}{3}\int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)\Big(\frac{1}{A_2^5}\tilde{\mathcal{U}}_2^3\tilde{\mathcal{U}}_{2,\eta} - \frac{1}{A_1^5}\tilde{\mathcal{U}}_1^3\tilde{\mathcal{U}}_{1,\eta}\Big)(t,\eta)\,d\eta$$
$$+ 2\int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)\Big(\frac{1}{A_2^6}\tilde{\mathcal{S}}_2\tilde{\mathcal{U}}_{2,\eta} - \frac{1}{A_1^6}\tilde{\mathcal{S}}_1\tilde{\mathcal{U}}_{1,\eta}\Big)(t,\eta)\,d\eta$$
$$= I_1 + I_2 + I_3.$$

The strategy is to use integration by parts for the last two integrals I_2 and I_3, while we want to use straightforward estimates for I_1, which will finally yield that

$$\frac{d}{dt}\|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2$$
$$\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + |A_1 - A_2|^2),$$

where $\mathcal{O}(1)$ denotes some constant which only depends on $A = \max_j(A_j)$ and which remains bounded as $A \to 0$.

The first integral I_1: Note that we can split I_1 as follows:

$$I_1 = \mathbb{1}_{A_1 \leqslant A_2} 2\frac{A_1^2 - A_2^2}{A_1^2 A_2^2}\int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)\tilde{\mathcal{Q}}_1(t,\eta)\,d\eta$$
$$+ \mathbb{1}_{A_2 < A_1} 2\frac{A_1^2 - A_2^2}{A_1^2 A_2^2}\int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)\tilde{\mathcal{Q}}_2(t,\eta)\,d\eta$$
$$+ 2\frac{1}{A^2}\int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(\tilde{\mathcal{Q}}_2 - \tilde{\mathcal{Q}}_1)(t,\eta)\,d\eta$$
$$= I_{11} + I_{12} + I_{13}.$$

As far as I_{11} is concerned (a similar argument works for I_{12}), we have

$$|I_{11}| \leqslant \mathbb{1}_{A_1 \leqslant A_2} 4\frac{|A_1 - A_2|}{A_1 A_2}\int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2|\tilde{\mathcal{P}}_1(t,\eta)\,d\eta$$
$$\leqslant A^2|A_1 - A_2|\int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2|(t,\eta)\,d\eta$$
$$\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + |A_1 - A_2|^2).$$

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Note that

$$|I_{13}| = \left|2\frac{1}{A^2}\int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(\tilde{\mathcal{Q}}_1 - \tilde{\mathcal{Q}}_2)(t,\eta)\,d\eta\right|$$
$$\leqslant \int_0^1 \left((\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2 + \frac{1}{A^4}(\tilde{\mathcal{Q}}_1 - \tilde{\mathcal{Q}}_2)^2\right)(t,\eta)\,d\eta,$$

and hence it suffices to show that

$$\|\tilde{\mathcal{Q}}_1 - \tilde{\mathcal{Q}}_2\| \leqslant \mathcal{O}(1)A^2(\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\| + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\| + |A_1 - A_2|), \tag{5.29}$$

which is equivalent to

$$\|\tilde{\mathcal{Q}}_1 - \tilde{\mathcal{Q}}_2\|^2 \leqslant \mathcal{O}(1)A^4(\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + |A_1 - A_2|^2).$$

To begin with, we observe that we can write

$$\begin{aligned}(\tilde{\mathcal{Q}}_1 - \tilde{\mathcal{Q}}_2)(t,\eta) &= (A_1\tilde{\mathcal{P}}_1 - A_2\tilde{\mathcal{P}}_2)(t,\eta) \\ &\quad + (\tilde{\mathcal{D}}_2 - \tilde{\mathcal{D}}_1)(t,\eta) \\ &= (A_1 - A_2)\tilde{\mathcal{P}}_1 \\ &\quad + A_2(\tilde{\mathcal{P}}_1^{1/2} + \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t,\eta) \\ &\quad + (\tilde{\mathcal{D}}_2 - \tilde{\mathcal{D}}_1)(t,\eta) \\ &= K_1(t,\eta) + K_2(t,\eta) + K_3(t,\eta).\end{aligned}$$

As far as $K_1(t,\eta)$ is concerned, we have

$$|A_1 - A_2|\|\tilde{\mathcal{P}}_1\|_{L^\infty} \leqslant \frac{A^4}{4}|A_1 - A_2|.$$

As far as $K_2(t,\eta)$ is concerned, we have

$$\|\tilde{\mathcal{P}}_1 - \tilde{\mathcal{P}}_2\| \leqslant A^2\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|,$$

since $\|\tilde{\mathcal{P}}_i^{1/2}\|_{L^\infty}$ can be bounded by a constant, which only depends on A.

As far as $K_3(t,\eta)$ is concerned, Lemma A.9 implies immediately that

$$\|\tilde{\mathcal{D}}_1 - \tilde{\mathcal{D}}_2\| \leqslant \mathcal{O}(1)A^2(\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\| + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\| + |A_1 - A_2|).$$

This finishes the proof of (5.29).

The second integral I_2: Note that we can write

$$
\begin{aligned}
\frac{3}{4}I_2 &= \int_0^1 (\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)\Big(\frac{1}{A_2^5}\tilde{\mathcal{U}}_2^3\tilde{\mathcal{U}}_{2,\eta}-\frac{1}{A_1^5}\tilde{\mathcal{U}}_1^3\tilde{\mathcal{U}}_{1,\eta}\Big)(t,\eta)\,d\eta\\
&= \frac{1}{A^5}\int_0^1 (\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)(\tilde{\mathcal{U}}_2^3\tilde{\mathcal{U}}_{2,\eta}-\tilde{\mathcal{U}}_1^3\tilde{\mathcal{U}}_{1,\eta})(t,\eta)\,d\eta\\
&\quad+\frac{A_1^5-A_2^5}{A_1^5A_2^5}\int_0^1 (\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)(\tilde{\mathcal{U}}_1^3\tilde{\mathcal{U}}_{1,\eta}\mathbb{1}_{A_1\leqslant A_2}+\tilde{\mathcal{U}}_2^3\tilde{\mathcal{U}}_{2,\eta}\mathbb{1}_{A_2<A_1})(t,\eta)\,d\eta\\
&= \frac{1}{A^5}\int_0^1 (\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)(\tilde{\mathcal{U}}_2-\tilde{\mathcal{U}}_1)\tilde{\mathcal{U}}_2^2\tilde{\mathcal{U}}_{2,\eta}(t,\eta)\,d\eta\\
&\quad+\frac{1}{A^5}\int_0^1 (\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)(\tilde{\mathcal{U}}_2^2\tilde{\mathcal{U}}_{2,\eta}-\tilde{\mathcal{U}}_1^2\tilde{\mathcal{U}}_{1,\eta})\tilde{\mathcal{U}}_1(t,\eta)\,d\eta\\
&\quad+\frac{A_1^5-A_2^5}{A_1^5A_2^5}\int_0^1 (\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)(\tilde{\mathcal{U}}_1^3\tilde{\mathcal{U}}_{1,\eta}\mathbb{1}_{A_1\leqslant A_2}+\tilde{\mathcal{U}}_2^3\tilde{\mathcal{U}}_{2,\eta}\mathbb{1}_{A_2<A_1})(t,\eta)\,d\eta\\
&= \frac{1}{A^5}\int_0^1 (\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)(\tilde{\mathcal{U}}_2-\tilde{\mathcal{U}}_1)\tilde{\mathcal{U}}_2^2\tilde{\mathcal{U}}_{2,\eta}(t,\eta)\,d\eta\\
&\quad+\frac{1}{A^5}\int_0^1 (\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)(\tilde{\mathcal{U}}_2^2-\tilde{\mathcal{U}}_1^2)\tilde{\mathcal{U}}_{2,\eta}\tilde{\mathcal{U}}_1(t,\eta)\mathbb{1}_{\tilde{\mathcal{U}}_1^2\leqslant\tilde{\mathcal{U}}_2^2}(t,\eta)\,d\eta\\
&\quad+\frac{1}{A^5}\int_0^1 (\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)(\tilde{\mathcal{U}}_2^2-\tilde{\mathcal{U}}_1^2)\tilde{\mathcal{U}}_1\tilde{\mathcal{U}}_{1,\eta}(t,\eta)\mathbb{1}_{\tilde{\mathcal{U}}_2^2<\tilde{\mathcal{U}}_1^2}(t,\eta)\,d\eta\\
&\quad+\frac{1}{A^5}\int_0^1 (\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)(\tilde{\mathcal{U}}_{2,\eta}-\tilde{\mathcal{U}}_{1,\eta})\tilde{\mathcal{U}}_1\min_j(\tilde{\mathcal{U}}_j^2)(t,\eta)\,d\eta\\
&\quad+\frac{A_1^5-A_2^5}{A_1^5A_2^5}\int_0^1 (\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)(\tilde{\mathcal{U}}_1^3\tilde{\mathcal{U}}_{1,\eta}\mathbb{1}_{A_1\leqslant A_2}+\tilde{\mathcal{U}}_2^3\tilde{\mathcal{U}}_{2,\eta}\mathbb{1}_{A_2<A_1})(t,\eta)\,d\eta\\
&= \frac{1}{A^5}\int_0^1 (\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)(\tilde{\mathcal{U}}_2-\tilde{\mathcal{U}}_1)\tilde{\mathcal{U}}_2^2\tilde{\mathcal{U}}_{2,\eta}(t,\eta)\,d\eta\\
&\quad+\frac{1}{A^5}\int_0^1 (\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)(\tilde{\mathcal{U}}_2^2-\tilde{\mathcal{U}}_1^2)\tilde{\mathcal{U}}_{2,\eta}\tilde{\mathcal{U}}_1(t,\eta)\mathbb{1}_{\tilde{\mathcal{U}}_1^2\leqslant\tilde{\mathcal{U}}_2^2}(t,\eta)\,d\eta\\
&\quad+\frac{1}{A^5}\int_0^1 (\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)(\tilde{\mathcal{U}}_2^2-\tilde{\mathcal{U}}_1^2)\tilde{\mathcal{U}}_1\tilde{\mathcal{U}}_{1,\eta}(t,\eta)\mathbb{1}_{\tilde{\mathcal{U}}_2^2<\tilde{\mathcal{U}}_1^2}(t,\eta)\,d\eta\\
&\quad+\frac{1}{2A^5}\int_0^1 (\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)^2\frac{d}{d\eta}(\tilde{\mathcal{U}}_1\min_j(\tilde{\mathcal{U}}_j^2))(t,\eta)\,d\eta\\
&\quad+\frac{A_1^5-A_2^5}{A_1^5A_2^5}\int_0^1 (\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)(\tilde{\mathcal{U}}_1^3\tilde{\mathcal{U}}_{1,\eta}\mathbb{1}_{A_1\leqslant A_2}+\tilde{\mathcal{U}}_2^3\tilde{\mathcal{U}}_{2,\eta}\mathbb{1}_{A_2<A_1})(t,\eta)\,d\eta,
\end{aligned}
$$

where we used integration by parts in the last step together with $\tilde{\mathcal{U}}_i(t,\eta) \to 0$ as $\eta \to 0, 1$. As far as the derivative in the last integral is concerned, observe that

$$|\tilde{\mathcal{U}}_{1,\eta} \min_j(\tilde{\mathcal{U}}_j^2)(t,\eta)| \leqslant |\tilde{\mathcal{U}}_1^2\tilde{\mathcal{U}}_{1,\eta}(t,\eta)| \leqslant \frac{A^4}{2}\|\tilde{\mathcal{U}}_1\|_{L^\infty} \leqslant \frac{A^6}{2\sqrt{2}}, \tag{5.30}$$

since $2|\tilde{\mathcal{U}}_i\tilde{\mathcal{U}}_{i,\eta}(t,\eta)| \leqslant A_i^4 \leqslant A^4$ for all t and η. Furthermore, we established before that the function $\theta \mapsto \min_j(\tilde{\mathcal{U}}_j^2(t,\eta))$ is Lipschitz continuous with Lipschitz constant at most A^4; see (5.8). Thus

$$\left|\frac{d}{d\eta}\tilde{\mathcal{U}}_1 \min_j(\tilde{\mathcal{U}}_j^2)(t,\eta)\right| \leqslant \frac{3}{2}A^4\|\tilde{\mathcal{U}}_1\|_{L^\infty} \leqslant \frac{3}{2\sqrt{2}}A^6 \tag{5.31}$$

and

$$\left|\frac{1}{A_1^5A_2^5}\tilde{\mathcal{U}}_i^3\tilde{\mathcal{U}}_{i,n}\mathbb{1}_{A_i=a}\right| \leqslant \frac{1}{A_1^5A_2^5}\frac{1}{4}A_i^8\mathbb{1}_{A_i=a} \leqslant \frac{1}{A^2}.$$

Finally, we get that

$$\begin{aligned}
\frac{3}{4}|I_2| &= \left|\int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)\left(\frac{1}{A_2^5}\tilde{\mathcal{U}}_2^3\tilde{\mathcal{U}}_{2,\eta} - \frac{1}{A_1^5}\tilde{\mathcal{U}}_1^3\tilde{\mathcal{U}}_{1,\eta}\right)(t,\eta)\,d\eta\right| \\
&\leqslant \mathcal{O}(1)\|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \frac{2}{A^2}|A_1^5 - A_2^5|\int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2|(t,\eta)\,d\eta \\
&\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + |A_1 - A_2|^2).
\end{aligned}$$

The third integral I_3: We will split I_3 into several terms that we treat separately, and combine them in the end.

Recall that we introduced the functions $\tilde{\mathcal{U}}_i^- = \min(0, \tilde{\mathcal{U}}_i)$ and $\tilde{\mathcal{U}}_i^+ = \max(0, \tilde{\mathcal{U}}_i)$ with properties (5.2) and (5.3). We write

$$I_3 = \frac{4}{3}I_{31} - 4I_{32} - 2I_{33},$$

where

$$\begin{aligned}
I_{31} &= \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t,\eta)\left(\frac{1}{A_2^6}\tilde{\mathcal{U}}_{2,\eta}(t,\eta)\int_0^1 e^{-\frac{1}{A_2}|\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta)|}\tilde{\mathcal{U}}_2^3\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right. \\
&\quad \left. - \frac{1}{A_1^6}\tilde{\mathcal{U}}_{1,\eta}(t,\eta)\int_0^1 e^{-\frac{1}{A_1}|\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta)|}\tilde{\mathcal{U}}_1^3\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\right)d\eta, \qquad (5.32a) \\
I_{32} &= \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t,\eta)\left(\frac{1}{A_2^6}\tilde{\mathcal{U}}_{2,\eta}(t,\eta)\int_0^1 e^{-\frac{1}{A_2}|\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta)|}\tilde{\mathcal{P}}_2\tilde{\mathcal{U}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right.
\end{aligned}$$

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$$- \frac{1}{A_1^6}\tilde{\mathcal{U}}_{1,\eta}(t,\eta)\int_0^1 e^{-\frac{1}{A_1}|\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta)|}\tilde{\mathcal{P}}_1\tilde{\mathcal{U}}_1\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\bigg)\,d\eta, \tag{5.32b}$$

$$I_{33} = \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t,\eta)\bigg(\frac{1}{A_2^6}\tilde{\mathcal{U}}_{2,\eta}(t,\eta)\int_0^1 e^{-\frac{1}{A_2}|\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta)|}\tilde{\mathcal{Q}}_2\tilde{\mathcal{U}}_{2,\eta}(t,\theta)\,d\theta$$

$$- \frac{1}{A_1^6}\tilde{\mathcal{U}}_{1,\eta}(t,\eta)\int_0^1 e^{-\frac{1}{A_1}|\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta)|}\tilde{\mathcal{Q}}_1\tilde{\mathcal{U}}_{1,\eta}(t,\theta)\,d\theta\bigg)\,d\eta. \tag{5.32c}$$

Thus we have

$$\begin{aligned}
I_{31} &= \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t,\eta)\\
&\quad \times \bigg(\frac{1}{A_2^6}\tilde{\mathcal{U}}_{2,\eta}(t,\eta)\int_0^1 e^{-\frac{1}{A_2}|\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta)|}\tilde{\mathcal{U}}_2^3\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\\
&\quad - \frac{1}{A_1^6}\tilde{\mathcal{U}}_{1,\eta}(t,\eta)\int_0^1 e^{-\frac{1}{A_1}|\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta)|}\tilde{\mathcal{U}}_1^3\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\bigg)\,d\eta\\
&= \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t,\eta)\\
&\quad \times \bigg(\frac{1}{A_2^6}\tilde{\mathcal{U}}_{2,\eta}(t,\eta)\int_0^1 e^{-\frac{1}{A_2}|\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta)|}(\tilde{\mathcal{U}}_2^-)^3\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\\
&\quad - \frac{1}{A_1^6}\tilde{\mathcal{U}}_{1,\eta}(t,\eta)\int_0^1 e^{-\frac{1}{A_1}|\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta)|}(\tilde{\mathcal{U}}_1^-)^3\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\bigg)\,d\eta\\
&\quad + \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t,\eta)\\
&\quad \times \bigg(\frac{1}{A_2^6}\tilde{\mathcal{U}}_{2,\eta}(t,\eta)\int_0^1 e^{-\frac{1}{A_2}|\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta)|}(\tilde{\mathcal{U}}_2^+)^3\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\\
&\quad - \frac{1}{A_1^6}\tilde{\mathcal{U}}_{1,\eta}(t,\eta)\int_0^1 e^{-\frac{1}{A_1}|\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta)|}(\tilde{\mathcal{U}}_1^+)^3\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\bigg)\,d\eta.
\end{aligned}$$

Since both inner integrals have the same structure, it suffices to consider the second integral. Recall rewrite (5.4). Thus we need to estimate the following term:

$$\begin{aligned}
&\frac{1}{A_2^6}\tilde{\mathcal{U}}_{2,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{U}}_2^+)^3\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\\
&\qquad - \frac{1}{A_1^6}\tilde{\mathcal{U}}_{1,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}(\tilde{\mathcal{U}}_1^+)^3\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\\
&\quad = \frac{1}{A^6}\bigg(\tilde{\mathcal{U}}_{2,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{U}}_2^+)^3\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta
\end{aligned}$$

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$$
\begin{aligned}
&-\tilde{\mathcal{U}}_{1,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}(\tilde{\mathcal{U}}_1^+)^3\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\Big)\\
&+\mathbb{1}_{A_1\leqslant A_2}\Big(\frac{1}{A_2^6}-\frac{1}{A_1^6}\Big)\Big(\tilde{\mathcal{U}}_{1,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}(\tilde{\mathcal{U}}_1^+)^3\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\Big)\\
&+\mathbb{1}_{A_2<A_1}\Big(\frac{1}{A_2^6}-\frac{1}{A_1^6}\Big)\Big(\tilde{\mathcal{U}}_{2,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{U}}_2^+)^3\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\Big)\\
=&\ \frac{1}{A^6}\tilde{\mathcal{U}}_{2,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}((\tilde{\mathcal{U}}_2^+)^3-(\tilde{\mathcal{U}}_1^+)^3)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\mathbb{1}_{\tilde{\mathcal{U}}_1^+\leqslant\tilde{\mathcal{U}}_2^+}(t,\theta)\,d\theta\\
&+\frac{1}{A^6}\tilde{\mathcal{U}}_{1,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}((\tilde{\mathcal{U}}_2^+)^3-(\tilde{\mathcal{U}}_1^+)^3)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\mathbb{1}_{\tilde{\mathcal{U}}_2^+<\tilde{\mathcal{U}}_1^+}(t,\theta)\,d\theta\\
&+\frac{1}{A^6}\tilde{\mathcal{U}}_{2,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\min_j(\tilde{\mathcal{U}}_j^+)^3\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\\
&-\frac{1}{A^6}\tilde{\mathcal{U}}_{1,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\min_j(\tilde{\mathcal{U}}_j^+)^3\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\\
&+\mathbb{1}_{A_1\leqslant A_2}\Big(\frac{1}{A_2^6}-\frac{1}{A_1^6}\Big)\Big(\tilde{\mathcal{U}}_{1,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}(\tilde{\mathcal{U}}_1^+)^3\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\Big)\\
&+\mathbb{1}_{A_2<A_1}\Big(\frac{1}{A_2^6}-\frac{1}{A_1^6}\Big)\Big(\tilde{\mathcal{U}}_{2,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{U}}_2^+)^3\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\Big)\\
=&\ \frac{1}{A^6}\tilde{\mathcal{U}}_{2,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}((\tilde{\mathcal{U}}_2^+)^3-(\tilde{\mathcal{U}}_1^+)^3)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\mathbb{1}_{\tilde{\mathcal{U}}_1^+\leqslant\tilde{\mathcal{U}}_2^+}(t,\theta)d\theta\\
&+\frac{1}{A^6}\tilde{\mathcal{U}}_{1,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}((\tilde{\mathcal{U}}_2^+)^3-(\tilde{\mathcal{U}}_1^+)^3)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\mathbb{1}_{\tilde{\mathcal{U}}_2^+<\tilde{\mathcal{U}}_1^+}(t,\theta)\,d\theta\\
&+\frac{1}{A^6}\mathbb{1}_{A_1\leqslant A_2}\tilde{\mathcal{U}}_{2,\eta}(t,\eta)\int_0^\eta(e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}-e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))})\\
&\times\min_j(\tilde{\mathcal{U}}_j^+)^3\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\\
&+\frac{1}{A^6}\mathbb{1}_{A_2<A_1}\tilde{\mathcal{U}}_{1,\eta}(t,\eta)\int_0^\eta(e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}-e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))})\\
&\times\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\\
&+\frac{1}{A^6}\tilde{\mathcal{U}}_{2,\eta}(t,\eta)\int_0^\eta(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}-e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))})\\
&\times\min_j(\tilde{\mathcal{U}}_j^+)^3\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\mathbb{1}_{B(\eta)}(t,\theta)\,d\theta\\
&-\frac{1}{A^6}\tilde{\mathcal{U}}_{1,\eta}(t,\eta)\int_0^\eta(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}-e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))})\\
&\times\min_j(\tilde{\mathcal{U}}_j^+)^3\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\mathbb{1}_{B^c(\eta)}(t,\theta)\,d\theta)
\end{aligned}
$$

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$$+\frac{1}{A^6}\tilde{\mathcal{U}}_{2,\eta}(t,\eta)\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{U}}_j^+)^3\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta$$
$$-\frac{1}{A^6}\tilde{\mathcal{U}}_{1,\eta}(t,\eta)\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{U}}_j^+)^3\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta$$
$$+\mathbb{1}_{A_1\leqslant A_2}\left(\frac{1}{A_2^6}-\frac{1}{A_1^6}\right)\left(\tilde{\mathcal{U}}_{1,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}(\tilde{\mathcal{U}}_1^+)^3\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\right)$$
$$+\mathbb{1}_{A_2<A_1}\left(\frac{1}{A_2^6}-\frac{1}{A_1^6}\right)\left(\tilde{\mathcal{U}}_{2,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{U}}_2^+)^3\hat{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)$$
$$=(J_1+J_2+J_3+J_4+J_5+J_6+J_7+J_8+J_9+J_{10})(t,\eta), \tag{5.33}$$

where $B(\eta)$ is given by (5.11).

As far as the integral that contains J_1 (a similar argument works for J_2) is concerned, we have

$$\left|\int_0^1 J_1(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)(t,\eta)\,d\eta\right|$$
$$\leqslant \|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2\|^2+\int_0^1 J_1^2(t,\eta)\,d\eta$$
$$\leqslant \|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2\|_{L^2}^2+\frac{1}{A^{12}}\int_0^1 \tilde{\mathcal{U}}_{2,\eta}^2(t,\eta)$$
$$\times\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}((\tilde{\mathcal{U}}_2^+)^3-(\tilde{\mathcal{U}}_1^+)^3)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\mathbb{1}_{\tilde{\mathcal{U}}_1^+\leqslant\tilde{\mathcal{U}}_2^+}(t,\theta)\,d\theta\right)^2 d\eta$$
$$\leqslant \|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2\|^2$$
$$+\frac{1}{A^9}\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}3(\tilde{\mathcal{U}}_2^+)^2|\tilde{\mathcal{U}}_2^+-\tilde{\mathcal{U}}_1^+|\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)^2 d\eta$$
$$\leqslant \|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2\|^2$$
$$+\frac{9}{A^4}\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)d\eta\|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2\|^2$$
$$\leqslant \left(1+\frac{36}{A^3}\int_0^1 \tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\,d\eta\right)\|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2\|^2$$
$$\leqslant \mathcal{O}(1)\|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2\|^2, \tag{5.34}$$

where we used (4.15e), (4.15g), and (4.15h).

As far as the third and fourth terms J_3 and J_4 are concerned, they again have the same structure, and hence we only consider the integral corresponding to J_3.

$$\left|\int_0^1 J_3(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)(t,\eta)\,d\eta\right|$$
$$=\mathbb{1}_{A_1\leqslant A_2}\frac{1}{A^6}\left|\int_0^1(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)\tilde{\mathcal{U}}_{2,\eta}(t,\eta)\right.$$

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$$
\begin{aligned}
&\times \int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}) \min_j(\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\, d\theta\, d\eta \Big| \\
&\leqslant \frac{2\sqrt{2}a}{A^6 e} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2||\tilde{\mathcal{U}}_{2,\eta}|(t,\eta) \\
&\quad \times \int_0^\eta e^{-\frac{3}{4A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\, d\theta\, d\eta |A_1 - A_2| \\
&\leqslant \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \frac{8}{A^{10}e^2} \int_0^1 \tilde{\mathcal{U}}_{2,\eta}^2(t,\eta) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\, d\theta \right) \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\, d\theta \right) d\eta |A_1 - A_2|^2 \\
&\leqslant \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \frac{32}{Ae^2} \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\, d\eta |A_1 - A_2|^2 \\
&\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + |A_1 - A_2|^2).
\end{aligned}
$$

As far as the third and fourth terms J_5 and J_6 are concerned, they again have the same structure, and hence we only consider the integral corresponding to J_5. Thus, using (5.13),

$$
\begin{aligned}
&\left| \int_0^1 J_5(\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t,\eta)\, d\eta \right| \\
&= \frac{1}{A^6} \left| \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)\tilde{\mathcal{U}}_{2,\eta}(t,\eta) \left(\int_0^\eta (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}) \right.\right. \\
&\quad \left.\left. \times \min_j(\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t,\theta) \mathbb{1}_{B(\eta)}(t,\theta)\, d\theta \right) d\eta \right| \\
&\leqslant \frac{1}{aA^6} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2||\tilde{\mathcal{U}}_{2,\eta}|(t,\eta) \\
&\quad \times \left(\int_0^\eta (|\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_1(t,\eta)| + |\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_1(t,\theta)|) \right. \\
&\quad \left. \times e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} \min_j(\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\, d\theta \right) d\eta \\
&\leqslant \frac{1}{aA^6} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2||\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2||\tilde{\mathcal{U}}_{2,\eta}|(t,\eta) \\
&\quad \times \int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} \min_j(\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\, d\theta\, d\eta \\
&\quad + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \frac{1}{a^2A^{12}} \int_0^1 \tilde{\mathcal{U}}_{2,\eta}^2(t,\eta)
\end{aligned}
$$

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$$
\begin{aligned}
&\times \left(\int_0^\eta |\tilde{\mathcal{Y}}_1(t,\theta)-\tilde{\mathcal{Y}}_2(t,\theta)| e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} \min_j(\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)^2 d\eta \\
&\leqslant \frac{4a}{A^5}\int_0^1 |\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2||\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|\tilde{\mathcal{P}}_2|\tilde{\mathcal{U}}_{2,\eta}|(t,\eta)\,d\eta + \|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2\|^2 \\
&\quad + \frac{1}{a^2A^9}\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right) \\
&\quad \times \left(\int_0^\eta (\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,\theta) e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} \min_j(\tilde{\mathcal{U}}_j^4)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right) d\eta \\
&\leqslant (\sqrt{2}A^2+1)(\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2+\|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2\|^2) + \frac{2}{A^6}\int_0^1 \tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\eta) \\
&\times \left(\int_0^\eta (\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,\theta) e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right) d\eta \\
&\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2+\|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2\|^2).
\end{aligned} \tag{5.35}
$$

Next are the terms J_7 and J_8. Therefore recall (5.14), which implies

$$
\begin{aligned}
J_7+J_8 &= \frac{1}{A^6}\tilde{\mathcal{U}}_{2,\eta}(t,\eta) \\
&\quad \times \left(\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{U}}_j^+)^3\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right) \\
&\quad - \frac{1}{A^6}\tilde{\mathcal{U}}_{1,\eta}(t,\eta) \\
&\quad \times \left(\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{U}}_j^+)^3\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\right) \\
&= \frac{1}{A^6}(\tilde{\mathcal{U}}_{2,\eta}-\tilde{\mathcal{U}}_{1,\eta})(t,\eta) \\
&\quad \times \min_k\left[\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{U}}_j^+)^3\tilde{\mathcal{Y}}_{k,\eta}(t,\theta)\,d\theta\right] \\
&\quad + \frac{1}{A^6}\tilde{\mathcal{U}}_{1,\eta}(t,\eta)\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}) \\
&\quad \times \min_j(\tilde{\mathcal{U}}_j^+)^3(\tilde{\mathcal{Y}}_{2,\eta}-\tilde{\mathcal{Y}}_{1,\eta})(t,\theta)\,d\theta\,\mathbb{1}_E(t,\eta) \\
&\quad + \frac{1}{A^6}\tilde{\mathcal{U}}_{2,\eta}(t,\eta)\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}) \\
&\quad \times \min_j(\tilde{\mathcal{U}}_j^+)^3(\tilde{\mathcal{Y}}_{2,\eta}-\tilde{\mathcal{Y}}_{1,\eta})(t,\theta)\,d\theta\,\mathbb{1}_{E^c}(t,\eta) \\
&= L_1+L_2+L_3.
\end{aligned}
$$

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As far as the first term L_1 is concerned, the corresponding integral can be estimated as follows, using (A.11),

$$\begin{aligned}
&\left|\int_0^1 L_1(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)(t,\eta)\,d\eta\right| \\
&= \frac{1}{A^6}\bigg|\int_0^1 (\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)(\tilde{\mathcal{U}}_{2,\eta}-\tilde{\mathcal{U}}_{1,\eta})(t,\eta) \\
&\quad\times \min_k\bigg[\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{U}}_j^+)^3\tilde{\mathcal{Y}}_{k,\eta}(t,\theta)\,d\theta\bigg]\,d\eta\bigg| \\
&= \bigg|-\frac{1}{2A^6}(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)^2(t,\eta) \\
&\quad\times \min_k\bigg[\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{U}}_j^+)^3\tilde{\mathcal{Y}}_{k,\eta}(t,\theta)\,d\theta\bigg]\bigg|_{\eta=0}^1 \\
&\quad+\frac{1}{2A^6}\int_0^1(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)^2(t,\eta) \\
&\quad\times\frac{d}{d\eta}\min_k\bigg[\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{U}}_j^+)^3\tilde{\mathcal{Y}}_{k,\eta}(t,\theta)\,d\theta\bigg]\,d\eta\bigg| \\
&\leqslant \mathcal{O}(1)\|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2\|^2, \qquad (5.36)
\end{aligned}$$

where $\mathcal{O}(1)$ denotes some constant only depending on A, which remains bounded as $A\to 0$.

As far as the last term L_3 (a similar argument works for L_2) is concerned, the corresponding integral can be estimated as follows:

$$\begin{aligned}
&\left|\int_0^1 L_3(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)(t,\eta)\,d\eta\right| \\
&= \frac{1}{A^6}\bigg|\int_0^1(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)\tilde{\mathcal{U}}_{2,\eta}(t,\eta)\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}) \\
&\quad\times\min_j(\tilde{\mathcal{U}}_j^+)^3(\tilde{\mathcal{Y}}_{2,\eta}-\tilde{\mathcal{Y}}_{1,\eta})(t,\theta)\,d\theta\,\mathbb{1}_{E^c}(t,\eta)\,d\eta\bigg| \\
&= \frac{1}{A^6}\bigg|\int_0^1(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)\tilde{\mathcal{U}}_{2,\eta}(t,\eta)\mathbb{1}_{E^c}(t,\eta) \\
&\quad\times\bigg[(\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1)(t,\theta)\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{U}}_j^+)^3(t,\theta)\bigg|_{\theta=0}^\eta \\
&\quad-\int_0^\eta(\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1)(t,\theta)\bigg[\bigg(\frac{d}{d\theta}\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\bigg)\min_j(\tilde{\mathcal{U}}_j^+)^3(t,\theta) \\
&\quad+\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\bigg(\frac{d}{d\theta}\min_j(\tilde{\mathcal{U}}_j^+)^3\bigg)(t,\theta)\bigg]\,d\theta\bigg]\mathbb{1}_{E^c}(t,\eta)\,d\eta\bigg|
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{A^6}\bigg| - \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2) \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{U}}_{2,\eta}(t,\eta) \mathbb{1}_{E^c}(t,\eta)\, d\eta \\
&\quad + \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)\tilde{\mathcal{U}}_{2,\eta}(t,\eta) \mathbb{1}_{E^c}(t,\eta) \\
&\quad \times \int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t,\theta) \bigg[\bigg(\frac{d}{d\theta} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \bigg) \min_j (\tilde{\mathcal{U}}_j^+)^3(t,\theta) \\
&\quad + \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \bigg(\frac{d}{d\theta} \min_j (\tilde{\mathcal{U}}_j^+)^3 \bigg)(t,\theta) \bigg] d\theta\, d\eta \bigg| \\
&\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2) + \frac{1}{A^{12}} \int_0^1 \tilde{\mathcal{U}}_{2,\eta}^2(t,\eta) \bigg(\int_0^\eta |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t,\theta) \\
&\quad \times \bigg[\frac{1}{a} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{U}}_j^+)^3 \max_j (\tilde{\mathcal{Y}}_{j,\eta})(t,\theta) \\
&\quad + 2A^4 \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{U}}_j^+)(t,\theta) \bigg] d\theta \bigg)^2 d\eta \\
&\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2) + \frac{54}{A} \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\, d\eta \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 \\
&\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2),
\end{aligned}
\tag{5.37}
$$

where we used (A.4), (4.16a), and (A.6).

Finally, we have a look at the integral, which contains J_9 (a similar argument works for J_{10}), where we can assume $A_1 \leqslant A_2$. Thus

$$
\begin{aligned}
&\bigg| \int_0^1 J_9 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t,\eta)\, d\eta \bigg| \\
&\quad = \bigg(\frac{1}{A_1^6} - \frac{1}{A_2^6} \bigg) \bigg| \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)\tilde{\mathcal{U}}_{1,\eta}(t,\eta) \\
&\qquad \times \bigg(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} (\tilde{\mathcal{U}}_1^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\, d\theta \bigg) d\eta \bigg| \\
&\quad = \frac{A_2^6 - A_1^6}{A_1^6 A_2^6} \bigg| \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)\tilde{\mathcal{U}}_{1,\eta}(t,\eta) \\
&\qquad \times \bigg(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} (\tilde{\mathcal{U}}_1^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\, d\theta \bigg) d\eta \bigg| \\
&\quad \leqslant 3 \frac{A_2 - A_1}{A_1^2 A_2} \bigg| \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)\tilde{\mathcal{U}}_{1,\eta}(t,\eta) \\
&\qquad \times \bigg(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\, d\theta \bigg) d\eta \bigg|
\end{aligned}
$$

$$\leqslant |A_1 - A_2|^2 + \frac{9}{A_1^4 A_2^2} \int_0^1 (\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1)^2 \tilde{\mathcal{U}}_{1,\eta}^2(t,\eta)$$
$$\times \left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\, d\theta \right)^2 d\eta$$
$$\leqslant |A_1 - A_2|^2 + \frac{9}{A_1 A_2^2} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2 \tilde{\mathcal{Y}}_{1,\eta}(t,\eta)$$
$$\times \left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\, d\theta \right)$$
$$\times \left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} (\tilde{\mathcal{U}}_1^+)^2 \tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\, d\theta \right) d\eta$$
$$\leqslant |A_1 - A_2|^2 + \frac{9}{A_2^2} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2 \tilde{\mathcal{Y}}_{1,\eta}(t,\eta)$$
$$\times \left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{U}}_1^2 \tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\, d\theta \right) d\eta$$
$$\leqslant |A_1 - A_2|^2 + \frac{36}{A} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2 \tilde{\mathcal{P}}_1 \tilde{\mathcal{Y}}_{1,\eta}(t,\eta)\, d\eta$$
$$\leqslant |A_1 - A_2|^2 + 18A^4 \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2. \tag{5.38}$$

We now turn to the integral I_{32}. Recall equation (5.32b), namely,

$$I_{32} = \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t,\eta) \left(\frac{1}{A_2^6} \tilde{\mathcal{U}}_{2,\eta}(t,\eta) \int_0^1 e^{-\frac{1}{A_2}|\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta)|} \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\, d\theta \right.$$
$$\left. - \frac{1}{A_1^6} \tilde{\mathcal{U}}_{1,\eta}(t,\eta) \int_0^1 e^{-\frac{1}{A_1}|\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta)|} \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1 \tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\, d\theta \right) d\eta.$$

By first using the decomposition $\tilde{\mathcal{U}}_j = \tilde{\mathcal{U}}_j^+ + \tilde{\mathcal{U}}_j^-$, and then involving (5.4), we need to estimate terms of the type

$$\tilde{I}_{32} = \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t,\eta) \left(\frac{1}{A_2^6} \tilde{\mathcal{U}}_{2,\eta}(t,\eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\, d\theta \right.$$
$$\left. - \frac{1}{A_1^6} \tilde{\mathcal{U}}_{1,\eta}(t,\eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\, d\theta \right) d\eta.$$

We invoke Lemma 5.1 and find

$$\tilde{I}_{32} = \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t,\eta) \left(\tilde{\mathcal{U}}_{2,\eta}(t,\eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\, d\theta \right.$$
$$\left. - \tilde{\mathcal{U}}_{1,\eta}(t,\eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\, d\theta \right) d\eta$$

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$$
\begin{aligned}
&+ \frac{A_1^6 - A_2^6}{A_1^6 A_2^6} \mathbb{1}_{A_2 \leqslant A_1} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)\tilde{\mathcal{U}}_{2,\eta}(t,\eta) \\
&\times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\, d\theta \right) d\eta \\
&+ \frac{A_1^6 - A_2^6}{A_1^6 A_2^6} \mathbb{1}_{A_1 < A_2} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)\tilde{\mathcal{U}}_{1,\eta}(t,\eta) \\
&\times \left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\, d\theta \right) d\eta \\
= {} & N_1 + N_2 + N_3.
\end{aligned}
$$

We consider first N_1, where we get

$$
\begin{aligned}
N_1 = {} & \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t,\eta) \bigg[\tilde{\mathcal{U}}_{2,\eta}(t,\eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\, d\theta \\
& - \tilde{\mathcal{U}}_{1,\eta}(t,\eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\, d\theta \bigg] d\eta \\
= {} & \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t,\eta) \\
& \times \bigg[\tilde{\mathcal{U}}_{2,\eta}(t,\eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{P}}_2 - \tilde{\mathcal{P}}_1) \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{\tilde{\mathcal{P}}_1 \leqslant \tilde{\mathcal{P}}_2}(t,\theta)\, d\theta \\
& + \tilde{\mathcal{U}}_{1,\eta}(t,\eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} (\tilde{\mathcal{P}}_2 - \tilde{\mathcal{P}}_1) \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1,\eta} \mathbb{1}_{\tilde{\mathcal{P}}_2 < \tilde{\mathcal{P}}_1}(t,\theta)\, d\theta \\
& + \tilde{\mathcal{U}}_{2,\eta}(t,\eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \min_j(\tilde{\mathcal{P}}_j)(\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+) \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{\tilde{\mathcal{U}}_1^+ \leqslant \tilde{\mathcal{U}}_2^+}(t,\theta)\, d\theta \\
& + \tilde{\mathcal{U}}_{1,\eta}(t,\eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \min_j(\tilde{\mathcal{P}}_j)(\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+) \tilde{\mathcal{Y}}_{1,\eta} \mathbb{1}_{\tilde{\mathcal{U}}_2^+ < \tilde{\mathcal{U}}_1^+}(t,\theta)\, d\theta \\
& + \tilde{\mathcal{U}}_{2,\eta}(t,\eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\, d\theta \\
& - \tilde{\mathcal{U}}_{1,\eta}(t,\eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\, d\theta \bigg] d\eta \\
= {} & N_{11} + N_{12} + N_{13} + N_{14} + N_{15} + N_{16}.
\end{aligned} \tag{5.39}
$$

The terms N_{11} and N_{12} can be treated similarly. To that end, we find

$$
\begin{aligned}
|N_{11}| = {} & \frac{1}{A^6} \bigg| \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t,\eta) \\
& \times \bigg[\tilde{\mathcal{U}}_{2,\eta}(t,\eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{P}}_2 - \tilde{\mathcal{P}}_1) \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{\tilde{\mathcal{P}}_1 \leqslant \tilde{\mathcal{P}}_2}(t,\theta)\, d\theta \bigg] d\eta \bigg|
\end{aligned}
$$

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$$
\begin{aligned}
&\leqslant \|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2\|^2+\frac{1}{A^{12}}\int_0^1 \tilde{\mathcal{U}}_{2,\eta}^2(t,\eta)\\
&\quad\times\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{P}}_2-\tilde{\mathcal{P}}_1)\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}\mathbb{1}_{\tilde{\mathcal{P}}_1\leqslant\tilde{\mathcal{P}}_2}(t,\theta)\,d\theta\right)^2 d\eta\\
&\leqslant \|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2\|^2\\
&\quad+\frac{4}{A^9}\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{U}}_2^+)^2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)\\
&\quad\times\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}|\tilde{\mathcal{P}}_2^{1/2}-\tilde{\mathcal{P}}_1^{1/2}|^2\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)d\eta\\
&\leqslant \|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2\|^2+\frac{16}{A^8}\int_0^1 \tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\\
&\quad\times\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}|\tilde{\mathcal{P}}_2^{1/2}-\tilde{\mathcal{P}}_1^{1/2}|^2\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)d\eta\\
&\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2\|^2+\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2),
\end{aligned}
$$

where we have used (4.15e), (4.15h), and (4.16c).

The terms N_{13} and N_{14} follow the same estimates. More precisely,

$$
\begin{aligned}
|N_{13}|&=\frac{1}{A^6}\bigg|\int_0^1(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)\tilde{\mathcal{U}}_{2,\eta}(t,\eta)\\
&\qquad\times\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\min_j(\tilde{\mathcal{P}}_j)(\tilde{\mathcal{U}}_2^+-\tilde{\mathcal{U}}_1^+)\tilde{\mathcal{Y}}_{2,\eta}\mathbb{1}_{\tilde{\mathcal{U}}_1^+\leqslant\tilde{\mathcal{U}}_2^+}(t,\theta)\,d\theta\,d\eta\bigg|\\
&\leqslant\frac{1}{A^6}\int_0^1|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2|\,|\tilde{\mathcal{U}}_{2,\eta}|(t,\eta)\\
&\quad\times\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\min_j(\tilde{\mathcal{P}}_j)|\tilde{\mathcal{U}}_2^+-\tilde{\mathcal{U}}_1^+|\tilde{\mathcal{Y}}_{2,\eta}\mathbb{1}_{\tilde{\mathcal{U}}_1^+\leqslant\tilde{\mathcal{U}}_2^+}(t,\theta)\,d\theta\,d\eta\\
&\leqslant\|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2\|^2+\frac{1}{A^{12}}\int_0^1\tilde{\mathcal{U}}_{2,\eta}^2(t,\eta)\\
&\quad\times\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\min_j(\tilde{\mathcal{P}}_j)|\tilde{\mathcal{U}}_2^+-\tilde{\mathcal{U}}_1^+|\tilde{\mathcal{Y}}_{2,\eta}\mathbb{1}_{\tilde{\mathcal{U}}_1^+\leqslant\tilde{\mathcal{U}}_2^+}(t,\theta)\,d\theta\right)^2 d\eta\\
&\leqslant\|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2\|^2\\
&\quad+\frac{1}{A^{12}}\int_0^1\tilde{\mathcal{U}}_{2,\eta}^2(t,\eta)\left(\int_0^\eta e^{-\frac{3}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)\\
&\quad\times\left(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{U}}_2^+-\tilde{\mathcal{U}}_1^+)^2\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)d\eta
\end{aligned}
$$

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$$
\begin{aligned}
&\leqslant \|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2\|^2+\frac{2}{A^{11}}\int_0^1 \tilde{\mathcal{P}}_2\tilde{\mathcal{U}}_{2,\eta}^2(t,\eta)\\
&\quad\times\left(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{U}}_2-\tilde{\mathcal{U}}_1)^2\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)d\eta\\
&\leqslant \mathcal{O}(1)\|\tilde{\mathcal{U}}_2-\tilde{\mathcal{U}}_1\|^2,
\end{aligned}
$$

by applying (4.15e), (4.15q), and (4.16e).

The term $N_{15}+N_{16}$ can be estimated as follows:

$$
\begin{aligned}
N_{15}+N_{16}&=\frac{1}{A^6}\int_0^1(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)(t,\eta)\\
&\quad\times\Bigg[\tilde{\mathcal{U}}_{2,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\\
&\quad-\tilde{\mathcal{U}}_{1,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\Bigg]d\eta\\
&=\frac{1}{A^6}\mathbb{1}_{A_1\leqslant A_2}\int_0^1(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)\tilde{\mathcal{U}}_{2,\eta}(t,\eta)\\
&\quad\times\Bigg(\int_0^\eta(e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}-e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))})\\
&\quad\times\min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\Bigg)d\eta\\
&\quad+\frac{1}{A^6}\mathbb{1}_{A_2<A_1}\int_0^1(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)\tilde{\mathcal{U}}_{1,\eta}(t,\eta)\\
&\quad\times\Bigg(\int_0^\eta(e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}-e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))})\\
&\quad\times\min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\Bigg)d\eta\\
&\quad+\frac{1}{A^6}\int_0^1(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)\tilde{\mathcal{U}}_{2,\eta}(t,\eta)\\
&\quad\times\Bigg(\int_0^\eta(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}-e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))})\\
&\quad\times\min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\mathbb{1}_{B(\eta)}(t,\theta)\,d\theta\Bigg)d\eta\\
&\quad+\frac{1}{A^6}\int_0^1(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)\tilde{\mathcal{U}}_{1,\eta}(t,\eta)\\
&\quad\times\Bigg(\int_0^\eta(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}-e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))})
\end{aligned}
$$

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$$\times \min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\mathbb{1}_{B^c(\eta)}(t,\theta)\,d\theta\Big)\,d\eta$$
$$+\frac{1}{A^6}\int_0^1(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)(t,\eta)\Big[\tilde{\mathcal{U}}_{2,\eta}(t,\eta)\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})$$
$$\times \min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta$$
$$-\tilde{\mathcal{U}}_{1,\eta}(t,\eta)\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})$$
$$\times \min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\Big]\,d\eta$$
$$= N_{151}+N_{152}+N_{153}+N_{154}+N_{155}.$$

Unfortunately, each of these terms needs a special treatment.

The terms N_{151} and N_{152} can be handled as follows:

$$|N_{151}| = \frac{1}{A^6}\mathbb{1}_{A_1\leqslant A_2}\Big|\int_0^1(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)\tilde{\mathcal{U}}_{2,\eta}(t,\eta)$$
$$\times\Big(\int_0^\eta(e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}-e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))})$$
$$\times \min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\Big)\,d\eta\Big|$$
$$\leqslant \frac{4}{A^6A_1e}\int_0^1|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2||\tilde{\mathcal{U}}_{2,\eta}|(t,\eta)$$
$$\times\Big(\int_0^\eta e^{-\frac{3}{4A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\Big)\,d\eta|A_1-A_2|$$
$$\leqslant \|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2\|^2+\frac{16}{A^{12}A_1^2e^2}\int_0^1\tilde{\mathcal{U}}_{2,\eta}^2(t,\eta)$$
$$\times\Big(\int_0^\eta e^{-\frac{3}{4A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}|\tilde{\mathcal{U}}_1|\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\Big)^2\,d\eta|A_1-A_2|^2$$
$$\leqslant \|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2\|^2+\frac{8}{A^4e^2}\int_0^1\tilde{\mathcal{P}}_2\tilde{\mathcal{U}}_{2,\eta}^2(t,\eta)\,d\eta|A_1-A_2|^2$$
$$\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2\|^2+|A_1-A_2|^2),$$

where we used (4.16e).

The terms N_{153} and N_{154} can be handled as follows:

$$|N_{153}| = \frac{1}{A^6}\Big|\int_0^1(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)\tilde{\mathcal{U}}_{2,\eta}(t,\eta)$$
$$\times\int_0^\eta(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}-e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))})$$

$$
\begin{aligned}
&\times \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\mathbb{1}_{B(\eta)}(t,\theta)\,d\theta\,d\eta\Bigg|\\
&\leqslant \frac{1}{A^6 a}\int_0^1 |\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2||\tilde{\mathcal{U}}_{2,\eta}|(t,\eta)\\
&\quad\times \int_0^\eta (|\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_1(t,\eta)|+|\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_1(t,\theta)|)e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\\
&\quad\times \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\,d\eta\\
&= \frac{1}{A^6 a}\int_0^1 |\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2|\,|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1||\tilde{\mathcal{U}}_{2,\eta}|(t,\eta)\\
&\quad\times \int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\,d\eta\\
&\quad+\frac{1}{A^6 a}\int_0^1 |\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2||\tilde{\mathcal{U}}_{2,\eta}|(t,\eta)\\
&\quad\times \int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1|\min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\,d\eta\\
&\leqslant \frac{1}{A^6 a}\int_0^1 |\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2|\,|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1||\tilde{\mathcal{U}}_{2,\eta}|(t,\eta)\\
&\quad\times\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2^2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)^{1/2}\\
&\quad\times\left(\int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\min_j(\tilde{\mathcal{U}}_j^+)^2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)^{1/2}d\eta\\
&\quad+\|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2\|^2+\frac{1}{A^{12}a^2}\int_0^1 \tilde{\mathcal{U}}_{2,\eta}^2(t,\eta)\\
&\quad\times\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1|\tilde{\mathcal{P}}_2\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)^2d\eta\\
&\leqslant \frac{\sqrt{3}}{2A^2}\int_0^1 |\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2|\,|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1|\tilde{\mathcal{P}}_2^{1/2}|\tilde{\mathcal{U}}_{2,\eta}|(t,\eta)\,d\eta+\|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2\|^2\\
&\quad+\frac{9}{4A^4}\int_0^1 \tilde{\mathcal{P}}_2\tilde{\mathcal{U}}_{2,\eta}^2(t,\eta)\,d\eta\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2\\
&\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2\|^2+\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2),
\end{aligned}
$$

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where we used (4.16k). We consider now N_{155}:

$$\begin{aligned} N_{155} &= \frac{1}{A^6}\int_0^1 (\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)(t,\eta) \\ &\quad\times\Big[\tilde{\mathcal{U}}_{2,\eta}(t,\eta)\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta \\ &\quad-\tilde{\mathcal{U}}_{1,\eta}(t,\eta)\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\Big]\,d\eta \\ &= \frac{1}{A^6}\int_0^1 (\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)(\tilde{\mathcal{U}}_2-\tilde{\mathcal{U}}_1)_\eta(t,\eta) \\ &\quad\times\min_k\Big[\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{k,\eta}(t,\theta)\,d\theta\Big]\,d\eta \\ &\quad+\frac{1}{A^6}\int_0^1 (\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)\tilde{\mathcal{U}}_{2,\eta}(t,\eta)\mathbb{1}_{\tilde{E}^c}(t,\eta) \\ &\quad\times\Big(\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)(\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1)_\eta\,d\theta\Big)\,d\eta \\ &\quad+\frac{1}{A^6}\int_0^1 (\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)\tilde{\mathcal{U}}_{1,\eta}(t,\eta)\mathbb{1}_{\tilde{E}}(t,\eta) \\ &\quad\times\Big(\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)(\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1)_\eta\,d\theta\Big)\,d\eta \\ &= N_{1551}+N_{1552}+N_{1553}, \end{aligned}$$

which, yet again, requires separate treatment. Here $\tilde{E}$ is defined in (5.16).

The term N_{1551} can be handled as the term $\tilde{L}_{31}$ (cf. (5.17)),

$$|N_{1551}| \leqslant \mathcal{O}(1)\|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2\|^2.$$

The terms N_{1552} and N_{1553} can be treated in the same manner:

$$\begin{aligned} |N_{1552}| &\leqslant \frac{1}{A^6}\Big|\int_0^1 (\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)\tilde{\mathcal{U}}_{2,\eta}(t,\eta)\mathbb{1}_{\tilde{E}^c}(t,\eta) \\ &\quad\times\Big(\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)(\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1)_\eta(t,\theta)\,d\theta\Big)\,d\eta\Big| \\ &\leqslant \frac{1}{A^6}\int_0^1 |\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2|\,|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|\min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)|\tilde{\mathcal{U}}_{2,\eta}|\mathbb{1}_{\tilde{E}^c}(t,\eta)\,d\eta \\ &\quad+\frac{1}{A^6}\Big|\int_0^1 (\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)\tilde{\mathcal{U}}_{2,\eta}(t,\eta)\mathbb{1}_{\tilde{E}^c}(t,\eta) \\ &\quad\times\int_0^\eta (\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)\Big[\frac{d}{d\theta}\big(\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\big)\min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+) \end{aligned}$$

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$$+ \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\frac{d}{d\theta}\big(\min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\big)\Big](t,\theta)\,d\theta\,d\eta|$$
$$\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2\|^2+\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2).$$

The term N_2 (and also N_3) can be treated as follows, keeping in mind that $A_2 \leqslant A_1$.

$$\begin{aligned}
|N_2| &\leqslant \frac{A_1^6-A_2^6}{A_1^6A_2^6}\int_0^1 |\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2||\tilde{\mathcal{U}}_{2,\eta}|(t,\eta)\Big(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2^2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\Big)^{1/2}\\
&\quad\times\Big(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\Big)^{1/2}d\eta\\
&\leqslant \sqrt{6}\frac{A_1^6-A_2^6}{A^6A_2^3}\int_0^1 |\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2|\tilde{\mathcal{P}}_2|\tilde{\mathcal{U}}_{2,\eta}|(t,\eta)\,d\eta\\
&\leqslant 3\sqrt{3}A^2|A_1-A_2|\int_0^1|\tilde{\mathcal{U}}_2-\tilde{\mathcal{U}}_1|(t,\eta)\,d\eta\\
&\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2\|^2+|A_1-A_2|^2).
\end{aligned}$$

Finally, we now turn to the integral I_{33}. Recall equation (5.32c), namely,

$$\begin{aligned}
I_{33} &= \int_0^1(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)(t,\eta)\Big(\frac{1}{A_2^6}\tilde{\mathcal{U}}_{2,\eta}(t,\eta)\int_0^1 e^{-\frac{1}{A_2}|\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta)|}\tilde{\mathcal{Q}}_2\tilde{\mathcal{U}}_{2,\eta}(t,\theta)\,d\theta\\
&\quad-\frac{1}{A_1^6}\tilde{\mathcal{U}}_{1,\eta}(t,\eta)\int_0^1 e^{-\frac{1}{A_1}|\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta)|}\tilde{\mathcal{Q}}_1\tilde{\mathcal{U}}_{1,\eta}(t,\theta)\,d\theta\Big)\,d\eta.
\end{aligned}$$

By first involving (5.4) and then Lemma 5.1, we see that it suffices to estimate

$$\begin{aligned}
\tilde{I}_{33} &= \int_0^1(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)(t,\eta)\Big(\frac{1}{A_2^6}\tilde{\mathcal{U}}_{2,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{Q}}_2\tilde{\mathcal{U}}_{2,\eta}(t,\theta)\,d\theta\\
&\quad-\frac{1}{A_1^6}\tilde{\mathcal{U}}_{1,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\tilde{\mathcal{Q}}_1\tilde{\mathcal{U}}_{1,\eta}(t,\theta)\,d\theta\Big)\,d\eta\\
&= \frac{1}{A^6}\int_0^1(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)(t,\eta)\Big(\tilde{\mathcal{U}}_{2,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{Q}}_2\tilde{\mathcal{U}}_{2,\eta}(t,\theta)\,d\theta\\
&\quad-\tilde{\mathcal{U}}_{1,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\tilde{\mathcal{Q}}_1\tilde{\mathcal{U}}_{1,\eta}(t,\theta)\,d\theta\Big)\,d\eta\\
&\quad+\mathbb{1}_{A_2\leqslant A_1}\Big(\frac{1}{A_2^6}-\frac{1}{A_1^6}\Big)\int_0^1(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)\tilde{\mathcal{U}}_{2,\eta}(t,\eta)\\
&\quad\times\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{Q}}_2\tilde{\mathcal{U}}_{2,\eta}(t,\theta)\,d\theta\,d\eta
\end{aligned}$$

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$$
\begin{aligned}
&+ \mathbb{1}_{A_1<A_2}\left(\frac{1}{A_2^6}-\frac{1}{A_1^6}\right)\int_0^1 (\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)\tilde{\mathcal{U}}_{1,\eta}(t,\eta)\\
&\times \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\tilde{\mathcal{Q}}_1\tilde{\mathcal{U}}_{1,\eta}(t,\theta)\,d\theta\,d\eta\\
&= M_1+M_2+M_3.
\end{aligned}
$$

To our dismay, the estimate for M_1 is rather involved. We estimate, using first that $\tilde{\mathcal{Q}}_i = A_i\tilde{\mathcal{P}}_i - \tilde{\mathcal{D}}_i$ (cf. (4.7)):

$$
\begin{aligned}
M_1 &= \frac{1}{A^6}\int_0^1 (\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)(t,\eta)\Big(\tilde{\mathcal{U}}_{2,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{Q}}_2\tilde{\mathcal{U}}_{2,\eta}(t,\theta)\,d\theta\\
&\quad - \tilde{\mathcal{U}}_{1,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\tilde{\mathcal{Q}}_1\tilde{\mathcal{U}}_{1,\eta}(t,\theta)\,d\theta\Big)\,d\eta\\
&= \frac{1}{A^6}\int_0^1 (\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)(t,\eta)\Big(\tilde{\mathcal{U}}_{2,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}A_2\tilde{\mathcal{P}}_2\tilde{\mathcal{U}}_{2,\eta}(t,\theta)\,d\theta\\
&\quad - \tilde{\mathcal{U}}_{1,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}A_1\tilde{\mathcal{P}}_1\tilde{\mathcal{U}}_{1,\eta}(t,\theta)\,d\theta\Big)\,d\eta\\
&\quad - \frac{1}{A^6}\int_0^1 (\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)(t,\eta)\Big(\tilde{\mathcal{U}}_{2,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{D}}_2\tilde{\mathcal{U}}_{2,\eta}(t,\theta)\,d\theta\\
&\quad - \tilde{\mathcal{U}}_{1,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\tilde{\mathcal{D}}_1\tilde{\mathcal{U}}_{1,\eta}(t,\theta)\,d\theta\Big)\,d\eta\\
&= \frac{1}{A^6}\int_0^1 (\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)(A_2\tilde{\mathcal{P}}_2\tilde{\mathcal{U}}_2\tilde{\mathcal{U}}_{2,\eta} - A_1\tilde{\mathcal{P}}_1\tilde{\mathcal{U}}_1\tilde{\mathcal{U}}_{1,\eta})(t,\eta)\,d\eta\\
&\quad - \frac{1}{A^6}\int_0^1 (\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)(\tilde{\mathcal{D}}_2\tilde{\mathcal{U}}_2\tilde{\mathcal{U}}_{2,\eta} - \tilde{\mathcal{D}}_1\tilde{\mathcal{U}}_1\tilde{\mathcal{U}}_{1,\eta})(t,\eta)\,d\eta\\
&\quad + \frac{1}{A^6}\int_0^1 (\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)(t,\eta)\\
&\quad \times \Big(\tilde{\mathcal{U}}_{2,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\frac{1}{A_2}\tilde{\mathcal{D}}_2\tilde{\mathcal{U}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\\
&\quad - \tilde{\mathcal{U}}_{1,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\frac{1}{A_1}\tilde{\mathcal{D}}_1\tilde{\mathcal{U}}_1\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\Big)\,d\eta\\
&\quad - \frac{3}{A^6}\int_0^1 (\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)(t,\eta)\\
&\quad \times \Big(\tilde{\mathcal{U}}_{2,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2\tilde{\mathcal{U}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta
\end{aligned}
$$

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$$
\begin{aligned}
&- \tilde{\mathcal{U}}_{1,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\tilde{\mathcal{P}}_1\tilde{\mathcal{U}}_1\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\Big)\,d\eta\\
&+\frac{1}{A^6}\int_0^1(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)(t,\eta)\Big(\tilde{\mathcal{U}}_{2,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^3\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\\
&- \tilde{\mathcal{U}}_{1,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\tilde{\mathcal{U}}_1^3\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\Big)\,d\eta\\
&+\frac{1}{2A^6}\int_0^1(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)(t,\eta)\Big(\tilde{\mathcal{U}}_{2,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}A_2^5\tilde{\mathcal{U}}_2(t,\theta)\,d\theta\\
&- \tilde{\mathcal{U}}_{1,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}A_1^5\tilde{\mathcal{U}}_1(t,\theta)\,d\theta\Big)\,d\eta\\
&= W_1+W_2+W_3+W_4+W_5+W_6.
\end{aligned}
$$

Here we have used the rewrite employed when manipulating the term $\bar{K}_1$ from the expression (5.18) to (5.19). We start by considering the term W_1:

$$
\begin{aligned}
W_1 &= \frac{1}{A^6}\int_0^1(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)(A_2\tilde{\mathcal{P}}_2\tilde{\mathcal{U}}_2\tilde{\mathcal{U}}_{2,\eta}-A_1\tilde{\mathcal{P}}_1\tilde{\mathcal{U}}_1\tilde{\mathcal{U}}_{1,\eta})(t,\eta)\,d\eta\\
&= \frac{1}{A^6}\mathbb{1}_{A_1\leqslant A_2}(A_2-A_1)\int_0^1(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)\tilde{\mathcal{P}}_2\tilde{\mathcal{U}}_2\tilde{\mathcal{U}}_{2,\eta}(t,\eta)\,d\eta\\
&\quad+\frac{1}{A^6}\mathbb{1}_{A_2<A_1}(A_2-A_1)\int_0^1(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)\tilde{\mathcal{P}}_1\tilde{\mathcal{U}}_1\tilde{\mathcal{U}}_{1,\eta}(t,\eta)\,d\eta\\
&\quad+\frac{a}{A^6}\int_0^1(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)(\tilde{\mathcal{P}}_2-\tilde{\mathcal{P}}_1)\tilde{\mathcal{U}}_2\tilde{\mathcal{U}}_{2,\eta}(t,\eta)\mathbb{1}_{\tilde{\mathcal{P}}_1\leqslant\tilde{\mathcal{P}}_2}(t,\eta)\,d\eta\\
&\quad+\frac{a}{A^6}\int_0^1(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)(\tilde{\mathcal{P}}_2-\tilde{\mathcal{P}}_1)\tilde{\mathcal{U}}_1\tilde{\mathcal{U}}_{1,\eta}(t,\eta)\mathbb{1}_{\tilde{\mathcal{P}}_2<\tilde{\mathcal{P}}_1}(t,\eta)\,d\eta\\
&\quad+\frac{a}{A^6}\int_0^1(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)(\tilde{\mathcal{U}}_{2,\eta}-\tilde{\mathcal{U}}_{1,\eta})\min_j(\tilde{\mathcal{P}}_j)\tilde{\mathcal{U}}_2(t,\eta)\,d\eta\\
&\quad-\frac{a}{A^6}\int_0^1(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)^2\min_j(\tilde{\mathcal{P}}_j)\tilde{\mathcal{U}}_{1,\eta}(t,\eta)\,d\eta\\
&= W_{11}+W_{12}+W_{13}+W_{14}+W_{15}+W_{16}.
\end{aligned}
$$

For W_{11} we have (and similarly for W_{12}) that

$$
\begin{aligned}
|W_{11}| &\leqslant \frac{A^2}{8}|A_1-A_2|\int_0^1|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2|(t,\eta)\,d\eta\\
&\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2\|^2+|A_1-A_2|^2).
\end{aligned}
$$

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The terms W_{13} and W_{14} are similar:

$$\begin{aligned}
|W_{11}| &= \frac{a}{A^6}\left|\int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(\tilde{\mathcal{P}}_2 - \tilde{\mathcal{P}}_1)\tilde{\mathcal{U}}_2\tilde{\mathcal{U}}_{2,\eta}(t,\eta)\mathbb{1}_{\tilde{\mathcal{P}}_1\leqslant\tilde{\mathcal{P}}_2}(t,\eta)\,d\eta\right| \\
&\leqslant \frac{2a}{A^6}\int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2|\,|\tilde{\mathcal{P}}_2^{1/2} - \tilde{\mathcal{P}}_1^{1/2}|\tilde{\mathcal{P}}_2^{1/2}|\tilde{\mathcal{U}}_2\tilde{\mathcal{U}}_{2,\eta}|(t,\eta)\,d\eta \\
&\leqslant \frac{A}{2}\int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2|\,|\tilde{\mathcal{P}}_2^{1/2} - \tilde{\mathcal{P}}_1^{1/2}|\,d\eta \\
&\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \|\tilde{\mathcal{P}}_2^{1/2} - \tilde{\mathcal{P}}_1^{1/2}\|^2),
\end{aligned}$$

using (4.15a) and (4.15f).

The term W_{15} goes as follows:

$$\begin{aligned}
|W_{15}| &= \frac{a}{A^6}\left|\int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(\tilde{\mathcal{U}}_{2,\eta} - \tilde{\mathcal{U}}_{1,\eta})\min_j(\tilde{\mathcal{P}}_j)\tilde{\mathcal{U}}_2(t,\eta)\,d\eta\right| \\
&= \left|\frac{a}{2A^6}(\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2\min_j(\tilde{\mathcal{P}}_j)\tilde{\mathcal{U}}_2(t,\eta)\,\right|_{\eta=0}^1 \\
&\qquad - \frac{a}{2A^6}\int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2\frac{d}{d\eta}\big(\min_j(\tilde{\mathcal{P}}_j)\tilde{\mathcal{U}}_2\big)(t,\eta)\,d\eta\bigg| \\
&\leqslant \frac{a}{2A^6}\int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2\left|\frac{d}{d\eta}\big(\min_j(\tilde{\mathcal{P}}_j)\tilde{\mathcal{U}}_2\big)\right|(t,\eta)\,d\eta \\
&\leqslant \mathcal{O}(1)\|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2;
\end{aligned}$$

see the estimates for $\bar{B}_{15}$ (cf. (5.20)). As for the term W_{16}, we get

$$\begin{aligned}
|W_{16}| &= \frac{a}{A^6}\int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2\min_j(\tilde{\mathcal{P}}_j)|\tilde{\mathcal{U}}_{1,\eta}|(t,\eta)\,d\eta \\
&\leqslant \mathcal{O}(1)\|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2,
\end{aligned}$$

using (4.15p).

As for the term W_2, we find

$$\begin{aligned}
-W_2 &= \frac{1}{A^6}\int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(\tilde{\mathcal{D}}_2\tilde{\mathcal{U}}_2\tilde{\mathcal{U}}_{2,\eta} - \tilde{\mathcal{D}}_1\tilde{\mathcal{U}}_1\tilde{\mathcal{U}}_{1,\eta})(t,\eta)\,d\eta \\
&= \frac{1}{A^6}\int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(\tilde{\mathcal{D}}_2 - \tilde{\mathcal{D}}_1)\tilde{\mathcal{U}}_1\tilde{\mathcal{U}}_{1,\eta}(t,\eta)\mathbb{1}_{\tilde{\mathcal{D}}_2\leqslant\tilde{\mathcal{D}}_1}(t,\eta)\,d\eta \\
&\quad + \frac{1}{A^6}\int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(\tilde{\mathcal{D}}_2 - \tilde{\mathcal{D}}_1)\tilde{\mathcal{U}}_2\tilde{\mathcal{U}}_{2,\eta}(t,\eta)\mathbb{1}_{\tilde{\mathcal{D}}_1<\tilde{\mathcal{D}}_2}(t,\eta)\,d\eta
\end{aligned}$$

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$$+\frac{1}{A^6}\int_0^1(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)(\tilde{\mathcal{U}}_{2,\eta}-\tilde{\mathcal{U}}_{1,\eta})\min_j(\tilde{\mathcal{D}}_j)\tilde{\mathcal{U}}_2(t,\eta)\,d\eta$$
$$-\frac{1}{A^6}\int_0^1(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)^2\min_j(\tilde{\mathcal{D}}_j)\tilde{\mathcal{U}}_{1,\eta}(t,\eta)\,d\eta$$
$$=W_{21}+W_{22}+W_{23}+W_{24}.$$

The terms W_{21} and W_{22} can be treated similarly. We need to estimate $\tilde{\mathcal{D}}_2-\tilde{\mathcal{D}}_1$. Applying Lemma A.9, we have

$$\begin{aligned}
|W_{21}| &= \frac{1}{A^6}\left|\int_0^1(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)(\tilde{\mathcal{D}}_2-\tilde{\mathcal{D}}_1)\tilde{\mathcal{U}}_1\tilde{\mathcal{U}}_{1,\eta}(t,\eta)\mathbb{1}_{\tilde{\mathcal{D}}_2\leqslant\tilde{\mathcal{D}}_1}(t,\eta)\,d\eta\right|\\
&\leqslant \frac{2}{A^{9/2}}\int_0^1|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2|\,|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|\tilde{\mathcal{D}}_1^{1/2}|\tilde{\mathcal{U}}_1\tilde{\mathcal{U}}_{1,\eta}|(t,\eta)\,d\eta\\
&\quad+\frac{1}{A^6}\int_0^1|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2|\,|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|(\tilde{\mathcal{U}}_1^2+\tilde{\mathcal{P}}_1)|\tilde{\mathcal{U}}_1\tilde{\mathcal{U}}_{1,\eta}|(t,\eta)\,d\eta\\
&\quad+\frac{2\sqrt{2}}{A^{9/2}}\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|\int_0^1|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2|\tilde{\mathcal{D}}_1^{1/2}|\tilde{\mathcal{U}}_1\tilde{\mathcal{U}}_{1,\eta}|(t,\eta)\,d\eta\\
&\quad+\frac{4}{A^3}\int_0^1|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2|\\
&\quad\times\left(\int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)^2(t,\theta)\,d\theta\right)^{1/2}|\tilde{\mathcal{U}}_1\tilde{\mathcal{U}}_{1,\eta}|(t,\eta)\,d\eta\\
&\quad+\frac{2\sqrt{2}}{A^3}\int_0^1|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2|\\
&\quad\times\left(\int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})^2(t,\theta)\,d\theta\right)^{1/2}|\tilde{\mathcal{U}}_1\tilde{\mathcal{U}}_{1,\eta}|(t,\eta)\,d\eta\\
&\quad+\frac{3}{\sqrt{2}A^2}\int_0^1|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2|\\
&\quad\times\left(\int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,\theta)\,d\theta\right)^{1/2}|\tilde{\mathcal{U}}_1\tilde{\mathcal{U}}_{1,\eta}|(t,\eta)\,d\eta\\
&\quad+\frac{12\sqrt{2}}{\sqrt{3}eA^2}\int_0^1|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2|\\
&\quad\times\left(\int_0^\eta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\,d\theta\right)^{1/2}|\tilde{\mathcal{U}}_1\tilde{\mathcal{U}}_{1,\eta}|(t,\eta)\,d\eta|A_1-A_2|\\
&\quad+\frac{3}{2A^2}\int_0^1|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2|\\
&\quad\times\left(\int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|(t,\theta)\,d\theta\right)|\tilde{\mathcal{U}}_1\tilde{\mathcal{U}}_{1,\eta}|(t,\eta)\,d\eta
\end{aligned}$$

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$$
\begin{aligned}
&+\frac{6}{A^2}|A_1-A_2|\int_0^1|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2|\Big(\int_0^\eta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\,d\theta\Big)|\tilde{\mathcal{U}}_1\tilde{\mathcal{U}}_{1,\eta}|(t,\eta)\,d\eta\\
&\leqslant\frac{2}{A^{9/2}}\int_0^1|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2|\,|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|(t,\eta)\frac{A^{13/2}}{2\sqrt{2}}\,d\eta\\
&\quad+\frac{1}{A^6}\int_0^1|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2|\,|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|(t,\eta)\frac{3A^8}{8}\,d\eta\\
&\quad+\frac{2\sqrt{2}}{A^{9/2}}\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|\int_0^1|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2|(t,\eta)\frac{A^{13/2}}{2\sqrt{2}}\,d\eta\\
&\quad+\frac{4}{A^3}\int_0^1|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2|(t,\eta)\Big(\int_0^\eta(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)^2(t,\theta)\,d\theta\Big)^{1/2}\frac{A^4}{2}\,d\eta\\
&\quad+\frac{2\sqrt{2}}{A^3}\int_0^1|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2|(t,\eta)\Big(\int_0^\eta(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})^2(t,\theta)\,d\theta\Big)^{1/2}\frac{A^4}{2}\,d\eta\\
&\quad+\frac{3}{\sqrt{2}A^2}\int_0^1|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2|(t,\eta)\Big(\int_0^\eta(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,\theta)\,d\theta\Big)^{1/2}\frac{A^4}{2}\,d\eta\\
&\quad+\frac{12\sqrt{2}}{\sqrt{3}eA^2}\int_0^1|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2|\frac{A^4}{2}\,d\eta|A_1-A_2|\\
&\quad+\frac{3}{2A^2}\int_0^1|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2|(t,\eta)\Big(\int_0^\eta|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|(t,\theta)\,d\theta\Big)\frac{A^4}{2}\,d\eta\\
&\quad+\frac{6}{A^2}|A_1-A_2|\int_0^1|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2|(t,\eta)\frac{A^4}{2}\,d\eta\\
&\leqslant\mathcal{O}(1)(\|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2\|^2+\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2+|A_1-A_2|^2),
\end{aligned}
$$

where we have used estimates (4.15a), (4.15b), (4.15f), and (4.15n). The term W_{23} goes as follows:

$$
\begin{aligned}
|W_{23}|&=\frac{1}{A^6}\Big|\int_0^1(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)(\tilde{\mathcal{U}}_{2,\eta}-\tilde{\mathcal{U}}_{1,\eta})\min_j(\tilde{\mathcal{D}}_j)\tilde{\mathcal{U}}_2(t,\eta)\,d\eta\Big|\\
&\leqslant\Big|\frac{1}{2A^6}(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)^2\min_j(\tilde{\mathcal{D}}_j)\tilde{\mathcal{U}}_2(t,\eta)\Big|_{\eta=0}^1\\
&\qquad-\frac{1}{2A^6}\int_0^1(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)^2\frac{d}{d\theta}\big(\min_j(\tilde{\mathcal{D}}_j)\tilde{\mathcal{U}}_2\big)(t,\eta)\,d\eta\Big|\\
&\leqslant\frac{1}{2A^6}\int_0^1(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)^2\Big|\frac{d}{d\theta}\big(\min_j(\tilde{\mathcal{D}}_j)\tilde{\mathcal{U}}_2\big)\Big|(t,\eta)\,d\eta\\
&\leqslant\mathcal{O}(1)\|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2\|^2,
\end{aligned}
$$

by applying Lemma A.4 (ii).

The term W_{24} goes as follows:

$$\begin{aligned}|W_{24}| &\leqslant \frac{1}{A^6}\int_0^1 (\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)^2 \min_j(\tilde{\mathcal{D}}_j)|\tilde{\mathcal{U}}_{1,\eta}|(t,\eta)\,d\eta\\ &\leqslant \mathcal{O}(1)\|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2\|^2,\end{aligned}$$

using

$$\min_j(\tilde{\mathcal{D}}_j)|\tilde{\mathcal{U}}_{1,\eta}| \leqslant 2A_1\tilde{\mathcal{P}}_1|\tilde{\mathcal{U}}_{1,\eta}| \leqslant \frac{A^7}{\sqrt{2}},$$

from (4.15n) and (4.15p).

Next, we turn to the term W_3:

$$\begin{aligned}W_3 &= \frac{1}{A^6}\int_0^1 (\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)(t,\eta)\\ &\quad\times\bigg(\tilde{\mathcal{U}}_{2,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\frac{1}{A_2}\tilde{\mathcal{D}}_2\tilde{\mathcal{U}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\\ &\quad - \tilde{\mathcal{U}}_{1,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\frac{1}{A_1}\tilde{\mathcal{D}}_1\tilde{\mathcal{U}}_1\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\bigg)\,d\eta\\ &= \frac{1}{A^6}\int_0^1 (\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)(t,\eta)\\ &\quad\times\bigg(\tilde{\mathcal{U}}_{2,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\frac{1}{A_2}\tilde{\mathcal{D}}_2\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\\ &\quad - \tilde{\mathcal{U}}_{1,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\frac{1}{A_1}\tilde{\mathcal{D}}_1\tilde{\mathcal{U}}_1^+\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\bigg)\,d\eta\\ &\quad + \frac{1}{A^6}\int_0^1 (\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)(t,\eta)\\ &\quad\times\bigg(\tilde{\mathcal{U}}_{2,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\frac{1}{A_2}\tilde{\mathcal{D}}_2\tilde{\mathcal{U}}_2^-\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\\ &\quad - \tilde{\mathcal{U}}_{1,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\frac{1}{A_1}\tilde{\mathcal{D}}_1\tilde{\mathcal{U}}_1^-\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\bigg)\,d\eta\\ &= W_{31}+W_{32}.\end{aligned}$$

These terms can be treated similarly:

$$\begin{aligned}W_{31} &= \frac{1}{A^6}\int_0^1 (\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)(t,\eta)\\ &\quad\times\bigg(\tilde{\mathcal{U}}_{2,\eta}(t,\eta)\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\frac{1}{A_2}\tilde{\mathcal{D}}_2\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\end{aligned}$$

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$$
\begin{aligned}
&- \tilde{\mathcal{U}}_{1,\eta}(t,\eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))} \frac{1}{A_1} \tilde{\mathcal{D}}_1 \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\, d\theta \Big) \, d\eta \\
= {}& \mathbb{1}_{A_1 \leqslant A_2} \frac{1}{A^6} \left(\frac{1}{A_2} - \frac{1}{A_1} \right) \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{1,\eta}(t,\eta) \\
&\times \left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{D}}_1 \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\, d\theta \right) d\eta \\
&+ \mathbb{1}_{A_2 < A_1} \frac{1}{A^6} \left(\frac{1}{A_2} - \frac{1}{A_1} \right) \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{2,\eta}(t,\eta) \\
&\times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{D}}_2 \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\, d\theta \right) d\eta \\
&+ \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{2,\eta}(t,\eta) \\
&\times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{D}}_2 - \tilde{\mathcal{D}}_1) \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{\tilde{\mathcal{D}}_1 \leqslant \tilde{\mathcal{D}}_2}(t,\theta)\, d\theta \right) d\eta \\
&+ \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{1,\eta}(t,\eta) \\
&\times \left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))} (\tilde{\mathcal{D}}_2 - \tilde{\mathcal{D}}_1) \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1,\eta} \mathbb{1}_{\tilde{\mathcal{D}}_2 < \tilde{\mathcal{D}}_1}(t,\theta)\, d\theta \right) d\eta \\
&+ \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{2,\eta}(t,\eta) \\
&\times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+) \min_j(\tilde{\mathcal{D}}_j) \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{\tilde{\mathcal{U}}_1^+ \leqslant \tilde{\mathcal{U}}_2^+}(t,\theta)\, d\theta \right) d\eta \\
&+ \frac{1}{A^7} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{1,\eta}(t,\eta) \\
&\times \left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))} (\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+) \min_j(\tilde{\mathcal{D}}_j) \tilde{\mathcal{Y}}_{1,\eta} \mathbb{1}_{\tilde{\mathcal{U}}_2^+ < \tilde{\mathcal{U}}_1^+}(t,\theta)\, d\theta \right) d\eta \\
&+ \frac{1}{A^7} \mathbb{1}_{A_1 \leqslant A_2} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{2,\eta}(t,\eta) \\
&\quad \times \Big(\int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}) \\
&\times \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\, d\theta \Big) d\eta \\
&+ \frac{1}{A^7} \mathbb{1}_{A_2 < A_1} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{1,\eta}(t,\eta) \\
&\times \Big(\int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))})
\end{aligned}
$$

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$$
\begin{aligned}
&\times \min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\Big)\,d\eta\\
&+\frac{1}{A^7}\int_0^1(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)\tilde{\mathcal{U}}_{2,\eta}(t,\eta)\\
&\times\Big(\int_0^\eta(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}-e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))})\\
&\times \min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{2,\eta}\mathbb{1}_{B(\eta)}(t,\theta)\,d\theta\Big)\,d\eta\\
&+\frac{1}{A^7}\int_0^1(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)\tilde{\mathcal{U}}_{1,\eta}(t,\eta)\\
&\times\Big(\int_0^\eta(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}-e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))})\\
&\times \min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{1,\eta}\mathbb{1}_{B^c(\eta)}(t,\theta)\,d\theta\Big)\,d\eta\\
&-\frac{1}{A^7}\int_0^1(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)_\eta(t,\eta)\\
&\times\min_k\Big(\int_0^\eta\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{k,\eta}(t,\theta)\,d\theta\Big)\,d\eta\\
&+\frac{1}{A^7}\int_0^1(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)\tilde{\mathcal{U}}_{2,\eta}\mathbb{1}_{D^c}(t,\eta)\\
&\times\Big(\int_0^\eta\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)(\tilde{\mathcal{Y}}_{2,\eta}-\tilde{\mathcal{Y}}_{1,\eta})(t,\theta)\,d\theta\Big)\,d\eta\\
&+\frac{1}{A^7}\int_0^1(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)\tilde{\mathcal{U}}_{1,\eta}\mathbb{1}_{D}(t,\eta)\\
&\times\Big(\int_0^\eta\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)(\tilde{\mathcal{Y}}_{2,\eta}-\tilde{\mathcal{Y}}_{1,\eta})(t,\theta)\,d\theta\Big)\,d\eta\\
=\;& Z_1+Z_2+Z_3+Z_4+Z_5+Z_6+Z_7+Z_8+Z_9+Z_{10}+Z_{11}+Z_{12}+Z_{13},
\end{aligned}
$$

where the set D is defined by (5.21). At this point, it cannot come as a surprise that we have to treat these terms separately.

Let us start with Z_1 (Z_2 is similar):

$$
\begin{aligned}
|Z_1|\leqslant\frac{2}{A^7}|A_1-A_2|&\int_0^1|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2||\tilde{\mathcal{U}}_{1,\eta}|(t,\eta)\\
&\times\Big(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\tilde{\mathcal{U}}_1^2\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\Big)^{1/2}\\
&\times\Big(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\tilde{\mathcal{P}}_1^2\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\Big)^{1/2}\,d\eta
\end{aligned}
$$

$$\leqslant \frac{2\sqrt{6}}{A^4}|A_1 - A_2| \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2| \tilde{\mathcal{P}}_1 |\tilde{\mathcal{U}}_{1,\eta}|(t,\eta)\, d\eta$$
$$\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + |A_1 - A_2|^2).$$

The terms Z_3 and Z_4 go as follows:

$$|Z_3| = \frac{1}{A^7}\bigg|\int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)\tilde{\mathcal{U}}_{2,\eta}(t,\eta)$$
$$\times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{D}}_2 - \tilde{\mathcal{D}}_1)\tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{\tilde{\mathcal{D}}_1 \leqslant \tilde{\mathcal{D}}_2}(t,\theta)\, d\theta\right) d\eta\bigg|$$
$$\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + |A_1 - A_2|^2),$$

by Lemma A.9. The terms Z_5 and Z_6 can be treated similarly:

$$|Z_5| = \frac{1}{A^7}\bigg|\int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)\tilde{\mathcal{U}}_{2,\eta}(t,\eta)$$
$$\times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+)\min_j(\tilde{\mathcal{D}}_j)\tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{\tilde{\mathcal{U}}_1^+ \leqslant \tilde{\mathcal{U}}_2^+}(t,\theta)\, d\theta\right) d\eta\bigg|$$
$$\leqslant \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \frac{1}{A^{14}}\int_0^1 \tilde{\mathcal{U}}_{2,\eta}^2(t,\eta)$$
$$\times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+)\min_j(\tilde{\mathcal{D}}_j)\tilde{\mathcal{Y}}_{2,\eta} \mathbb{1}_{\tilde{\mathcal{U}}_1^+ \leqslant \tilde{\mathcal{U}}_2^+}(t,\theta)\, d\theta\right)^2 d\eta$$
$$\leqslant \mathcal{O}(1)\|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2,$$

by applying (4.16e), estimating $\tilde{\mathcal{D}}_2 \leqslant 2A_2\tilde{\mathcal{P}}_2$ (cf. (4.15n)), and subsequently $2\sqrt{2}\tilde{\mathcal{P}}_2\tilde{\mathcal{U}}_{2,\eta}^2 \leqslant A_2^6$ (cf. (4.15q)). The terms Z_7 and Z_8 follow this pattern:

$$|Z_8| \leqslant \frac{4}{A^7 ae}\int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2||\tilde{\mathcal{U}}_{1,\eta}|(t,\eta)$$
$$\times \left(\int_0^\eta e^{-\frac{3}{4A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} 2A_1\tilde{\mathcal{P}}_1\tilde{\mathcal{Y}}_{1,\eta}|\tilde{\mathcal{U}}_2|(t,\theta)\, d\theta\right) d\eta |A_1 - A_2|$$
$$\leqslant \frac{4\sqrt{2}}{A^5 e}\int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2||\tilde{\mathcal{U}}_{1,\eta}|(t,\eta)\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_2(t,\eta))}\tilde{\mathcal{P}}_1^2\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\, d\theta\right)^{1/2}$$
$$\times \left(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))}\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\, d\theta\right)^{1/2} d\eta |A_1 - A_2|$$

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$$\leqslant \frac{4\sqrt{6}}{A^2 e}\int_0^1 |\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2|\tilde{\mathcal{P}}_1^{1/2}|\tilde{\mathcal{U}}_{1,\eta}|(t,\eta)\,d\eta|A_1-A_2|$$
$$\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2\|^2+|A_1-A_2|^2).$$

The terms Z_9 and Z_{10} follow this pattern:

$$\begin{aligned}
|Z_9| &= \frac{1}{A^7}\bigg|\int_0^1 (\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)\tilde{\mathcal{U}}_{2,\eta}(t,\eta)\bigg(\int_0^\eta (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}-e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}) \\
&\qquad \times \min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{2,\eta}\mathbb{1}_{B(\eta)}(t,\theta)\,d\theta\bigg)\,d\eta\bigg| \\
&\leqslant \frac{a}{\sqrt{2}A^7}\int_0^1 |\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2||\tilde{\mathcal{U}}_{2,\eta}|(t,\eta) \\
&\qquad \times \bigg(\int_0^\eta (|\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_1(t,\eta)|+|\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_1(t,\theta)|) \\
&\qquad \times e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{D}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\bigg)\,d\eta \\
&\leqslant \frac{1}{\sqrt{2}A^6}\int_0^1 |\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2|\,|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1||\tilde{\mathcal{U}}_{2,\eta}|(t,\eta) \\
&\qquad \times \bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{D}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\bigg)\,d\eta \\
&\qquad + \frac{1}{\sqrt{2}A^6}\int_0^1 |\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2||\tilde{\mathcal{U}}_{2,\eta}|(t,\eta) \\
&\qquad \times \bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{D}}_2\tilde{\mathcal{Y}}_{2,\eta}|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1|(t,\theta)\,d\theta\bigg)\,d\eta \\
&\leqslant \frac{\sqrt{2}}{A^5}\int_0^1 |\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2|\,|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1||\tilde{\mathcal{U}}_{2,\eta}|(t,\eta) \\
&\qquad \times \bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2^2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\bigg)^{1/2} \\
&\qquad \times \bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\bigg)^{1/2}\,d\eta \\
&\qquad + \|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2\|^2+\frac{2}{A^{10}}\int_0^1 \tilde{\mathcal{U}}_{2,\eta}^2(t,\eta)\bigg(\int_0^\eta e^{-\frac{3}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\bigg) \\
&\qquad \times \bigg(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1|^2(t,\theta)\,d\theta\bigg)\,d\eta \\
&\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2\|^2+\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2).
\end{aligned}$$

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The term Z_{11}:

$$\begin{aligned}
|Z_{11}| &= \frac{1}{A^7}\bigg|\int_0^1 (\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)_\eta(t,\eta)\\
&\quad\times \min_k\bigg(\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{k,\eta}(t,\theta)\,d\theta\bigg)\,d\eta\bigg|\\
&\leqslant \mathcal{O}(1)\|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2\|_2^2,
\end{aligned}$$

following the estimates employed for the term $\bar{B}_{37}$; see (5.23).

The terms Z_{12} and Z_{13} may be treated as follows:

$$\begin{aligned}
|Z_{12}| &= \frac{1}{A^7}\bigg|\int_0^1 (\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)\tilde{\mathcal{U}}_{2,\eta}\mathbb{1}_{D^c}(t,\eta)\bigg(\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\\
&\quad\times \min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)(\tilde{\mathcal{Y}}_{2,\eta}-\tilde{\mathcal{Y}}_{1,\eta})(t,\theta)\,d\theta\bigg)\,d\eta\bigg|\\
&= \frac{1}{A^7}\bigg|\int_0^1 (\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)\tilde{\mathcal{U}}_{2,\eta}\mathbb{1}_{D^c}(t,\eta)\\
&\quad\times\bigg[\bigg(\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)(\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1)(t,\theta)\bigg)\bigg|_{\theta=0}^{\eta}\\
&\quad-\int_0^\eta(\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1)\frac{d}{d\theta}\big(\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\big)(t,\theta)\,d\theta\bigg]\,d\eta\bigg|\\
&\leqslant \frac{1}{A^7}\bigg|\int_0^1 (\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)\tilde{\mathcal{U}}_{2,\eta}\mathbb{1}_{D^c}\min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)(\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1)(t,\eta)\,d\eta\bigg|\\
&\quad+\frac{1}{A^7}\bigg|\int_0^1 (\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)\tilde{\mathcal{U}}_{2,\eta}\mathbb{1}_{D^c}(t,\eta)\\
&\quad\times\bigg(\int_0^\eta(\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1)\frac{d}{d\theta}\big(\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\big)(t,\theta)\,d\theta\bigg)\,d\eta\bigg|\\
&\leqslant \frac{A^2}{4}\int_0^1|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2|\,|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1|(t,\eta)\,d\eta\\
&\quad+\frac{1}{A^7}\int_0^1|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2|\,|\tilde{\mathcal{U}}_{2,\eta}|(t,\eta)\bigg(\int_0^\eta|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|\bigg|\frac{d}{d\theta}\big(\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\\
&\quad\times\min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\big)\bigg|(t,\theta)\,d\theta\bigg)\,d\eta\\
&= \tilde{M}_1+\tilde{M}_2.
\end{aligned}$$

Here we find

$$\tilde{M}_1 \leqslant \mathcal{O}(1)(\|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2\|^2+\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2),$$

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while $\tilde{M}_2$ requires more care:

$$\begin{aligned}
\tilde{M}_2 &= \frac{1}{A^7}\int_0^1 |\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2|\,|\tilde{\mathcal{U}}_{2,\eta}|(t,\eta)\bigg(\int_0^\eta |\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|\bigg|\frac{d}{d\theta}\big(\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\\
&\quad\times \min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\big)(t,\theta)\bigg|\,d\theta\bigg)\,d\eta\\
&\leqslant \frac{1}{A^7}\int_0^1 |\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2|\,|\tilde{\mathcal{U}}_{2,\eta}|(t,\eta)\bigg(\int_0^\eta |\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|\\
&\quad\times\bigg|\frac{d}{d\theta}\big(\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\big)\min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\\
&\quad+\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\frac{d}{d\theta}\big(\min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\big)\bigg|(t,\theta)\,d\theta\bigg)\,d\eta\\
&\leqslant \frac{1}{A^7}\int_0^1 |\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2|\,|\tilde{\mathcal{U}}_{2,\eta}|(t,\eta)\bigg(\int_0^\eta |\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|\bigg|\frac{d}{d\theta}\big(\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\big)\\
&\quad\times \min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)(t,\theta)\bigg|\,d\theta\bigg)\,d\eta\\
&\quad+\frac{1}{A^7}\int_0^1 |\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2|\,|\tilde{\mathcal{U}}_{2,\eta}|(t,\eta)\bigg(\int_0^\eta |\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|\big|\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\\
&\quad\times\frac{d}{d\theta}\big(\min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\big)(t,\theta)\big|\,d\theta\bigg)\,d\eta\\
&\leqslant \frac{1}{aA^7}\int_0^1 |\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2|\,|\tilde{\mathcal{U}}_{2,\eta}|(t,\eta)\bigg(\int_0^\eta |\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\\
&\quad\times\max_j(\tilde{\mathcal{Y}}_{j,\eta})\min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)(t,\theta)\,d\theta\bigg)\,d\eta\\
&\quad+\mathcal{O}(1)\frac{1}{A^{5/2}}\int_0^1 |\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2|\,|\tilde{\mathcal{U}}_{2,\eta}|(t,\eta)\\
&\quad\times\bigg(\int_0^\eta |\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\big(\min_j(\tilde{\mathcal{D}}_j)^{1/2}+|\tilde{\mathcal{U}}_2|\big)(t,\theta)\,d\theta\bigg)\,d\eta\\
&= \tilde{M}_{21}+\tilde{M}_{22},
\end{aligned}$$

where estimates for the derivatives come from Lemmas A.2 and A.4. We find for the term $\tilde{M}_{21}$ that

$$\begin{aligned}
\tilde{M}_{21} &= \frac{1}{aA^7}\int_0^1 |\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2|\,|\tilde{\mathcal{U}}_{2,\eta}|(t,\eta)\bigg(\int_0^\eta |\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\\
&\quad\times\max_j(\tilde{\mathcal{Y}}_{j,\eta})\min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)(t,\theta)\,d\theta\bigg)\,d\eta
\end{aligned}$$

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$$
\begin{aligned}
&\leqslant \|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2\|^2+\frac{1}{a^2A^{14}}\int_0^1 \tilde{\mathcal{U}}_{2,\eta}^2(t,\eta)\Big(\int_0^\eta |\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\\
&\quad\times \max_j(\tilde{\mathcal{Y}}_{j,\eta})\min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)(t,\theta)\,d\theta\Big)^2 d\eta\\
&\leqslant \|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2\|^2+\frac{1}{2}\int_0^1 \tilde{\mathcal{U}}_{2,\eta}^2(t,\eta)\\
&\quad\times \Big(\int_0^\eta |\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\,d\theta\Big)^2 d\eta\\
&\leqslant \|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2\|^2+\frac{A^3}{2}\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\\
&\quad\times \Big(\int_0^\eta (\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2 e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\,d\theta\Big)\Big(\int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\,d\theta\Big)d\eta\\
&\leqslant \|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2\|^2+\frac{A^3}{2}\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t,\eta)e^{-\frac{1}{a}\tilde{\mathcal{Y}}_2(t,\eta)}\Big(\int_0^\eta e^{\frac{1}{a}\tilde{\mathcal{Y}}_2(t,\theta)}\,d\theta\Big)d\eta\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2\\
&\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2\|^2+\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2),
\end{aligned}
$$

where we used the estimate

$$
\max_j(\tilde{\mathcal{Y}}_{j,\eta})\min_j(\tilde{\mathcal{D}}_j)\leqslant 2\max_j(\tilde{\mathcal{Y}}_{j,\eta})\min_j(A_j\tilde{\mathcal{P}}_j)\leqslant 2A\max_j(\tilde{\mathcal{P}}_j\tilde{\mathcal{Y}}_{j,\eta})\leqslant A^6,
$$

using (4.15e) and (4.15n), as well as (4.15q) and (4.16a). The term $\tilde{M}_{22}$ reads as

$$
\begin{aligned}
\tilde{M}_{22}&=\mathcal{O}(1)\frac{1}{A^{5/2}}\int_0^1 |\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2|\,|\tilde{\mathcal{U}}_{2,\eta}|(t,\eta)\\
&\quad\times\Big(\int_0^\eta |\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\big(\min_j(\tilde{\mathcal{D}}_j)^{1/2}+|\tilde{\mathcal{U}}_2|\big)(t,\theta)\,d\theta\Big)d\eta\\
&\leqslant \|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2\|^2+\mathcal{O}(1)\frac{1}{A^5}\int_0^1 \tilde{\mathcal{U}}_{2,\eta}^2(t,\eta)\\
&\quad\times\Big(\int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|\tilde{\mathcal{P}}_2^{1/2}(t,\theta)\,d\theta\Big)^2 d\eta\\
&\leqslant \|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2\|^2+\mathcal{O}(1)\frac{1}{A^5}\int_0^1 \tilde{\mathcal{U}}_{2,\eta}^2(t,\eta)\Big(\int_0^\eta e^{-\frac{3}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2(t,\theta)\,d\theta\Big)\\
&\quad\times\Big(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,\theta)\,d\theta\Big)d\eta\\
&\leqslant \|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2\|^2+\mathcal{O}(1)\frac{1}{A^5}\int_0^1 \tilde{\mathcal{P}}_2\tilde{\mathcal{U}}_{2,\eta}^2(t,\eta)
\end{aligned}
$$

$$\times \left(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t,\theta)\, d\theta \right) d\eta$$
$$\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2).$$

Here we employed

$$\min_j (\tilde{\mathcal{D}}_j)^{1/2} + |\tilde{\mathcal{U}}_2| \leqslant \sqrt{2A_2}\tilde{\mathcal{P}}_2^{1/2} + \sqrt{2}\tilde{\mathcal{P}}_2^{1/2} \leqslant \sqrt{2}(1+\sqrt{A_2})\tilde{\mathcal{P}}_2^{1/2},$$

as well as (4.15q) and (4.16i).

Next, we turn to the term W_4:

$$\begin{aligned} -W_4 &= \frac{3}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t,\eta) \bigg(\tilde{\mathcal{U}}_{2,\eta}(t,\eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\, d\theta \\ &\quad - \tilde{\mathcal{U}}_{1,\eta}(t,\eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1 \tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\, d\theta \bigg) d\eta \\ &= W_4^+ + W_4^-, \end{aligned}$$

with

$$\begin{aligned} W_4^\pm &= \frac{3}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t,\eta) \bigg(\tilde{\mathcal{U}}_{2,\eta}(t,\eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2^\pm \tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\, d\theta \\ &\quad - \tilde{\mathcal{U}}_{1,\eta}(t,\eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1^\pm \tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\, d\theta \bigg) d\eta \\ &\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + |A_1 - A_2|^2), \end{aligned}$$

using that $W_4^+ = 3N_1$ (see (5.39)) and similarly for W_4^-.

Next, we turn to the term W_5:

$$\begin{aligned} W_5 &= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t,\eta) \bigg(\tilde{\mathcal{U}}_{2,\eta}(t,\eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^3 \tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\, d\theta \\ &\quad - \tilde{\mathcal{U}}_{1,\eta}(t,\eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{U}}_1^3 \tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\, d\theta \bigg) d\eta. \end{aligned}$$

We take positive and negative parts of the term $\tilde{\mathcal{U}}_j^3$, and thus it suffices to study the term

$$\begin{aligned} W_5^+ &= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \bigg[\tilde{\mathcal{U}}_{2,\eta}(t,\eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{U}}_2^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\, d\theta \\ &\quad - \tilde{\mathcal{U}}_{1,\eta}(t,\eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))} (\tilde{\mathcal{U}}_1^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\, d\theta \bigg] d\eta. \end{aligned}$$

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Having a close look at W_5^+ one has

$$W_5^+ = \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(J_1 + J_2 + J_3 + J_4 + J_5 + J_6 + J_7 + J_8)(t, \eta)\, d\eta,$$

where $J_1, \ldots, J_6$ are defined in (5.33). Thus we can conclude immediately that

$$\begin{aligned} |W_5^+| \leqslant \mathcal{O}(1)(\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 \\ + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + |A_1 - A_2|^2). \end{aligned}$$

Next, we turn to the term W_6:

$$\begin{aligned} 2W_6 &= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(t, \eta) \bigg(\tilde{\mathcal{U}}_{2,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} A_2^5 \tilde{\mathcal{U}}_2(t, \theta)\, d\theta \\ &\quad - \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} A_1^5 \tilde{\mathcal{U}}_1(t, \theta)\, d\theta \bigg) d\eta \\ &= \frac{A_2^5 - A_1^5}{A^6} \mathbb{1}_{A_1 \leqslant A_2} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \\ &\quad \times \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2(t, \theta)\, d\theta\, d\eta \\ &\quad + \frac{A_2^5 - A_1^5}{A^6} \mathbb{1}_{A_2 < A_1} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \\ &\quad \times \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{U}}_1(t, \theta)\, d\theta\, d\eta \\ &\quad + \frac{a^5}{A^6} \mathbb{1}_{A_1 \leqslant A_2} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{U}_{2,\eta}(t, \eta) \\ &\quad \times \int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))}) \tilde{\mathcal{U}}_2(t, \theta)\, d\theta\, d\eta \\ &\quad + \frac{a^5}{A^5} \mathbb{1}_{A_2 < A_1} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \\ &\quad \times \int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))}) \tilde{\mathcal{U}}_1(t, \theta)\, d\theta\, d\eta \\ &\quad + \frac{a^5}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{2,\eta}(t, \eta) \\ &\quad \times \int_0^\eta (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))}) \tilde{\mathcal{U}}_2 \mathbb{1}_{B(\eta)}(t, \theta)\, d\theta\, d\eta \\ &\quad + \frac{a^5}{A^6} \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{1,\eta}(t, \eta) \end{aligned}$$

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$$
\begin{aligned}
&\times \int_0^\eta (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))})\tilde{\mathcal{U}}_1 \mathbb{1}_{B(\eta)^c}(t,\theta) d\theta\, d\eta \\
&+ \frac{a^5}{A^6}\int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)\tilde{\mathcal{U}}_{2,\eta}(t,\eta) \\
&\times \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})(\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+)\mathbb{1}_{\tilde{\mathcal{U}}_1^+\leqslant \tilde{\mathcal{U}}_2^+}(t,\theta)\, d\theta\, d\eta \\
&+ \frac{a^5}{A^6}\int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)\tilde{\mathcal{U}}_{2,\eta}(t,\eta) \\
&\times \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})(\tilde{\mathcal{U}}_2^- - \tilde{\mathcal{U}}_1^-)\mathbb{1}_{\tilde{\mathcal{U}}_2^-\leqslant \tilde{\mathcal{U}}_1^-}(t,\theta)\, d\theta\, d\eta \\
&+ \frac{a^5}{A^6}\int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)\tilde{\mathcal{U}}_{1,\eta}(t,\eta) \\
&\times \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})(\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+)\mathbb{1}_{\tilde{\mathcal{U}}_2^+ < \tilde{\mathcal{U}}_1^+}(t,\theta)\, d\theta\, d\eta \\
&+ \frac{a^5}{A^6}\int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)\tilde{\mathcal{U}}_{1,\eta}(t,\eta) \\
&\times \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})(\tilde{\mathcal{U}}_2^- - \tilde{\mathcal{U}}_1^-)\mathbb{1}_{\tilde{\mathcal{U}}_1^- < \tilde{\mathcal{U}}_2^-}(t,\theta)\, d\theta\, d\eta \\
&- \frac{a^5}{A^6}\int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)_\eta(t,\eta) \\
&\times \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{U}}_j^+)(t,\theta)\, d\theta\, d\eta \\
&- \frac{a^5}{A^6}\int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)_\eta(t,\eta) \\
&\times \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}) \max_j(\tilde{\mathcal{U}}_j^-)(t,\theta)\, d\theta\, d\eta \\
= {}& W_{61} + W_{62} + W_{63} + W_{64} + W_{65} + W_{66} + W_{67}^{\pm} + W_{68}^{\pm} + W_{69}^{\pm}.
\end{aligned}
$$

Here we go again! The terms W_{61} and W_{62}:

$$
\begin{aligned}
|W_{61}| &= \frac{A_2^5 - A_1^5}{A^6}\mathbb{1}_{A_1\leqslant A_2}\bigg|\int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)\tilde{\mathcal{U}}_{2,\eta}(t,\eta) \\
&\quad \times \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2(t,\theta)\, d\theta\, d\eta\bigg| \\
&\leqslant \frac{A_2^5 - A_1^5}{A^6}\mathbb{1}_{A_1\leqslant A_2}\int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2||\tilde{\mathcal{U}}_{2,\eta}|(t,\eta)
\end{aligned}
$$

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$$\times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^2(t,\theta)\,d\theta\right)^{1/2}$$
$$\times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\,d\theta\right)^{1/2} d\eta$$
$$\leqslant \frac{\sqrt{65}}{A^2}|A_2-A_1|\int_0^1 |\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2|\tilde{\mathcal{P}}_2^{1/2}|\tilde{\mathcal{U}}_{2,\eta}|(t,\eta)\,d\eta$$
$$\leqslant \sqrt{35}A^2|A_2-A_1|\|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2\|$$
$$\leqslant \mathcal{O}(1)(|A_1-A_2|^2+\|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2\|^2),$$

using (4.15q) and (4.16a).

The terms W_{63} and W_{64}:

$$|W_{63}| \leqslant \frac{4a^4}{A^6e}\int_0^1 |\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2||\tilde{\mathcal{U}}_{2,\eta}|(t,\eta)$$
$$\times \int_0^\eta e^{-\frac{3}{4A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}|\tilde{\mathcal{U}}_2|(t,\theta)\,d\theta\,d\eta|A_1-A_2|$$
$$\leqslant \frac{4}{A^2e}\int_0^1 |\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2||\tilde{\mathcal{U}}_{2,\eta}|(t,\eta)\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^2(t,\theta)\,d\theta\right)^{1/2}$$
$$\times \left(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\,d\theta\right)^{1/2} d\eta|A_1-A_2|$$
$$\leqslant \frac{4\sqrt{6}}{A^2e}\int_0^1 |\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2|\tilde{\mathcal{P}}_2^{1/2}|\tilde{\mathcal{U}}_{2,\eta}|(t,\eta)\,d\eta|A_1-A_2|$$
$$\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2\|^2+|A_1-A_2|^2).$$

Next comes the terms W_{65} and W_{66}:

$$|W_{65}| = \frac{a^5}{A^6}\bigg|\int_0^1 (\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)\tilde{\mathcal{U}}_{2,\eta}(t,\eta)$$
$$\times \int_0^\eta (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}-e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))})\tilde{\mathcal{U}}_2\mathbb{1}_{B(\eta)}(t,\theta)\,d\theta\,d\eta\bigg|$$
$$\leqslant \frac{1}{A^2}\int_0^1 |\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2||\tilde{\mathcal{U}}_{2,\eta}|(t,\eta)$$
$$\times \int_0^\eta (|\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_1(t,\eta)|+|\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_1(t,\theta)|)$$
$$\times e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}|\tilde{\mathcal{U}}_2|(t,\theta)\,d\theta\,d\eta$$

$$
\begin{aligned}
&\leqslant \frac{1}{A^2}\int_0^1 |\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2||\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1||\tilde{\mathcal{U}}_{2,\eta}|(t,\eta)\\
&\quad\times\int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}|\tilde{\mathcal{U}}_2|(t,\theta)\,d\theta\,d\eta\\
&\quad+\frac{1}{A^2}\int_0^1 |\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2||\tilde{\mathcal{U}}_{2,\eta}|(t,\eta)\\
&\quad\times\int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1|\,|\tilde{\mathcal{U}}_2|(t,\theta)\,d\theta\,d\eta\\
&\leqslant \frac{1}{A^2}\int_0^1 |\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2||\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1||\tilde{\mathcal{U}}_{2,\eta}|(t,\eta)\\
&\quad\times\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^2(t,\theta)\,d\theta\right)^{1/2}d\eta\\
&\quad+\frac{1}{A^2}\int_0^1 |\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2||\tilde{\mathcal{U}}_{2,\eta}|(t,\eta)\\
&\quad\times\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^2(t,\theta),d\theta\right)^{1/2}d\eta\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|\\
&\leqslant \frac{\sqrt{6}}{A^2}\int_0^1 |\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2||\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1|\tilde{\mathcal{P}}_2^{1/2}|\tilde{\mathcal{U}}_{2,\eta}|(t,\eta)\,d\eta\\
&\quad+\frac{\sqrt{6}}{A^2}\int_0^1 |\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2|\tilde{\mathcal{P}}_2^{1/2}|\tilde{\mathcal{U}}_{2,\eta}|(t,\eta)\,d\eta\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|\\
&\leqslant \mathcal{O}(1)\|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2\|\,\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|\\
&\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2\|^2+\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2),
\end{aligned}
$$

using (4.15q), (4.16a), and (5.13).

The terms $W_{67}^{\pm}$ and $W_{68}^{\pm}$ have a similar structure:

$$
\begin{aligned}
|W_{67}^{+}| &= \frac{a^5}{A^6}\bigg|\int_0^1 (\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)\tilde{\mathcal{U}}_{2,\eta}(t,\eta)\\
&\quad\times\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})(\tilde{\mathcal{U}}_2^{+}-\tilde{\mathcal{U}}_1^{+})\mathbb{1}_{\tilde{\mathcal{U}}_1^{+}\leqslant\tilde{\mathcal{U}}_2^{+}}(t,\theta)\,d\theta\,d\eta\bigg|\\
&\leqslant \|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2\|^2+\frac{1}{A^2}\int_0^1 \tilde{\mathcal{U}}_{2,\eta}^2(t,\eta)\\
&\quad\times\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{U}}_2^{+}-\tilde{\mathcal{U}}_1^{+})(t,\theta)\,d\theta\right)^2 d\eta
\end{aligned}
$$

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$$
\begin{aligned}
&\leqslant \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + A \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t,\eta) \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1)^2(t,\theta)\, d\theta \right) d\eta \\
&\leqslant \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 \\
&\quad + A \int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t,\eta) e^{-\frac{1}{A_2}\tilde{\mathcal{Y}}_2(t,\eta)} \left(\int_0^\eta e^{\frac{1}{A_2}\tilde{\mathcal{Y}}_2(t,\theta)} (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2(t,\theta)\, d\theta \right) d\eta \\
&\leqslant \mathcal{O}(1) \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2,
\end{aligned}
$$

using (4.15h).

Finally, the terms $W_{69}^{\pm}$:

$$
\begin{aligned}
|W_{69}^+| &= \frac{a^5}{A^6} \bigg| \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)(\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)_\eta(t,\eta) \\
&\quad \times \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{U}}_j^+)(t,\theta)\, d\theta\, d\eta \bigg| \\
&\leqslant \frac{1}{2A} \bigg| \left((\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2(t,\eta) \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{U}}_j^+)(t,\theta)\, d\theta \right) \bigg|_{\eta=0}^1 \\
&\quad - \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2(t,\eta) \frac{d}{d\eta} \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{U}}_j^+)(t,\theta)\, d\theta\, d\eta \bigg| \\
&\leqslant \mathcal{O}(1) \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2;
\end{aligned}
$$

see estimates for $\bar{B}_{67}^{\pm}$ (cf. (5.25)) and Lemma A.5.

The terms M_2 and M_3 can be treated similarly. More precisely,

$$
\begin{aligned}
|M_2| &\leqslant \mathbb{1}_{A_2 \leqslant A_1} \left(\frac{1}{A_2^6} - \frac{1}{A_1^6} \right) \bigg| \int_0^1 (\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2) \tilde{\mathcal{U}}_{2,\eta}(t,\eta) \\
&\quad \times \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{Q}}_2 \tilde{\mathcal{U}}_{2,\eta}(t,\theta)\, d\theta\, d\eta \bigg| \\
&\leqslant \mathbb{1}_{A_2 \leqslant A_1} \frac{|A_1^6 - A_2^6|}{A^6 A_2^5} \int_0^1 |\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2|\, |\tilde{\mathcal{U}}_{2,\eta}|(t,\eta) \\
&\quad \times \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 |\tilde{\mathcal{U}}_{2,\eta}|(t,\theta)\, d\theta\, d\eta \\
&\leqslant \mathbb{1}_{A_2 \leqslant A_1} 6 \frac{|A_1 - A_2|}{A A_2^5} \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\| \\
&\quad \times \left(\int_0^1 \tilde{\mathcal{U}}_{2,\eta}^2(t,\eta) \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 |\tilde{\mathcal{U}}_{2,\eta}|(t,\theta)\, d\theta \right)^2 d\eta \right)^{1/2}
\end{aligned}
$$

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$$\begin{aligned}
&\leqslant \mathbb{1}_{A_2\leqslant A_1} 6\frac{|A_1-A_2|}{AA_2^6}\|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2\| \\
&\quad\times\bigg(\int_0^1 \tilde{\mathcal{U}}_{2,\eta}^2(t,\eta)\Big(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2^2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\Big) \\
&\quad\times\Big(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{H}}_{2,\eta}(t,\theta)\,d\theta\Big)\,d\eta\bigg)^{1/2} \\
&\leqslant \mathbb{1}_{A_2\leqslant A_1} 6\sqrt{6}\frac{|A_1-A_2|}{AA_2^3}\|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2\|\Big(\int_0^1 \tilde{\mathcal{U}}_{2,\eta}^2\tilde{\mathcal{P}}_2^2(t,\eta)\,d\eta\Big)^{1/2} \\
&\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2\|^2+|A_1-A_2|^2).
\end{aligned}$$

Here we used

$$\tilde{\mathcal{P}}_2|\tilde{\mathcal{U}}_{2,\eta}|\leqslant\frac{1}{A_2}\tilde{\mathcal{P}}_2\sqrt{\tilde{\mathcal{Y}}_{2,\eta}\tilde{\mathcal{H}}_{2,\eta}}$$

(cf. (4.15m)) as well as (4.15a) and (4.15q). In addition, we applied (4.16b) and (4.16d).

We have shown the anticipated result.

LEMMA 5.4. *Let $\tilde{\mathcal{U}}_i$ be two solutions of (5.28). Then we have*

$$\begin{aligned}
&\frac{d}{dt}\|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2\|^2 \\
&\quad\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2+\|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2\|^2+\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+|A_1-A_2|^2),
\end{aligned}$$

where $\mathcal{O}(1)$ denotes some constant which only depends on $A=\max_j(A_j)$ and which remains bounded as $A\to 0$.

5.3. Lipschitz estimates for $\tilde{\mathcal{P}}$ (or $\tilde{\mathcal{P}}^{1/2}$). From the system of differential equations we recall

$$(\tilde{\mathcal{P}}_i^{1/2})_t+\Big(\frac{2}{3}\frac{1}{A_i^5}\tilde{\mathcal{U}}_i^3+\frac{1}{A_i^6}\tilde{\mathcal{S}}_i\Big)(\tilde{\mathcal{P}}_i^{1/2})_\eta=\frac{1}{2A_i^2}\frac{\tilde{\mathcal{Q}}_i\tilde{\mathcal{U}}_i}{\tilde{\mathcal{P}}_i^{1/2}}+\frac{1}{2A_i^3}\frac{\tilde{\mathcal{R}}_i}{\tilde{\mathcal{P}}_i^{1/2}}, \tag{5.40}$$

where

$$\tilde{\mathcal{Q}}_i(t,\eta)=-\frac{1}{4}\int_0^1 \operatorname{sign}(\eta-\theta)e^{-\frac{1}{A_i}|\tilde{\mathcal{Y}}_i(t,\eta)-\tilde{\mathcal{Y}}_i(t,\theta)|}(2(\tilde{\mathcal{U}}_i^2-\tilde{\mathcal{P}}_i)\tilde{\mathcal{Y}}_{i,\eta}(t,\theta)+A_i^5)\,d\theta, \tag{5.41a}$$

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$$\tilde{\mathcal{S}}_i(t,\eta) = \int_0^1 e^{-\frac{1}{A_i}|\tilde{\mathcal{Y}}_i(t,\eta)-\tilde{\mathcal{Y}}_i(t,\theta)|}\Big(\frac{2}{3}\tilde{\mathcal{U}}_i^3\tilde{\mathcal{Y}}_{i,\eta} - \tilde{\mathcal{Q}}_i\tilde{\mathcal{U}}_{i,\eta} - 2\tilde{\mathcal{P}}_i\tilde{\mathcal{U}}_i\tilde{\mathcal{Y}}_{i,\eta}\Big)(t,\theta)\,d\theta, \tag{5.41b}$$

$$\begin{aligned}\tilde{\mathcal{R}}_i(t,\eta) &= \frac{1}{4}\int_0^1 \operatorname{sign}(\eta-\theta)e^{-\frac{1}{A_i}|\tilde{\mathcal{Y}}_i(t,\eta)-\tilde{\mathcal{Y}}_i(t,\theta)|}\Big(\frac{2}{3}A_i\tilde{\mathcal{U}}_i^3\tilde{\mathcal{Y}}_{i,\eta} + A_i^6\tilde{\mathcal{U}}_i\Big)(t,\theta)\,d\theta\\ &\quad - \frac{1}{2}\int_0^1 e^{-\frac{1}{A_i}|\tilde{\mathcal{Y}}_i(t,\eta)-\tilde{\mathcal{Y}}_i(t,\theta)|}\tilde{\mathcal{U}}_i\tilde{\mathcal{Q}}_i\tilde{\mathcal{Y}}_{i,\eta}(t,\theta)\,d\theta. \end{aligned}\tag{5.41c}$$

Thus we have

$$\begin{aligned}&\frac{d}{dt}\int_0^1(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})^2(t,\eta)\,d\eta\\
&= 2\int_0^1(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})((\tilde{\mathcal{P}}_1^{1/2})_t-(\tilde{\mathcal{P}}_2^{1/2})_t)(t,\eta)\,d\eta\\
&= -\frac{2}{3}\int_0^1(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})\Big(\frac{1}{A_1^5}\tilde{\mathcal{U}}_1^3(\tilde{\mathcal{P}}_1^{1/2})_\eta - \frac{1}{A_2^5}\tilde{\mathcal{U}}_2^3(\tilde{\mathcal{P}}_2^{1/2})_\eta\Big)(t,\eta)\,d\eta\\
&\quad - \int_0^1(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})\Big(\frac{1}{A_1^6}\tilde{\mathcal{S}}_1(\tilde{\mathcal{P}}_1^{1/2})_\eta - \frac{1}{A_2^6}\tilde{\mathcal{S}}_2(\tilde{\mathcal{P}}_2^{1/2})_\eta\Big)(t,\eta)\,d\eta\\
&\quad + \frac{1}{2}\int_0^1(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})\Big(\frac{1}{A_1^2}\frac{\tilde{\mathcal{Q}}_1\tilde{\mathcal{U}}_1}{\tilde{\mathcal{P}}_1^{1/2}} - \frac{1}{A_2^2}\frac{\tilde{\mathcal{Q}}_2\tilde{\mathcal{U}}_2}{\tilde{\mathcal{P}}_2^{1/2}}\Big)(t,\eta)\,d\eta\\
&\quad + \frac{1}{2}\int_0^1(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})\Big(\frac{1}{A_1^3}\frac{\tilde{\mathcal{R}}_1}{\tilde{\mathcal{P}}_1^{1/2}} - \frac{1}{A_2^3}\frac{\tilde{\mathcal{R}}_2}{\tilde{\mathcal{P}}_2^{1/2}}\Big)(t,\eta)\,d\eta\\
&= \frac{2}{3}I_1 + I_2 + \frac{1}{2}I_3 + \frac{1}{2}I_4.\end{aligned}$$

We will estimate each of these terms, yielding that

$$\begin{aligned}&\frac{d}{dt}\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2\\
&\quad\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2 + \|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2\|^2 + \|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2 + |A_1-A_2|^2),\end{aligned}$$

where $\mathcal{O}(1)$ denotes some constant which only depends on $A = \max_j(A_j)$ and which remains bounded as $A \to 0$.

The term I_1: Here we do as follows:

$$\begin{aligned}I_1 &= \int_0^1(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})\Big(\frac{1}{A_2^5}\tilde{\mathcal{U}}_2^3(\tilde{\mathcal{P}}_2^{1/2})_\eta - \frac{1}{A_1^5}\tilde{\mathcal{U}}_1^3(\tilde{\mathcal{P}}_1^{1/2})_\eta\Big)(t,\eta)\,d\eta\\
&= \frac{1}{A^5}\int_0^1(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{U}}_2^3(\tilde{\mathcal{P}}_2^{1/2})_\eta - \tilde{\mathcal{U}}_1^3(\tilde{\mathcal{P}}_1^{1/2})_\eta)(t,\eta)\,d\eta\end{aligned}$$

$$
\begin{aligned}
&+ \frac{A_1^5 - A_2^5}{A_1^5 A_2^5} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \\
&\times (\tilde{\mathcal{U}}_2^3 (\tilde{\mathcal{P}}_2^{1/2})_\eta \mathbb{1}_{A_2 < A_1} + \tilde{\mathcal{U}}_1^3 (\tilde{\mathcal{P}}_1^{1/2})_\eta \mathbb{1}_{A_1 \leqslant A_2})(t,\eta)\, d\eta \\
= &\frac{1}{A^5} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1)\tilde{\mathcal{U}}_2^2 (\tilde{\mathcal{P}}_2^{1/2})_\eta (t,\eta)\, d\eta \\
&+ \frac{1}{A^5} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{U}}_2^2 (\tilde{\mathcal{P}}_2^{1/2})_\eta - \tilde{\mathcal{U}}_1^2 (\tilde{\mathcal{P}}_1^{1/2})_\eta)\tilde{\mathcal{U}}_1 (t,\eta)\, d\eta \\
&+ \frac{A_1^5 - A_2^5}{A_1^5 A_2^5} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \\
&\times (\tilde{\mathcal{U}}_2^3 (\tilde{\mathcal{P}}_2^{1/2})_\eta \mathbb{1}_{A_2 < A_1} + \tilde{\mathcal{U}}_1^3 (\tilde{\mathcal{P}}_1^{1/2})_\eta \mathbb{1}_{A_1 \leqslant A_2})(t,\eta)\, d\eta \\
= &\frac{1}{A^5} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1)\tilde{\mathcal{U}}_2^2 (\tilde{\mathcal{P}}_2^{1/2})_\eta (t,\eta)\, d\eta \\
&+ \frac{1}{A^5} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{U}}_2^2 - \tilde{\mathcal{U}}_1^2)(\tilde{\mathcal{P}}_2^{1/2})_\eta \tilde{\mathcal{U}}_1 \mathbb{1}_{\tilde{\mathcal{U}}_1^2 \leqslant \tilde{\mathcal{U}}_2^2} (t,\eta)\, d\eta \\
&+ \frac{1}{A^5} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{U}}_2^2 - \tilde{\mathcal{U}}_1^2)(\tilde{\mathcal{P}}_1^{1/2})_\eta \tilde{\mathcal{U}}_1 \mathbb{1}_{\tilde{\mathcal{U}}_2^2 < \tilde{\mathcal{U}}_1^2} (t,\eta)\, d\eta \\
&+ \frac{1}{A^5} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})((\tilde{\mathcal{P}}_2^{1/2})_\eta - (\tilde{\mathcal{P}}_1^{1/2})_\eta)\tilde{\mathcal{U}}_1 \min_j (\tilde{\mathcal{U}}_j^2)(t,\eta)\, d\eta \\
&+ \frac{A_1^5 - A_2^5}{A_1^5 A_2^5} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \\
&\times (\tilde{\mathcal{U}}_1^3 (\tilde{\mathcal{P}}_1^{1/2})_\eta \mathbb{1}_{A_1 \leqslant A_2} + \tilde{\mathcal{U}}_2^3 (\tilde{\mathcal{P}}_2^{1/2})_\eta \mathbb{1}_{A_2 < A_1})(t,\eta)\, d\eta \\
= &I_{11} + I_{12} + I_{13} + I_{14} + I_{15}.
\end{aligned}
$$

We first estimate

$$
|(\tilde{\mathcal{P}}_i^{1/2})_\eta| = \frac{|\tilde{\mathcal{P}}_{i,\eta}|}{2\tilde{\mathcal{P}}_i^{1/2}} = \frac{|\tilde{\mathcal{Q}}_i \tilde{\mathcal{Y}}_{i,\eta}|}{2A_i^2 \tilde{\mathcal{P}}_i^{1/2}} \leqslant \frac{1}{2A_i} \tilde{\mathcal{P}}_i^{1/2} \tilde{\mathcal{Y}}_{i,\eta},
$$

and thus

$$
|\tilde{\mathcal{U}}_i \tilde{\mathcal{P}}_i^{1/2} \tilde{\mathcal{Y}}_{i,\eta}| \leqslant \frac{1}{2}(\tilde{\mathcal{U}}_i^2 \tilde{\mathcal{Y}}_{i,\eta} + \tilde{\mathcal{P}}_i \tilde{\mathcal{Y}}_{i,\eta}) \leqslant \frac{3}{4} A_i^5.
$$

We start with the term I_{11}:

$$
\begin{aligned}
|I_{11}| &\leqslant \mathcal{O}(1) \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|\, |\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1|(t,\eta)\, d\eta \\
&\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1\|^2).
\end{aligned}
$$

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Subsequently (and similarly for I_{13}),

$$\begin{aligned}
|I_{12}| &\leqslant \frac{1}{A^5}\int_0^1 |\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}|\,|\tilde{\mathcal{U}}_2-\tilde{\mathcal{U}}_1|\,|\tilde{\mathcal{U}}_2+\tilde{\mathcal{U}}_1|\,|(\tilde{\mathcal{P}}_2^{1/2})_\eta\tilde{\mathcal{U}}_1|\mathbb{1}_{\tilde{\mathcal{U}}_1^2\leqslant\tilde{\mathcal{U}}_2^2}(t,\eta)\,d\eta\\
&\leqslant \frac{2}{A^5}\int_0^1 |\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}|\,|\tilde{\mathcal{U}}_2-\tilde{\mathcal{U}}_1|\tilde{\mathcal{U}}_2^2|(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t,\eta)\,d\eta\\
&\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+\|\tilde{\mathcal{U}}_2-\tilde{\mathcal{U}}_1\|^2).
\end{aligned}$$

The term I_{14} requires more estimates (see (5.31)):

$$\begin{aligned}
|I_{14}| &= \frac{1}{A^5}\left|\int_0^1 (\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})((\tilde{\mathcal{P}}_2^{1/2})_\eta-(\tilde{\mathcal{P}}_1^{1/2})_\eta)\tilde{\mathcal{U}}_1\min_j(\tilde{\mathcal{U}}_j^2)(t,\eta)\,d\eta\right|\\
&= \left|\frac{1}{2A^5}(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})^2\tilde{\mathcal{U}}_1\min_j(\tilde{\mathcal{U}}_j^2)(t,\eta)\,\right|_{\eta=0}^1\\
&\quad -\frac{1}{2A^5}\int_0^1(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})^2\left(\frac{d}{d\eta}\tilde{\mathcal{U}}_1\min_j(\tilde{\mathcal{U}}_j^2)\right)(t,\eta)\,d\eta\Bigg|\\
&= \frac{1}{2A^5}\left|\int_0^1(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})^2\left(\frac{d}{d\eta}\tilde{\mathcal{U}}_1\min_j(\tilde{\mathcal{U}}_j^2)\right)(t,\eta)\,d\eta\right|\\
&\leqslant \mathcal{O}(1)\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2.
\end{aligned}$$

The last term can be estimated as follows:

$$\begin{aligned}
|I_{15}| \leqslant &\left|\frac{A_1^5-A_2^5}{A_1^5A_2^5}\int_0^1(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})\right.\\
&\left.\times(\tilde{\mathcal{U}}_1^3(\tilde{\mathcal{P}}_1^{1/2})_\eta\mathbb{1}_{A_1\leqslant A_2}+\tilde{\mathcal{U}}_2^3(\tilde{\mathcal{P}}_2^{1/2})_\eta\mathbb{1}_{A_2<A_1})(t,\eta)\,d\eta\right|\\
\leqslant\ & 5\frac{|A_1-A_2|A^4}{A_1^5A_2^5}\\
&\times\int_0^1(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})(|\tilde{\mathcal{U}}_1^3(\tilde{\mathcal{P}}_1^{1/2})_\eta|\mathbb{1}_{A_1\leqslant A_2}+|\tilde{\mathcal{U}}_2^3(\tilde{\mathcal{P}}_2^{1/2})_\eta|\mathbb{1}_{A_2<A_1})(t,\eta)\,d\eta.
\end{aligned}$$

We consider, in particular, the term

$$\frac{A^4}{A_1^5A_2^5}|\tilde{\mathcal{U}}_1^3(\tilde{\mathcal{P}}_1^{1/2})_\eta|\mathbb{1}_{A_1\leqslant A_2}\leqslant\frac{A^4}{A_1^5A_2^5}\frac{3}{16}A_1^8\mathbb{1}_{A_1\leqslant A_2}\leqslant\frac{3}{16}A^2.$$

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Thus

$$|I_{15}| \leqslant \frac{30}{16}A^2|A_1 - A_2| \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t,\eta)\,d\eta$$
$$\leqslant \mathcal{O}(1)(|A_1 - A_2|^2 + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2).$$

The term I_2: Recall that the term reads as follows:

$$I_2 = \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})\Big(\frac{1}{A_2^6}\tilde{\mathcal{S}}_2(\tilde{\mathcal{P}}_2^{1/2})_\eta - \frac{1}{A_1^6}\tilde{\mathcal{S}}_1(\tilde{\mathcal{P}}_1^{1/2})_\eta\Big)(t,\eta)\,d\eta.$$

Here we can follow the estimates of the term I_3; cf. (5.6) and (5.9). Here we go. Define

$$\begin{aligned}
I_{21} &= \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t,\eta)\Big(\frac{1}{A_2^6}(\tilde{\mathcal{P}}_2^{1/2})_\eta(t,\eta)\int_0^1 e^{-\frac{1}{A_2}|\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta)|}\tilde{\mathcal{U}}_2^3\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta \\
&\quad - \frac{1}{A_1^6}(\tilde{\mathcal{P}}_1^{1/2})_\eta(t,\eta)\int_0^1 e^{-\frac{1}{A_1}|\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta)|}\tilde{\mathcal{U}}_1^3\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\Big)\,d\eta, \\
I_{22} &= \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t,\eta) \\
&\quad \times \Big(\frac{1}{A_2^6}(\tilde{\mathcal{P}}_2^{1/2})_\eta(t,\eta)\int_0^1 e^{-\frac{1}{A_2}|\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta)|}\tilde{\mathcal{P}}_2\tilde{\mathcal{U}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta \\
&\quad - \frac{1}{A_1^6}(\tilde{\mathcal{P}}_1^{1/2})_\eta(t,\eta)\int_0^1 e^{-\frac{1}{A_1}|\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta)|}\tilde{\mathcal{P}}_1\tilde{\mathcal{U}}_1\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\Big)\,d\eta, \\
I_{23} &= \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t,\eta) \\
&\quad \times \Big(\frac{1}{A_2^6}(\tilde{\mathcal{P}}_2^{1/2})_\eta(t,\eta)\int_0^1 e^{-\frac{1}{A_2}|\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta)|}\tilde{\mathcal{Q}}_2\tilde{\mathcal{U}}_{2,\eta}(t,\theta)\,d\theta \\
&\quad - \frac{1}{A_1^6}(\tilde{\mathcal{P}}_1^{1/2})_\eta(t,\eta)\int_0^1 e^{-\frac{1}{A_1}|\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta)|}\tilde{\mathcal{Q}}_1\tilde{\mathcal{U}}_{1,\eta}(t,\theta)\,d\theta\Big)\,d\eta.
\end{aligned}$$

Using the definition (5.41b) of $\tilde{\mathcal{S}}$, we can write

$$I_2 = \frac{2}{3}I_{21} - 2I_{22} - I_{23}.$$

We commence with the estimates for I_{21}:

$$I_{21} = \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t,\eta)$$

$$
\begin{aligned}
&\times \bigg(\frac{1}{A_2^6}(\tilde{\mathcal{P}}_2^{1/2})_\eta(t,\eta)\int_0^1 e^{-\frac{1}{A_2}|\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta)|}\tilde{\mathcal{U}}_2^3\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\\
&- \frac{1}{A_1^6}(\tilde{\mathcal{P}}_1^{1/2})_\eta(t,\eta)\int_0^1 e^{-\frac{1}{A_1}|\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta)|}\tilde{\mathcal{U}}_1^3\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\bigg)\,d\eta\\
=&\int_0^1(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})(t,\eta)\\
&\times \bigg(\frac{1}{A_2^6}(\tilde{\mathcal{P}}_2^{1/2})_\eta(t,\eta)\int_0^1 e^{-\frac{1}{A_2}|\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta)|}(\tilde{\mathcal{U}}_2^-)^3\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\\
&- \frac{1}{A_1^6}(\tilde{\mathcal{P}}_1^{1/2})_\eta(t,\eta)\int_0^1 e^{-\frac{1}{A_1}|\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta)|}(\tilde{\mathcal{U}}_1^-)^3\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\bigg)\,d\eta\\
&+\int_0^1(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})(t,\eta)\\
&\times \bigg(\frac{1}{A_2^6}(\tilde{\mathcal{P}}_2^{1/2})_\eta(t,\eta)\int_0^1 e^{-\frac{1}{A_2}|\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta)|}(\tilde{\mathcal{U}}_2^+)^3\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\\
&- \frac{1}{A_1^6}(\tilde{\mathcal{P}}_1^{1/2})_\eta(t,\eta)\int_0^1 e^{-\frac{1}{A_1}|\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta)|}(\tilde{\mathcal{U}}_1^+)^3\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\bigg)\,d\eta.
\end{aligned}
$$

The two terms are similar, and we only discuss the second one. Next, we use trick (5.4). Thus we have to consider the term

$$
\begin{aligned}
&\frac{1}{A_2^6}(\tilde{\mathcal{P}}_2^{1/2})_\eta(t,\eta)\int_0^\eta e^{-\frac{1}{A_2}|\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta)|}(\tilde{\mathcal{U}}_2^+)^3\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\\
&\quad-\frac{1}{A_1^6}(\tilde{\mathcal{P}}_1^{1/2})_\eta(t,\eta)\int_0^\eta e^{-\frac{1}{A_1}|\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta)|}(\tilde{\mathcal{U}}_1^+)^3\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\\
&=\frac{1}{A^6}\bigg((\tilde{\mathcal{P}}_2^{1/2})_\eta(t,\eta)\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{U}}_2^+)^3\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\\
&\quad-(\tilde{\mathcal{P}}_1^{1/2})_\eta(t,\eta)\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}(\tilde{\mathcal{U}}_1^+)^3\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\bigg)\\
&\quad+\mathbb{1}_{A_1\leqslant A_2}\bigg(\frac{1}{A_2^6}-\frac{1}{A_1^6}\bigg)(\tilde{\mathcal{P}}_1^{1/2})_\eta(t,\eta)\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}(\tilde{\mathcal{U}}_1^+)^3\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\\
&\quad+\mathbb{1}_{A_2< A_1}\bigg(\frac{1}{A_2^6}-\frac{1}{A_1^6}\bigg)(\tilde{\mathcal{P}}_2^{1/2})_\eta(t,\eta)\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{U}}_2^+)^3\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\\
&=\frac{1}{A^6}(\tilde{\mathcal{P}}_2^{1/2})_\eta(t,\eta)\\
&\quad\times\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}((\tilde{\mathcal{U}}_2^+)^3-(\tilde{\mathcal{U}}_1^+)^3)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\mathbb{1}_{\tilde{\mathcal{U}}_1^+\leqslant\tilde{\mathcal{U}}_2^+}(t,\theta)\,d\theta
\end{aligned}
$$

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$$
\begin{aligned}
&+\frac{1}{A^6}(\tilde{\mathcal{P}}_1^{1/2})_\eta(t,\eta)\\
&\times\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}((\tilde{\mathcal{U}}_2^+)^3-(\tilde{\mathcal{U}}_1^+)^3)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\mathbb{1}_{\tilde{\mathcal{U}}_2^+<\tilde{\mathcal{U}}_1^+}(t,\theta)\,d\theta\\
&+\frac{1}{A^6}(\tilde{\mathcal{P}}_2^{1/2})_\eta(t,\eta)\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\min_j(\tilde{\mathcal{U}}_j^+)^3\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\\
&-\frac{1}{A^6}(\tilde{\mathcal{P}}_1^{1/2})_\eta(t,\eta)\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\min_j(\tilde{\mathcal{U}}_j^+)^3\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\\
&+\mathbb{1}_{A_1\leqslant A_2}\left(\frac{1}{A_2^6}-\frac{1}{A_1^6}\right)(\tilde{\mathcal{P}}_1^{1/2})_\eta(t,\eta)\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}(\tilde{\mathcal{U}}_1^+)^3\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\\
&+\mathbb{1}_{A_2<A_1}\left(\frac{1}{A_2^6}-\frac{1}{A_1^6}\right)(\tilde{\mathcal{P}}_2^{1/2})_\eta(t,\eta)\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{U}}_2^+)^3\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\\
=\ &\frac{1}{A^6}(\tilde{\mathcal{P}}_2^{1/2})_\eta(t,\eta)\\
&\times\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}((\tilde{\mathcal{U}}_2^+)^3-(\tilde{\mathcal{U}}_1^+)^3)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\mathbb{1}_{\tilde{\mathcal{U}}_1^+\leqslant\tilde{\mathcal{U}}_2^+}(t,\theta)\,d\theta\\
&+\frac{1}{A^6}(\tilde{\mathcal{P}}_1^{1/2})_\eta(t,\eta)\\
&\times\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}((\tilde{\mathcal{U}}_2^+)^3-(\tilde{\mathcal{U}}_1^+)^3)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\mathbb{1}_{\tilde{\mathcal{U}}_2^+<\tilde{\mathcal{U}}_1^+}(t,\theta)\,d\theta\\
&+\mathbb{1}_{A_1\leqslant A_2}\frac{1}{A^6}(\tilde{\mathcal{P}}_2^{1/2})_\eta(t,\eta)\\
&\times\int_0^\eta(e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}-e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))})\min_j(\tilde{\mathcal{U}}_j^+)^3\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\,d\eta\\
&+\mathbb{1}_{A_2<A_1}\frac{1}{A^6}(\tilde{\mathcal{P}}_1^{1/2})_\eta(t,\eta)\\
&\times\int_0^\eta(e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}-e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))})\min_j(\tilde{\mathcal{U}}_j^+)^3\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\,d\eta\\
&+\frac{1}{A^6}\,(\tilde{\mathcal{P}}_2^{1/2})_\eta(t,\eta)\Bigg(\int_0^\eta(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}-e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))})\\
&\times\min_j(\tilde{\mathcal{U}}_j^+)^3\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\mathbb{1}_{B(\eta)}(t,\theta)\,d\theta\Bigg)\\
&-\frac{1}{A^6}(\tilde{\mathcal{P}}_1^{1/2})_\eta(t,\eta)\Bigg(\int_0^\eta(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}-e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))})\\
&\times\min_j(\tilde{\mathcal{U}}_j^+)^3\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\mathbb{1}_{B^c(\eta)}(t,\theta)\,d\theta\Bigg)\\
&+\frac{1}{A^6}(\tilde{\mathcal{P}}_2^{1/2})_\eta(t,\eta)\left(\int_0^\eta\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{U}}_j^+)^3\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)
\end{aligned}
$$

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$$
\begin{aligned}
&- \frac{1}{A^6}(\tilde{\mathcal{P}}_1^{1/2})_\eta(t,\eta)\Big(\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{U}}_j^+)^3\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\Big)\\
&+ \mathbb{1}_{A_1\leqslant A_2}\Big(\frac{1}{A_2^6}-\frac{1}{A_1^6}\Big)(\tilde{\mathcal{P}}_1^{1/2})_\eta(t,\eta)\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}(\tilde{\mathcal{U}}_1^+)^3\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\\
&+ \mathbb{1}_{A_2< A_1}\Big(\frac{1}{A_2^6}-\frac{1}{A_1^6}\Big)(\tilde{\mathcal{P}}_2^{1/2})_\eta(t,\eta)\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{U}}_2^+)^3\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\\
&= (I_{211}+I_{212}+I_{213}+I_{214}+I_{215}+I_{216}+I_{217}+I_{218}+I_{219}+I_{220})(t,\eta).
\end{aligned}
\tag{5.42}
$$

The same estimate works for I_{211} and I_{212} (cf. (5.34)):

$$
\Big|\int_0^1 I_{211}(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})(t,\eta)\,d\eta\Big| \leqslant \|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+\int_0^1 I_{211}^2(t,\eta)\,d\eta,
$$

which shows that it suffices to estimate

$$
\begin{aligned}
\int_0^1 I_{211}^2(t,\eta)\,d\eta &\leqslant \frac{1}{A^{12}}\int_0^1(\tilde{\mathcal{P}}_2^{1/2})_\eta^2(t,\eta)\Big(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\\
&\qquad\times((\tilde{\mathcal{U}}_2^+)^3-(\tilde{\mathcal{U}}_1^+)^3)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\mathbb{1}_{\tilde{\mathcal{U}}_1^+\leqslant\tilde{\mathcal{U}}_2^+}(t,\theta)\,d\theta\Big)^2 d\eta\\
&\leqslant \mathcal{O}(1)\|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2\|^2,
\end{aligned}
$$

where we used that $((\tilde{\mathcal{P}}_2^{1/2})_\eta)^2 \leqslant \frac{1}{8}A_2^3\tilde{\mathcal{Y}}_{2,\eta}$.

Regarding the term I_{213} (and I_{214}), we find

$$
\begin{aligned}
&\Big|\int_0^1 I_{213}(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})(t,\eta)\,d\eta\Big|\\
&= \mathbb{1}_{A_1\leqslant A_2}\frac{1}{A^6}\Big|\int_0^1(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_2^{1/2})_\eta(t,\eta)\\
&\quad\times\int_0^\eta(e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}-e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))})\min_j(\tilde{\mathcal{U}}_j^+)^3\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\,d\eta\Big|\\
&\leqslant \frac{2\sqrt{2}a}{A^6 e}\int_0^1|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}||(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t,\eta)\\
&\quad\times\int_0^\eta e^{-\frac{3}{4A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\,d\eta|A_1-A_2|\\
&\leqslant \|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2\\
&\quad+\frac{8}{A^{10}e^2}\int_0^1(\tilde{\mathcal{P}}_2^{1/2})_\eta^2(t,\eta)\Big(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\Big)
\end{aligned}
$$

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$$\times \left(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\, d\theta \right) d\eta |A_1 - A_2|^2$$

$$\leqslant \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \frac{4}{Ae^2} \int_0^1 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\, d\eta |A_1 - A_2|^2$$

$$\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + |A_1 - A_2|^2).$$

Regarding the term I_{215} (and I_{216}), we find

$$\left| \int_0^1 I_{215}(\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t,\eta)\, d\eta \right|$$

$$= \frac{1}{A^6} \left| \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_2^{1/2})_\eta(t,\eta) \right.$$

$$\times \left(\int_0^\eta (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}) \right.$$

$$\left. \left. \times \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t,\theta) \mathbb{1}_{B(\eta)}(t,\theta)\, d\theta \right) d\eta \right|$$

$$\leqslant \frac{1}{aA^6} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}||(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t,\eta)$$

$$\times \left(\int_0^\eta (|\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_1(t,\eta)| + |\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_1(t,\theta)|) \right.$$

$$\left. \times e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\, d\theta \right) d\eta$$

$$\leqslant \frac{1}{aA^6} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|\, |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|\, |(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t,\eta)$$

$$\times \int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\, d\theta\, d\eta$$

$$+ \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \frac{1}{a^2 A^{12}} \int_0^1 (\tilde{\mathcal{P}}_2^{1/2})_\eta^2(t,\eta)$$

$$\times \left(\int_0^\eta |\tilde{\mathcal{Y}}_1(t,\theta) - \tilde{\mathcal{Y}}_2(t,\theta)| e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\, d\theta \right)^2 d\eta$$

$$\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2),$$

using the estimates in (5.35) and $((\tilde{\mathcal{P}}_2^{1/2})_\eta)^2 \leqslant \frac{1}{8} A^3 \tilde{\mathcal{Y}}_{2,\eta}$.

The next terms are $I_{217} + I_{218}$:

$$I_{217} + I_{218} = \frac{1}{A^6} (\tilde{\mathcal{P}}_2^{1/2})_\eta(t,\eta)$$

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$$\begin{aligned}
&\times \left(\int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\, d\theta \right) \\
&- \frac{1}{A^6} (\tilde{\mathcal{P}}_1^{1/2})_\eta (t,\eta) \\
&\times \left(\int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\, d\theta \right) \\
= {}& \frac{1}{A^6} ((\tilde{\mathcal{P}}_2^{1/2})_\eta - (\tilde{\mathcal{P}}_1^{1/2})_\eta)(t,\eta) \\
&\times \min_k \left[\int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}) \min_j (\tilde{\mathcal{U}}_j^+)^3 \tilde{\mathcal{Y}}_{k,\eta}(t,\theta)\, d\theta \right] \\
&+ \frac{1}{A^6} (\tilde{\mathcal{P}}_1^{1/2})_\eta (t,\eta) \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}) \\
&\times \min_j (\tilde{\mathcal{U}}_j^+)^3 (\tilde{\mathcal{Y}}_{2,\eta} - \tilde{\mathcal{Y}}_{1,\eta})(t,\theta)\, d\theta \mathbb{1}_E(t,\eta) \\
&+ \frac{1}{A^6} (\tilde{\mathcal{P}}_2^{1/2})_\eta (t,\eta) \int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}) \\
&\times \min_j (\tilde{\mathcal{U}}_j^+)^3 (\tilde{\mathcal{Y}}_{2,\eta} - \tilde{\mathcal{Y}}_{1,\eta})(t,\theta)\, d\theta \mathbb{1}_{E^c}(t,\eta) \\
= {}& L_{211} + L_{212} + L_{213}.
\end{aligned}$$

The term L_{211} can be estimated as in (5.36), yielding the result

$$\left| \int_0^1 L_{211} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t,\eta)\, d\eta \right| \leqslant \mathcal{O}(1) \| \tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2} \|^2.$$

The terms L_{212} and L_{213} can be treated similarly; we list L_{213}:

$$\left| \int_0^1 L_{213} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t,\eta)\, d\eta \right| \leqslant \mathcal{O}(1) (\| \tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2 \|^2 + \| \tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2} \|^2);$$

see (5.37). The terms I_{219} and I_{220} share the same behavior. We find, when $A_1 \leqslant A_2$ (otherwise I_{219} vanishes; see (5.38)), that

$$\begin{aligned}
&\left| \int_0^1 I_{219} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t,\eta)\, d\eta \right| \\
&= \left(\frac{1}{A_1^6} - \frac{1}{A_2^6} \right) \left| \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_1^{1/2})_\eta (t,\eta) \right. \\
&\quad \left. \times \left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))} (\tilde{\mathcal{U}}_1^+)^3 \tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\, d\theta \right) d\eta \right|
\end{aligned}$$

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$$\leqslant \mathcal{O}(1)(|A_1 - A_2|^2 + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2).$$

Next, we consider the term I_{22}:

$$\begin{aligned} I_{22} &= \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t,\eta) \\ &\quad \times \left(\frac{1}{A_2^6} (\tilde{\mathcal{P}}_2^{1/2})_\eta(t,\eta) \int_0^1 e^{-\frac{1}{A_2}|\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta)|} \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\, d\theta \right. \\ &\quad \left. - \frac{1}{A_1^6} (\tilde{\mathcal{P}}_1^{1/2})_\eta(t,\eta) \int_0^1 e^{-\frac{1}{A_1}|\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta)|} \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1 \tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\, d\theta \right) d\eta. \end{aligned}$$

We start by replacing the integrals $\int_0^1 (\cdots)\, d\theta$ with $\int_0^\eta (\cdots)\, d\theta$; cf. (5.4). Furthermore, we write $\tilde{\mathcal{U}}_j = \tilde{\mathcal{U}}_j^+ + \tilde{\mathcal{U}}_j^-$, and hence it suffices to consider

$$\begin{aligned} \tilde{I}_{22} &= \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t,\eta) \\ &\quad \times \left(\frac{1}{A_2^6} (\tilde{\mathcal{P}}_2^{1/2})_\eta(t,\eta) \int_0^\eta e^{-\frac{1}{A_2}|\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta)|} \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\, d\theta \right. \\ &\quad \left. - \frac{1}{A_1^6} (\tilde{\mathcal{P}}_1^{1/2})_\eta(t,\eta) \int_0^\eta e^{-\frac{1}{A_1}|\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta)|} \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\, d\theta \right) d\eta \\ &= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t,\eta) \\ &\quad \times \left((\tilde{\mathcal{P}}_2^{1/2})_\eta(t,\eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\, d\theta \right. \\ &\quad \left. - (\tilde{\mathcal{P}}_1^{1/2})_\eta(t,\eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\, d\theta \right) d\eta \\ &\quad + \frac{A_1^6 - A_2^6}{A_1^6 A_2^6} \mathbb{1}_{A_2 \leqslant A_1} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t,\eta) (\tilde{\mathcal{P}}_2^{1/2})_\eta(t,\eta) \\ &\quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\, d\theta \right) d\eta \\ &\quad + \frac{A_1^6 - A_2^6}{A_1^6 A_2^6} \mathbb{1}_{A_1 < A_2} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t,\eta) (\tilde{\mathcal{P}}_1^{1/2})_\eta(t,\eta) \\ &\quad \times \left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\, d\theta \right) d\eta \\ &= N_1 + N_2 + N_3. \end{aligned}$$

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We consider first N_1, where we get

$$\begin{aligned}
N_1 &= \frac{1}{A^6}\int_0^1 (\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})(t,\eta)\\
&\quad\times\Big[(\tilde{\mathcal{P}}_2^{1/2})_\eta(t,\eta)\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\\
&\quad-(\tilde{\mathcal{P}}_1^{1/2})_\eta(t,\eta)\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\tilde{\mathcal{P}}_1\tilde{\mathcal{U}}_1^+\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\Big]\,d\eta\\
&= \frac{1}{A^6}\int_0^1 (\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})(t,\eta)\\
&\quad\times\Big[(\tilde{\mathcal{P}}_2^{1/2})_\eta(t,\eta)\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{P}}_2-\tilde{\mathcal{P}}_1)\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}\mathbb{1}_{\tilde{\mathcal{P}}_1\leqslant\tilde{\mathcal{P}}_2}(t,\theta)\,d\theta\\
&\quad+(\tilde{\mathcal{P}}_1^{1/2})_\eta(t,\eta)\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}(\tilde{\mathcal{P}}_2-\tilde{\mathcal{P}}_1)\tilde{\mathcal{U}}_1^+\tilde{\mathcal{Y}}_{1,\eta}\mathbb{1}_{\tilde{\mathcal{P}}_2<\tilde{\mathcal{P}}_1}(t,\theta)\,d\theta\\
&\quad+(\tilde{\mathcal{P}}_2^{1/2})_\eta(t,\eta)\\
&\quad\times\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\min_j(\tilde{\mathcal{P}}_j)(\tilde{\mathcal{U}}_2^+-\tilde{\mathcal{U}}_1^+)\tilde{\mathcal{Y}}_{2,\eta}\mathbb{1}_{\tilde{\mathcal{U}}_1^+\leqslant\tilde{\mathcal{U}}_2^+}(t,\theta)\,d\theta\\
&\quad+(\tilde{\mathcal{P}}_1^{1/2})_\eta(t,\eta)\\
&\quad\times\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\min_j(\tilde{\mathcal{P}}_j)(\tilde{\mathcal{U}}_2^+-\tilde{\mathcal{U}}_1^+)\tilde{\mathcal{Y}}_{1,\eta}\mathbb{1}_{\tilde{\mathcal{U}}_2^+<\tilde{\mathcal{U}}_1^+}(t,\theta)\,d\theta\\
&\quad+(\tilde{\mathcal{P}}_2^{1/2})_\eta(t,\eta)\\
&\quad\times\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\\
&\quad-(\tilde{\mathcal{P}}_1^{1/2})_\eta(t,\eta)\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\Big]\,d\eta\\
&= N_{11}+N_{12}+N_{13}+N_{14}+N_{15}+N_{16}.
\end{aligned}\tag{5.43}$$

The terms N_{11} and N_{12} can be treated similarly. To that end, we find

$$\begin{aligned}
|N_{11}| &= \frac{1}{A^6}\Big|\int_0^1 (\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_2^{1/2})_\eta(t,\eta)\\
&\quad\times\Big(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{P}}_2-\tilde{\mathcal{P}}_1)\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}\mathbb{1}_{\tilde{\mathcal{P}}_1\leqslant\tilde{\mathcal{P}}_2}(t,\theta)\,d\theta\Big)\,d\eta\Big|\\
&\leqslant \|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+\frac{1}{A^{12}}\int_0^1((\tilde{\mathcal{P}}_2^{1/2})_\eta)^2(t,\eta)
\end{aligned}$$

$$\times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{P}}_2-\tilde{\mathcal{P}}_1)\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}\mathbb{1}_{\tilde{\mathcal{P}}_1\leqslant\tilde{\mathcal{P}}_2}(t,\theta)\,d\theta\right)^2 d\eta$$
$$\leqslant \|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2$$
$$+\frac{1}{2A^9}\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{U}}_2^+)^2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)$$
$$\times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}|\tilde{\mathcal{P}}_2^{1/2}-\tilde{\mathcal{P}}_1^{1/2}|^2\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)d\eta$$
$$\leqslant \|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+\frac{2}{A^8}\int_0^1 \tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)$$
$$\times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}|\tilde{\mathcal{P}}_2^{1/2}-\tilde{\mathcal{P}}_1^{1/2}|^2\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)d\eta$$
$$\leqslant \mathcal{O}(1)\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2,$$

where we have used (4.15e) and (4.17c). The terms N_{13} and N_{14} follow the same lines. More precisely,

$$|N_{13}|=\frac{1}{A^6}\left|\int_0^1(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_2^{1/2})_\eta(t,\eta)\right.$$
$$\left.\times\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\min_j(\tilde{\mathcal{P}}_j)(\tilde{\mathcal{U}}_2^+-\tilde{\mathcal{U}}_1^+)\tilde{\mathcal{Y}}_{2,\eta}\mathbb{1}_{\tilde{\mathcal{U}}_1^+\leqslant\tilde{\mathcal{U}}_2^+}(t,\theta)\,d\theta\,d\eta\right|$$
$$\leqslant\frac{1}{A^6}\int_0^1|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}|\,|(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t,\eta)$$
$$\times\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\min_j(\tilde{\mathcal{P}}_j)|\tilde{\mathcal{U}}_2^+-\tilde{\mathcal{U}}_1^+|\tilde{\mathcal{Y}}_{2,\eta}\mathbb{1}_{\tilde{\mathcal{U}}_1^+\leqslant\tilde{\mathcal{U}}_2^+}(t,\theta)\,d\theta\,d\eta$$
$$\leqslant\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+\frac{1}{A^{12}}\int_0^1((\tilde{\mathcal{P}}_2^{1/2})_\eta)^2(t,\eta)$$
$$\times\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\min_j(\tilde{\mathcal{P}}_j)|\tilde{\mathcal{U}}_2^+-\tilde{\mathcal{U}}_1^+|\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)^2d\eta$$
$$\leqslant\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+\frac{1}{A^{12}}\int_0^1((\tilde{\mathcal{P}}_2^{1/2})_\eta)^2(t,\eta)$$
$$\times\left(\int_0^\eta e^{-\frac{3}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)$$
$$\times\left(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{U}}_2^+-\tilde{\mathcal{U}}_1^+)^2\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)d\eta$$

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$$
\begin{aligned}
&\leqslant \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \frac{2}{A^{11}} \int_0^1 \tilde{\mathcal{P}}_2((\tilde{\mathcal{P}}_2^{1/2})_\eta)^2(t,\eta) \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} (\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1)^2 \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\, d\theta \right) d\eta \\
&\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{P}}_2^{1/2} - \tilde{\mathcal{P}}_1^{1/2}\|^2 + \|\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1\|^2),
\end{aligned}
$$

by applying (4.15e), (4.16e), and (4.17c). The term $N_{15} + N_{16}$ can be estimated as follows:

$$
\begin{aligned}
N_{15} + N_{16} &= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t,\eta) \\
&\quad \times \Bigg[(\tilde{\mathcal{P}}_2^{1/2})_\eta(t,\eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\, d\theta \\
&\quad - (\tilde{\mathcal{P}}_1^{1/2})_\eta(t,\eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))} \\
&\quad \times \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\, d\theta \Bigg] d\eta \\
&= \mathbb{1}_{A_1 \leqslant A_2} \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t,\eta) (\tilde{\mathcal{P}}_2^{1/2})_\eta(t,\eta) \\
&\quad \times \Bigg(\int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}) \\
&\quad \times \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\, d\theta \Bigg) d\eta \\
&\quad + \mathbb{1}_{A_2 < A_1} \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t,\eta) (\tilde{\mathcal{P}}_1^{1/2})_\eta(t,\eta) \\
&\quad \times \Bigg(\int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}) \\
&\quad \times \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\, d\theta \Bigg) d\eta \\
&\quad + \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_2^{1/2})_\eta(t,\eta) \\
&\quad \times \int_0^\eta (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}) \\
&\quad \times \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t,\theta) \mathbb{1}_{B(\eta)}(t,\theta)\, d\theta \\
&\quad + \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_1^{1/2})_\eta(t,\eta) \\
&\quad \times \int_0^\eta (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))})
\end{aligned}
$$

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$$
\begin{aligned}
&\times \min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\mathbb{1}_{B^c(\eta)}(t,\theta)\,d\theta\\
&+\frac{1}{A^6}\int_0^1(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})(t,\eta)\bigg[(\tilde{\mathcal{P}}_2^{1/2})_\eta(t,\eta)\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\\
&\times \min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\\
&-(\tilde{\mathcal{P}}_1^{1/2})_\eta(t,\eta)\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\\
&\times \min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\bigg]\,d\eta\\
=\,&N_{151}+N_{152}+N_{153}+N_{154}+N_{155},
\end{aligned}
$$

which all, unfortunately, need a special treatment.

The terms N_{151} and N_{152} can be handled as follows:

$$
\begin{aligned}
|N_{151}| &\leqslant \mathbb{1}_{A_1\leqslant A_2}\frac{2\sqrt{2}a}{A^6 e}\int_0^1 |\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}||(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t,\eta)\\
&\quad\times\left(\int_0^\eta e^{-\frac{3}{4A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)d\eta|A_1-A_2|\\
&\leqslant \frac{2\sqrt{2}a}{A^6 e}\int_0^1 |\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}||(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t,\eta)\\
&\quad\times\left(\int_0^\eta e^{-\frac{3}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)^{1/2}\\
&\quad\times\left(\int_0^\eta \tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)^{1/2}d\eta|A_1-A_2|\\
&\leqslant \frac{2\sqrt{2}a}{A^3 e}\int_0^1 |\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}||(\tilde{\mathcal{P}}_2^{1/2})_\eta|\tilde{\mathcal{P}}_2^{1/2}(t,\eta)\,d\eta|A_1-A_2|\\
&\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+|A_1-A_2|^2).
\end{aligned}
$$

The terms N_{153} and N_{154} can be handled as follows:

$$
\begin{aligned}
|N_{153}| &= \frac{1}{A^6}\bigg|\int_0^1(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_2^{1/2})_\eta(t,\eta)\\
&\quad\times\int_0^\eta(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}-e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))})\\
&\quad\times\min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\mathbb{1}_{B(\eta)}(t,\theta)\,d\theta\,d\eta\bigg|\\
&\leqslant \frac{1}{aA^6}\int_0^1|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}|\,|(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t,\eta)
\end{aligned}
$$

$$
\begin{aligned}
&\times \int_0^\eta (|\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_1(t,\eta)|+|\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_1(t,\theta)|)e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\\
&\times \min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\,d\eta\\
=&\ \frac{1}{aA^6}\int_0^1 |\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}|\,|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1|\,|(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t,\eta)\int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\\
&\times \min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\,d\eta\\
&+\frac{1}{aA^6}\int_0^1 |\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}|\,|(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t,\eta)\int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1|\\
&\times \min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\,d\eta\\
\leqslant&\ \frac{1}{\sqrt{2}A^5}\int_0^1 |\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}|\,|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1|\,|(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t,\eta)\\
&\times \left(\int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2^2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)^{1/2}\\
&\times \left(\int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)^{1/2}d\eta\\
&+\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+\frac{1}{2A^{10}}\int_0^1((\tilde{\mathcal{P}}_2^{1/2})_\eta)^2(t,\eta)\\
&\times \left(\int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1|\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)^2d\eta\\
\leqslant&\ \frac{\sqrt{3}}{2A^2}\int_0^1 |\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}|\,|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1|\tilde{\mathcal{P}}_2^{1/2}|(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t,\eta)d\eta+\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2\\
&+\frac{1}{2A^{10}}\int_0^1((\tilde{\mathcal{P}}_2^{1/2})_\eta)^2(t,\eta)\left(\int_0^\eta e^{-\frac{3}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)\\
&\times \left(\int_0^\eta e^{-\frac{1}{2a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1)^2\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)d\eta\\
\leqslant&\ \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2),
\end{aligned}
$$

using (4.15e), (4.16b), (4.16e), and (4.17a). We consider now N_{155}:

$$
\begin{aligned}
N_{155}=&\ \frac{1}{A^6}\int_0^1(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})(t,\eta)\\
&\times\Big[(\tilde{\mathcal{P}}_2^{1/2})_\eta(t,\eta)\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)d\theta\\
&-(\tilde{\mathcal{P}}_1^{1/2})_\eta(t,\eta)\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\Big]d\eta
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{1}{A^6}\int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})_\eta(t,\eta) \\
&\quad \times \min_k\left[\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{k,\eta}(t,\theta)\, d\theta\right] d\eta \\
&\quad + \frac{1}{A^6}\int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_2^{1/2})_\eta(t,\eta)\mathbb{1}_{\tilde{E}^c}(t,\eta) \\
&\quad \times \left(\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+)(\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)_\eta(t,\theta)\, d\theta\right) d\eta \\
&\quad + \frac{1}{A^6}\int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_1^{1/2})_\eta(t,\eta)\mathbb{1}_{\tilde{E}}(t,\eta) \\
&\quad \times \left(\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+)(\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)_\eta(t,\theta)\, d\theta\right) d\eta \\
&= N_{1551} + N_{1552} + N_{1553},
\end{aligned}
$$

which, yet again, requires separate treatment. Here $\tilde{E}$ is defined in (5.16). The term N_{1551} can be handled as the term $\tilde{L}_{31}$ (cf. (5.17)),

$$
|N_{1551}| \leqslant \mathcal{O}(1)\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2.
$$

The terms N_{1552} and N_{1553} can be treated in the same manner:

$$
\begin{aligned}
|N_{1552}| &\leqslant \frac{1}{A^6}\int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|\,|(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t,\eta)\mathbb{1}_{\tilde{E}^c}(t,\eta) \\
&\qquad \times \left|\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+)(\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)_\eta(t,\theta)\, d\theta\right| d\eta \\
&\leqslant \frac{1}{A^6}\int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|\,|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+)|(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t,\eta)\, d\eta \\
&\qquad + \frac{1}{A^6}\int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}||(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t,\eta) \\
&\qquad \times \left|\int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)\left[\frac{d}{d\theta}\left(\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\right) \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+)\right.\right. \\
&\qquad \left.\left. + \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\frac{d}{d\theta}\left(\min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+)\right)\right](t,\theta)\, d\theta\right| d\eta \\
&\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2).
\end{aligned}
$$

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The term N_2 (and also N_3) can be treated as follows:

$$
\begin{aligned}
|N_2| &\leqslant \frac{A_1^6 - A_2^6}{A_1^6 A_2^6} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}||(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t,\eta) \\
&\qquad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{P}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\, d\theta \right)^{1/2} \\
&\qquad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^2 \tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\, d\theta \right)^{1/2} d\eta \\
&\leqslant \sqrt{6} \frac{A_1^6 - A_2^6}{A^6 A_2^3} \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|\tilde{\mathcal{P}}_2|(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t,\eta)\, d\eta \\
&\leqslant \mathcal{O}(1)|A_1 - A_2| \int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t,\eta)\, d\eta \\
&\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + |A_1 - A_2|^2).
\end{aligned}
$$

This concludes the discussion of the term I_{22}.

The term I_{23} is handled as follows:

$$
\begin{aligned}
I_{23} = \int_0^1 &(\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t,\eta) \\
&\times \left(\frac{1}{A_2^6} (\tilde{\mathcal{P}}_2^{1/2})_\eta(t,\eta) \int_0^1 e^{-\frac{1}{A_2}|\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta)|} \tilde{\mathcal{Q}}_2 \tilde{\mathcal{U}}_{2,\eta}(t,\theta)\, d\theta \right. \\
&\left. - \frac{1}{A_1^6} (\tilde{\mathcal{P}}_1^{1/2})_\eta(t,\eta) \int_0^1 e^{-\frac{1}{A_1}|\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta)|} \tilde{\mathcal{Q}}_1 \tilde{\mathcal{U}}_{1,\eta}(t,\theta)\, d\theta \right) d\eta.
\end{aligned}
$$

We start by replacing the integrals $\int_0^1 (\cdots)\, d\theta$ with $\int_0^\eta (\cdots)\, d\theta$; cf. (5.4). Thus

$$
\begin{aligned}
\tilde{I}_{23} &= \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \left(\frac{1}{A_2^6} (\tilde{\mathcal{P}}_2^{1/2})_\eta(t,\eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{Q}}_2 \tilde{\mathcal{U}}_{2,\eta}(t,\theta)\, d\theta \right. \\
&\qquad \left. - \frac{1}{A_1} (\tilde{\mathcal{P}}_1^{1/2})_\eta(t,\eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{Q}}_1 \tilde{\mathcal{U}}_{1,\eta}(t,\theta)\, d\theta \right) d\eta \\
&= \frac{1}{A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \left((\tilde{\mathcal{P}}_2^{1/2})_\eta(t,\eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{Q}}_2 \tilde{\mathcal{U}}_{2,\eta}(t,\theta)\, d\theta \right. \\
&\qquad \left. - (\tilde{\mathcal{P}}_1^{1/2})_\eta(t,\eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{Q}}_1 \tilde{\mathcal{U}}_{1,\eta}(t,\theta)\, d\theta \right) d\eta \\
&\qquad + \mathbb{1}_{A_2 \leqslant A_1} \left(\frac{1}{A_2^6} - \frac{1}{A_1^6} \right) \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_2^{1/2})_\eta(t,\eta)
\end{aligned}
$$

$$
\begin{aligned}
&\times \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{Q}}_2\tilde{\mathcal{U}}_{2,\eta}(t,\theta)\,d\theta\,d\eta\\
&+\mathbb{1}_{A_1<A_2}\left(\frac{1}{A_2^6}-\frac{1}{A_1^6}\right)\int_0^1(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_1^{1/2})_\eta(t,\eta)\\
&\times \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\tilde{\mathcal{Q}}_1\tilde{\mathcal{U}}_{1,\eta}(t,\theta)\,d\theta\,d\eta\\
=\;& M_1+M_2+M_3.
\end{aligned}
$$

We estimate, using first that $\tilde{\mathcal{Q}}_i = A_i\tilde{\mathcal{P}}_i - \tilde{\mathcal{D}}_i$ (cf. (4.7)):

$$
\begin{aligned}
M_1 &= \frac{1}{A^6}\int_0^1(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})\bigg((\tilde{\mathcal{P}}_2^{1/2})_\eta(t,\eta)\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{Q}}_2\tilde{\mathcal{U}}_{2,\eta}(t,\theta)\,d\theta\\
&\quad -(\tilde{\mathcal{P}}_1^{1/2})_\eta(t,\eta)\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\tilde{\mathcal{Q}}_1\tilde{\mathcal{U}}_{1,\eta}(t,\theta)\,d\theta\bigg)\,d\eta\\
&= \frac{1}{A^6}\int_0^1(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})\bigg(A_2(\tilde{\mathcal{P}}_2^{1/2})_\eta(t,\eta)\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2\tilde{\mathcal{U}}_{2,\eta}(t,\theta)\,d\theta\\
&\quad -A_1(\tilde{\mathcal{P}}_1^{1/2})_\eta(t,\eta)\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\tilde{\mathcal{P}}_1\tilde{\mathcal{U}}_{1,\eta}(t,\theta)\,d\theta\bigg)\,d\eta\\
&\quad -\frac{1}{A^6}\int_0^1(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})\bigg((\tilde{\mathcal{P}}_2^{1/2})_\eta(t,\eta)\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{D}}_2\tilde{\mathcal{U}}_{2,\eta}(t,\theta)\,d\theta\\
&\quad -(\tilde{\mathcal{P}}_1^{1/2})_\eta(t,\eta)\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\tilde{\mathcal{D}}_1\tilde{\mathcal{U}}_{1,\eta}(t,\theta)\,d\theta\bigg)\,d\eta\\
&= \frac{1}{A^6}\int_0^1(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})(A_2\tilde{\mathcal{P}}_2\tilde{\mathcal{U}}_2(\tilde{\mathcal{P}}_2^{1/2})_\eta - A_1\tilde{\mathcal{P}}_1\tilde{\mathcal{U}}_1(\tilde{\mathcal{P}}_1^{1/2})_\eta)(t,\eta)\,d\eta\\
&\quad -\frac{1}{A^6}\int_0^1(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{D}}_2\tilde{\mathcal{U}}_2(\tilde{\mathcal{P}}_2^{1/2})_\eta - \tilde{\mathcal{D}}_1\tilde{\mathcal{U}}_1(\tilde{\mathcal{P}}_1^{1/2})_\eta)(t,\eta)\,d\eta\\
&\quad +\frac{1}{A^6}\int_0^1(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})\\
&\quad \times\bigg((\tilde{\mathcal{P}}_2^{1/2})_\eta(t,\eta)\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\frac{1}{A_2}\tilde{\mathcal{D}}_2\tilde{\mathcal{U}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\\
&\quad -(\tilde{\mathcal{P}}_1^{1/2})_\eta(t,\eta)\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\frac{1}{A_1}\tilde{\mathcal{D}}_1\tilde{\mathcal{U}}_1\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\bigg)\,d\eta\\
&\quad -\frac{3}{A^6}\int_0^1(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})\\
&\quad \times\bigg((\tilde{\mathcal{P}}_2^{1/2})_\eta(t,\eta)\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2\tilde{\mathcal{U}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta
\end{aligned}
$$

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$$
\begin{aligned}
&- (\tilde{\mathcal{P}}_1^{1/2})_\eta(t,\eta)\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\tilde{\mathcal{P}}_1\tilde{\mathcal{U}}_1\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\Big)\,d\eta\\
&+ \frac{1}{A^6}\int_0^1(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})\Big((\tilde{\mathcal{P}}_2^{1/2})_\eta(t,\eta)\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^3\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\\
&- (\tilde{\mathcal{P}}_1^{1/2})_\eta(t,\eta)\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\tilde{\mathcal{U}}_1^3\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\Big)\,d\eta\\
&+ \frac{1}{2A^6}\int_0^1(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})\Big((\tilde{\mathcal{P}}_2^{1/2})_\eta(t,\eta)\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}A_2^5\tilde{\mathcal{U}}_2(t,\theta)\,d\theta\\
&- (\tilde{\mathcal{P}}_1^{1/2})_\eta(t,\eta)\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}A_1^5\tilde{\mathcal{U}}_1(t,\theta)\,d\theta\Big)\,d\eta\\
&= W_1+W_2+W_3+W_4+W_5+W_6.
\end{aligned}
$$

Here we have used the rewrite employed when manipulating the term $\bar{K}_1$ from the expression (5.18) to (5.19). We start by considering the term W_1:

$$
\begin{aligned}
W_1 &= \frac{1}{A^6}\int_0^1(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})(A_2\tilde{\mathcal{P}}_2\tilde{\mathcal{U}}_2(\tilde{\mathcal{P}}_2^{1/2})_\eta - A_1\tilde{\mathcal{P}}_1\tilde{\mathcal{U}}_1(\tilde{\mathcal{P}}_1^{1/2})_\eta)(t,\eta)\,d\eta\\
&= \mathbb{1}_{A_1\leqslant A_2}\frac{A_2-A_1}{A^6}\int_0^1(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})\tilde{\mathcal{P}}_2\tilde{\mathcal{U}}_2(\tilde{\mathcal{P}}_2^{1/2})_\eta(t,\eta)\,d\eta\\
&\quad+ \mathbb{1}_{A_2<A_1}\frac{A_2-A_1}{A^6}\int_0^1(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})\tilde{\mathcal{P}}_1\tilde{\mathcal{U}}_1(\tilde{\mathcal{P}}_1^{1/2})_\eta(t,\eta)\,d\eta\\
&\quad+ \frac{a}{A^6}\int_0^1(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_2-\tilde{\mathcal{P}}_1)\tilde{\mathcal{U}}_2(\tilde{\mathcal{P}}_2^{1/2})_\eta(t,\eta)\mathbb{1}_{\tilde{\mathcal{P}}_1\leqslant\tilde{\mathcal{P}}_2}(t,\eta)\,d\eta\\
&\quad+ \frac{a}{A^6}\int_0^1(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_2-\tilde{\mathcal{P}}_1)\tilde{\mathcal{U}}_1(\tilde{\mathcal{P}}_1^{1/2})_\eta(t,\eta)\mathbb{1}_{\tilde{\mathcal{P}}_2<\tilde{\mathcal{P}}_1}(t,\eta)\,d\eta\\
&\quad+ \frac{a}{A^6}\int_0^1(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})((\tilde{\mathcal{P}}_2^{1/2})_\eta-(\tilde{\mathcal{P}}_1^{1/2})_\eta)\min_j(\tilde{\mathcal{P}}_j)\tilde{\mathcal{U}}_2(t,\eta)\,d\eta\\
&\quad+ \frac{a}{A^6}\int_0^1(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{U}}_2-\tilde{\mathcal{U}}_1)\min_j(\tilde{\mathcal{P}}_j)(\tilde{\mathcal{P}}_1^{1/2})_\eta(t,\eta)\,d\eta\\
&= W_{11}+W_{12}+W_{13}+W_{14}+W_{15}+W_{16}.
\end{aligned}
$$

Using (4.15a), (4.15b), (4.15e), (4.17a), and (4.17b), we find that

$$
\begin{aligned}
&W_{11}+W_{12}+W_{13}+W_{14}+W_{16}\\
&\quad\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2\|^2+\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+|A_1-A_2|^2).
\end{aligned}
$$

Regarding the term W_{15}:

$$\begin{aligned}
|W_{15}| &= \frac{a}{A^6}\left|\int_0^1 (\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})((\tilde{\mathcal{P}}_2^{1/2})_\eta-(\tilde{\mathcal{P}}_1^{1/2})_\eta)\min_j(\tilde{\mathcal{P}}_j)\tilde{\mathcal{U}}_2(t,\eta)\,d\eta\right|\\
&= \left|\frac{a}{2A^6}(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})^2\min_j(\tilde{\mathcal{P}}_j)\tilde{\mathcal{U}}_2(t,\eta)\,\right|_{\eta=0}^{1}\\
&\quad-\frac{a}{2A^6}\int_0^1(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})^2\frac{d}{d\eta}\big(\min_j(\tilde{\mathcal{P}}_j)\tilde{\mathcal{U}}_2\big)(t,\eta)\,d\eta\bigg|\\
&= \frac{a}{2A^6}\left|\int_0^1(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})^2\frac{d}{d\eta}\big(\min_j(\tilde{\mathcal{P}}_j)\tilde{\mathcal{U}}_2\big)(t,\eta)\,d\eta\right|\\
&\leqslant \mathcal{O}(1)\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2,
\end{aligned}$$

using the same estimates as for $\bar{B}_{13}$ (cf. (5.20)) and Lemma A.3 (ii).

As for the term W_2, we find

$$\begin{aligned}
-W_2 &= \frac{1}{A^6}\int_0^1(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{D}}_2\tilde{\mathcal{U}}_2(\tilde{\mathcal{P}}_2^{1/2})_\eta-\tilde{\mathcal{D}}_1\tilde{\mathcal{U}}_1(\tilde{\mathcal{P}}_1^{1/2})_\eta)(t,\eta)\,d\eta\\
&= \frac{1}{A^6}\int_0^1(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{D}}_2-\tilde{\mathcal{D}}_1)\tilde{\mathcal{U}}_1(\tilde{\mathcal{P}}_1^{1/2})_\eta\mathbb{1}_{\tilde{\mathcal{D}}_2\leqslant\tilde{\mathcal{D}}_1}(t,\eta)\,d\eta\\
&\quad+\frac{1}{A^6}\int_0^1(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{D}}_2-\tilde{\mathcal{D}}_1)\tilde{\mathcal{U}}_2(\tilde{\mathcal{P}}_2^{1/2})_\eta\mathbb{1}_{\tilde{\mathcal{D}}_1<\tilde{\mathcal{D}}_2}(t,\eta)\,d\eta\\
&\quad+\frac{1}{A^6}\int_0^1(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})((\tilde{\mathcal{P}}_2^{1/2})_\eta-(\tilde{\mathcal{P}}_1^{1/2})_\eta)\min_j(\tilde{\mathcal{D}}_j)\tilde{\mathcal{U}}_2(t,\eta)\,d\eta\\
&\quad-\frac{1}{A^6}\int_0^1(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)\min_j(\tilde{\mathcal{D}}_j)(\tilde{\mathcal{P}}_1^{1/2})_\eta(t,\eta)\,d\eta\\
&= W_{21}+W_{22}+W_{23}+W_{24}.
\end{aligned}$$

The terms W_{21} and W_{22} can be treated similarly. We need to estimate $\tilde{\mathcal{D}}_2-\tilde{\mathcal{D}}_1$. Applying Lemma A.9, we have

$$\begin{aligned}
|W_{21}| &= \frac{1}{A^6}\left|\int_0^1(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{D}}_2-\tilde{\mathcal{D}}_1)\tilde{\mathcal{U}}_1(\tilde{\mathcal{P}}_1^{1/2})_\eta(t,\eta)\mathbb{1}_{\tilde{\mathcal{D}}_2\leqslant\tilde{\mathcal{D}}_1}(t,\eta)\,d\eta\right|\\
&\leqslant \frac{2}{A^{9/2}}\int_0^1|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}|\,|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|\tilde{\mathcal{D}}_1^{1/2}|\tilde{\mathcal{U}}_1(\tilde{\mathcal{P}}_1^{1/2})_\eta|(t,\eta)\,d\eta\\
&\quad+\frac{1}{A^6}\int_0^1|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}|\,|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|(\tilde{\mathcal{U}}_1^2+\tilde{\mathcal{P}}_1)|\tilde{\mathcal{U}}_1(\tilde{\mathcal{P}}_1^{1/2})_\eta|(t,\eta)\,d\eta\\
&\quad+\frac{2\sqrt{2}}{A^{9/2}}\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|\int_0^1|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}|\tilde{\mathcal{D}}_1^{1/2}|\tilde{\mathcal{U}}_1(\tilde{\mathcal{P}}_1^{1/2})_\eta|(t,\eta)\,d\eta
\end{aligned}$$

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$$
\begin{aligned}
&+ \frac{4}{A^3}\int_0^1 |\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}| \\
&\times \left(\int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)^2(t,\theta)\,d\theta\right)^{1/2}|\tilde{\mathcal{U}}_1(\tilde{\mathcal{P}}_1^{1/2})_\eta|(t,\eta)\,d\eta \\
&+ \frac{2\sqrt{2}}{A^3}\int_0^1 |\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}| \\
&\times \left(\int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})^2(t,\theta)\,d\theta\right)^{1/2}|\tilde{\mathcal{U}}_1(\tilde{\mathcal{P}}_1^{1/2})_\eta|(t,\eta)\,d\eta \\
&+ \frac{3}{\sqrt{2}A^2}\int_0^1 |\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}| \\
&\times \left(\int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,\theta)\,d\theta\right)^{1/2}|\tilde{\mathcal{U}}_1(\tilde{\mathcal{P}}_1^{1/2})_\eta|(t,\eta)\,d\eta \\
&+ \frac{3}{2A^2}\int_0^1 |\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}| \\
&\times \left(\int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|(t,\theta)\,d\theta\right)|\tilde{\mathcal{U}}_1(\tilde{\mathcal{P}}_1^{1/2})_\eta|(t,\eta)\,d\eta \\
&+ \frac{6}{A^2}|A_1-A_2|\int_0^1 |\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}| \\
&\times \left(\int_0^\eta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\,d\theta\right)|\tilde{\mathcal{U}}_1(\tilde{\mathcal{P}}_1^{1/2})_\eta|(t,\eta)\,d\eta \\
&+ \frac{12\sqrt{2}}{\sqrt{3e}A^2}|A_1-A_2|\int_0^1 |\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}| \\
&\times \left(\int_0^\eta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\,d\theta\right)^{1/2}|\tilde{\mathcal{U}}_1(\tilde{\mathcal{P}}_1^{1/2})_\eta|(t,\eta)\,d\eta \\
\leqslant{}& \frac{2}{A^{9/2}}\int_0^1 |\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}|\,|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|(t,\eta)\frac{A^{13/2}}{4}\,d\eta \\
&+ \frac{1}{A^6}\int_0^1 |\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}|\,|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|(t,\eta)\frac{3A^8}{8\sqrt{2}}\,d\eta \\
&+ \frac{2\sqrt{2}}{A^{9/2}}\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|\int_0^1 |\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}|(t,\eta)\frac{A^{13/2}}{4}\,d\eta \\
&+ \frac{4}{A^3}\int_0^1 |\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}|(t,\eta)\left(\int_0^\eta (\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)^2(t,\theta)\,d\theta\right)^{1/2}\frac{A^4}{2\sqrt{2}}\,d\eta \\
&+ \frac{2\sqrt{2}}{A^3}\int_0^1 |\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}|(t,\eta)\left(\int_0^\eta (\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})^2(t,\theta)\,d\theta\right)^{1/2}\frac{A^4}{2\sqrt{2}}\,d\eta
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{3}{\sqrt{2}A^2}\int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t,\eta)\bigg(\int_0^\eta (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t,\theta)\,d\theta\bigg)^{1/2}\frac{A^4}{2\sqrt{2}}\,d\eta \\
&+ \frac{3}{2A^2}\int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t,\eta)\bigg(\int_0^\eta |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t,\theta)\,d\theta\bigg)\frac{A^4}{2\sqrt{2}}\,d\eta \\
&+ \frac{6}{A^2}|A_1 - A_2|\int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t,\eta)\frac{A^4}{2\sqrt{2}}\,d\eta \\
&+ \frac{12\sqrt{2}}{\sqrt{3e}A^2}|A_1 - A_2|\int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t,\eta)\frac{A^4}{2\sqrt{2}}\,d\eta \\
&\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + |A_1 - A_2|^2),
\end{aligned}
$$

where we have used estimates (4.15a), (4.15b), (4.15n), and (4.17b). Furthermore,

$$
\begin{aligned}
|W_{23}| &= \frac{1}{A^6}\bigg|\int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})((\tilde{\mathcal{P}}_2^{1/2})_\eta - (\tilde{\mathcal{P}}_1^{1/2})_\eta)\min_j(\tilde{\mathcal{D}}_j)\tilde{\mathcal{U}}_2(t,\eta)\,d\eta\bigg| \\
&\leqslant \bigg|\frac{1}{2A^6}(\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2\min_j(\tilde{\mathcal{D}}_j)\tilde{\mathcal{U}}_2(t,\eta)\ \bigg|_{\eta=0}^{1} \\
&\quad - \frac{1}{2A^6}\int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2\frac{d}{d\theta}\big(\min_j(\tilde{\mathcal{D}}_j)\tilde{\mathcal{U}}_2\big)(t,\eta)\,d\eta\bigg| \\
&= \frac{1}{2A^6}\bigg|\int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2\frac{d}{d\theta}\big(\min_j(\tilde{\mathcal{D}}_j)\tilde{\mathcal{U}}_2\big)(t,\eta)\,d\eta\bigg| \\
&\leqslant \mathcal{O}(1)\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2,
\end{aligned}
$$

by applying Lemma A.4 (ii). The term W_{24} goes as follows:

$$
\begin{aligned}
|W_{24}| &= \frac{1}{A^6}\bigg|\int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)\min_j(\tilde{\mathcal{D}}_j)(\tilde{\mathcal{P}}_1^{1/2})_\eta(t,\eta)\,d\eta\bigg| \\
&\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2)
\end{aligned}
$$

using

$$
\min_j(\tilde{\mathcal{D}}_j)|(\tilde{\mathcal{P}}_1^{1/2})_\eta| \leqslant 2A_1\tilde{\mathcal{P}}_1|(\tilde{\mathcal{P}}_1^{1/2})_\eta| \leqslant \tilde{\mathcal{P}}_1^{1/2}\tilde{\mathcal{P}}_1\tilde{\mathcal{Y}}_{1,\eta} \leqslant \frac{A^7}{4}
$$

from (4.15a), (4.15e), (4.15n), and (4.17a).

Next comes W_3, namely,

$$
\begin{aligned}
W_3 = \frac{1}{A^6}\int_0^1 &(\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \\
&\times \bigg((\tilde{\mathcal{P}}_2^{1/2})_\eta(t,\eta)\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))}\frac{1}{A_2}\tilde{\mathcal{D}}_2\tilde{\mathcal{U}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta
\end{aligned}
$$

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$$
\begin{aligned}
&- (\tilde{\mathcal{P}}_1^{1/2})_\eta(t,\eta)\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\frac{1}{A_1}\tilde{\mathcal{D}}_1\tilde{\mathcal{U}}_1\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\Big)\,d\eta\\
&= \frac{1}{A^6}\int_0^1(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})\\
&\quad\times\Big((\tilde{\mathcal{P}}_2^{1/2})_\eta(t,\eta)\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\frac{1}{A_2}\tilde{\mathcal{D}}_2\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\\
&\quad- (\tilde{\mathcal{P}}_1^{1/2})_\eta(t,\eta)\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\frac{1}{A_1}\tilde{\mathcal{D}}_1\tilde{\mathcal{U}}_1^+\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\Big)\,d\eta\\
&\quad+ \frac{1}{A^6}\int_0^1(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})\\
&\quad\times\Big((\tilde{\mathcal{P}}_2^{1/2})_\eta(t,\eta)\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\frac{1}{A_2}\tilde{\mathcal{D}}_2\tilde{\mathcal{U}}_2^-\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\\
&\quad- (\tilde{\mathcal{P}}_1^{1/2})_\eta(t,\eta)\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\frac{1}{A_1}\tilde{\mathcal{D}}_1\tilde{\mathcal{U}}_1^-\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\Big)\,d\eta\\
&= W_{31}+W_{32},
\end{aligned}
$$

and the two terms can be treated in the same manner. Thus

$$
\begin{aligned}
W_{31} &= \frac{1}{A^6}\int_0^1(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})\\
&\quad\times\Big((\tilde{\mathcal{P}}_2^{1/2})_\eta(t,\eta)\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\frac{1}{A_2}\tilde{\mathcal{D}}_2\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\\
&\quad- (\tilde{\mathcal{P}}_1^{1/2})_\eta(t,\eta)\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\frac{1}{A_1}\tilde{\mathcal{D}}_1\tilde{\mathcal{U}}_1^+\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\Big)\,d\eta\\
&= \mathbb{1}_{A_1\leqslant A_2}\frac{1}{A^6}\Big(\frac{1}{A_2}-\frac{1}{A_1}\Big)\int_0^1(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_1^{1/2})_\eta(t,\eta)\\
&\quad\times\Big(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\tilde{\mathcal{D}}_1\tilde{\mathcal{U}}_1^+\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\Big)\,d\eta\\
&\quad+ \mathbb{1}_{A_2<A_1}\frac{1}{A^6}\Big(\frac{1}{A_2}-\frac{1}{A_1}\Big)\int_0^1(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_2^{1/2})_\eta(t,\eta)\\
&\quad\times\Big(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{D}}_2\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\Big)\,d\eta\\
&\quad+ \frac{1}{A^7}\int_0^1(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_2^{1/2})_\eta(t,\eta)\\
&\quad\times\Big(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{D}}_2-\tilde{\mathcal{D}}_1)\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\mathbb{1}_{\tilde{\mathcal{D}}_1\leqslant\tilde{\mathcal{D}}_2}(t,\theta)\,d\theta\Big)\,d\eta
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{A^7}\int_0^1(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_1^{1/2})_\eta(t,\eta)\\
&\times\left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}(\tilde{\mathcal{D}}_2-\tilde{\mathcal{D}}_1)\tilde{\mathcal{U}}_1^+\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\mathbb{1}_{\tilde{\mathcal{D}}_2<\tilde{\mathcal{D}}_1}(t,\theta)\,d\theta\right)d\eta\\
&+\frac{1}{A^7}\int_0^1(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_2^{1/2})_\eta(t,\eta)\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\right.\\
&\left.\times\min_j(\tilde{\mathcal{D}}_j)(\tilde{\mathcal{U}}_2^+-\tilde{\mathcal{U}}_1^+)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\mathbb{1}_{\tilde{\mathcal{U}}_1^+\leqslant\tilde{\mathcal{U}}_2^+}(t,\theta)\,d\theta\right)d\eta\\
&+\frac{1}{A^7}\int_0^1(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_1^{1/2})_\eta(t,\eta)\left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\right.\\
&\left.\times\min_j(\tilde{\mathcal{D}}_j)(\tilde{\mathcal{U}}_2^+-\tilde{\mathcal{U}}_1^+)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\mathbb{1}_{\tilde{\mathcal{U}}_2^+<\tilde{\mathcal{U}}_1^+}(t,\theta)\,d\theta\right)d\eta\\
&+\mathbb{1}_{A_1\leqslant A_2}\frac{1}{A^7}\int_0^1(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_2^{1/2})_\eta(t,\eta)\\
&\times\left(\int_0^\eta(e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}-e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))})\right.\\
&\left.\times\min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)d\eta\\
&+\mathbb{1}_{A_2<A_1}\frac{1}{A^7}\int_0^1(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_1^{1/2})_\eta(t,\eta)\\
&\times\left(\int_0^\eta(e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}-e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))})\right.\\
&\left.\times\min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\right)d\eta\\
&+\frac{1}{A^7}\int_0^1(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_2^{1/2})_\eta(t,\eta)\\
&\times\left(\int_0^\eta(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}-e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))})\right.\\
&\left.\times\min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\mathbb{1}_{B(\eta)}(t,\theta)\,d\theta\right)d\eta\\
&+\frac{1}{A^7}\int_0^1(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_1^{1/2})_\eta(t,\eta)\\
&\times\left(\int_0^\eta(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}-e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))})\right.\\
&\left.\times\min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\mathbb{1}_{B(\eta)^c}(t,\theta)\,d\theta\right)d\eta
\end{aligned}
$$

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$$+\frac{1}{A^7}\int_0^1(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})((\tilde{\mathcal{P}}_2^{1/2})_\eta-(\tilde{\mathcal{P}}_1^{1/2})_\eta)(t,\eta)$$
$$\times\min_k\left(\int_0^\eta\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{k,\eta}(t,\theta)\,d\theta\right)d\eta$$
$$+\frac{1}{A^7}\int_0^1(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_2^{1/2})_\eta\mathbb{1}_{D^c}(t,\eta)$$
$$\times\left(\int_0^\eta\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)(\tilde{\mathcal{Y}}_{2,\eta}-\tilde{\mathcal{Y}}_{1,\eta})(t,\theta)\,d\theta\right)d\eta$$
$$+\frac{1}{A^7}\int_0^1(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_1^{1/2})_\eta\mathbb{1}_e(t,\eta)$$
$$\times\left(\int_0^\eta\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)(\tilde{\mathcal{Y}}_{2,\eta}-\tilde{\mathcal{Y}}_{1,\eta})(t,\theta)\,d\theta\right)d\eta$$
$$=Z_1+Z_2+Z_3+Z_4+Z_5+Z_6+Z_7+Z_8+Z_9+Z_{10}+Z_{11}+Z_{12}+Z_{13},$$

where the set e is given by (5.21). Here there is no alternative but to treat these terms more or less separately. For the terms Z_1 and Z_2, we find

$$|Z_1|\leqslant\frac{2}{A^7}|A_1-A_2|\int_0^1|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}||(\tilde{\mathcal{P}}_1^{1/2})_\eta|$$
$$\times\left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\tilde{\mathcal{U}}_1^2\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\right)^{1/2}$$
$$\times\left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\tilde{\mathcal{P}}_1^2\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\right)^{1/2}d\eta$$
$$\leqslant\mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+|A_1-A_2|^2).$$

For the terms Z_3 and Z_4, we find

$$|Z_3|=\frac{1}{A^7}\bigg|\int_0^1(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_2^{1/2})_\eta(t,\eta)$$
$$\times\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{D}}_2-\tilde{\mathcal{D}}_1)\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\mathbb{1}_{\tilde{\mathcal{D}}_1\leqslant\tilde{\mathcal{D}}_2}(t,\theta)\,d\theta\right)d\eta\bigg|$$
$$\leqslant\mathcal{O}(1)(\|\tilde{\mathcal{U}}_2-\tilde{\mathcal{U}}_1\|^2+\|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1\|^2+\|\tilde{\mathcal{P}}_2^{1/2}-\tilde{\mathcal{P}}_1^{1/2}\|^2+|A_2-A_1|^2)$$

by Lemma A.9.

For the terms Z_5 and Z_6, we find

$$|Z_5|=\frac{1}{A^7}\bigg|\int_0^1(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_2^{1/2})_\eta(t,\eta)$$
$$\times\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\min_j(\tilde{\mathcal{D}}_j)(\tilde{\mathcal{U}}_2^+-\tilde{\mathcal{U}}_1^+)\tilde{\mathcal{Y}}_{2,\eta}\mathbb{1}_{\tilde{\mathcal{U}}_1^+\leqslant\tilde{\mathcal{U}}_2^+}(t,\theta)\,d\theta\right)d\eta\bigg|$$

$$
\begin{aligned}
&\leqslant \|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+\frac{1}{A^{14}}\int_0^1((\tilde{\mathcal{P}}_2^{1/2})_\eta)^2(t,\eta)\\
&\quad\times\bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\min_j(\tilde{\mathcal{D}}_j)(\tilde{\mathcal{U}}_2^+-\tilde{\mathcal{U}}_1^+)\tilde{\mathcal{Y}}_{2,\eta}\mathbb{1}_{\tilde{\mathcal{U}}_1^+\leqslant\tilde{\mathcal{U}}_2^+}(t,\theta)\,d\theta\bigg)^2\,d\eta\\
&\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+\|\tilde{\mathcal{U}}_2-\tilde{\mathcal{U}}_1\|^2),
\end{aligned}
$$

by applying (4.16e), estimating $\tilde{\mathcal{D}}_2\leqslant 2A_2\tilde{\mathcal{P}}_1$ (cf. (4.15n)), and subsequently (4.17a).

For the terms Z_7 and Z_8, we find

$$
\begin{aligned}
|Z_7| &= \mathbb{1}_{A_1\leqslant A_2}\frac{1}{A^7}\bigg|\int_0^1(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_2^{1/2})_\eta(t,\eta)\\
&\qquad\times\bigg(\int_0^\eta(e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}-e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))})\\
&\qquad\times\min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\bigg)\,d\eta\bigg|\\
&\leqslant\frac{4}{A_1A^7e}\bigg|\int_0^1|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}||(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t,\eta)\\
&\qquad\times\bigg(\int_0^\eta e^{-\frac{3}{4A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{D}}_2|\tilde{\mathcal{U}}_1|\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\bigg)\,d\eta\bigg||A_1-A_2|\\
&\leqslant\frac{4\sqrt{2}a}{A^6e}\int_0^1|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}||(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t,\eta)\\
&\qquad\times\bigg(\int_0^\eta e^{-\frac{3}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\bigg)^{1/2}\\
&\qquad\times\bigg(\int_0^\eta\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\bigg)^{1/2}\,d\eta|A_1-A_2|\\
&\leqslant\mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+|A_1-A_2|^2).
\end{aligned}
$$

For the terms Z_9 and Z_{10}, we find

$$
\begin{aligned}
|Z_9| &= \frac{1}{A^7}\bigg|\int_0^1(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_2^{1/2})_\eta(t,\eta)\\
&\quad\times\bigg(\int_0^\eta(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}-e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))})\\
&\quad\times\min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\mathbb{1}_{B(\eta)}(t,\theta)\,d\theta\bigg)\,d\eta\bigg|\\
&\leqslant\frac{1}{aA^7}\int_0^1|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}|\,|(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t,\eta)
\end{aligned}
$$

$$
\begin{aligned}
&\quad \times \left(\int_0^\eta (|\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1|(t,\eta) + |\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1|(t,\theta))\right.\\
&\quad \left.\times\, e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{D}}_2 \min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right) d\eta\\
&\leqslant \frac{1}{\sqrt{2}A^6}\int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}||\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1||(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t,\eta)\\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{D}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right) d\eta\\
&\quad + \frac{1}{\sqrt{2}A^6}\int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}||(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t,\eta)\\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}|\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1|\tilde{\mathcal{D}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right) d\eta\\
&\leqslant \frac{\sqrt{2}}{A^5}\int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}||\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1||(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t,\eta)\\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2^2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)^{1/2}\\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)^{1/2} d\eta\\
&\quad + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2\\
&\quad + \frac{2}{A^{10}}\int_0^1 ((\tilde{\mathcal{P}}_2^{1/2})_\eta)^2(t,\eta)\left(\int_0^\eta e^{-\frac{3}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2^2\tilde{\mathcal{Y}}_{2,\eta}^2(t,\theta)\,d\theta\right)\\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1)^2(t,\theta)\,d\theta\right) d\eta\\
&\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2),
\end{aligned}
$$

applying the same set of estimates applied when studying $\bar{B}_{35}$; see (5.22). For the term Z_{11}, we find

$$
\begin{aligned}
|Z_{11}| &= \frac{1}{A^7}\left|\int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})((\tilde{\mathcal{P}}_2^{1/2})_\eta - (\tilde{\mathcal{P}}_1^{1/2})_\eta)(t,\eta)\right.\\
&\qquad \left.\times \min_k\left(\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{k,\eta}(t,\theta)\,d\theta\right) d\eta\right|\\
&\leqslant \mathcal{O}(1)\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2;
\end{aligned}
$$

see estimates for $\bar{B}_{37}$ (cf. (5.23)).

For the terms Z_{12} and Z_{13}, we find

$$
\begin{aligned}
|Z_{12}| &= \frac{1}{A^7}\bigg|\int_0^1 (\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_2^{1/2})_\eta \mathbb{1}_{D^c}(t,\eta)\\
&\quad\times\bigg(\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)(\tilde{\mathcal{Y}}_{2,\eta}-\tilde{\mathcal{Y}}_{1,\eta})(t,\theta)\,d\theta\bigg)\,d\eta\bigg|\\
&= \frac{1}{A^7}\bigg|\int_0^1 (\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_2^{1/2})_\eta \mathbb{1}_{D^c}(t,\eta)\\
&\quad\times\bigg[\bigg(\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)(\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1)(t,\theta)\bigg)\bigg|_{\theta=0}^{\eta}\\
&\quad-\int_0^\eta(\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1)\frac{d}{d\theta}\big(\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\big)(t,\theta)\,d\theta\bigg]\,d\eta\bigg|\\
&= \bigg|\frac{1}{A^7}\int_0^1 (\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_2^{1/2})_\eta \mathbb{1}_{D^c}\min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)(\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1)(t,\eta)\,d\eta\\
&\quad-\frac{1}{A^7}\int_0^1 (\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_2^{1/2})_\eta \mathbb{1}_{D^c}(t,\eta)\bigg(\int_0^\eta(\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1)\\
&\quad\times\frac{d}{d\theta}\big(\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\big)(t,\theta)\,d\theta\bigg)\,d\eta\bigg|\\
&\leqslant \frac{A^2}{4\sqrt{2}}\int_0^1 |\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}||\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1|(t,\eta)\,d\eta\\
&\quad+\frac{1}{A^7}\int_0^1 |\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}||(\tilde{\mathcal{P}}_2^{1/2})_\eta|\mathbb{1}_{D^c}(t,\eta)\bigg(\int_0^\eta|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1|\\
&\quad\times\bigg|\frac{d}{d\theta}\big(\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\big)\bigg|(t,\theta)\,d\theta\bigg)\,d\eta\\
&= \tilde{M}_1+\tilde{M}_2.
\end{aligned}
$$

We find directly

$$|\tilde{M}_1|\leqslant\mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2).$$

For the other term $\tilde{M}_2$, we proceed as follows:

$$
\begin{aligned}
|\tilde{M}_2| &\leqslant \frac{1}{A^7}\int_0^1 |\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}||(\tilde{\mathcal{P}}_2^{1/2})_\eta|\mathbb{1}_{D^c}(t,\eta)\bigg(\int_0^\eta|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1|\\
&\quad\times\bigg|\frac{d}{d\theta}\big(\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\big)\bigg|(t,\theta)\,d\theta\bigg)\,d\eta\\
&\leqslant \frac{1}{A^7}\int_0^1 |\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}||(\tilde{\mathcal{P}}_2^{1/2})_\eta|\mathbb{1}_{D^c}(t,\eta)
\end{aligned}
$$

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$$
\begin{aligned}
&\times \bigg(\int_0^\eta |\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1|\bigg[\Big|\Big(\frac{d}{d\theta}\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\Big)\Big|\,\big|\min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\big|\\
&+\big|\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\big|\,\Big|\frac{d}{d\theta}\big(\min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\big)\Big|\bigg](t,\theta)\,d\theta\bigg)\,d\eta\\
&\leqslant \frac{1}{A^7}\int_0^1 |\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}||(\tilde{\mathcal{P}}_2^{1/2})_\eta|\mathbb{1}_{D^c}(t,\eta)\bigg(\int_0^\eta |\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1|(t,\theta)\\
&\times\bigg[\frac{1}{a}\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\max_j(\tilde{\mathcal{Y}}_{j,\eta})\min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)(t,\theta)\\
&+\mathcal{O}(1)A^{9/2}\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\big(\min_j(\tilde{\mathcal{D}}_j)^{1/2}+|\tilde{\mathcal{U}}_2|\big)(t,\theta)\bigg]\,d\theta\bigg)\,d\eta\\
&=\tilde{M}_{21}+\tilde{M}_{22},
\end{aligned}
$$

by using Lemmas A.2 and A.4. For $\tilde{M}_{21}$, we find

$$
\begin{aligned}
\tilde{M}_{21}&=\frac{1}{aA^7}\int_0^1 |\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}||(\tilde{\mathcal{P}}_2^{1/2})_\eta|\mathbb{1}_{D^c}(t,\eta)\bigg(\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\\
&\quad\times|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1|\big|\max_j(\tilde{\mathcal{Y}}_{j,\eta})\min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\big|(t,\theta)\,d\theta\bigg)\,d\eta\\
&\leqslant \|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+\frac{1}{a^2A^{14}}\int_0^1((\tilde{\mathcal{P}}_2^{1/2})_\eta)^2(t,\eta)\bigg(\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\\
&\quad\times|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1|\max_j(\tilde{\mathcal{Y}}_{j,\eta})\min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)(t,\theta)\,d\theta\bigg)^2\,d\eta\\
&\leqslant \|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2\\
&\quad+\frac{1}{2}\int_0^1((\tilde{\mathcal{P}}_2^{1/2})_\eta)^2(t,\eta)\bigg(\int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1|(t,\theta)\,d\theta\bigg)^2\,d\eta\\
&\leqslant \|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2\\
&\quad+\frac{1}{8A^2}\int_0^1\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\bigg(\int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1)^2(t,\theta)\,d\theta\bigg)\\
&\quad\times\bigg(\int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\,d\theta\bigg)\,d\eta\\
&\leqslant \|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+\mathcal{O}(1)\int_0^1\int_0^\eta(\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1)^2(t,\theta)\,d\theta\,d\eta\\
&\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+\|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1\|^2),
\end{aligned}
$$

using

$$\max_j(\tilde{\mathcal{Y}}_{j,\eta})\min_j(\tilde{\mathcal{D}}_j) \leqslant 2\max_j(\tilde{\mathcal{Y}}_{j,\eta})\min_j(A_i\tilde{\mathcal{P}}_j) \leqslant 2\max_j(A_i\tilde{\mathcal{P}}_j\tilde{\mathcal{Y}}_{j,\eta}) \leqslant A^6,$$

and (4.15e), (4.16a), and (4.17c). For $\tilde{M}_{22}$, we find

$$\begin{aligned}
\tilde{M}_{22} &= \frac{\mathcal{O}(1)}{A^{5/2}}\int_0^1 |\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}||(\tilde{\mathcal{P}}_2^{1/2})_\eta|\mathbb{1}_{D^c}(t,\eta)\\
&\quad\times\left(\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1|\big(\min_j(\tilde{\mathcal{D}}_j)^{1/2}+|\tilde{\mathcal{U}}_2|\big)(t,\theta)\,d\theta\right)d\eta\\
&\leqslant \|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+\frac{\mathcal{O}(1)}{A^5}\int_0^1((\tilde{\mathcal{P}}_2^{1/2})_\eta)^2(t,\eta)\\
&\quad\times\left(\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1|(\min_j(\tilde{\mathcal{D}}_j)^{1/2}+|\tilde{\mathcal{U}}_2|)(t,\theta)\,d\theta\right)^2 d\eta\\
&\leqslant \|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+\frac{\mathcal{O}(1)}{A^5}\int_0^1((\tilde{\mathcal{P}}_2^{1/2})_\eta)^2(t,\eta)\\
&\quad\times\left(\int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1|\tilde{\mathcal{P}}_2^{1/2}(t,\theta)\,d\theta\right)^2 d\eta\\
&\leqslant \|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2\\
&\quad+\frac{\mathcal{O}(1)}{A^7}\int_0^1 \tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\left(\int_0^\eta e^{-\frac{3}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2(t,\theta)\,d\theta\right)\\
&\quad\times\left(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1|(t,\theta)\,d\theta\right)d\eta\\
&\leqslant \|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2\\
&\quad+\frac{\mathcal{O}(1)}{A^7}\int_0^1 \tilde{\mathcal{P}}_2^2\tilde{\mathcal{Y}}_{2,\eta}^2(t,\eta)\left(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1)^2(t,\theta)\,d\theta\right)d\eta\\
&\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+\|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1\|^2),
\end{aligned}$$

using that (cf. (4.15d) and (4.15n))

$$\min_j(\tilde{\mathcal{D}}_j)^{1/2}+|\tilde{\mathcal{U}}_2| \leqslant \sqrt{2A_2}\tilde{\mathcal{P}}_2^{1/2}+|\tilde{\mathcal{U}}_2| \leqslant \sqrt{2}(1+\sqrt{A_2})\tilde{\mathcal{P}}_2^{1/2},$$

and (4.15e), (4.16i), and (4.17a).

We now consider the term W_4:

$$\begin{aligned}
W_4 &= -\frac{3}{A^6}\int_0^1 (\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})\\
&\quad\times\Bigg((\tilde{\mathcal{P}}_2^{1/2})_\eta(t,\eta)\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2\tilde{\mathcal{U}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta
\end{aligned}$$

$$
\begin{aligned}
&- (\tilde{\mathcal{P}}_1^{1/2})_\eta(t,\eta)\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\tilde{\mathcal{P}}_1\tilde{\mathcal{U}}_1\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\Big)\,d\eta\\
&= -\frac{3}{A^6}\int_0^1(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})\\
&\quad\times\Big((\tilde{\mathcal{P}}_2^{1/2})_\eta(t,\eta)\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\\
&\quad- (\tilde{\mathcal{P}}_1^{1/2})_\eta(t,\eta)\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\tilde{\mathcal{P}}_1\tilde{\mathcal{U}}_1^+\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\Big)\,d\eta\\
&\quad-\frac{3}{A^6}\int_0^1(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})\\
&\quad\times\Big((\tilde{\mathcal{P}}_2^{1/2})_\eta(t,\eta)\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2\tilde{\mathcal{U}}_2^-\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\\
&\quad- (\tilde{\mathcal{P}}_1^{1/2})_\eta(t,\eta)\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\tilde{\mathcal{P}}_1\tilde{\mathcal{U}}_1^-\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\Big)\,d\eta\\
&= W_4^+ + W_4^-.
\end{aligned}
$$

Since $W_4^+ = -3N_1$ (see (5.43)), we find that

$$|W_4| \leqslant \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2\|^2 + \|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1\|^2 + |A_1-A_2|^2).$$

The next term W_5 would be laborious:

$$
\begin{aligned}
W_5 &= \frac{1}{A^6}\int_0^1(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})\Big((\tilde{\mathcal{P}}_2^{1/2})_\eta(t,\eta)\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^3\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\\
&\quad- (\tilde{\mathcal{P}}_1^{1/2})_\eta(t,\eta)\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\tilde{\mathcal{U}}_1^3\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\Big)\,d\eta\\
&= \frac{1}{A^6}\int_0^1(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})\Big((\tilde{\mathcal{P}}_2^{1/2})_\eta(t,\eta)\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{U}}_2^+)^3\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\\
&\quad- (\tilde{\mathcal{P}}_1^{1/2})_\eta(t,\eta)\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}(\tilde{\mathcal{U}}_1^+)^3\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\Big)\,d\eta\\
&\quad+\frac{1}{A^6}\int_0^1(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})\Big((\tilde{\mathcal{P}}_2^{1/2})_\eta(t,\eta)\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{U}}_2^-)^3\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\\
&\quad- (\tilde{\mathcal{P}}_1^{1/2})_\eta(t,\eta)\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}(\tilde{\mathcal{U}}_1^-)^3\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\Big)\,d\eta\\
&= W_5^+ + W_5^-.
\end{aligned}
$$

Unfortunately, having a close look at W_5^+ one has

$$W_5^+ = \int_0^1(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})(I_{211}+I_{212}+I_{213}+I_{214}+I_{215}+I_{216}+I_{217}+I_{218})(t,\eta)\,d\eta,$$

where $I_{211}, \ldots, I_{218}$ are defined in (5.42). Thus we can conclude immediately that

$$|W_5^+| \leqslant \mathcal{O}(1)(\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + |A_1 - A_2|^2).$$

For the term W_6, we find

$$\begin{aligned}
W_6 &= \frac{1}{2A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}) \bigg((\tilde{\mathcal{P}}_2^{1/2})_\eta(t,\eta) \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} A_2^5 \tilde{\mathcal{U}}_2(t,\theta)\, d\theta \\
&\quad - (\tilde{\mathcal{P}}_1^{1/2})_\eta(t,\eta) \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} A_1^5 \tilde{\mathcal{U}}_1(t,\theta)\, d\theta \bigg)\, d\eta \\
&= \frac{A_2^5 - A_1^5}{2A^6} \mathbb{1}_{A_1 \leqslant A_2} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_2^{1/2})_\eta(t,\eta) \\
&\quad \times \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2(t,\theta)\, d\theta\, d\eta \\
&\quad + \frac{A_2^5 - A_1^5}{2A^6} \mathbb{1}_{A_2 < A_1} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_1^{1/2})_\eta(t,\eta) \\
&\quad \times \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_1(t,\theta)\, d\theta\, d\eta \\
&\quad + \mathbb{1}_{A_1 \leqslant A_2} \frac{a^5}{2A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_2^{1/2})_\eta(t,\eta) \\
&\quad \times \bigg(\int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))}) \tilde{\mathcal{U}}_2(t,\theta)\, d\theta \bigg)\, d\eta \\
&\quad + \mathbb{1}_{A_2 < A_1} \frac{a^5}{2A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_1^{1/2})_\eta(t,\eta) \\
&\quad \times \bigg(\int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))}) \tilde{\mathcal{U}}_1(t,\theta)\, d\theta \bigg)\, d\eta \\
&\quad + \frac{a^5}{2A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_2^{1/2})_\eta(t,\eta) \\
&\quad \times \bigg(\int_0^\eta (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))}) \tilde{\mathcal{U}}_2 \mathbb{1}_{B(\eta)}(t,\theta)\, d\theta \bigg)\, d\eta \\
&\quad + \frac{a^5}{2A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_1^{1/2})_\eta(t,\eta) \\
&\quad \times \bigg(\int_0^\eta (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))}) \tilde{\mathcal{U}}_1 \mathbb{1}_{B(\eta)^c}(t,\theta)\, d\theta \bigg)\, d\eta \\
&\quad + \frac{a^5}{2A^6} \int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_2^{1/2})_\eta(t,\eta)
\end{aligned}$$

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$$
\begin{aligned}
&\times \left(\int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})(\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+)\mathbb{1}_{\tilde{\mathcal{U}}_1^+ \leqslant \tilde{\mathcal{U}}_2^+}(t,\theta)\,d\theta\right) d\eta \\
&+ \frac{a^5}{2A^6}\int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_2^{1/2})_\eta(t,\eta) \\
&\times \left(\int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})(\tilde{\mathcal{U}}_2^- - \tilde{\mathcal{U}}_1^-)\mathbb{1}_{\tilde{\mathcal{U}}_2^- \leqslant \tilde{\mathcal{U}}_1^-}(t,\theta)\,d\theta\right) d\eta \\
&+ \frac{a^5}{2A^6}\int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_1^{1/2})_\eta(t,\eta) \\
&\times \left(\int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})(\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+)\mathbb{1}_{\tilde{\mathcal{U}}_2^+ < \tilde{\mathcal{U}}_1^+}(t,\theta)\,d\theta\right) d\eta \\
&+ \frac{a^5}{2A^6}\int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_1^{1/2})_\eta(t,\eta) \\
&\times \left(\int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})(\tilde{\mathcal{U}}_2^- - \tilde{\mathcal{U}}_1^-)\mathbb{1}_{\tilde{\mathcal{U}}_1^- < \tilde{\mathcal{U}}_2^-}(t,\theta)\,d\theta\right) d\eta \\
&+ \frac{a^5}{2A^6}\int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})((\tilde{\mathcal{P}}_2^{1/2})_\eta - (\tilde{\mathcal{P}}_1^{1/2})_\eta)(t,\eta) \\
&\times \left(\int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{U}}_j^+)(t,\theta)\,d\theta\right) d\eta \\
&+ \frac{a^5}{2A^6}\int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})((\tilde{\mathcal{P}}_2^{1/2})_\eta - (\tilde{\mathcal{P}}_1^{1/2})_\eta)(t,\eta) \\
&\times \left(\int_0^\eta \max_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{U}}_j^-)(t,\theta)\,d\theta\right) d\eta \\
= {} & W_{61} + W_{62} + W_{63} + W_{64} + W_{65} + W_{66} \\
&+ W_{67}^+ + W_{67}^- + W_{68}^+ + W_{68}^- + W_{69}^+ + W_{69}^-.
\end{aligned}
$$

The terms W_{61} and W_{62}:

$$
\begin{aligned}
|W_{61}| &= \frac{A_2^5 - A_1^5}{2A^6}\mathbb{1}_{A_1 \leqslant A_2}\bigg|\int_0^1 (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_2^{1/2})_\eta(t,\eta) \\
&\qquad \times \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2(t,\theta)\,d\theta\,d\eta\bigg| \\
&\leqslant 5\frac{A_2 - A_1}{2A^2}\int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}||(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t,\eta) \\
&\qquad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^2(t,\theta)\,d\theta\right)^{1/2}
\end{aligned}
$$

$$
\begin{aligned}
&\times\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\,d\theta\right)^{1/2} d\eta\\
&\leqslant 5\sqrt{6}\frac{A_2-A_1}{2A^2}\int_0^1 |\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}|\tilde{\mathcal{P}}_2^{1/2}|(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t,\eta)\,d\eta\\
&\leqslant \mathcal{O}(1)(|A_1-A_2|^2+\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2),
\end{aligned}
$$

using (4.15e), (4.16a), and (4.17a).

The terms W_{63} and W_{64}:

$$
\begin{aligned}
|W_{63}| &= \mathbb{1}_{A_1\leqslant A_2}\frac{a^5}{2A^6}\int_0^1 |\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}||(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t,\eta)\\
&\quad\times\left|\int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}-e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))})\tilde{\mathcal{U}}_2(t,\theta)\,d\theta\right| d\eta\\
&\leqslant \frac{4a^5}{2aA^6e}\int_0^1 |\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}||(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t,\eta)\\
&\quad\times\left(\int_0^\eta e^{-\frac{3}{4A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}|\tilde{\mathcal{U}}_2|(t,\theta)\,d\theta\right) d\eta|A_1-A_2|\\
&\leqslant \frac{2}{A^2e}\int_0^1 |\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}||(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t,\eta)\\
&\quad\times\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^2(t,\theta)\,d\theta\right)^{1/2} d\eta|A_1-A_2|\\
&\leqslant \frac{2\sqrt{6}}{A^2e}\int_0^1 |\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}|\tilde{\mathcal{P}}_2^{1/2}|(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t,\eta)\,d\eta|A_1-A_2|\\
&\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+|A_1-A_2|^2).
\end{aligned}
$$

The terms W_{65} and W_{66}:

$$
\begin{aligned}
|W_{65}| &= \frac{a^5}{2A^6}\left|\int_0^1 (\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_2^{1/2})_\eta(t,\eta)\right.\\
&\quad\left.\times\int_0^\eta (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}-e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))})\tilde{\mathcal{U}}_2\mathbb{1}_{B(\eta)}(t,\theta)\,d\theta\,d\eta\right|\\
&\leqslant \frac{1}{2A^2}\int_0^1 |\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}||(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t,\eta)\\
&\quad\times\int_0^\eta (|\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_1(t,\eta)|+|\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_1(t,\theta)|)\\
&\quad\times e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}|\tilde{\mathcal{U}}_2|(t,\theta)\,d\theta\,d\eta
\end{aligned}
$$

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$$
\begin{aligned}
&\leqslant \frac{1}{2A^2}\int_0^1 |\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}||\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1||(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t,\eta)\\
&\quad\times\int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}|\tilde{\mathcal{U}}_2|(t,\theta)\,d\theta\,d\eta\\
&\quad+\frac{1}{2A^2}\int_0^1 |\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}||(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t,\eta)\\
&\quad\times\int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1|\,|\tilde{\mathcal{U}}_2|(t,\theta)\,d\theta\,d\eta\\
&\leqslant \frac{1}{2A^2}\int_0^1 |\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}||\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1||(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t,\eta)\\
&\quad\times\left(\int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^2(t,\theta)\,d\theta\right)^{1/2}d\eta\\
&\quad+\frac{1}{2A^2}\int_0^1 |\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}||(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t,\eta)\\
&\quad\times\left(\int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^2(t,\theta)\,d\theta\right)^{1/2}d\eta\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|\\
&\leqslant \frac{\sqrt{3}}{\sqrt{2}A^2}\int_0^1 |\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}||\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1|\tilde{\mathcal{P}}_2^{1/2}|(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t,\eta)\,d\eta\\
&\quad+\frac{\sqrt{3}}{\sqrt{2}A^2}\int_0^1 |\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}|\tilde{\mathcal{P}}_2^{1/2}|(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t,\eta)\,d\eta\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|\\
&\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2),
\end{aligned}
$$

using (4.15e), (4.16a), (4.17a), and (5.13).

The terms $W_{67}^{\pm}$ and $W_{68}^{\pm}$:

$$
\begin{aligned}
|W_{67}^+| &= \frac{a^5}{2A^6}\left|\int_0^1 (\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_2^{1/2})_\eta(t,\eta)\right.\\
&\quad\times\left.\left(\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})(\tilde{\mathcal{U}}_2^+-\tilde{\mathcal{U}}_1^+)\mathbb{1}_{\tilde{\mathcal{U}}_1^+\leqslant\tilde{\mathcal{U}}_2^+}(t,\theta)\,d\theta\right)d\eta\right|\\
&\leqslant \|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+\frac{1}{4A^2}\int_0^1((\tilde{\mathcal{P}}_2^{1/2})_\eta)^2(t,\eta)\\
&\quad\times\left(\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})(\tilde{\mathcal{U}}_2^+-\tilde{\mathcal{U}}_1^+)\mathbb{1}_{\tilde{\mathcal{U}}_1^+\leqslant\tilde{\mathcal{U}}_2^+}(t,\theta)\,d\theta\right)^2d\eta\\
&\leqslant \|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2
\end{aligned}
$$

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$$
\begin{aligned}
&+\frac{1}{16A^4}\int_0^1 \tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_2^2(t,\eta)\bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{U}}_2^+-\tilde{\mathcal{U}}_1)^2(t,\theta)\,d\theta\bigg)\,d\eta\\
&\leqslant \|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2\\
&\quad+\frac{A}{32}\int_0^1 \tilde{\mathcal{Y}}_{2,\eta}(t,\eta)\bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{U}}_2-\tilde{\mathcal{U}}_1)^2(t,\theta)\,d\theta\bigg)\,d\eta\\
&\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+\|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2\|^2)
\end{aligned}
$$

following the estimates used for $\bar{B}_{65}^+$; see (5.24). The term $W_{69}^\pm$:

$$
\begin{aligned}
|W_{69}^+| &= \frac{a^5}{2A^6}\bigg|\int_0^1 (\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})((\tilde{\mathcal{P}}_2^{1/2})_\eta-(\tilde{\mathcal{P}}_1^{1/2})_\eta)(t,\eta)\\
&\quad\times\bigg(\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{U}}_j^+)(t,\theta)\,d\theta\bigg)\,d\eta\bigg|\\
&= \frac{a^5}{4A^6}\bigg|\bigg((\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})^2\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{U}}_j^+)(t,\theta)\,d\theta\bigg)\bigg|_{\eta=0}^1\\
&\quad-\int_0^1(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})^2\frac{d}{d\eta}\bigg(\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{U}}_j^+)(t,\theta)\,d\theta\bigg)\,d\eta\bigg|\\
&\leqslant \frac{a^5}{4A^6}\int_0^1(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})^2\\
&\quad\times\bigg|\frac{d}{d\eta}\bigg(\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{U}}_j^+)(t,\theta)\,d\theta\bigg)\bigg|\,d\eta\\
&\leqslant \mathcal{O}(1)\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2;
\end{aligned}
$$

see estimates employed for $\bar{B}_{67}$ (cf. (5.25)) and Lemma A.5.

The terms M_2 and M_3 can be treated similarly. More precisely,

$$
\begin{aligned}
|M_2| &\leqslant \mathbb{1}_{A_2\leqslant A_1}\bigg(\frac{1}{A_2^6}-\frac{1}{A_1^6}\bigg)\bigg|\int_0^1(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{P}}_2^{1/2})_\eta(t,\eta)\\
&\quad\times\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{Q}}_2\tilde{\mathcal{U}}_{2,\eta}(t,\theta)\,d\theta\,d\eta\bigg|\\
&\leqslant \mathbb{1}_{A_2\leqslant A_1}\frac{|A_1^6-A_2^6|}{A^6A_2^5}\int_0^1|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}|\,|(\tilde{\mathcal{P}}_2^{1/2})_\eta|(t,\eta)\\
&\quad\times\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2|\tilde{\mathcal{U}}_{2,\eta}|(t,\theta)\,d\theta\,d\eta\\
&\leqslant \mathbb{1}_{A_2\leqslant A_1}6\frac{|A_1-A_2|}{AA_2^5}\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|\\
&\quad\times\bigg(\int_0^1((\tilde{\mathcal{P}}_2^{1/2})_\eta)^2(t,\eta)\bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2|\tilde{\mathcal{U}}_{2,\eta}|(t,\theta)\,d\theta\bigg)^2\,d\eta\bigg)^{1/2}
\end{aligned}
$$

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$$
\begin{aligned}
&\leqslant \mathbb{1}_{A_2\leqslant A_1} 6\frac{|A_1-A_2|}{AA_2^6}\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|\\
&\quad\times\bigg(\int_0^1((\tilde{\mathcal{P}}_2^{1/2})_\eta)^2(t,\eta)\bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2^2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\bigg)\\
&\quad\times\bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{H}}_{2,\eta}(t,\theta)\,d\theta\bigg)\,d\eta\bigg)^{1/2}\\
&\leqslant \mathbb{1}_{A_2\leqslant A_1}6\sqrt{6}\frac{|A_1-A_2|}{AA_2^3}\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|\bigg(\int_0^1((\tilde{\mathcal{P}}_2^{1/2})_\eta)^2\tilde{\mathcal{P}}_2^2(t,\eta)\,d\eta\bigg)^{1/2}\\
&\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+|A_1-A_2|^2).
\end{aligned}
$$

Here we used

$$
\tilde{\mathcal{P}}_2|\tilde{\mathcal{U}}_{2,\eta}|\leqslant\frac{1}{A_2}\tilde{\mathcal{P}}_2\sqrt{\tilde{\mathcal{Y}}_{2,\eta}\tilde{\mathcal{H}}_{2,\eta}},
$$

cf. (4.15m), as well as (4.15c), (4.15e), and (4.17d). In addition, we applied (4.16b) and (4.16d).

The term I_3: We have the following estimates:

$$
\begin{aligned}
|I_3|&\leqslant\int_0^1|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}|\bigg|\frac{1}{A_1^2}\frac{\tilde{\mathcal{Q}}_1\tilde{\mathcal{U}}_1}{\tilde{\mathcal{P}}_1^{1/2}}-\frac{1}{A_2^2}\frac{\tilde{\mathcal{Q}}_2\tilde{\mathcal{U}}_2}{\tilde{\mathcal{P}}_2^{1/2}}\bigg|(t,\eta)\,d\eta\\
&\leqslant\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+\bigg\|\frac{1}{A_1^2}\frac{\tilde{\mathcal{Q}}_1\tilde{\mathcal{U}}_1}{\tilde{\mathcal{P}}_1^{1/2}}-\frac{1}{A_2^2}\frac{\tilde{\mathcal{Q}}_2\tilde{\mathcal{U}}_2}{\tilde{\mathcal{P}}_2^{1/2}}\bigg\|^2.
\end{aligned}
$$

We consider the latter term and find

$$
\begin{aligned}
&\bigg\|\frac{1}{A_1^2}\frac{\tilde{\mathcal{Q}}_1\tilde{\mathcal{U}}_1}{\tilde{\mathcal{P}}_1^{1/2}}-\frac{1}{A_2^2}\frac{\tilde{\mathcal{Q}}_2\tilde{\mathcal{U}}_2}{\tilde{\mathcal{P}}_2^{1/2}}\bigg\|^2\\
&\quad\leqslant 3\mathbb{1}_{A_1\leqslant A_2}\bigg|\frac{1}{A_1^2}-\frac{1}{A_2^2}\bigg|^2\bigg\|\frac{\tilde{\mathcal{Q}}_1\tilde{\mathcal{U}}_1}{\tilde{\mathcal{P}}_1^{1/2}}\bigg\|^2+3\mathbb{1}_{A_2<A_1}\bigg|\frac{1}{A_1^2}-\frac{1}{A_2}\bigg|^2\bigg\|\frac{\tilde{\mathcal{Q}}_2\tilde{\mathcal{U}}_2}{\tilde{\mathcal{P}}_2^{1/2}}\bigg\|^2\\
&\quad+3\frac{1}{A^4}\bigg\|\frac{\tilde{\mathcal{Q}}_1\tilde{\mathcal{U}}_1}{\tilde{\mathcal{P}}_1^{1/2}}-\frac{\tilde{\mathcal{Q}}_2\tilde{\mathcal{U}}_2}{\tilde{\mathcal{P}}_2^{1/2}}\bigg\|^2\\
&\quad\leqslant\mathcal{O}(1)|A_1-A_2|^2+3\frac{1}{A^4}\bigg\|\frac{\tilde{\mathcal{Q}}_1\tilde{\mathcal{U}}_1}{\tilde{\mathcal{P}}_1^{1/2}}-\frac{\tilde{\mathcal{Q}}_2\tilde{\mathcal{U}}_2}{\tilde{\mathcal{P}}_2^{1/2}}\bigg\|^2,
\end{aligned}
$$

where we have used that

$$
\bigg|\frac{\tilde{\mathcal{Q}}_i\tilde{\mathcal{U}}_i}{\tilde{\mathcal{P}}_i^{1/2}}\bigg|\leqslant\frac{A_i\tilde{\mathcal{P}}_i\tilde{\mathcal{U}}_i}{\tilde{\mathcal{P}}_i^{1/2}}|=A_i\tilde{\mathcal{P}}_i^{1/2}|\tilde{\mathcal{U}}_i|\leqslant\mathcal{O}(1)A_i^5.
$$

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We consider the latter term and find

$$\begin{aligned}
&\frac{1}{A^4}\left\|\frac{\tilde{\mathcal{Q}}_1\tilde{\mathcal{U}}_1}{\tilde{\mathcal{P}}_1^{1/2}}-\frac{\tilde{\mathcal{Q}}_2\tilde{\mathcal{U}}_2}{\tilde{\mathcal{P}}_2^{1/2}}\right\|^2\\
&=\frac{1}{A^4}\int_0^1\left|\frac{\tilde{\mathcal{Q}}_1\tilde{\mathcal{U}}_1}{\tilde{\mathcal{P}}_1^{1/2}}-\frac{\tilde{\mathcal{Q}}_2\tilde{\mathcal{U}}_2}{\tilde{\mathcal{P}}_2^{1/2}}\right|^2(\mathbb{1}_{\tilde{\mathcal{P}}_1\leqslant\tilde{\mathcal{P}}_2}+\mathbb{1}_{\tilde{\mathcal{P}}_2<\tilde{\mathcal{P}}_1})(t,\eta)\,d\eta\\
&\leqslant\frac{2}{A^4}\int_0^1\left|\frac{1}{\tilde{\mathcal{P}}_2^{1/2}}(\tilde{\mathcal{Q}}_1\tilde{\mathcal{U}}_1-\tilde{\mathcal{Q}}_2\tilde{\mathcal{U}}_2)+\left(\frac{1}{\tilde{\mathcal{P}}_1^{1/2}}-\frac{1}{\tilde{\mathcal{P}}_2^{1/2}}\right)\tilde{\mathcal{Q}}_1\tilde{\mathcal{U}}_1\right|^2\mathbb{1}_{\tilde{\mathcal{P}}_1\leqslant\tilde{\mathcal{P}}_2}(t,\eta)\,d\eta\\
&\quad+\frac{2}{A^4}\int_0^1\left|\frac{1}{\tilde{\mathcal{P}}_1^{1/2}}(\tilde{\mathcal{Q}}_1\tilde{\mathcal{U}}_1-\tilde{\mathcal{Q}}_2\tilde{\mathcal{U}}_2)+\left(\frac{1}{\tilde{\mathcal{P}}_1^{1/2}}-\frac{1}{\tilde{\mathcal{P}}_2^{1/2}}\right)\tilde{\mathcal{Q}}_2\tilde{\mathcal{U}}_2\right|^2\mathbb{1}_{\tilde{\mathcal{P}}_2<\tilde{\mathcal{P}}_1}(t,\eta)\,d\eta\\
&\leqslant\frac{4}{A^4}\int_0^1\left|\frac{1}{\max_i(\tilde{\mathcal{P}}_i^{1/2})}(\tilde{\mathcal{Q}}_1(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)+\tilde{\mathcal{U}}_2(\tilde{\mathcal{Q}}_1-\tilde{\mathcal{Q}}_2))\right|^2\mathbb{1}_{\tilde{\mathcal{P}}_1\leqslant\tilde{\mathcal{P}}_2}(t,\eta)\,d\eta\\
&\quad+\frac{4}{A^4}\int_0^1\left|\frac{\tilde{\mathcal{Q}}_1\tilde{\mathcal{U}}_1}{\tilde{\mathcal{P}}_1^{1/2}\tilde{\mathcal{P}}_2^{1/2}}(\tilde{\mathcal{P}}_2^{1/2}-\tilde{\mathcal{P}}_1^{1/2})\right|^2\mathbb{1}_{\tilde{\mathcal{P}}_1\leqslant\tilde{\mathcal{P}}_2}(t,\eta)\,d\eta\\
&\quad+\frac{4}{A^4}\int_0^1\left|\frac{1}{\max_i(\tilde{\mathcal{P}}_i^{1/2})}(\tilde{\mathcal{Q}}_1(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)+\tilde{\mathcal{U}}_2(\tilde{\mathcal{Q}}_1-\tilde{\mathcal{Q}}_2))\right|^2\mathbb{1}_{\tilde{\mathcal{P}}_2<\tilde{\mathcal{P}}_1}(t,\eta)\,d\eta\\
&\quad+\frac{4}{A^4}\int_0^1\left|\frac{\tilde{\mathcal{Q}}_2\tilde{\mathcal{U}}_2}{\tilde{\mathcal{P}}_1^{1/2}\tilde{\mathcal{P}}_2^{1/2}}(\tilde{\mathcal{P}}_2^{1/2}-\tilde{\mathcal{P}}_1^{1/2})\right|^2\mathbb{1}_{\tilde{\mathcal{P}}_2<\tilde{\mathcal{P}}_1}(t,\eta)\,d\eta\\
&=4(I_{31}+I_{32}+I_{33}+I_{34}).
\end{aligned}$$

We treat terms I_{31} and I_{32}; the others are similar.

$$\begin{aligned}
I_{31}&\leqslant\frac{2}{A^4}\int_0^1\left|\left(\frac{\tilde{\mathcal{Q}}_1}{\max_i(\tilde{\mathcal{P}}_i^{1/2})}\right)^2|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2|^2\right.\\
&\qquad\left.+\left(\frac{\tilde{\mathcal{U}}_2}{\max_i(\tilde{\mathcal{P}}_i^{1/2})}\right)^2|\tilde{\mathcal{Q}}_1-\tilde{\mathcal{Q}}_2|^2\right|\mathbb{1}_{\tilde{\mathcal{P}}_1\leqslant\tilde{\mathcal{P}}_2}(t,\eta)\,d\eta\\
&\leqslant\mathcal{O}(1)(\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2+\|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2\|^2+\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+|A_1-A_2|^2),
\end{aligned}$$

where we have used (5.29), that

$$\begin{aligned}
\left|\frac{\tilde{\mathcal{Q}}_1}{\max_i(\tilde{\mathcal{P}}_i^{1/2})}\right|\mathbb{1}_{\tilde{\mathcal{P}}_1\leqslant\tilde{\mathcal{P}}_2}&\leqslant\frac{A_1\tilde{\mathcal{P}}_1}{\max_i(\tilde{\mathcal{P}}_i^{1/2})}\mathbb{1}_{\tilde{\mathcal{P}}_1\leqslant\tilde{\mathcal{P}}_2}\\
&\leqslant A\frac{\max_i(\tilde{\mathcal{P}}_i)}{\max_i(\tilde{\mathcal{P}}_i^{1/2})}\mathbb{1}_{\tilde{\mathcal{P}}_1\leqslant\tilde{\mathcal{P}}_2}\leqslant A\max_i(\tilde{\mathcal{P}}_i^{1/2})\leqslant\frac{A^3}{2},
\end{aligned}$$

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and

$$\left|\frac{\tilde{\mathcal{U}}_2}{\max_i(\tilde{\mathcal{P}}_i^{1/2})}\right|\mathbb{1}_{\tilde{\mathcal{P}}_1\leqslant\tilde{\mathcal{P}}_2}\leqslant\frac{|\tilde{\mathcal{U}}_2|}{\tilde{\mathcal{P}}_2^{1/2}}\mathbb{1}_{\tilde{\mathcal{P}}_1\leqslant\tilde{\mathcal{P}}_2}\leqslant\sqrt{2}.$$

Next follows

$$\begin{aligned}I_{32}&\leqslant\frac{1}{A^4}\int_0^1\left|\frac{\tilde{\mathcal{Q}}_1\tilde{\mathcal{U}}_1}{\tilde{\mathcal{P}}_1^{1/2}\tilde{\mathcal{P}}_2^{1/2}}(\tilde{\mathcal{P}}_2^{1/2}-\tilde{\mathcal{P}}_1^{1/2})\right|^2\mathbb{1}_{\tilde{\mathcal{P}}_1\leqslant\tilde{\mathcal{P}}_2}(t,\eta)\,d\eta\\&\leqslant\mathcal{O}(1)\|\tilde{\mathcal{P}}_2^{1/2}-\tilde{\mathcal{P}}_1^{1/2}\|^2,\end{aligned}$$

using that

$$\left|\frac{\tilde{\mathcal{Q}}_1\tilde{\mathcal{U}}_1}{\tilde{\mathcal{P}}_1^{1/2}\tilde{\mathcal{P}}_2^{1/2}}\right|\mathbb{1}_{\tilde{\mathcal{P}}_1\leqslant\tilde{\mathcal{P}}_2}\leqslant\left|\frac{\tilde{\mathcal{Q}}_1\tilde{\mathcal{U}}_1}{\tilde{\mathcal{P}}_1}\right|\leqslant|A_1\tilde{\mathcal{U}}_1|\leqslant\frac{A^3}{\sqrt{2}}.$$

This proves that

$$\begin{aligned}&\frac{1}{A^4}\left\|\frac{\tilde{\mathcal{Q}}_1\tilde{\mathcal{U}}_1}{\tilde{\mathcal{P}}_1^{1/2}}-\frac{\tilde{\mathcal{Q}}_2\tilde{\mathcal{U}}_2}{\tilde{\mathcal{P}}_2^{1/2}}\right\|^2\\&\quad\leqslant\mathcal{O}(1)(\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2+\|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2\|^2+\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+|A_1-A_2|^2),\end{aligned}$$

and thus I_3 has the right form.

The term I_4:

$$\begin{aligned}I_4&=\int_0^1(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})\left(\frac{1}{A_1^3}\frac{\tilde{\mathcal{R}}_1}{\tilde{\mathcal{P}}_1^{1/2}}-\frac{1}{A_2^3}\frac{\tilde{\mathcal{R}}_2}{\tilde{\mathcal{P}}_2^{1/2}}\right)(t,\eta)\,d\eta\\&=\mathbb{1}_{A_1\leqslant A_2}\left(\frac{1}{A_1^3}-\frac{1}{A_2^3}\right)\int_0^1(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})\frac{\tilde{\mathcal{R}}_1}{\tilde{\mathcal{P}}_1^{1/2}}(t,\eta)\,d\eta\\&\quad+\mathbb{1}_{A_2<A_1}\left(\frac{1}{A_1^3}-\frac{1}{A_2^3}\right)\int_0^1(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})\frac{\tilde{\mathcal{R}}_2}{\tilde{\mathcal{P}}_2^{1/2}}(t,\eta)\,d\eta\\&\quad+\frac{1}{A^3}\int_0^1(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})\bigg(\frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}(\tilde{\mathcal{R}}_1-\tilde{\mathcal{R}}_2)\\&\quad-\left(\frac{1}{\tilde{\mathcal{P}}_2^{1/2}}-\frac{1}{\tilde{\mathcal{P}}_1^{1/2}}\right)(\tilde{\mathcal{R}}_2\mathbb{1}_{\tilde{\mathcal{P}}_2<\tilde{\mathcal{P}}_1}+\tilde{\mathcal{R}}_1\mathbb{1}_{\tilde{\mathcal{P}}_1\leqslant\tilde{\mathcal{P}}_2})\bigg)(t,\eta)\,d\eta\\&=\mathbb{1}_{A_1\leqslant A_2}\left(\frac{1}{A_1^3}-\frac{1}{A_2^3}\right)\int_0^1(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})\frac{\tilde{\mathcal{R}}_1}{\tilde{\mathcal{P}}_1^{1/2}}(t,\eta)\,d\eta\end{aligned}$$

$$
\begin{aligned}
&+ \mathbb{1}_{A_2<A_1}\left(\frac{1}{A_1^3}-\frac{1}{A_2^3}\right)\int_0^1 (\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})\frac{\tilde{\mathcal{R}}_2}{\tilde{\mathcal{P}}_2^{1/2}}(t,\eta)\,d\eta\\
&+ \frac{1}{A^3}\int_0^1\Bigg[\frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{R}}_1-\tilde{\mathcal{R}}_2)-(\tilde{\mathcal{P}}_1^{1/2}\\
&-\tilde{\mathcal{P}}_2^{1/2})^2\frac{1}{\tilde{\mathcal{P}}_1^{1/2}\tilde{\mathcal{P}}_2^{1/2}}(\tilde{\mathcal{R}}_2\mathbb{1}_{\tilde{\mathcal{P}}_2<\tilde{\mathcal{P}}_1}+\tilde{\mathcal{R}}_1\mathbb{1}_{\tilde{\mathcal{P}}_1\leqslant\tilde{\mathcal{P}}_2})\Bigg](t,\eta)\,d\eta\\
={}& J_1+J_2+J_3-J_4-J_5,
\end{aligned}
$$

where

$$
\begin{aligned}
J_1 &= \mathbb{1}_{A_1\leqslant A_2}\left(\frac{1}{A_1^3}-\frac{1}{A_2^3}\right)\int_0^1 (\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})\frac{\tilde{\mathcal{R}}_1}{\tilde{\mathcal{P}}_1^{1/2}}(t,\eta)\,d\eta,\\
J_2 &= \mathbb{1}_{A_2<A_1}\left(\frac{1}{A_1^3}-\frac{1}{A_2^3}\right)\int_0^1 (\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})\frac{\tilde{\mathcal{R}}_2}{\tilde{\mathcal{P}}_2^{1/2}}(t,\eta)\,d\eta,\\
J_3 &= \frac{1}{A^3}\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{R}}_1-\tilde{\mathcal{R}}_2)(t,\eta)\,d\eta,\\
J_4 &= \frac{1}{A^3}\int_0^1 (\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})^2\frac{1}{\tilde{\mathcal{P}}_1^{1/2}\tilde{\mathcal{P}}_2^{1/2}}\tilde{\mathcal{R}}_2\mathbb{1}_{\tilde{\mathcal{P}}_2<\tilde{\mathcal{P}}_1}(t,\eta)\,d\eta,\\
J_5 &= \frac{1}{A^3}\int_0^1 (\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})^2\frac{1}{\tilde{\mathcal{P}}_1^{1/2}\tilde{\mathcal{P}}_2^{1/2}}\tilde{\mathcal{R}}_1\mathbb{1}_{\tilde{\mathcal{P}}_1\leqslant\tilde{\mathcal{P}}_2}(t,\eta)\,d\eta.
\end{aligned}
$$

For J_1 (and similarly for J_2), we find

$$
\begin{aligned}
|J_1| &\leqslant \frac{A_2^3-A_1^3}{a^3A^3}\int_0^1 |\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}|(t,\eta)\mathcal{O}(1)a^5\,d\eta\\
&\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+|A_1-A_2|^2),
\end{aligned}
$$

where we used that

$$
\left|\frac{\tilde{\mathcal{R}}_i}{\tilde{\mathcal{P}}_i^{1/2}}\right| \leqslant \mathcal{O}(1)A_i^3\tilde{\mathcal{P}}_i^{1/2}\leqslant \mathcal{O}(1)A_i^5.
$$

For J_3, we find

$$
\begin{aligned}
J_3 &= \frac{1}{A^3}\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{R}}_1-\tilde{\mathcal{R}}_2)(t,\eta)\,d\eta\\
&= \frac{1}{6A^3}\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})(t,\eta)\bigg(\int_0^1 \operatorname{sign}(\eta-\theta)
\end{aligned}
$$

$$\times \left(e^{-\frac{1}{A_1}|\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta)|}A_1\tilde{\mathcal{U}}_1^3\tilde{\mathcal{Y}}_{1,\eta} - e^{-\frac{1}{A_2}|\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta)|}A_2\tilde{\mathcal{U}}_2^3\tilde{\mathcal{Y}}_{2,\eta}\right)(t,\theta)\,d\theta\Big)\,d\eta$$

$$+ \frac{1}{4A^3}\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}(\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t,\eta)$$

$$\times \Big(\int_0^1 \operatorname{sign}(\eta-\theta)(e^{-\frac{1}{A_1}|\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta)|}A_1^6\tilde{\mathcal{U}}_1$$

$$- e^{-\frac{1}{A_2}|\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta)|}A_2^6\tilde{\mathcal{U}}_2)(t,\theta)\,d\theta\Big)\,d\eta$$

$$+ \frac{1}{2A^3}\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}(\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t,\eta)$$

$$\times \Big(\int_0^1 (e^{-\frac{1}{A_2}|\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta)|}\tilde{\mathcal{U}}_2\tilde{\mathcal{Q}}_2\tilde{\mathcal{Y}}_{2,\eta}$$

$$- e^{-\frac{1}{A_1}|\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta)|}\tilde{\mathcal{U}}_1\tilde{\mathcal{Q}}_1\tilde{\mathcal{Y}}_{1,\eta})(t,\theta)\,d\theta\Big)\,d\eta$$

$$= J_{11} + J_{12} + J_{13}.$$

Next we replace all the inner integrals $\int_0^1 \cdots d\theta$ by $(\int_0^\eta + \int_\eta^1)\cdots d\theta$ (cf. (5.4)), consider only the terms with $\int_0^\eta \cdots d\theta$, and call the corresponding quantities $\tilde{J}_{11}$, $\tilde{J}_{12}$, $\tilde{J}_{13}$. Thus

$$\tilde{J}_{11} = \frac{1}{6A^3}\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}(\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t,\eta)\Big(\int_0^\eta (e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}A_1\tilde{\mathcal{U}}_1^3\tilde{\mathcal{Y}}_{1,\eta}$$
$$- e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}A_2\tilde{\mathcal{U}}_2^3\tilde{\mathcal{Y}}_{2,\eta})(t,\theta)\,d\theta\Big)\,d\eta, \tag{5.44}$$

$$\tilde{J}_{12} = \frac{1}{4A^3}\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}(\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t,\eta)\Big(\int_0^\eta (e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}A_1^6\tilde{\mathcal{U}}_1$$
$$- e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}A_2^6\tilde{\mathcal{U}}_2)(t,\theta)\,d\theta\Big)\,d\eta, \tag{5.45}$$

$$\tilde{J}_{13} = \frac{1}{2A^3}\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}(\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t,\eta)\Big(\int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2\tilde{\mathcal{Q}}_2\tilde{\mathcal{Y}}_{2,\eta}$$
$$- e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\tilde{\mathcal{U}}_1\tilde{\mathcal{Q}}_1\tilde{\mathcal{Y}}_{1,\eta})(t,\theta)\,d\theta\Big)\,d\eta. \tag{5.46}$$

For the term $\tilde{J}_{11}$ we write $\tilde{\mathcal{U}}_j = \tilde{\mathcal{U}}_j^+ + \tilde{\mathcal{U}}_j^-$, collect terms, and study the positive part and the negative part separately.

With a slight abuse of notation, we need to consider the term

$$\begin{aligned}
\tilde{J}_{11} &= \frac{1}{6A^3}\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})(t,\eta) \\
&\quad\times\bigg(\int_0^\eta (e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}A_1(\tilde{\mathcal{U}}_1^+)^3\tilde{\mathcal{Y}}_{1,\eta} \\
&\quad - e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}A_2(\tilde{\mathcal{U}}_2^+)^3\tilde{\mathcal{Y}}_{2,\eta})(t,\theta)\,d\theta\bigg)\,d\eta \\
&= \frac{1}{6A^3}\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})(t,\eta) \\
&\quad\times\bigg(\mathbb{1}_{A_2<A_1}(A_1-A_2)\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}(\tilde{\mathcal{U}}_1^+)^3\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta \\
&\quad + \mathbb{1}_{A_1\leqslant A_2}(A_1-A_2)\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{U}}_2^+)^3\tilde{\mathcal{Y}}_{2,\eta}(t,\theta),\,d\theta \\
&\quad + a\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}((\tilde{\mathcal{U}}_1^+)^3-(\tilde{\mathcal{U}}_2^+)^3)\mathbb{1}_{\tilde{\mathcal{U}}_2^+<\tilde{\mathcal{U}}_1^+}\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta \\
&\quad + a\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}((\tilde{\mathcal{U}}_1^+)^3-(\tilde{\mathcal{U}}_2^+)^3)\mathbb{1}_{\tilde{\mathcal{U}}_1^+\leqslant\tilde{\mathcal{U}}_2^+}\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta \\
&\quad - a\mathbb{1}_{A_2<A_1}\int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}-e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}) \\
&\quad\times\min_j((\tilde{\mathcal{U}}_j^+)^3)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta \\
&\quad - a\mathbb{1}_{A_1\leqslant A_2}\int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}-e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}) \\
&\quad\times\min_j((\tilde{\mathcal{U}}_j^+)^3)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta \\
&\quad - a\int_0^\eta (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}-e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}) \\
&\quad\times\min_j((\tilde{\mathcal{U}}_j^+)^3)\tilde{\mathcal{Y}}_{2,\eta}\mathbb{1}_{B(\eta)}(t,\theta)\,d\theta \\
&\quad - a\int_0^\eta (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}-e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}) \\
&\quad\times\min_j\big((\tilde{\mathcal{U}}_j^+)^3\big)\tilde{\mathcal{Y}}_{1,\eta}\mathbb{1}_{B(\eta)^c}(t,\theta)\,d\theta
\end{aligned}$$

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$$
\begin{aligned}
&- a\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j((\tilde{\mathcal{U}}_j^+)^3)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\\
&+ a\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j((\tilde{\mathcal{U}}_j^+)^3)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\bigg)\,d\eta\\
&= K_1+K_2+K_3+K_4+K_5+K_6+K_7+K_8+K_9+K_{10}.
\end{aligned}
$$

It never stops, but we need to study the terms in groups.

The term K_1 can be estimated as follows (K_2 is similar):

$$
\begin{aligned}
|K_1| &= \frac{1}{6A^3}\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}|(t,\eta)\\
&\quad\times\left|\mathbb{1}_{A_2<A_1}(A_1-A_2)\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}(\tilde{\mathcal{U}}_1^+)^3\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\right|d\eta\\
&\leqslant \frac{\sqrt{2}}{3}\int_0^1 \frac{\tilde{\mathcal{P}}_1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}|(t,\eta)\,d\eta|A_1-A_2|\\
&\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+|A_1-A_2|^2).
\end{aligned}
$$

The term K_3 can be estimated as follows (K_4 is similar):

$$
\begin{aligned}
|K_3| &= \frac{a}{6A^3}\bigg|\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})(t,\eta)\\
&\quad\times\left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}((\tilde{\mathcal{U}}_1^+)^3-(\tilde{\mathcal{U}}_2^+)^3)\mathbb{1}_{\tilde{\mathcal{U}}_2^+<\tilde{\mathcal{U}}_1^+}\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\right)d\eta\bigg|\\
&\leqslant \frac{1}{2A^2}\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}|(t,\eta)\\
&\quad\times\left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}(\tilde{\mathcal{U}}_1^+)^2|\tilde{\mathcal{U}}_1^+-\tilde{\mathcal{U}}_2^+|\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\right)d\eta\\
&\leqslant \frac{1}{2}\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+\frac{1}{2A^4}\int_0^1\frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})^2}\\
&\quad\times\left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}(\tilde{\mathcal{U}}_1^+)^2|\tilde{\mathcal{U}}_1^+-\tilde{\mathcal{U}}_2^+|\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\right)^2d\eta\\
&\leqslant \frac{1}{2}\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2\\
&\quad+\frac{1}{2A^4}\int_0^1\frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})^2}\left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}(\tilde{\mathcal{U}}_1^+)^2\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\right)
\end{aligned}
$$

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$$\times \left(\int_0^{\eta} e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}|\tilde{\mathcal{U}}_1^+ - \tilde{\mathcal{U}}_2^+|^2(\tilde{\mathcal{U}}_1^+)^2\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\right)d\eta$$
$$\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2),$$

using (4.15g), (4.16c), and (5.3).

As for K_5 (K_6 is similar), we traverse the following path:

$$|K_5| = \frac{1}{6A^3}\left|\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}(\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t,\eta)\right.$$
$$\times \left(-a\mathbb{1}_{A_2<A_1}\int_0^{\eta}(e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))})\right.$$
$$\left.\left.\times \min_j((\tilde{\mathcal{U}}_j^+)^3)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\right)d\eta\right|$$
$$\leqslant \frac{2}{3A^3e}\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t,\eta)$$
$$\times \left(\int_0^{\eta} e^{-\frac{3}{4A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\min_j((\tilde{\mathcal{U}}_j^+)^3)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\right)d\eta|A_1 - A_2|$$
$$\leqslant \frac{2\sqrt{2}A^2}{3}\int_0^1 \frac{\tilde{\mathcal{P}}_1^{1/2}}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t,\eta)\,d\eta|A_1 - A_2|$$
$$\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + |A_1 - A_2|^2).$$

As for K_7 (K_8 is similar), we traverse the following path:

$$|K_7| = \frac{a}{6A^3}\left|\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}(\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t,\eta)\right.$$
$$\times \left(\int_0^{\eta}(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))})\right.$$
$$\left.\left.\times \min_j((\tilde{\mathcal{U}}_j^+)^3)\tilde{\mathcal{Y}}_{2,\eta}\mathbb{1}_{B(\eta)}(t,\theta)\,d\theta\right)d\eta\right|$$
$$\leqslant \frac{1}{6A^3}\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t,\eta)\left(\int_0^{\eta} e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\right.$$
$$\left.\times (|\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_1(t,\eta)| + |\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_1(t,\theta)|)(\tilde{\mathcal{U}}_2^+)^3\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)d\eta$$
$$\leqslant \frac{1}{6A^3}\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}||\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1|(t,\eta)$$
$$\times \left(\int_0^{\eta} e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{U}}_2^+)^3\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)d\eta$$

$$
\begin{aligned}
&+\frac{1}{6A^3}\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}|(t,\eta)\\
&\times\left(\int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1|(\tilde{\mathcal{U}}_2^+)^3\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)d\eta\\
\leqslant{}&\frac{\sqrt{2}}{3}\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}||\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1|\tilde{\mathcal{P}}_2(t,\eta)\,d\eta\\
&+\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2\\
&+\frac{1}{36A^6}\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})^2}(t,\eta)\\
&\times\left(\int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1|(\tilde{\mathcal{U}}_2^+)^3\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)^2 d\eta\\
\leqslant{}&\mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+\|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1\|^2)\\
&+\frac{1}{36A^6}\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})^2}(t,\eta)\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{U}}_2^+)^2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)\\
&\times\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1|^2(\tilde{\mathcal{U}}_2^+)^4\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)d\eta\\
\leqslant{}&\mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+\|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1\|^2)\\
&+\frac{1}{9A^5}\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})^2}\tilde{\mathcal{P}}_2(t,\eta)\\
&\times\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1|^2(\tilde{\mathcal{U}}_2^+)^4\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)d\eta\\
\leqslant{}&\mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+\|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1\|^2).
\end{aligned}
$$

Here we have applied (4.15b), (4.15g), (4.16c), and (5.13). Lo and behold, we can do K_9 and K_{10} in one sweep:

$$
\begin{aligned}
K_9+K_{10}={}&-\frac{a}{6A^3}\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})(t,\eta)\\
&\times\left(\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j((\tilde{\mathcal{U}}_j^+)^3)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right.\\
&\left.-\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j((\tilde{\mathcal{U}}_j^+)^3)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\right)d\eta\\
={}&-\frac{a}{6A^3}\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})(t,\eta)
\end{aligned}
$$

$$
\begin{aligned}
&\times \left(\int_0^{\eta} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}) \min_j ((\tilde{\mathcal{U}}_j^+)^3)(\tilde{\mathcal{Y}}_{2,\eta} - \tilde{\mathcal{Y}}_{1,\eta})(t,\theta)\, d\theta \right) d\eta \\
&= -\frac{a}{6A^3} \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t,\eta) \\
&\quad \times \left[\left(\min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}) \min_j ((\tilde{\mathcal{U}}_j^+)^3)(\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1) \right)(t,\theta) \Big|_{\theta=0}^{\eta} \right. \\
&\quad \left. - \int_0^{\eta} (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1) \frac{d}{d\theta} \left(\min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}) \min_j ((\tilde{\mathcal{U}}_j^+)^3) \right)(t,\theta)\, d\theta \right] d\eta \\
&= -\frac{a}{6A^3} \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1) \min_j ((\tilde{\mathcal{U}}_j^+)^3)(t,\eta)\, d\eta \\
&\quad + \frac{a}{6A^3} \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t,\eta) \\
&\quad \times \left(\int_0^{\eta} (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1) \min_j ((\tilde{\mathcal{U}}_j^+)^3) \frac{d}{d\theta} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})(t,\theta)\, d\theta \right) d\eta \\
&\quad + \frac{a}{6A^3} \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t,\eta) \\
&\quad \times \left(\int_0^{\eta} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})(\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1) \frac{d}{d\theta} \min_j ((\tilde{\mathcal{U}}_j^+)^3)(t,\theta)\, d\theta \right) d\eta \\
&= L_1 + L_2 + L_3,
\end{aligned}
$$

where

$$
\begin{aligned}
L_1 &= -\frac{a}{6A^3} \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1) \min_j ((\tilde{\mathcal{U}}_j^+)^3)(t,\eta)\, d\eta, \\
L_2 &= \frac{a}{6A^3} \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t,\eta) \\
&\quad \times \left(\int_0^{\eta} (\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1) \min_j ((\tilde{\mathcal{U}}_j^+)^3) \frac{d}{d\theta} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})(t,\theta)\, d\theta \right) d\eta, \\
L_3 &= \frac{a}{6A^3} \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t,\eta) \\
&\quad \times \left(\int_0^{\eta} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})(\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1) \frac{d}{d\theta} \min_j ((\tilde{\mathcal{U}}_j^+)^3)(t,\theta)\, d\theta \right) d\eta.
\end{aligned}
$$

We easily find

$$
\begin{aligned}
|L_1| &\leqslant \frac{a}{6A^3}\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}||\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1|\min_j((\tilde{\mathcal{U}}_j^+)^3)(t,\eta)\,d\eta\\
&\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+\|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1\|^2),
\end{aligned}
$$

using (4.15b) and (4.15d). Furthermore, applying Lemma A.2,

$$
\begin{aligned}
|L_2| &\leqslant \frac{1}{6A^3}\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}|(t,\eta)\\
&\quad\times\bigg(\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1|\\
&\quad\times\min_j((\tilde{\mathcal{U}}_j^+)^3)\max_j(\tilde{\mathcal{Y}}_{j,\eta})(t,\theta)\,d\theta\bigg)\,d\eta\\
&\leqslant \frac{1}{6A^3}\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}|(t,\eta)\\
&\quad\times\bigg(\int_0^\eta \min_j(e^{-\frac{1}{A_j}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}\tilde{\mathcal{U}}_j^2)(t,\theta)\,d\theta\bigg)^{\frac{1}{2}}\\
&\quad\times\bigg(\int_0^\eta |\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1|^2\max_j(\tilde{\mathcal{U}}_j^4\tilde{\mathcal{Y}}_{j,\eta})^2(t,\theta)\,d\theta\bigg)^{\frac{1}{2}}\,d\eta\\
&\leqslant \mathcal{O}(1)\int_0^1 |\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}|(t,\eta)\,d\eta\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|\\
&\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+\|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1\|^2),
\end{aligned}
$$

using (4.15g), (4.16a), and

$$
\min_j((\tilde{\mathcal{U}}_j^+)^3)\max_j(\tilde{\mathcal{Y}}_{j,\eta})\leqslant \min_j(\tilde{\mathcal{U}}_j^+)\max_j(\tilde{\mathcal{U}}_j^2\tilde{\mathcal{Y}}_{j,\eta})\leqslant \frac{A^7}{\sqrt{2}}.
$$

Another application of Lemma A.2 yields

$$
\begin{aligned}
|L_3| &\leqslant \frac{a}{6A^3}\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}|(t,\eta)\\
&\quad\times\bigg(\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1|\left|\frac{d}{d\theta}\min_j((\tilde{\mathcal{U}}_j^+)^3)\right|(t,\theta)\,d\theta\bigg)\,d\eta\\
&\leqslant \frac{A^2}{3}\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}|(t,\eta)
\end{aligned}
$$

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$$
\begin{aligned}
&\times \left(\int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1| \min_j(\tilde{\mathcal{U}}_j^+)(t,\theta)\,d\theta\right)d\eta\\
&\leqslant \frac{A^2}{3}\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}|(t,\eta)\\
&\quad\times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^2(t,\theta)\,d\theta\right)^{1/2}\\
&\quad\times \left(\int_0^\eta \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1|^2(t,\theta)\,d\theta\right)^{1/2}d\eta\\
&\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+\|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1\|^2),
\end{aligned}
$$

using (4.16a). We find

$$
\begin{aligned}
|K_9+K_{10}| &\leqslant |L_1|+|L_2|+|L_3|\\
&\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+\|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1\|^2).
\end{aligned}
$$

Thus we can sum up the estimates for $\tilde{J}_{11}$, and we find

$$
|\tilde{J}_{11}| \leqslant \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+\|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2\|^2+\|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1\|^2+|A_1-A_2|^2).
$$

We now proceed to the next term from (5.45):

$$
\begin{aligned}
\tilde{J}_{12} &= \frac{1}{4A^3}\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})(t,\eta)\\
&\quad\times \left(\int_0^\eta (e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}A_1^6\tilde{\mathcal{U}}_1 - e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}A_2^6\tilde{\mathcal{U}}_2)(t,\theta)\,d\theta\right)d\eta\\
&= \frac{1}{4A^3}\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})(t,\eta)\\
&\quad\times \Bigg((A_1^6-A_2^6)\mathbb{1}_{A_1<A_2}\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2(t,\theta)\,d\theta\\
&\quad+(A_1^6-A_2^2)\mathbb{1}_{A_2\leqslant A_1}\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\tilde{\mathcal{U}}_1(t,\theta)\,d\theta\\
&\quad+a^6\int_0^\eta (e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\tilde{\mathcal{U}}_1 - e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2)(t,\theta)\,d\theta\Bigg)d\eta\\
&= M_1+M_2+M_3.
\end{aligned}
$$

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The term M_1 (M_2 is similar) can be estimated as follows:

$$
\begin{aligned}
|M_1| &\leqslant \frac{1}{4A^3}|A_1^6 - A_2^6|\mathbb{1}_{A_1<A_2}\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t,\eta)\\
&\quad\times\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}|\tilde{\mathcal{U}}_2|(t,\theta)\,d\theta\right)d\eta\\
&\leqslant \frac{3A^2}{2}|A_1 - A_2|\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t,\eta)\\
&\quad\times\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^2(t,\theta)\,d\theta\right)^{1/2}\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\,d\theta\right)^{1/2}d\eta\\
&\leqslant \frac{3\sqrt{3}A^2}{\sqrt{2}}|A_1 - A_2|\int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t,\eta)\,d\eta\\
&\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + |A_1 - A_2|^2),
\end{aligned}
$$

using (4.16a). As for the next term M_3, we first write $\tilde{\mathcal{U}}_j = \tilde{\mathcal{U}}_j^+ + \tilde{\mathcal{U}}_j^-$, collect terms, and study the positive part and the negative part separately. In the interest of the reader, we do not change the notation. Thus

$$
\begin{aligned}
M_3 &= \frac{a^6}{4A^3}\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}(\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t,\eta)\\
&\quad\times\int_0^\eta (e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\tilde{\mathcal{U}}_1^+ - e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^+)(t,\theta)\,d\theta\,d\eta\\
&= \frac{a^6}{4A^3}\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}(\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t,\eta)\\
&\quad\times\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\mathbb{1}_{\tilde{\mathcal{U}}_1^+<\tilde{\mathcal{U}}_2^+}(\tilde{\mathcal{U}}_1^+ - \tilde{\mathcal{U}}_2^+))(t,\theta)\,d\theta\right.\\
&\quad+\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}(\mathbb{1}_{\tilde{\mathcal{U}}_2^+\leqslant\tilde{\mathcal{U}}_1^+}(\tilde{\mathcal{U}}_1^+ - \tilde{\mathcal{U}}_2^+))(t,\theta)\,d\theta\\
&\quad\left.+\int_0^\eta (e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))} - e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))})\min_j(\tilde{\mathcal{U}}_j^+)(t,\theta)\,d\theta\right)d\eta\\
&= M_{31} + M_{32} + M_{33}.
\end{aligned}
$$

The terms M_{31} and M_{32} can be treated similarly. Thus

$$
\begin{aligned}
|M_{31}| &= \frac{a^6}{4A^3}\left|\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}(\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t,\eta)\right.\\
&\quad\left.\times\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\mathbb{1}_{\tilde{\mathcal{U}}_1^+<\tilde{\mathcal{U}}_2^+}(\tilde{\mathcal{U}}_1^+ - \tilde{\mathcal{U}}_2^+))(t,\theta)\,d\theta\right)d\eta\right|
\end{aligned}
$$

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$$\leqslant \frac{1}{4A^2}\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}|(t,\eta)$$
$$\times\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}A_2^5|\tilde{\mathcal{U}}_1^+-\tilde{\mathcal{U}}_2^+|(t,\theta)\,d\theta\right)d\eta$$
$$\leqslant \mathcal{O}(1)\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}|\tilde{\mathcal{P}}_2^{1/2}(t,\eta)\|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2\|\,d\eta$$
$$\leqslant \mathcal{O}(1)\int_0^1 |\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}|(t,\eta)\|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2\|\,d\eta$$
$$\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+\|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2\|^2),$$

using

$$a^5\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}|\tilde{\mathcal{U}}_1^+-\tilde{\mathcal{U}}_2^+|(t,\theta)\,d\theta$$
$$\leqslant \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}A_2^5|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2|(t,\theta)\,d\theta$$
$$=\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}((2\tilde{\mathcal{P}}_2-\tilde{\mathcal{U}}_2^2)\tilde{\mathcal{Y}}_{2,\eta}+\tilde{\mathcal{H}}_{2,\eta})|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2|(t,\theta)\,d\theta$$
$$=2\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2|(t,\theta)\,d\theta$$
$$+\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{H}}_{2,\eta}|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2|(t,\theta)\,d\theta$$
$$-\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^2\tilde{\mathcal{Y}}_{2,\eta}|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2|(t,\theta)\,d\theta$$
$$\leqslant 2\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2|(t,\theta)\,d\theta$$
$$+\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{H}}_{2,\eta}|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2|(t,\theta)\,d\theta$$
$$\leqslant 2\left(\int_0^\eta e^{-\frac{3}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)^{1/2}$$
$$\times\left(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)^2\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)^{1/2}$$
$$+\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{H}}_{2,\eta}(t,\theta)\,d\theta\right)^{1/2}$$

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$$\times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2)^2 \tilde{\mathcal{H}}_{2,\eta}(t,\theta)\, d\theta \right)^{1/2}$$
$$\leqslant \mathcal{O}(1) A^3 \tilde{\mathcal{P}}_2^{1/2}(t,\eta) \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|,$$

using (4.13), (4.15e), (4.15k), (4.16d), and (4.16e). The term M_{33} goes as follows:

$$\begin{aligned}
M_{33} &= \frac{a^6}{4A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t,\eta) \\
&\quad \times \left(\int_0^\eta (e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))} - e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}) \right. \\
&\quad \left. \times \min_j(\tilde{\mathcal{U}}_j^+)(t,\theta)\, d\theta \right) d\eta \\
&= \frac{a^6}{4A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t,\eta) \\
&\quad \times \mathbb{1}_{A_1 \leqslant A_2} \left(\int_0^\eta (e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}) \right. \\
&\quad \left. \times \min_j(\tilde{\mathcal{U}}_j^+)(t,\theta)\, d\theta \right) d\eta \\
&\quad + \frac{a^6}{4A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t,\eta) \\
&\quad \times \mathbb{1}_{A_2 < A_1} \left(\int_0^\eta (e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))} - e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}) \right. \\
&\quad \left. \times \min_j(\tilde{\mathcal{U}}_j^+)(t,\theta)\, d\theta \right) d\eta \\
&\quad + \frac{a^6}{4A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t,\eta) \\
&\quad \times \left(\int_0^\eta (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}) \mathbb{1}_{B(\eta)} \right. \\
&\quad \left. \times \min_j(\tilde{\mathcal{U}}_j^+)(t,\theta)\, d\theta \right) d\eta \\
&\quad + \frac{a^6}{4A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t,\eta) \\
&\quad \times \left(\int_0^\eta (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}) \mathbb{1}_{B(\eta)^c} \right.
\end{aligned}$$

$$\times \min_j(\tilde{\mathcal{U}}_j^+)(t,\theta)\,d\theta\Big)\,d\eta$$
$$= M_{331} + M_{332} + M_{333} + M_{334},$$

where $B(\eta)$ is defined by (5.11). The terms M_{331} and M_{332} can be treated in the same manner. More specifically,

$$\begin{aligned}
|M_{331}| &\leqslant \frac{A^2}{e}\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t,\eta) \\
&\quad \times \left(\int_0^\eta e^{-\frac{3}{4A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^+(t,\theta)\,d\theta\right)d\eta|A_1 - A_2| \\
&\leqslant \frac{A^2}{e}\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t,\eta) \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^2(t,\theta)\,d\theta\right)^{1/2} d\eta|A_1 - A_2| \\
&\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + |A_1 - A_2|^2).
\end{aligned}$$

The terms M_{333} and M_{334} can be treated in a similar manner. More specifically,

$$\begin{aligned}
|M_{333}| &\leqslant \frac{a^6}{4A^3}\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t,\eta) \\
&\quad \times \left(\int_0^\eta |e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}|\mathbb{1}_{B(\eta)}\right. \\
&\quad \times \left.\min_j(\tilde{\mathcal{U}}_j^+)(t,\theta)\,d\theta\right)d\eta \\
&\leqslant \frac{A^2}{4}\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t,\eta)\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\right. \\
&\quad \times \left.(|\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_1(t,\eta)| + |\tilde{\mathcal{Y}}_2(t,\theta) - \tilde{\mathcal{Y}}_1(t,\theta)|)\min_j(\tilde{\mathcal{U}}_j^+)(t,\theta)\,d\theta\right)d\eta \\
&\leqslant \frac{A^2}{4}\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|\,|\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1|(t,\eta) \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\mathbb{1}_{B(\eta)}\min_j(\tilde{\mathcal{U}}_j^+)(t,\theta)\,d\theta\right)d\eta \\
&\quad + \frac{A^2}{4}\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t,\eta) \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}|\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1|\min_j(\tilde{\mathcal{U}}_j^+)(t,\theta)\,d\theta\right)d\eta
\end{aligned}$$

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$$
\begin{aligned}
&\leqslant \frac{A^2}{4}\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}|\,|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1|(t,\eta)\\
&\quad\times\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^2(t,\theta)\,d\theta\right)^{1/2}\\
&\quad\times\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\,d\theta\right)^{1/2}d\eta\\
&\quad+\frac{A^2}{4}\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}|(t,\eta)\\
&\quad\times\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^2(t,\theta)\,d\theta\right)^{1/2}\\
&\quad\times\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1)^2(t,\theta)\right)^{1/2}d\eta\\
&\leqslant \mathcal{O}(1)\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}|\,|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1|\tilde{\mathcal{P}}_2^{1/2}(t,\eta)\,d\eta\\
&\quad+\mathcal{O}(1)\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}|\,\|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1\|\tilde{\mathcal{P}}_2^{1/2}(t,\eta)\,d\eta\\
&\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+\|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1\|^2),
\end{aligned}
$$

using (4.16a) and (5.13).

And the final term from J_3 (cf. (5.46)) can be estimated as follows:

$$
\begin{aligned}
\tilde{J}_{13}&=\frac{1}{2A^3}\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})(t,\eta)\bigg(\int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2\tilde{\mathcal{Q}}_2\tilde{\mathcal{Y}}_{2,\eta}\\
&\quad-e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\tilde{\mathcal{U}}_1\tilde{\mathcal{Q}}_1\tilde{\mathcal{Y}}_{1,\eta})(t,\theta)\,d\theta\bigg)\,d\eta.
\end{aligned}
$$

We introduce the positive and negative parts of $\tilde{\mathcal{U}}_j$, that is, $\tilde{\mathcal{U}}_j=\tilde{\mathcal{U}}_j^+ + \tilde{\mathcal{U}}_j^-$ (see (5.1)) and introduce $\tilde{\mathcal{Q}}_j = A_j\tilde{\mathcal{P}}_j - \tilde{\mathcal{D}}_j$. We study the term with $\tilde{\mathcal{P}}_j$ first:

$$
\begin{aligned}
\tilde{J}_{131}&=\frac{1}{2A^3}\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})(t,\eta)\\
&\quad\times\bigg(\int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}A_2\tilde{\mathcal{U}}_2^+\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}\\
&\quad-e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}A_1\tilde{\mathcal{U}}_1^+\tilde{\mathcal{P}}_1\tilde{\mathcal{Y}}_{1,\eta})(t,\theta)\,d\theta\bigg)\,d\eta
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2A^3}\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})(t,\eta)\\
&\quad\times\Bigg[\mathbb{1}_{A_1\leqslant A_2}(A_2-A_1)\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^+\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\\
&\quad+\mathbb{1}_{A_2<A_1}(A_2-A_1)\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\tilde{\mathcal{U}}_1^+\tilde{\mathcal{P}}_1\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\\
&\quad+a\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{P}}_2-\tilde{\mathcal{P}}_1)\mathbb{1}_{\tilde{\mathcal{P}}_1<\tilde{\mathcal{P}}_2}\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\\
&\quad+a\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}(\tilde{\mathcal{P}}_2-\tilde{\mathcal{P}}_1)\mathbb{1}_{\tilde{\mathcal{P}}_2\leqslant\tilde{\mathcal{P}}_1}\tilde{\mathcal{U}}_1^+\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\\
&\quad+a\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\min_j(\tilde{\mathcal{P}}_j)(\tilde{\mathcal{U}}_2^+-\tilde{\mathcal{U}}_1^+)\mathbb{1}_{\tilde{\mathcal{U}}_1^+<\tilde{\mathcal{U}}_2^+}\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\\
&\quad+a\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\min_j(\tilde{\mathcal{P}}_j)(\tilde{\mathcal{U}}_2^+-\tilde{\mathcal{U}}_1^+)\mathbb{1}_{\tilde{\mathcal{U}}_2^+\leqslant\tilde{\mathcal{U}}_1^+}\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\\
&\quad+a\mathbb{1}_{A_1\leqslant A_2}\int_0^\eta(e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}-e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))})\\
&\quad\times\min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\\
&\qquad+a\mathbb{1}_{A_2<A_1}\int_0^\eta(e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}-e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))})\\
&\quad\times\min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\\
&\quad+a\int_0^\eta(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}-e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))})\mathbb{1}_{B(\eta)}\\
&\quad\times\min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\\
&\quad+a\int_0^\eta(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}-e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))})\mathbb{1}_{B(\eta)^c}\\
&\quad\times\min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\\
&\quad+a\int_0^\eta\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\\
&\quad\times\min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)(\tilde{\mathcal{Y}}_{2,\eta}-\tilde{\mathcal{Y}}_{1,\eta})(t,\theta)\,d\theta\Bigg]\,d\eta\\
&= K_1+K_2+K_3+K_4+K_5+K_6+K_7+K_8+K_9+K_{10}+K_{11}.
\end{aligned}
$$

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The terms K_1 and K_2 can be treated similarly. Thus

$$\begin{aligned}
|K_1| &= \frac{1}{2A^3}\bigg|\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})(t,\eta)\\
&\quad\times\left(\mathbb{1}_{A_1\leqslant A_2}(A_2-A_1)\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^+\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)d\eta\bigg|\\
&\leqslant \frac{1}{2A^3}\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}|(t,\eta)\\
&\quad\times\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2^2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)^{1/2}\\
&\quad\times\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)^{1/2}d\eta|A_1-A_2|\\
&\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+|A_1-A_2|^2).
\end{aligned}$$

The terms K_3 and K_4 can be treated similarly. Thus

$$\begin{aligned}
|K_3| &= \frac{a}{2A^3}\bigg|\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})(t,\eta)\\
&\quad\times\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{P}}_1-\tilde{\mathcal{P}}_2)\mathbb{1}_{\tilde{\mathcal{P}}_1<\tilde{\mathcal{P}}_2}\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)d\eta\bigg|\\
&\leqslant \frac{1}{A^2}\bigg|\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}|(t,\eta)\\
&\quad\times\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}|\tilde{\mathcal{P}}_2^{1/2}\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)d\eta\bigg|\\
&\leqslant \frac{1}{A^2}\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}|(t,\eta)\\
&\quad\times\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})^2(t,\theta)\,d\theta\right)^{1/2}\\
&\quad\times\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2\tilde{\mathcal{U}}_2^2\tilde{\mathcal{Y}}_{2,\eta}^2(t,\theta)\,d\theta\right)^{1/2}d\eta\\
&\leqslant \mathcal{O}(1)\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}|\tilde{\mathcal{P}}_2^{1/2}(t,\eta)
\end{aligned}$$

$$\times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})^2(t,\theta)\,d\theta\right)^{1/2} d\eta$$
$$\leqslant \mathcal{O}(1)\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2,$$

using (4.15e) and (4.16c). As for K_5 and K_6, we find

$$\begin{aligned}
|K_5| &= \frac{a}{2A^3}\bigg|\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})(t,\eta)\\
&\quad+\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\min_j(\tilde{\mathcal{P}}_j)(\tilde{\mathcal{U}}_1^+-\tilde{\mathcal{U}}_2^+)\mathbb{1}_{\tilde{\mathcal{U}}_1^+<\tilde{\mathcal{U}}_2^+}\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)d\eta\bigg|\\
&\leqslant \frac{1}{2A^2}\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}|(t,\eta)\\
&\quad+\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\min_j(\tilde{\mathcal{P}}_j)|\tilde{\mathcal{U}}_1^+-\tilde{\mathcal{U}}_2^+|\mathbb{1}_{\tilde{\mathcal{U}}_1^+<\tilde{\mathcal{U}}_2^+}\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)d\eta\\
&\leqslant \|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+\frac{1}{4A^4}\int_0^1 \frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2}(t,\eta)\\
&\quad\times\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\min_j(\tilde{\mathcal{P}}_j)|\tilde{\mathcal{U}}_1^+-\tilde{\mathcal{U}}_2^+|\mathbb{1}_{\tilde{\mathcal{U}}_1^+<\tilde{\mathcal{U}}_2^+}\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)^2 d\eta\\
&\leqslant \|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2\\
&\quad+\frac{1}{4A^4}\int_0^1 \frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2}(t,\eta)\left(\int_0^\eta e^{-\frac{3}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)\\
&\quad\times\left(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(\tilde{\mathcal{U}}_1^+-\tilde{\mathcal{U}}_2^+)^2(t,\theta)\,d\theta\right)d\eta\\
&\leqslant \|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+\frac{1}{2A^3}\int_0^1 \frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2}\tilde{\mathcal{P}}_2(t,\eta)\\
&\quad\times\left(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(\tilde{\mathcal{U}}_1^+-\tilde{\mathcal{U}}_2^+)^2(t,\theta)\,d\theta\right)d\eta\\
&\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+\|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2\|^2),
\end{aligned}$$

using (4.15e) and (4.16e).

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The terms K_7 and K_8 allow for the following estimates:

$$
\begin{aligned}
|K_7| &= \frac{a}{2A^3}\bigg|\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})(t,\eta)\\
&\quad\times \mathbb{1}_{A_1\leqslant A_2}\int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}-e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))})\\
&\quad\times \min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\,d\eta\bigg|\\
&\leqslant \frac{2}{A^3e}\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}|(t,\eta)\\
&\quad\times \int_0^\eta e^{-\frac{3}{4A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2|\tilde{\mathcal{U}}_2|\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\,d\eta|A_1-A_2|\\
&\leqslant \frac{2}{A^3e}\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}|(t,\eta)\\
&\quad\times \bigg(\int_0^1 e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\bigg)^{1/2}\\
&\quad\times \bigg(\int_0^1 e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2^2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\bigg)^{1/2}d\eta|A_1-A_2|\\
&\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+|A_1-A_2|^2).
\end{aligned}
$$

The terms K_9 and K_{10} allow for the following estimates:

$$
\begin{aligned}
|K_9| &= \frac{a}{2A^3}\bigg|\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})(t,\eta)\\
&\quad+\int_0^\eta (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}-e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))})\mathbb{1}_{B(\eta)}\\
&\quad\times \min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\,d\eta\bigg|\\
&\leqslant \frac{1}{2A^3}\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}|(t,\eta)\\
&\quad\times \int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(|\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_1(t,\eta)|+|\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_1(t,\theta)|)\\
&\quad\times \min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\,d\eta
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant \frac{1}{2A^3}\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}||\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1|(t,\eta)\\
&\quad\times\left(\int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)d\eta\\
&\quad+\frac{1}{2A^3}\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}|(t,\eta)\\
&\quad\times\left(\int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1|\min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)d\eta\\
&\leqslant \frac{1}{2A^3}\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}||\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1|(t,\eta)\\
&\quad\times\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2^2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)^{1/2}\\
&\quad\times\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)^{1/2}d\eta\\
&\quad+\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+\frac{1}{4A^6}\int_0^1\frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2}(t,\eta)\\
&\quad\times\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1|\min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)^2d\eta\\
&\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+\|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1\|^2)+\frac{1}{4A^6}\int_0^1\frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2}(t,\eta)\\
&\quad\times\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)\\
&\quad\times\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1)^2\tilde{\mathcal{P}}_2^2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)d\eta\\
&\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+\|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1\|^2),
\end{aligned}
$$

using (4.15a), (4.15e), and (4.16c). The last term K_{11} receives special treatment

$$
\begin{aligned}
|K_{11}| &= \frac{a}{2A^3}\left|\int_0^1\frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})(t,\eta)\left(\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\right.\right.\\
&\qquad\left.\left.\times\min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)(\tilde{\mathcal{Y}}_{1,\eta}-\tilde{\mathcal{Y}}_{2,\eta})(t,\theta)\,d\theta\right)d\eta\right|\\
&= \frac{a}{2A^3}\left|\int_0^1\frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})(t,\eta)\right.
\end{aligned}
$$

$$
\begin{aligned}
&\times \bigg[\Big(\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)(t,\theta)\Big)\bigg|_{\theta=0}^{1}\\
&-\int_0^\eta(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)\\
&\times\frac{d}{d\theta}\Big(\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\Big)(t,\theta)\,d\theta\bigg]\,d\eta\bigg|\\
\leqslant{}& \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+\|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1\|^2)\\
&+\frac{a}{2A^3}\int_0^1\frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}|(t,\eta)\\
&\times\int_0^\eta|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|\bigg|\bigg(\frac{d}{d\theta}\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\bigg)\min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\\
&+\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\bigg(\frac{d}{d\theta}\min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\bigg)\bigg|(t,\theta)\,d\theta\,d\eta\\
\leqslant{}& \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2)\\
&+\frac{1}{2A^3}\int_0^1\frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}|(t,\eta)\\
&\times\int_0^\eta|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|\Big(\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\max_j(\tilde{\mathcal{Y}}_{j,\eta})\min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\\
&+2aA^4\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})(\min_j(\tilde{\mathcal{P}}_j^{1/2})+|\tilde{\mathcal{U}}_2|)\Big)(t,\theta)\,d\theta\,d\eta\\
\leqslant{}& \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2)\\
&+\frac{1}{2A^3}\int_0^1\frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}|(t,\eta)\\
&\times\bigg(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|\tilde{\mathcal{Y}}_{1,\eta}\tilde{\mathcal{P}}_1\tilde{\mathcal{U}}_1^+(t,\theta)\,d\theta\bigg)\,d\eta\\
&+\frac{1}{2A^3}\int_0^1\frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}|(t,\eta)\\
&\times\bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|\tilde{\mathcal{Y}}_{2,\eta}\tilde{\mathcal{P}}_2\tilde{\mathcal{U}}_2^+(t,\theta)\,d\theta\bigg)\,d\eta\\
&+A^2\int_0^1\frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}|(t,\eta)
\end{aligned}
$$

$$\times \left(\int_0^\eta |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \min_j (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))})(\tilde{\mathcal{P}}_2^{1/2} + |\tilde{\mathcal{U}}_2|)(t,\theta)\, d\theta \right) d\eta$$
$$= \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2) + K_{71} + K_{72} + K_{73},$$

using Lemmas A.2 and A.3, and

$$\min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}) \max_j (\tilde{\mathcal{Y}}_{j,\eta}) \min_j (\tilde{\mathcal{P}}_j) \min_j (\tilde{\mathcal{U}}_j^+)(t,\theta)$$
$$\leqslant e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{Y}}_{1,\eta} \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1^+(t,\theta) + e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{Y}}_{2,\eta} \tilde{\mathcal{P}}_2 \tilde{\mathcal{U}}_2^+(t,\theta).$$

Here K_{71} and K_{72} allow for the same treatment:

$$\begin{aligned}
2K_{71} &= \frac{1}{A^3} \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t,\eta) \\
&\quad \times \left(\int_0^\eta |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{Y}}_{1,\eta} \tilde{\mathcal{P}}_1 \tilde{\mathcal{U}}_1^+(t,\theta)\, d\theta \right) d\eta \\
&\leqslant \frac{1}{A^3} \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t,\eta) \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))} (\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)^2(t,\theta)\, d\theta \right)^{1/2} \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{Y}}_{1,\eta}^2 \tilde{\mathcal{P}}_1^2 (\tilde{\mathcal{U}}_1^+)^2(t,\theta)\, d\theta \right)^{1/2} d\eta \\
&\leqslant A^2\sqrt{2} \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}| \tilde{\mathcal{P}}_1^{1/2}(t,\eta) \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|\, d\eta \\
&\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2),
\end{aligned}$$

using (4.15g) and (4.16b). Furthermore,

$$\begin{aligned}
K_{73} &= A^2 \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t,\eta) \\
&\quad \times \left(\int_0^\eta |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \min_j (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))})(\tilde{\mathcal{P}}_2^{1/2} + |\tilde{\mathcal{U}}_2|)(t,\theta)\, d\theta \right) d\eta \\
&\leqslant \mathcal{O}(1) \int_0^1 \frac{1}{\max_j (\tilde{\mathcal{P}}_j^{1/2})} |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t,\eta) \\
&\quad \times \left(\int_0^\eta |\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2| \min_j (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}) \tilde{\mathcal{P}}_2^{1/2}(t,\theta)\, d\theta \right) d\eta
\end{aligned}$$

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$$\begin{aligned}
&\leqslant \mathcal{O}(1)\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+\int_0^1 \frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2}(t,\eta)\\
&\quad\times\bigg(\int_0^\eta |\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|\min_j(e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))})\tilde{\mathcal{P}}_2^{1/2}(t,\theta)\,d\theta\bigg)^2\,d\eta\\
&\leqslant \mathcal{O}(1)\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2\\
&\quad+\int_0^1 \frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2}(t,\eta)\bigg(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,\theta)\,d\theta\bigg)\\
&\quad\times\bigg(\int_0^\eta e^{-\frac{3}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2(t,\theta)\,d\theta\bigg)\,d\eta\\
&\leqslant \mathcal{O}(1)\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+\mathcal{O}(1)\int_0^1 \frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2}\tilde{\mathcal{P}}_2(t,\eta)\\
&\quad\times\bigg(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,\theta)\,d\theta\bigg)\,d\eta\\
&\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2),
\end{aligned}$$

using (4.15d) and (4.16i). This proves that

$$\tilde{J}_{131}\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+\|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2\|^2+\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2+|A_1-A_2|^2).$$

It remains to consider $\tilde{J}_{132}$:

$$\begin{aligned}
-\tilde{J}_{132}&=\frac{1}{2A^3}\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})(t,\eta)\\
&\qquad\times\bigg(\int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^+\tilde{\mathcal{D}}_2\tilde{\mathcal{Y}}_{2,\eta}\\
&\qquad-e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\tilde{\mathcal{U}}_1^+\tilde{\mathcal{D}}_1\tilde{\mathcal{Y}}_{1,\eta})(t,\theta)\,d\theta\bigg)\,d\eta\\
&=\frac{1}{2A^3}\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})(t,\eta)\\
&\qquad\times\bigg[\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}(\tilde{\mathcal{D}}_2-\tilde{\mathcal{D}}_1)\mathbb{1}_{\tilde{\mathcal{D}}_1<\tilde{\mathcal{D}}_2}(t,\theta)\,d\theta\\
&\qquad+\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\tilde{\mathcal{U}}_1^+\tilde{\mathcal{Y}}_{1,\eta}(\tilde{\mathcal{D}}_2-\tilde{\mathcal{D}}_1)\mathbb{1}_{\tilde{\mathcal{D}}_2\leqslant\tilde{\mathcal{D}}_1}(t,\theta)\,d\theta\\
&\qquad+\int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}-e^{-(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\tilde{\mathcal{U}}_1^+\tilde{\mathcal{Y}}_{1,\eta})
\end{aligned}$$

$$\times \min_j(\tilde{\mathcal{D}}_j)(t,\theta)\,d\theta\Big]\,d\eta$$
$$= \tilde{A}_1 + \tilde{A}_2 + \tilde{A}_3.$$

The terms $\tilde{A}_1$ and $\tilde{A}_2$ allow for the same treatment. Specifically, (see Lemma A.9)

$$\begin{aligned}
|\tilde{A}_1| &= \frac{1}{2A^3}\bigg|\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}(\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t,\eta)\\
&\quad \times \bigg[\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}(\tilde{\mathcal{D}}_2 - \tilde{\mathcal{D}}_1)\mathbb{1}_{\tilde{\mathcal{D}}_1<\tilde{\mathcal{D}}_2}(t,\theta)\,d\theta\bigg]\,d\eta\bigg|\\
&\leqslant \frac{1}{2A^3}\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t,\eta)\\
&\quad \times \bigg[\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}(\bar{d}_{11} + \bar{d}_{12})\mathbb{1}_{\tilde{\mathcal{D}}_1<\tilde{\mathcal{D}}_2}(t,\theta)\,d\theta\\
&\quad + \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}T_1\mathbb{1}_{\tilde{\mathcal{D}}_1<\tilde{\mathcal{D}}_2}(t,\theta)\,d\theta\\
&\quad + \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}T_2\mathbb{1}_{\tilde{\mathcal{D}}_1<\tilde{\mathcal{D}}_2}(t,\theta)\,d\theta\\
&\quad + \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}T_3\mathbb{1}_{\tilde{\mathcal{D}}_1<\tilde{\mathcal{D}}_2}(t,\theta)\,d\theta\bigg]\,d\eta\\
&= \frac{1}{2}(A_{11} + A_{12} + A_{13} + A_{14}).
\end{aligned}$$

Again we are forced to consider individual terms,

$$\begin{aligned}
A_{11} &= \frac{1}{A^3}\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t,\eta)\\
&\quad \times \bigg[\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}(\bar{d}_{11} + \bar{d}_{12})\mathbb{1}_{\tilde{\mathcal{D}}_1<\tilde{\mathcal{D}}_2}(t,\theta)\,d\theta\bigg]\,d\eta\\
&\leqslant \frac{1}{A^3}\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t,\eta)\\
&\quad \times \bigg[\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}(2A^{3/2}\tilde{\mathcal{D}}_2^{1/2}|\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1|\\
&\quad + 2\sqrt{2}A^{3/2}\tilde{\mathcal{D}}_2^{1/2}\|\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1\|)(t,\theta)\,d\theta\bigg]\,d\eta\\
&\leqslant \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \frac{4}{A^3}\int_0^1 \frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2}(t,\eta)
\end{aligned}$$

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$$
\begin{aligned}
&\times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}\tilde{\mathcal{D}}_2^{1/2}|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1|(t,\theta)\,d\theta\right)^2 d\eta \\
&+ \frac{2\sqrt{2}}{A^{3/2}}\|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1\|\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|\left(\int_0^1 \frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2}(t,\eta)\right. \\
&\left.\times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}\tilde{\mathcal{D}}_2^{1/2}(t,\theta)\,d\theta\right)^2 d\eta\right)^{1/2} \\
\leqslant\; & \|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2 \\
&+ \frac{4}{A^3}\int_0^1 \frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2}(t,\eta)\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right) \\
&\times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{D}}_2\tilde{\mathcal{Y}}_{2,\eta}|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1|^2(t,\theta)\,d\theta\right) d\eta \\
&+ \frac{2\sqrt{2}}{A^3}\|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1\|\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|\left[\int_0^1 \frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2}(t,\eta)\right. \\
&\times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right) \\
&\left.\times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}2A\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right) d\eta\right]^{1/2} \\
\leqslant\; & \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+\|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1\|^2),
\end{aligned}
$$

using (4.15e), (4.15n), and (4.16c). Next we find (see (A.14))

$$
\begin{aligned}
A_{12} &= \frac{1}{A^3}\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}|(t,\eta) \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}T_1\mathbb{1}_{\tilde{\mathcal{D}}_1<\tilde{\mathcal{D}}_2}(t,\theta)\,d\theta\right) d\eta \\
&\leqslant \frac{1}{A^3}\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}|(t,\eta) \\
&\quad \times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}|\tilde{\mathcal{U}}_2|^3\tilde{\mathcal{Y}}_{2,\eta}|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1|(t,\theta)\,d\theta\right) d\eta \\
&\quad + 4\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}|(t,\eta)\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\right. \\
&\quad \left.\times \left(\int_0^\theta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,l))}(\tilde{\mathcal{U}}_2-\tilde{\mathcal{U}}_1)^2(t,l)\,dl\right)^{1/2}\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right) d\eta
\end{aligned}
$$

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$$
\begin{aligned}
&+\sqrt{2}A\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}|(t,\eta)\bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\\
&\times\bigg(\int_0^\theta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,l))}(\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1)^2(t,l)\,dl\bigg)^{1/2}\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\bigg)\,d\eta\\
&+A\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}|(t,\eta)\bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\\
&\times\bigg(\int_0^\theta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,l))}|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1|(t,l)\,dl\bigg)\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\bigg)\,d\eta\\
&+\frac{8\sqrt{2}A}{\sqrt{3}e}\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}|(t,\eta)\bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\\
&\times\bigg(\int_0^\theta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,l))}\,dl\bigg)^{1/2}|A_1-A_2|\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\bigg)\,d\eta\\
&\leqslant 5\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+\frac{1}{A^6}\int_0^1\frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2}(t,\eta)\\
&\times\bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}|\tilde{\mathcal{U}}_2|^3\tilde{\mathcal{Y}}_{2,\eta}|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1|(t,\theta)\,d\theta\bigg)^2\,d\eta\\
&+16\int_0^1\frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2}(t,\eta)\bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\\
&\times\bigg(\int_0^\theta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,l))}(\tilde{\mathcal{U}}_2-\tilde{\mathcal{U}}_1)^2(t,l)\,dl\bigg)^{1/2}\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\bigg)^2\,d\eta\\
&+2A^2\int_0^1\frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2}(t,\eta)\bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\\
&\times\bigg(\int_0^\theta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,l))}(\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1)^2(t,l)\,dl\bigg)^{1/2}\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\bigg)^2\,d\eta\\
&+A^2\int_0^1\frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2}(t,\eta)\bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\\
&\times\bigg(\int_0^\theta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,l))}|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1|(t,l)\,dl\bigg)\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\bigg)^2\,d\eta\\
&+\frac{128A^2}{3e^2}\int_0^1\frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2}\bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\\
&\times\bigg(\int_0^\theta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,l))}\,dl\bigg)^{1/2}\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\bigg)^2\,d\eta|A_1-A_2|^2\\
&\leqslant 5\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2
\end{aligned}
$$

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$$
\begin{aligned}
&+ \frac{1}{A^6}\int_0^1 \frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2}(t,\eta)\Big(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\Big)\\
&\times\Big(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^4\tilde{\mathcal{Y}}_{2,\eta}(\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1)^2(t,\theta)\,d\theta\Big)\,d\eta\\
&+ 16\int_0^1 \frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2}(t,\eta)\Big(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^2\tilde{\mathcal{Y}}_{2,\eta}\,d\theta\Big)\\
&\times\Big(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\\
&\times\Big(\int_0^\theta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,l))}(\tilde{\mathcal{U}}_2-\tilde{\mathcal{U}}_1)^2(t,l)\,dl\Big)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\Big)\,d\eta\\
&+ 2A^2\int_0^1 \frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2}(t,\eta)\Big(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^2\tilde{\mathcal{Y}}_{2,\eta}\,d\theta\Big)\\
&\times\Big(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\\
&\times\Big(\int_0^\theta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,l))}(\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1)^2(t,l)\,dl\Big)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\Big)\,d\eta\\
&+ A^2\int_0^1 \frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2}(t,\eta)\Big(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\Big)\\
&\times\Big(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\\
&\times\Big(\int_0^\theta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,l))}|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1|(t,l)\,dl\Big)^2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\Big)\,d\eta\\
&+ \frac{128A^2}{3e^2}\int_0^1 \frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2}(t,\eta)\Big(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\Big)\\
&\times\Big(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\\
&\times\Big(\int_0^\theta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,l))}\,dl\Big)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\Big)\,d\eta|A_1-A_2|^2\\
\leqslant{}& \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+\|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1\|^2)+\mathcal{O}(1)\int_0^1\Big(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\\
&\times\Big(\int_0^\theta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,l))}(\tilde{\mathcal{U}}_2-\tilde{\mathcal{U}}_1)^2(t,l)\,dl\Big)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\Big)\,d\eta\\
&+ \mathcal{O}(1)\int_0^1\Big(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}
\end{aligned}
$$

$$
\begin{aligned}
&\times \bigg(\int_0^{\theta} e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,l))}(\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1)^2(t,l)\,dl\bigg)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\bigg)\,d\eta\\
&+\mathcal{O}(1)\int_0^1\bigg(\int_0^{\eta} e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\\
&\times \bigg(\int_0^{\theta} e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,l))}\,dl\bigg)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\bigg)\,d\eta|A_1-A_2|^2\\
\leqslant{}& \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+\|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1\|^2)+B_1+B_2+B_3,
\end{aligned}
$$

using (4.15b), (4.15g), and (4.16c). As for B_1, we find

$$
\begin{aligned}
B_1 ={}& \mathcal{O}(1)\int_0^1\bigg(\int_0^{\eta} e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\\
&\times \bigg(\int_0^{\theta} e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,l))}(\tilde{\mathcal{U}}_2-\tilde{\mathcal{U}}_1)^2(t,l)\,dl\bigg)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\bigg)\,d\eta\\
\leqslant{}& \mathcal{O}(1)\int_0^1\bigg(\int_0^{\eta} e^{-\frac{1}{2A}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\\
&\times \bigg(\int_0^{\theta} e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,l))}(\tilde{\mathcal{U}}_2-\tilde{\mathcal{U}}_1)^2(t,l)\,dl\bigg)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\bigg)\,d\eta\\
={}& \mathcal{O}(1)\int_0^1\bigg(\int_0^{\eta} e^{-\frac{1}{2A}\tilde{\mathcal{Y}}_2(t,\theta)}\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\\
&\times \bigg(\int_0^{\theta} e^{-(\frac{1}{2A}\tilde{\mathcal{Y}}_2(t,\eta)-\frac{1}{A}\tilde{\mathcal{Y}}_2(t,l))}(\tilde{\mathcal{U}}_2-\tilde{\mathcal{U}}_1)^2(t,l)\,dl\bigg)d\theta\bigg)\,d\eta\\
={}& \mathcal{O}(1)\int_0^1\bigg[\bigg(-2Ae^{-\frac{1}{2A}\tilde{\mathcal{Y}}_2(t,\theta)}\\
&\times \bigg(\int_0^{\theta} e^{-(\frac{1}{2A}\tilde{\mathcal{Y}}_2(t,\eta)-\frac{1}{A}\tilde{\mathcal{Y}}_2(t,l))}(\tilde{\mathcal{U}}_2-\tilde{\mathcal{U}}_1)^2(t,l)\,dl\bigg)\bigg)\bigg|_{\theta=0}^{\eta}\\
&+2A\int_0^{\eta} e^{-\frac{1}{2A}\tilde{\mathcal{Y}}_2(t,\theta)}e^{-(\frac{1}{2A}\tilde{\mathcal{Y}}_2(t,\eta)-\frac{1}{A}\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{U}}_2-\tilde{\mathcal{U}}_1)^2(t,\theta)\,d\theta\bigg]\,d\eta\\
={}& -2A\mathcal{O}(1)\int_0^1\int_0^{\eta} e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{U}}_2-\tilde{\mathcal{U}}_1)^2(t,\theta)\,d\theta\,d\eta\\
&+2A\mathcal{O}(1)\int_0^1\int_0^{\eta} e^{-\frac{1}{2A}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{U}}_2-\tilde{\mathcal{U}}_1)^2(t,\theta)\,d\theta\,d\eta\\
\leqslant{}& \mathcal{O}(1)\|\tilde{\mathcal{U}}_2-\tilde{\mathcal{U}}_1\|^2,
\end{aligned}
$$

while for B_2, we estimate as follows:

$$\begin{aligned}
B_2 &= \mathcal{O}(1)\int_0^1\bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\\
&\quad\times\bigg(\int_0^\theta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,l))}(\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1)^2(t,l)\,dl\bigg)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\bigg)\,d\eta\\
&\leqslant \mathcal{O}(1)\int_0^1\bigg(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\\
&\quad\times\bigg(\int_0^\theta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,l))}(\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1)^2(t,l)\,dl\bigg)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\bigg)\,d\eta\\
&= \mathcal{O}(1)\int_0^1\bigg(\int_0^\eta e^{-\frac{1}{2A_2}\tilde{\mathcal{Y}}_2(t,\theta)}\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\\
&\quad\times\bigg(\int_0^\theta e^{-(\frac{1}{2A_2}\tilde{\mathcal{Y}}_2(t,\eta)-\frac{1}{A_2}\tilde{\mathcal{Y}}_2(t,l))}(\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1)^2(t,l)\,dl\bigg)\,d\theta\bigg)\,d\eta\\
&= \mathcal{O}(1)\int_0^1\bigg[\bigg(-2A_2e^{-\frac{1}{2A_2}\tilde{\mathcal{Y}}_2(t,\theta)}\\
&\quad\times\bigg(\int_0^\theta e^{-(\frac{1}{2A_2}\tilde{\mathcal{Y}}_2(t,\eta)-\frac{1}{A_2}\tilde{\mathcal{Y}}_2(t,l))}(\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1)^2(t,l)\,dl\bigg)\bigg)\bigg|_{\theta=0}^{\eta}\\
&\quad+2A_2\int_0^\eta e^{-\frac{1}{2A_2}\tilde{\mathcal{Y}}_2(t,\theta)}e^{-(\frac{1}{2A_2}\tilde{\mathcal{Y}}_2(t,\eta)-\frac{1}{A_2}\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1)^2(t,\theta)\,d\theta\bigg]\,d\eta\\
&= -2A_2\mathcal{O}(1)\int_0^1\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1)^2(t,\theta)\,d\theta\,d\eta\\
&\quad+2A_2\mathcal{O}(1)\int_0^1\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1)^2(t,\theta)\,d\theta\,d\eta\\
&\leqslant \mathcal{O}(1)\|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1\|^2,
\end{aligned}$$

while for B_3, we estimate as follows:

$$\begin{aligned}
B_3 &= \mathcal{O}(1)\int_0^1\bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\\
&\quad\times\bigg(\int_0^\theta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,l))}\,dl\bigg)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\bigg)\,d\eta|A_1-A_2|^2\\
&\leqslant \mathcal{O}(1)\int_0^1\bigg(\int_0^\eta e^{-\frac{3}{8A}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\\
&\quad\times\bigg(\int_0^\theta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,l))}\,dl\bigg)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\bigg)\,d\eta|A_1-A_2|^2
\end{aligned}$$

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$$
\begin{aligned}
&= \mathcal{O}(1)\int_0^1 \bigg(\bigg(\int_0^\eta e^{-\frac{3}{8A}\tilde{\mathcal{Y}}_2(t,\theta)}\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\\
&\quad \times \bigg(\int_0^\theta e^{-(\frac{3}{8A}\tilde{\mathcal{Y}}_2(t,\eta)-\frac{3}{4A}\tilde{\mathcal{Y}}_2(t,l))}\,dl\bigg)\,d\theta\bigg)\,d\eta|A_1-A_2|^2\\
&= \mathcal{O}(1)\int_0^1 \bigg[\bigg(-\frac{8}{3}Ae^{-\frac{3}{8A}\tilde{\mathcal{Y}}_2(t,\theta)}\\
&\quad \times \bigg(\int_0^\theta e^{-(\frac{3}{8A}\tilde{\mathcal{Y}}_2(t,\eta)-\frac{3}{4A}\tilde{\mathcal{Y}}_2(t,l))}\,dl\bigg)\bigg)\bigg|_{\theta=0}^{\eta}\\
&\quad + \frac{8}{3}A\int_0^\eta e^{-\frac{3}{8A}\tilde{\mathcal{Y}}_2(t,\theta)}e^{-(\frac{3}{8A}\tilde{\mathcal{Y}}_2(t,\eta)-\frac{3}{4A}\tilde{\mathcal{Y}}_2(t,\theta))}\,d\theta\bigg]\,d\eta|A_1-A_2|^2\\
&= -\frac{8}{3}A\mathcal{O}(1)\int_0^1\int_0^\eta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\,d\theta\,d\eta|A_1-A_2|^2\\
&\quad + \frac{8}{3}A\mathcal{O}(1)\int_0^1\int_0^\eta e^{-\frac{3}{8A}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\,d\theta\,d\eta|A_1-A_2|^2\\
&\leqslant \mathcal{O}(1)|A_1-A_2|^2.
\end{aligned}
$$

Thus we find that

$$
A_{12} \leqslant \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+\|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1\|^2+\|\tilde{\mathcal{U}}_2-\tilde{\mathcal{U}}_1\|^2+|A_1-A_2|^2).
$$

Next we find (see (A.15))

$$
\begin{aligned}
A_{13} &= \frac{1}{A^3}\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}|(t,\eta)\\
&\quad \times \bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}T_2\mathbb{1}_{\tilde{\mathcal{D}}_1<\tilde{\mathcal{D}}_2}(t,\theta)\,d\theta\bigg)\,d\eta\\
&\leqslant \frac{1}{A^3}\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}|(t,\eta)\\
&\quad \times \bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\bigg(\tilde{\mathcal{P}}_j|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1|(t,\theta)\\
&\quad + 2\sqrt{2}A^3\bigg(\int_0^\theta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,l))}(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})^2(t,l)\,dl\bigg)^{1/2}\\
&\quad + \frac{A^4}{\sqrt{2}}\bigg(\int_0^\theta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,l))}(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,l)\,dl\bigg)^{1/2}\\
&\quad + \frac{A^4}{2}\bigg(\int_0^\theta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,l))}|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1|(t,l)\,dl\bigg)
\end{aligned}
$$

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$$
\begin{aligned}
&+\frac{4\sqrt{2}A^4}{\sqrt{3}e}\bigg(\int_0^\eta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,l))}\,dl\bigg)^{1/2}|A_1-A_2|\bigg)\,d\theta\bigg)\,d\eta\\
&\leqslant 5\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+\frac{1}{A^6}\int_0^1\frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2}(t,\eta)\\
&\times\bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}\tilde{\mathcal{P}}_j|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1|(t,\theta)\,d\theta\bigg)^2\,d\eta\\
&+8\int_0^1\frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2}(t,\eta)\bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\\
&\times\bigg(\int_0^\theta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,l))}(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})^2(t,l)\,dl\bigg)^{1/2}\,d\theta\bigg)^2\,d\eta\\
&+\frac{A^2}{2}\int_0^1\frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2}(t,\eta)\bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\\
&\times\bigg(\int_0^\theta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,l))}(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,l)\,dl\bigg)^{1/2}\,d\theta\bigg)^2\,d\eta\\
&+\frac{A^2}{4}\int_0^1\frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2}(t,\eta)\bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\\
&\times\bigg(\int_0^\theta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,l))}|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1|(t,l)\,dl\bigg)\,d\theta\bigg)^2\,d\eta\\
&+\frac{32A^2}{3e^2}\int_0^1\frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2}(t,\eta)\bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\\
&\times\bigg(\int_0^\theta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,l))}\,dl\bigg)^{1/2}\,d\theta\bigg)^2\,d\eta|A_1-A_2|^2\\
&\leqslant 5\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2\\
&+\frac{1}{A^6}\int_0^1\frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2}(t,\eta)\bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2^2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\bigg)\\
&\times\bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^2\tilde{\mathcal{Y}}_{2,\eta}(\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1)^2(t,\theta)\,d\theta\bigg)\,d\eta\\
&+8\int_0^1\frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2}(t,\eta)\bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\bigg)\\
&\times\bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)
\end{aligned}
$$

$$
\begin{aligned}
&\times \left(\int_0^\theta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,l))}(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})^2(t,l)\,dl\right)d\theta\bigg)\,d\eta\\
&+\frac{A^2}{2}\int_0^1 \frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2}(t,\eta)\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)\\
&\times \bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\\
&\times \left(\int_0^\theta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,l))}(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,l)\,dl\right)d\theta\bigg)\,d\eta\\
&+\frac{A^2}{4}\int_0^1 \frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2}(t,\eta)\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)\\
&\times \bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\\
&\times \left(\int_0^\theta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,l))}|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1|(t,l)\,dl\right)^2 d\theta\bigg)\,d\eta\\
&+\frac{32A^2}{3e^2}\int_0^1 \frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2}(t,\eta)\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)\\
&\times \bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\\
&\times \left(\int_0^\theta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,l))}\,dl\right)d\theta\bigg)\,d\eta|A_1-A_2|^2\\
\leqslant{}& \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+\|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1\|^2)\\
&+\mathcal{O}(1)\int_0^1\bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\\
&\quad\times\left(\int_0^\theta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,l))}(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})^2(t,l)\,dl\right)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\bigg)\,d\eta\\
&+\mathcal{O}(1)\int_0^1\bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\\
&\times\left(\int_0^\theta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,l))}(\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1)^2(t,l)\,dl\right)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\bigg)\,d\eta\\
&+\mathcal{O}(1)\bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\\
&\times\left(\int_0^\theta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,l))}\,dl\right)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\bigg)\,d\eta|A_1-A_2|^2
\end{aligned}
$$

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$$\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+\|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1\|^2)+B_4+B_2+B_3$$
$$\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+\|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1\|^2+|A_1-A_2|^2),$$

following the approach used for A_{12}. As for B_4, we find

$$\begin{aligned}
B_4 &= \mathcal{O}(1)\int_0^1\bigg(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\\
&\quad\times\bigg(\int_0^\theta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,l))}(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})^2(t,l)\,dl\bigg)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\bigg)\,d\eta\\
&\leqslant \mathcal{O}(1)\int_0^1\bigg(\int_0^\eta e^{-\frac{1}{2A}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\\
&\quad\times\bigg(\int_0^\theta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,l))}(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})^2(t,l)\,dl\bigg)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\bigg)\,d\eta\\
&= \mathcal{O}(1)\int_0^1\bigg(\int_0^\eta e^{-\frac{1}{2A}\tilde{\mathcal{Y}}_2(t,\theta)}\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\\
&\quad\times\bigg(\int_0^\theta e^{-(\frac{1}{2A}\tilde{\mathcal{Y}}_2(t,\eta)-\frac{1}{A}\tilde{\mathcal{Y}}_2(t,l))}(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})^2(t,l)\,dl\bigg)\,d\theta\bigg)\,d\eta\\
&= \mathcal{O}(1)\int_0^1\bigg[\bigg(-2Ae^{-\frac{1}{2A}\tilde{\mathcal{Y}}_2(t,\theta)}\\
&\quad\times\bigg(\int_0^\theta e^{-(\frac{1}{2A}\tilde{\mathcal{Y}}_2(t,\eta)-\frac{1}{A}\tilde{\mathcal{Y}}_2(t,l))}(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})^2(t,l)\,dl\bigg)\bigg)\bigg|_{\theta=0}^{\eta}\\
&\quad+2A\int_0^\eta e^{-\frac{1}{2A}\tilde{\mathcal{Y}}_2(t,\theta)}e^{-(\frac{1}{2A}\tilde{\mathcal{Y}}_2(t,\eta)-\frac{1}{A}\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})^2(t,\theta)\,d\theta\bigg]\,d\eta\\
&= -2A\mathcal{O}(1)\int_0^1\int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})^2(t,\theta)\,d\theta\,d\eta\\
&\quad+2A\mathcal{O}(1)\int_0^1\int_0^\eta e^{-\frac{1}{2A}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})^2(t,\theta)\,d\theta\,d\eta\\
&\leqslant \mathcal{O}(1)\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2.
\end{aligned}$$

Thus we find that

$$A_{12}\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+\|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1\|^2+\|\tilde{\mathcal{U}}_2-\tilde{\mathcal{U}}_1\|^2+|A_1-A_2|^2).$$

Next we find (see (A.16))

$$
\begin{aligned}
A_{14} &= \frac{1}{A^3}\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}|(t,\eta) \\
&\quad\times\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}T_3\mathbb{1}_{\tilde{\mathcal{D}}_1<\tilde{\mathcal{D}}_2}(t,\theta)\,d\theta\right)d\eta \\
&\leqslant 12A|A_1-A_2|\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}|(t,\eta) \\
&\quad\times\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\left(\int_0^\theta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,l))}\,dl\right)d\theta\right)d\eta \\
&\leqslant 12A|A_1-A_2|\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|\left(\int_0^1 \frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2}(t,\eta)\right. \\
&\quad\times\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^+\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\right. \\
&\quad\times\left.\left.\left(\int_0^\theta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,l))}\,dl\right)d\theta\right)^2 d\eta\right)^{1/2} \\
&\leqslant 12A|A_1-A_2|\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\| \\
&\quad\times\left(\int_0^1 \frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2}(t,\eta)\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)\right. \\
&\quad\times\left.\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\left(\int_0^\theta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,l))}\,dl\right)^2 d\theta\right)d\eta\right)^{1/2} \\
&\leqslant \mathcal{O}(1)|A_1-A_2|\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|\left(\int_0^1\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\right.\right. \\
&\quad\times\left.\left.\left(\int_0^\theta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,l))}\,dl\right)^2 d\theta\right)d\eta\right)^{1/2} \\
&\leqslant \mathcal{O}(1)|A_1-A_2|\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\| \\
&\quad\times\left(\int_0^1\left(\int_0^\eta e^{-\frac{3}{8A}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\right.\right. \\
&\quad\times\left.\left.\left(\int_0^\theta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,l))}\,dl\right)d\theta\right)d\eta\right)^{1/2} \\
&\leqslant \mathcal{O}(1)|A_1-A_2|\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\| \\
&\quad\times\left(\int_0^1\left(\int_0^\eta e^{-\frac{3}{8A}\tilde{\mathcal{Y}}_2(t,\theta)}\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\right.\right. \\
&\quad\times\left.\left.\left(\int_0^\theta e^{-(\frac{3}{8A}\tilde{\mathcal{Y}}_2(t,\eta)-\frac{3}{4A}\tilde{\mathcal{Y}}_2(t,l))}\,dl\right)d\theta\right)d\eta\right)^{1/2}
\end{aligned}
$$

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$$\begin{aligned}
&\leqslant \mathcal{O}(1)|A_1 - A_2|\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\| \\
&\quad \times \bigg(\int_0^1 \bigg[\bigg(-\frac{8}{3} A e^{-\frac{3}{8A}\tilde{\mathcal{Y}}_2(t,\theta)} \int_0^\theta e^{-(\frac{3}{8A}\tilde{\mathcal{Y}}_2(t,\eta) - \frac{3}{4A}\tilde{\mathcal{Y}}_2(t,l))}\, dl \bigg) \bigg|_{\theta=0}^{\eta} \\
&\quad + \frac{8}{3} A \int_0^\eta e^{-\frac{3}{8A}\tilde{\mathcal{Y}}_2(t,\theta)} e^{-(\frac{3}{8A}\tilde{\mathcal{Y}}_2(t,\eta) - \frac{3}{4A}\tilde{\mathcal{Y}}_2(t,\theta))}\, d\theta \bigg] d\eta \bigg)^{1/2} \\
&\leqslant \mathcal{O}(1)|A_1 - A_2|\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\| \\
&\quad \times \bigg(\int_0^1 \bigg[-\frac{8}{3} A \int_0^\eta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))}\, d\theta \\
&\quad + \frac{8}{3} A \int_0^\eta e^{-\frac{3}{8A}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))}\, d\theta \bigg] d\eta \bigg)^{1/2} \\
&\leqslant \mathcal{O}(1)(|A_1 - A_2|^2 + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2).
\end{aligned}$$

Thus we have shown that

$$\tilde{A}_1 + \tilde{A}_2 \leqslant \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{Y}}_2 - \tilde{\mathcal{Y}}_1\|^2 + \|\tilde{\mathcal{U}}_2 - \tilde{\mathcal{U}}_1\|^2 + |A_1 - A_2|^2).$$

We next consider the term $\tilde{A}_3$:

$$\begin{aligned}
\tilde{A}_3 &= \frac{1}{2A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t,\eta) \int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \tilde{\mathcal{U}}_2^+ \tilde{\mathcal{Y}}_{2,\eta} \\
&\quad - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \tilde{\mathcal{U}}_1^+ \tilde{\mathcal{Y}}_{1,\eta}) \min_j(\tilde{\mathcal{D}}_j)(t,\theta)\, d\theta\, d\eta \\
&= \frac{1}{2A^3} \int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})} (\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})(t,\eta) \\
&\quad \times \bigg[\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} \min_j(\tilde{\mathcal{D}}_j) \tilde{\mathcal{Y}}_{2,\eta} (\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+) \mathbb{1}_{\tilde{\mathcal{U}}_1^+ < \tilde{\mathcal{U}}_2^+}(t,\theta)\, d\theta \\
&\quad + \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} \min_j(\tilde{\mathcal{D}}_j) \tilde{\mathcal{Y}}_{1,\eta} (\tilde{\mathcal{U}}_2^+ - \tilde{\mathcal{U}}_1^+) \mathbb{1}_{\tilde{\mathcal{U}}_2^+ \leqslant \tilde{\mathcal{U}}_1^+}(t,\theta)\, d\theta \\
&\quad + \mathbb{1}_{A_1 \leqslant A_2} \int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))}) \\
&\quad \times \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\, d\theta \\
&\quad + \mathbb{1}_{A_2 < A_1} \int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))}) \\
&\quad \times \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+) \tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\, d\theta \\
&\quad + \int_0^\eta (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta) - \tilde{\mathcal{Y}}_2(t,\theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_1(t,\theta))})
\end{aligned}$$

$$
\begin{aligned}
&\times \min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{2,\eta}\mathbb{1}_{B(\eta)}(t,\theta)\,d\theta\\
&+\int_0^\eta (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}-e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))})\\
&\times \min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{1,\eta}\mathbb{1}_{B(\eta)^c}(t,\theta)\,d\theta\\
&+\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)(\tilde{\mathcal{Y}}_{2,\eta}-\tilde{\mathcal{Y}}_{1,\eta})(t,\theta)\,d\theta\Bigg]\,d\eta\\
=\;& A_{31}+A_{32}+A_{33}+A_{34}+A_{35}+A_{36}+A_{37}.
\end{aligned}\tag{5.47}
$$

Terms A_{31} and A_{32} allow for the same treatment:

$$
\begin{aligned}
|A_{31}| &= \frac{1}{2A^3}\bigg|\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})(t,\eta)\\
&\quad\times\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\min_j(\tilde{\mathcal{D}}_j)\tilde{\mathcal{Y}}_{2,\eta}(\tilde{\mathcal{U}}_2^+-\tilde{\mathcal{U}}_1^+)\mathbb{1}_{\tilde{\mathcal{U}}_1^+<\tilde{\mathcal{U}}_2^+}(t,\theta)\,d\theta\right)d\eta\bigg|\\
&\leqslant \frac{1}{A^2}\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}|(t,\eta)\\
&\quad\times\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}|\tilde{\mathcal{U}}_2^+-\tilde{\mathcal{U}}_1^+|(t,\theta)\,d\theta\right)d\eta\\
&\leqslant \|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+\frac{1}{A^4}\int_0^1\frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2}(t,\eta)\\
&\quad\times\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}|\tilde{\mathcal{U}}_2^+-\tilde{\mathcal{U}}_1^+|(t,\theta)\,d\theta\right)^2d\eta\\
&\leqslant \|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+\frac{1}{A^4}\int_0^1\frac{1}{(\max_j(\tilde{\mathcal{P}}_j^{1/2}))^2}(t,\eta)\\
&\quad\times\left(\int_0^\eta e^{-\frac{3}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)\\
&\quad\times\left(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(\tilde{\mathcal{U}}_2-\tilde{\mathcal{U}}_1)^2(t,\theta)\,d\theta\right)d\eta\\
&\leqslant \|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2\\
&\quad+\frac{2}{A^3}\int_0^1\left(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(\tilde{\mathcal{U}}_2-\tilde{\mathcal{U}}_1)^2(t,\theta)\,d\theta\right)d\eta\\
&\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+\|\tilde{\mathcal{U}}_2-\tilde{\mathcal{U}}_1\|^2),
\end{aligned}
$$

using (4.15e), (4.15n), and (4.16e).

The terms A_{33} and A_{34} can be treated as follows:

$$\begin{aligned}
|A_{33}| &= \frac{1}{2A^3}\bigg|\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})\\
&\quad\times \mathbb{1}_{A_1\leqslant A_2}\int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta)}-e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))})\\
&\quad\times \min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\,d\eta\bigg|\\
&\leqslant \frac{\sqrt{2}a}{A^3e}\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}|(t,\eta)\\
&\quad\times\left(\int_0^\eta e^{-\frac{3}{4A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{D}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)d\eta|A_1-A_2|\\
&\leqslant \frac{2\sqrt{2}}{A^2e}\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}|(t,\eta)\\
&\quad\times\left(\int_0^\eta e^{-\frac{3}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)^{1/2}\\
&\quad\times\left(\int_0^\eta \tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)^{1/2}d\eta|A_1-A_2|\\
&\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+|A_1-A_2|^2).
\end{aligned}$$

Furthermore, the terms A_{35} and A_{36} can be estimated like this:

$$\begin{aligned}
|A_{35}| &= \frac{1}{2A^3}\bigg|\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})(t,\eta)\\
&\quad\times\left(\int_0^\eta (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}-e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))})\right.\\
&\quad\left.\times \min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{2,\eta}\mathbb{1}_{B(\eta)}(t,\theta)\,d\theta\right)d\eta\bigg|\\
&\leqslant \frac{1}{2aA^3}\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}|(t,\eta)\\
&\quad\times\left(\int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(|\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_1(t,\eta)|+|\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_1(t,\theta)|)\right.\\
&\quad\left.\times \min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{2,\eta}\mathbb{1}_{B(\eta)}(t,\theta)\,d\theta\right)d\eta
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \frac{a}{2\sqrt{2}A^3}\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}||\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1|(t,\eta)\\
&\quad\times\left(\int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\min_j(\tilde{\mathcal{D}}_j)\tilde{\mathcal{Y}}_{2,\eta}\mathbb{1}_{B(\eta)}(t,\theta)\,d\theta\right)d\eta\\
&\quad+\frac{a}{2\sqrt{2}A^3}\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}|(t,\eta)\\
&\quad\times\left(\int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1|\min_j(\tilde{\mathcal{D}}_j)\tilde{\mathcal{Y}}_{2,\eta}\mathbb{1}_{B(\eta)}(t,\theta)\,d\theta\right)d\eta\\
&\leqslant \frac{1}{\sqrt{2}A}\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}||\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1|(t,\eta)\\
&\quad\times\left(\int_0^\eta e^{-\frac{3}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)^{1/2}\\
&\quad\times\left(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)^{1/2}d\eta\\
&\quad+\frac{1}{\sqrt{2}A}\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}|(t,\eta)\\
&\quad\times\left(\int_0^\eta e^{-\frac{3}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)^{1/2}\\
&\quad\times\left(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1)^2(t,\theta)\,d\theta\right)^{1/2}d\eta\\
&\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+\|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1\|^2),
\end{aligned}
$$

using (4.15e), (4.15n), and (4.16e). As for A_{37}, we follow this path:

$$
\begin{aligned}
|A_{37}| &= \frac{1}{2A^3}\left|\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})(t,\eta)\left(\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\right.\right.\\
&\quad\left.\left.\times\min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)(\tilde{\mathcal{Y}}_{2,\eta}-\tilde{\mathcal{Y}}_{1,\eta})(t,\theta)d\theta\right)d\eta\right|\\
&= \frac{1}{2A^3}\left|\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})(t,\eta)\right.\\
&\quad\times\left[\left(\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)(\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1)(t,\theta)\right)\Bigg|_{\theta=0}^{\eta}\right.\\
&\quad-\int_0^\eta(\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1)
\end{aligned}
$$

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$$
\begin{aligned}
&\times \frac{d}{d\theta}\Big(\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\Big)(t,\theta)\,d\theta\Big]\,d\eta\Big|\\
&\leqslant \frac{1}{2A^3}\Big|\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})\min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)(\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1)(t,\eta)\,d\eta\Big|\\
&\quad+\frac{1}{2A^3}\Big|\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})(t,\eta)\Big(\int_0^\eta(\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1)\\
&\quad\times \frac{d}{d\theta}\Big(\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\Big)(t,\theta)\,d\theta\Big)\,d\eta\Big|\\
&\leqslant \frac{1}{A^2}\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}|\tilde{\mathcal{P}}_2|\tilde{\mathcal{U}}_2||\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1|(t,\eta)\,d\eta\\
&\quad+\frac{1}{2A^3}\Big|\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})(t,\eta)\Big(\int_0^\eta(\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1)\\
&\quad\times\bigg(\Big(\frac{d}{d\theta}\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\Big)\min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\\
&\quad+\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\Big(\frac{d}{d\theta}\min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\Big)\bigg)(t,\theta)\,d\theta\Big)\,d\eta\Big|\\
&\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2)\\
&\quad+\frac{1}{2A^3}\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}|(t,\eta)\\
&\quad\times\Big(\int_0^\eta|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1|\frac{1}{a}\Big(\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\\
&\quad\times\max_j(\tilde{\mathcal{Y}}_{j,\eta})\min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\\
&\quad+\mathcal{O}(1)A^5\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\Big(\min_j(\tilde{\mathcal{D}}_j^{1/2})+|\tilde{\mathcal{U}}_2|\Big)\Big)(t,\theta)\,d\theta\Big)\,d\eta\\
&\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2)+\frac{1}{2}(A_{371}+A_{372}),
\end{aligned}
$$

using (4.15a), (4.15b), (4.15n), and Lemmas A.2 and A.4. Here

$$
\begin{aligned}
A_{371}=\frac{1}{aA^3}\int_0^1&\frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}|(t,\eta)\\
&\times\Big(\int_0^\eta|\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1|\Big(\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\\
&\times\max_j(\tilde{\mathcal{Y}}_{j,\eta})\min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)(t,\theta)\,d\theta\Big)\,d\eta
\end{aligned}
$$

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$$
\begin{aligned}
&\leqslant \frac{\sqrt{2}a}{A^2}\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}|(t,\eta)\bigg(\int_0^\eta |\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1|\\
&\quad\times (e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\tilde{\mathcal{P}}_1\tilde{\mathcal{Y}}_{1,\eta}+e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta})(t,\theta)\,d\theta\bigg)\,d\eta\\
&\leqslant \frac{\sqrt{2}}{A}\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}|(t,\eta)\\
&\quad\times\bigg[\bigg(\int_0^\eta e^{-\frac{3}{2A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\tilde{\mathcal{P}}_1\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\bigg)^{1/2}\\
&\quad\times\bigg(\int_0^\eta e^{-\frac{1}{2A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\tilde{\mathcal{P}}_1\tilde{\mathcal{Y}}_{1,\eta}(\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1)^2(t,\theta)\,d\theta\bigg)^{1/2}\\
&\quad+\bigg(\int_0^\eta e^{-\frac{3}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\bigg)^{1/2}\\
&\quad\times\bigg(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1)^2(t,\theta)\,d\theta\bigg)^{1/2}\bigg]\,d\eta\\
&\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2),
\end{aligned}
$$

using $\max_j(a_j)\leqslant a_1+a_2$ and $\min_j(b_j)\leqslant b_k$. Then

$$
\begin{aligned}
A_{372}&=\mathcal{O}(1)\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}|(t,\eta)\\
&\quad\times\bigg(\int_0^\eta |\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1|\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\big(\min_j(\tilde{\mathcal{D}}_j^{1/2})+|\tilde{\mathcal{U}}_2|\big)(t,\theta)\,d\theta\bigg)\,d\eta\\
&\leqslant \mathcal{O}(1)\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}|(t,\eta)\\
&\quad\times\bigg(\int_0^\eta |\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1|e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2^{1/2}(t,\theta)\,d\theta\bigg)\,d\eta\\
&\leqslant \mathcal{O}(1)\int_0^1 \frac{1}{\max_j(\tilde{\mathcal{P}}_j^{1/2})}|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}|(t,\eta)\\
&\quad\times\bigg(\int_0^\eta e^{-\frac{3}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2(t,\theta)\,d\theta\bigg)^{1/2}\\
&\quad\times\bigg(\int_0^\eta e^{-\frac{1}{2A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}(\tilde{\mathcal{Y}}_2-\tilde{\mathcal{Y}}_1)^2(t,\theta)\,d\theta\bigg)^{1/2}\,d\eta\\
&\leqslant \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2}\|^2+\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|^2),
\end{aligned}
$$

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using (4.15d), (4.15n), and (4.16i). This completes the estimate for $\tilde{A}_3$ (see (5.47)):

$$\tilde{A}_3 \leqslant \mathcal{O}(1)(\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + |A_1 - A_2|^2).$$

For J_4 (and similarly for J_5) the estimates read as

$$\begin{aligned} |J_4| &\leqslant \frac{1}{A^3}\int_0^1 |\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|^2 \frac{|\tilde{\mathcal{R}}_2|}{\tilde{\mathcal{P}}_2}(t,\eta)\,d\eta \\ &\leqslant \mathcal{O}(1)\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2, \end{aligned}$$

using (4.15j).

We have shown the anticipated result.

LEMMA 5.5. *Let $\tilde{\mathcal{P}}_i^{1/2}$ be two solutions of* (5.40) *for $i = 1, 2$. Then we have*

$$\begin{aligned} &\frac{d}{dt}\|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 \\ &\quad \leqslant \mathcal{O}(1)(\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|^2 + \|\tilde{\mathcal{U}}_1 - \tilde{\mathcal{U}}_2\|^2 + \|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}\|^2 + |A_1 - A_2|^2), \end{aligned}$$

where $\mathcal{O}(1)$ denotes some constant which only depends on $A = \max_j(A_j)$ and which remains bounded as $A \to 0$.

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Appendix A. Lipschitz continuity and uniformly bounded Lipschitz constants

We need to establish that a number of complicated functions are Lipschitz continuous with uniformly bounded Lipschitz constants. In a desperate attempt to ease the readability of the main estimates, we collect these results here together with some other estimates that are essential in Section 5.

LEMMA A.1. (i) *We have*

$$|e^{a_1-a_2} - e^{b_1-b_2}| \leqslant \max(e^{a_1-a_2}, e^{b_1-b_2})(|b_1-a_1| + |b_2-a_2|) \tag{A.1}$$

$$\leqslant |b_1-a_1| + |b_2-a_2|, \tag{A.2}$$

$$a_1 < a_2, \quad b_1 < b_2.$$

(ii) *Let* $0 < a \leqslant A_j \leqslant A$, $j = 1, 2$. *Then we have*

$$|e^{-\frac{1}{A_2}x} - e^{-\frac{1}{A_1}x}| \leqslant \frac{4}{ae} e^{-\frac{3}{4A}x}|A_2 - A_1|, \quad x \in [0, \infty). \tag{A.3}$$

Proof. (i) The result follows from the elementary inequality

$$|e^a - e^b| = \left|\int_b^a e^x\,dx\right| \leqslant e^b\left|\int_b^a dx\right| \leqslant e^b|b-a|, \quad a < b.$$

(ii) For $x \in [0, \infty)$ one can write

$$\begin{aligned}|e^{-\frac{1}{A_2}x} - e^{-\frac{1}{A_1}x}| &= \left|\int_{A_1}^{A_2} \frac{1}{s^2} x e^{-\frac{1}{s}x}\,ds\right| \\ &\leqslant e^{-\frac{3}{4A}x}\left|\int_{A_1}^{A_2} \frac{1}{s^2} x e^{-\frac{1}{4s}x}\,ds\right| \\ &\leqslant \frac{4}{ae} e^{-\frac{3}{4A}x}|A_2 - A_1|.\end{aligned}$$

Here we used in the last step that for $s \in [a, A]$, the function $f: [0,\infty) \to [0,\infty)$ with $f(x) = \frac{1}{s^2} x e^{-\frac{1}{4s}x}$ attains its maximum at $x = 4s$ and

$$0 \leqslant f(4s) = \frac{4}{se} \leqslant \frac{4}{ae}. \qquad \square$$

LEMMA A.2. (i) *The function* $\theta \mapsto \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))})$ *is nondecreasing for almost every* η *and thus differentiable almost everywhere. We have that*

$$\left|\frac{d}{d\theta}\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))})\right| \leqslant \frac{1}{a}\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta) - \tilde{\mathcal{Y}}_j(t,\theta))})\max_j(\tilde{\mathcal{Y}}_{j,\eta}(t,\theta)). \tag{A.4}$$

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(ii) *The function* $\theta \mapsto \min_j(\tilde{\mathcal{U}}_j^2)(t,\theta)$ *is differentiable almost everywhere with*

$$\left|\frac{d}{d\theta}\min_j(\tilde{\mathcal{U}}_j^2(t,\theta))\right| \leqslant A^4. \tag{A.5}$$

(iii) *The function* $\theta \mapsto \min_j(\tilde{\mathcal{U}}_j^+)^3(t,\theta)$ *is differentiable almost everywhere with*

$$\left|\frac{d}{d\theta}\min_j(\tilde{\mathcal{U}}_j^+)^3(t,\theta)\right| \leqslant 2A^4\min_j(\tilde{\mathcal{U}}_j^+)(t,\theta). \tag{A.6}$$

Proof. (i): First of all, note that the function $\theta \mapsto \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})$ is nondecreasing and hence differentiable almost everywhere. Consider in the following $\theta < \eta$. Assume that for fixed θ, such that the given function is differentiable, we have

$$\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}) = e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))} \tag{A.7}$$

and that there exists a sequence $\theta_n \uparrow \theta$ such that

$$\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta_n))}) = e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta_n))} \quad \text{for all } n.$$

Then we have

$$\begin{aligned}
&\left|\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}) - \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta_n))})\right| \\
&\quad = e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta_n))} \\
&\quad \leqslant \frac{1}{a}e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}(\tilde{\mathcal{Y}}_1(t,\theta)-\tilde{\mathcal{Y}}_1(t,\theta_n)),
\end{aligned}$$

since $\tilde{\mathcal{Y}}_i(t,\cdot)$ is nondecreasing.

$$\begin{aligned}
\left|\frac{d}{d\theta}\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\right| &\leqslant \lim_{\theta_n\uparrow\theta}\frac{1}{a}e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\frac{\tilde{\mathcal{Y}}_1(t,\theta)-\tilde{\mathcal{Y}}_1(t,\theta_n)}{\theta-\theta_n} \\
&= \frac{1}{a}e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\tilde{\mathcal{Y}}_{1,\eta}(t,\theta) \\
&= \frac{1}{a}\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\tilde{\mathcal{Y}}_{1,\eta}(t,\theta).
\end{aligned}$$

Assume, on the other hand, that for fixed θ we have

$$\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}) = e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}$$

and that there exists a sequence $\theta_n \downarrow \theta$ such that

$$\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta_n))}) = e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta_n))} \quad \text{for all } n.$$

Then we have

$$\begin{aligned}
&\left|\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}) - \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta_n))})\right| \\
&\quad = e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta_n))} - e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))} \\
&\quad \leqslant \frac{1}{a}e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta_n))}(\tilde{\mathcal{Y}}_1(t,\theta_n) - \tilde{\mathcal{Y}}_1(t,\theta)),
\end{aligned}$$

since $\tilde{\mathcal{Y}}_i(t,\,\cdot\,)$ is nondecreasing. Thus

$$\begin{aligned}
\left|\frac{d}{d\theta}\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\right| &\leqslant \lim_{\theta_n\downarrow\theta}\frac{1}{a}e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta_n))}\frac{\tilde{\mathcal{Y}}_1(t,\theta_n) - \tilde{\mathcal{Y}}_1(t,\theta)}{\theta_n-\theta} \\
&= \frac{1}{a}e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\tilde{\mathcal{Y}}_{1,\eta}(t,\theta) \\
&= \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\tilde{\mathcal{Y}}_{1,\eta}(t,\theta).
\end{aligned}$$

Thus in case (A.7) we find

$$\left|\frac{d}{d\theta}\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\right| \leqslant \frac{1}{a}\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\tilde{\mathcal{Y}}_{1,\eta}(t,\theta).$$

If we instead of (A.7) assume

$$\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}) = e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))},$$

a similar argument yields

$$\left|\frac{d}{d\theta}\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\right| \leqslant \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\tilde{\mathcal{Y}}_{2,\eta}(t,\theta).$$

Thus we conclude that in general we have

$$\left|\frac{d}{d\theta}\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\right| \leqslant \frac{1}{a}\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\max_j(\tilde{\mathcal{Y}}_{j,\eta})(t,\theta).$$

(ii): We have (cf. Lemma 5.1) that

$$\begin{aligned}
&\left|\min_j(\tilde{\mathcal{U}}_j^2(t,\eta)) - \min_j(\tilde{\mathcal{U}}_j^2(t,\theta))\right| \\
&\quad \leqslant \max(|\tilde{\mathcal{U}}_1^2(t,\eta) - \tilde{\mathcal{U}}_1^2(t,\theta)|, |\tilde{\mathcal{U}}_2^2(t,\eta) - \tilde{\mathcal{U}}_2^2(t,\eta)|) \\
&\quad \leqslant 2\max_j\|\tilde{\mathcal{U}}_j\tilde{\mathcal{U}}_{j\eta}\|_\infty|\eta-\theta| \leqslant A^4|\eta-\theta|, \qquad \text{(A.8)}
\end{aligned}$$

using (4.15f).

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(iii): Note that one has that for any positive function $m(x)$,

$$\begin{aligned}|m^3(x)-m^3(y)| &= (m^2(x)+m(x)m(y)+m^2(y))|m(x)-m(y)| \\ &\leqslant (m^2(x)+2m(x)m(y)+m^2(y))|m(x)-m(y)| \\ &= (m(x)+m(y))^2|m(x)-m(y)| \\ &= (m(x)+m(y))|m^2(x)-m^2(y)|.\end{aligned}$$

If we replace $m(x)$ by $\min_j(\tilde{\mathcal{U}}_j^+)(t,\eta)$, we have

$$\begin{aligned}&\big|\min_j(\tilde{\mathcal{U}}_j^+)^3(t,\eta)-\min_j(\tilde{\mathcal{U}}_j^+)^3(t,\theta)\big| \\ &\quad\leqslant \big(\min_j(\tilde{\mathcal{U}}_j^+)(t,\eta)+\min_j(\tilde{\mathcal{U}}_j^+)(t,\theta)\big)\big|\min_j(\tilde{\mathcal{U}}_j^+)^2(t,\eta)-\min_j(\tilde{\mathcal{U}}_j^+)^2(t,\theta)\big| \\ &\quad\leqslant \big(\min_j(\tilde{\mathcal{U}}_j^+)(t,\eta)+\min_j(\tilde{\mathcal{U}}_j^+)(t,\theta)\big) \\ &\quad\times \max(|(\tilde{\mathcal{U}}_1^+)^2(t,\eta)-(\tilde{\mathcal{U}}_1^+)^2(t,\theta)|,|(\tilde{\mathcal{U}}_2^+)^2(t,\eta)-(\tilde{\mathcal{U}}_2^+)^2(t,\theta)|) \\ &\quad\leqslant A^4(\min_j(\tilde{\mathcal{U}}_j^+)(t,\eta)+\min_j(\tilde{\mathcal{U}}_j^+)(t,\theta))|\eta-\theta|\end{aligned}$$

(see (A.8)), and hence $\min_j(\tilde{\mathcal{U}}_j^+)^3(t,\eta)$ is differentiable almost everywhere with

$$\left|\frac{d}{d\theta}\min_j(\tilde{\mathcal{U}}_j^+)^3(t,\theta)\right|\leqslant 2A^4\min_j(\tilde{\mathcal{U}}_j^+)(t,\theta). \qquad \square$$

LEMMA A.3. (i) *The function* $\eta\mapsto\min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)(t,\eta)$ *is Lipschitz continuous with a uniformly bounded Lipschitz constant and thus differentiable almost everywhere with*

$$\left|\frac{d}{d\eta}\big(\min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\big)(t,\eta)\right|\leqslant 2A^4(\min_j(\tilde{\mathcal{P}}_j)^{1/2}+|\tilde{\mathcal{U}}_k|)(t,\eta), \quad k=1,2.$$

(ii) *The function* $\eta\mapsto\min_j(\tilde{\mathcal{P}}_j)\tilde{\mathcal{U}}_k(t,\eta)$, $k=1,2$ *is Lipschitz continuous with a uniformly bounded Lipschitz constant and thus differentiable almost everywhere with*

$$\left|\frac{d}{d\eta}\big(\min_j(\tilde{\mathcal{P}}_j)\tilde{\mathcal{U}}_k\big)(t,\eta)\right|\leqslant\frac{1}{\sqrt{2}}A^6.$$

Proof. (i) We only present the proof for the case $k=2$, since the case $k=1$ is similar.

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Given $0 \leqslant \eta_1 < \eta_2 \leqslant 1$, we assume without loss of generality that

$$0 \leqslant \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+)(t, \eta_2) - \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+)(t, \eta_1).$$

To ease the notation, we introduce the function

$$d(t, \eta) = \min_j(\tilde{\mathcal{P}}_j) \min_j(\tilde{\mathcal{U}}_j^+)(t, \eta).$$

We will distinguish several cases:

(1): If $d(t, \eta_2) = 0$, then $d(t, \eta_1) = 0$ and one has

$$\begin{aligned}
0 &\leqslant \min_j(\tilde{\mathcal{U}}_j^+)(t, \eta_2) \big| \min_j(\tilde{\mathcal{P}}_j)(t, \eta_2) - \min_j(\tilde{\mathcal{P}}_j)(t, \eta_1) \big| \\
&\leqslant \min_j(\tilde{\mathcal{U}}_j^+)(t, \eta_2) |(|\tilde{\mathcal{P}}_1(t, \eta_2) - \tilde{\mathcal{P}}_1(t, \eta_1)| + |\tilde{\mathcal{P}}_2(t, \eta_2) - \tilde{\mathcal{P}}_2(t, \eta_1)|) \\
&\leqslant A^4 \min_j(\tilde{\mathcal{U}}_j^+)(t, \eta_2) |\eta_2 - \eta_1|.
\end{aligned}$$

(2): If $d(t, \eta_2) > 0$, then $\min_j(\tilde{\mathcal{P}}_j)(t, \eta_2) > 0$ and $\min_j(\tilde{\mathcal{U}}_j^+)(t, \eta_2) > 0$ or equivalently

$$\tilde{\mathcal{P}}_1(t, \eta_2) > 0, \quad \tilde{\mathcal{P}}_2(t, \eta_2) > 0, \quad \tilde{\mathcal{U}}_1(t, \eta_2) > 0, \quad \text{and} \quad \tilde{\mathcal{U}}_2(t, \eta_2) > 0.$$

(2a): Assume that $d(t, \eta_1) = 0$ and $\min_j(\tilde{\mathcal{P}}_j)(t, \eta_1) = 0$. Then

$$\begin{aligned}
0 &\leqslant d(t, \eta_2) - d(t, \eta_1) \\
&\leqslant \big(\min_j(\tilde{\mathcal{P}}_j)(t, \eta_2) - \min_j(\tilde{\mathcal{P}}_j)(t, \eta_1)\big) \min_j(\tilde{\mathcal{U}}_j^+)(t, \eta_2) \\
&\leqslant \min_j(\tilde{\mathcal{U}}_j^+)(t, \eta_2)(|\tilde{\mathcal{P}}_1(t, \eta_2) - \tilde{\mathcal{P}}_1(t, \eta_1)| + |\tilde{\mathcal{P}}_2(t, \eta_2) - \tilde{\mathcal{P}}_2(t, \eta_1)|) \\
&\leqslant A^4 \min_j(\tilde{\mathcal{U}}_j^+)(t, \eta_2) |\eta_2 - \eta_1|.
\end{aligned}$$

(2b): Assume that $d(t, \eta_1) = 0$ and $\min_j(\tilde{\mathcal{U}}_j^+)(t, \eta_1) = 0$. Then one either has that

(i): $\tilde{\mathcal{U}}_2(t, \eta_1) \leqslant \tilde{\mathcal{U}}_2^+(t, \eta_1) = \min_j(\tilde{\mathcal{U}}_j^+)(t, \eta_1)$, and we can write

$$\begin{aligned}
0 &\leqslant d(t, \eta_2) - d(t, \eta_1) \\
&\leqslant \tilde{\mathcal{U}}_2 \min_j(\tilde{\mathcal{P}}_j)(t, \eta_2) - \tilde{\mathcal{U}}_2 \min_j(\tilde{\mathcal{P}}_j)(t, \eta_1) \\
&\leqslant \int_{\eta_1}^{\eta_2} \Big(\tilde{\mathcal{U}}_{2,\eta} \min_j(\tilde{\mathcal{P}}_j) + \tilde{\mathcal{U}}_2 \frac{d}{d\theta} \min_j(\tilde{\mathcal{P}}_j)\Big)(t, s)\, ds
\end{aligned}$$

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$$\leqslant \int_{\eta_1}^{\eta_2} \left(|\tilde{\mathcal{U}}_{2,\eta}| \min_j(\tilde{\mathcal{P}}_j) + |\tilde{\mathcal{U}}_2| \left(\frac{1}{A_1} \tilde{\mathcal{P}}_1 \tilde{\mathcal{Y}}_{1,\eta} + \frac{1}{A_2} \tilde{\mathcal{P}}_2 \tilde{\mathcal{Y}}_{2,\eta} \right) \right)(t,s)\, ds$$
$$\leqslant \frac{A^4}{\sqrt{2}} \int_{\eta_1}^{\eta_2} \min_j(\tilde{\mathcal{P}}_j)^{1/2}(t,s)\, ds + A^4 \int_{\eta_1}^{\eta_2} |\tilde{\mathcal{U}}_2|(t,s)\, ds$$
$$\leqslant \frac{3}{2\sqrt{2}} A^6 |\eta_2 - \eta_1|,$$

or

(ii): $\min_j(\tilde{\mathcal{U}}_j^+)(t,\eta_1) = \tilde{\mathcal{U}}_1^+(t,\eta_1)$. Then there exists a maximal interval $[\eta_1, a]$ such that $\tilde{\mathcal{U}}_1^+(t,s) < \tilde{\mathcal{U}}_2^+(t,s)$ for all $s \in [\eta_1, a)$ and $\tilde{\mathcal{U}}_1^+(t,a) = \tilde{\mathcal{U}}_2^+(t,a)$. Moreover, there exists a maximal interval $[b,a] \subset [\eta_1, a]$ such that $\tilde{\mathcal{U}}_1^+(t,s) > 0$ for all $s \in (b,a]$ and $\tilde{\mathcal{U}}_1^+(t,b) = 0$. Hence we can write

$$\begin{aligned}
0 &\leqslant d(t,\eta_2) - d(t,\eta_1) \\
&\leqslant d(t,\eta_2) - d(t,a) + d(t,a) - d(t,b) \\
&\leqslant \tilde{\mathcal{U}}_2 \min_j(\tilde{\mathcal{P}}_j)(t,\eta_2) - \tilde{\mathcal{U}}_2 \min_j(\tilde{\mathcal{P}}_j)(t,a) \\
&\quad + \tilde{\mathcal{U}}_1 \min_j(\tilde{\mathcal{P}}_j)(t,a) - \tilde{\mathcal{U}}_1 \min_j(\tilde{\mathcal{P}}_j)(t,b) \\
&\leqslant \int_a^{\eta_2} \left(\tilde{\mathcal{U}}_{2,\eta} \min_j(\tilde{\mathcal{P}}_j) + \tilde{\mathcal{U}}_2 \frac{d}{d\eta} \min_j(\tilde{\mathcal{P}}_j) \right)(t,s)\, ds \\
&\quad + \int_b^a \left(\tilde{\mathcal{U}}_{1,\eta} \min_j(\tilde{\mathcal{P}}_j) + \tilde{\mathcal{U}}_1 \frac{d}{d\eta} \min_j(\tilde{\mathcal{P}}_j) \right)(t,s)\, ds \\
&\leqslant \int_a^{\eta_2} \left(\frac{A^4}{\sqrt{2}} \min_j(\tilde{\mathcal{P}}_j)^{1/2} + A^4 |\tilde{\mathcal{U}}_2| \right)(t,s)\, ds \\
&\quad + \int_b^a \left(\frac{A^4}{\sqrt{2}} \min_j(\tilde{\mathcal{P}}_j)^{1/2} + A^4 |\tilde{\mathcal{U}}_1| \right)(t,s)\, ds \\
&\leqslant \frac{A^4}{\sqrt{2}} \int_b^{\eta_2} \min_j(\tilde{\mathcal{P}}_j)^{1/2}(t,s)\, ds + A^4 \int_b^{\eta_2} |\tilde{\mathcal{U}}_2|(t,s)\, ds \\
&\leqslant \frac{A^4}{\sqrt{2}} \int_{\eta_1}^{\eta_2} \min_j(\tilde{\mathcal{P}}_j)^{1/2}(t,s)\, ds + A^4 \int_{\eta_1}^{\eta_2} |\tilde{\mathcal{U}}_2|(t,s)\, ds \\
&\leqslant \frac{3}{2\sqrt{2}} A^6 |\eta_2 - \eta_1|.
\end{aligned}$$

Note, in the case that $\tilde{\mathcal{U}}_1^+(t,s) < \tilde{\mathcal{U}}_2^+(t,s)$ for all $s \in [\eta_1, \eta_2]$, the estimate starts with

$$0 \leqslant d(t,\eta_2) - d(t,\eta_1) \leqslant \tilde{\mathcal{U}}_1 \min_j(\tilde{\mathcal{P}}_j)(t,\eta_2) - \tilde{\mathcal{U}}_1 \min_j(\tilde{\mathcal{P}}_j)(t,b),$$

where $(b, \eta_2]$ denotes the maximal interval such that $\tilde{\mathcal{U}}_1^+(t,s) > 0$ for all $s \in (b, \eta_2]$.

(2c): Assume that $d(t,\eta_1) > 0$; then $\min_j(\tilde{\mathcal{P}}_j)(t,\eta_1) > 0$ and $\min_j(\tilde{\mathcal{U}}_j^+)(t, \eta_1) = \min_j(\tilde{\mathcal{U}}_j)(t,\eta_1) > 0$. Then one either has that

(i): $\min_j(\tilde{\mathcal{U}}_j)(t,\eta_1) = \tilde{\mathcal{U}}_2(t,\eta_1)$, and we have (as before)

$$\begin{aligned}
0 &\leqslant d(t,\eta_2) - d(t,\eta_1)\\
&\leqslant \tilde{\mathcal{U}}_2 \min_j(\tilde{\mathcal{P}}_j)(t,\eta_2) - \tilde{\mathcal{U}}_2 \min_j(\tilde{\mathcal{P}}_j)(t,\eta_1)\\
&\leqslant \frac{A^4}{\sqrt{2}} \int_{\eta_1}^{\eta_2} \min_j(\tilde{\mathcal{P}}_j)^{1/2}(t,s)\,ds + A^4 \int_{\eta_1}^{\eta_2} |\tilde{\mathcal{U}}_2|(t,s)\,ds\\
&\leqslant \frac{3}{2\sqrt{2}} A^6 |\eta_2 - \eta_1|,
\end{aligned}$$

or

(ii): $\min_j(\tilde{\mathcal{U}}_j)(t,\eta_1) = \tilde{\mathcal{U}}_1(t,\eta_1)$. Then there exists a maximal interval $[\eta_1, a]$ such that $\tilde{\mathcal{U}}_1^+(t,s) < \tilde{\mathcal{U}}_2^+(t,s)$ for all $s \in [\eta_1, a)$ and $\tilde{\mathcal{U}}_1^+(t,a) = \tilde{\mathcal{U}}_2^+(t,a)$. Moreover, there exists a maximal interval $[b,a] \subset [\eta_1, a]$ such that $\tilde{\mathcal{U}}_1^+(t,s) > 0$ for all $s \in (b,a]$ and $\tilde{\mathcal{U}}_1^+(t,b) = 0$. Hence we can write

$$\begin{aligned}
0 &\leqslant d(t,\eta_2) - d(t,\eta_1)\\
&\leqslant d(t,\eta_2) - d(t,a) + d(t,a) - d(t,b)\\
&\leqslant \tilde{\mathcal{U}}_2 \min_j(\tilde{\mathcal{P}}_j)(t,\eta_2) - \tilde{\mathcal{U}}_2 \min_j(\tilde{\mathcal{P}}_j)(t,a)\\
&\quad + \tilde{\mathcal{U}}_1 \min_j(\tilde{\mathcal{P}}_j)(t,a) - \tilde{\mathcal{U}}_1 \min_j(\tilde{\mathcal{P}}_j)(t,b)\\
&\leqslant \int_a^{\eta_2} \left(\tilde{\mathcal{U}}_{2,\eta} \min_j(\tilde{\mathcal{P}}_j) + \tilde{\mathcal{U}}_2 \frac{d}{d\eta} \min_j(\tilde{\mathcal{P}}_j)\right)(t,s)\,ds\\
&\quad + \int_b^{a} \left(\tilde{\mathcal{U}}_{1,\eta} \min_j(\tilde{\mathcal{P}}_j) + \tilde{\mathcal{U}}_1 \frac{d}{d\eta} \min_j(\tilde{\mathcal{P}}_j)\right)(t,s)\,ds\\
&\leqslant \int_a^{\eta_2} \left(\frac{A^4}{\sqrt{2}} \min_j(\tilde{\mathcal{P}}_j)^{1/2} + A^4 |\tilde{\mathcal{U}}_2|\right)(t,s)\,ds\\
&\quad + \int_b^{a} \left(\frac{A^4}{\sqrt{2}} \min_j(\tilde{\mathcal{P}}_j)^{1/2} + A^4 |\tilde{\mathcal{U}}_1|\right)(t,s)\,ds\\
&\leqslant \frac{A^4}{\sqrt{2}} \int_b^{\eta_2} \min_j(\tilde{\mathcal{P}}_j)^{1/2}(t,s)\,ds + A^4 \int_b^{\eta_2} |\tilde{\mathcal{U}}_2|(t,s)\,ds
\end{aligned}$$

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$$\leqslant \frac{A^4}{\sqrt{2}} \int_{\eta_1}^{\eta_2} \min_j (\tilde{\mathcal{P}}_j)^{1/2}(t,s)\,ds + A^4 \int_{\eta_1}^{\eta_2} |\tilde{\mathcal{U}}_2|(t,s)\,ds$$
$$\leqslant \frac{3}{2\sqrt{2}} A^6 |\eta_2 - \eta_1|.$$

Note, in the case that $\tilde{\mathcal{U}}_1^+(t,s) < \tilde{\mathcal{U}}_2^+(t,s)$ for all $s \in [\eta_1, \eta_2]$, the estimate starts with

$$0 \leqslant d(t,\eta_2) - d(t,\eta_1) \leqslant \tilde{\mathcal{U}}_1 \min_j (\tilde{\mathcal{P}}_j)(t,\eta_2) - \tilde{\mathcal{U}}_1 \min_j (\tilde{\mathcal{P}}_j)(t,b),$$

where $(b, \eta_2]$ denotes the maximal interval such that $\tilde{\mathcal{U}}_1^+(t,s) > 0$ for all $s \in (b, \eta_2]$.

Thus we showed that

$$|d(t,\eta_2) - d(t,\eta_1)| \leqslant \frac{3}{2\sqrt{2}} A^6 |\eta_2 - \eta_1|,$$

or, in other words, $d(t,\cdot) = \min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)(t,\cdot)$ is Lipschitz continuous with Lipschitz constant $\frac{3}{2\sqrt{2}}A^6$, which is independent of time, and thus differentiable almost everywhere. Moreover, a close look reveals that

$$|d(t,\eta_2) - d(t,\eta_1)| \leqslant A^4 \min_j (\tilde{\mathcal{U}}_j^+)(t,\eta_2)|\eta_2 - \eta_1|$$
$$+ \frac{A^4}{\sqrt{2}} \left| \int_{\eta_1}^{\eta_2} \min_j (\tilde{\mathcal{P}}_j)^{1/2}(t,s)\,ds \right| + A^4 \left| \int_{\eta_1}^{\eta_2} |\tilde{\mathcal{U}}_2|(t,s)\,ds \right|.$$

Since both $|\tilde{\mathcal{U}}_2|(t,\cdot)$ and $\min_j(\tilde{\mathcal{P}}_j)^{1/2}(t,\cdot)$ are continuous, the fundamental theorem of calculus implies that

$$\left| \frac{d(t,\eta_2) - d(t,\eta_1)}{\eta_2 - \eta_1} \right| \leqslant A^4 \min_j (\tilde{\mathcal{U}}_j^+)(t,\eta_2)$$
$$+ \frac{A^4}{\sqrt{2}} \min_j (\tilde{\mathcal{P}}_j)^{1/2}(t,\tilde{\eta}) + A^4 |\tilde{\mathcal{U}}_2|(t,\tilde{\eta})$$

for some $\tilde{\eta}$ between η_1 and η_2. Letting $\eta_2 \to \eta_1$ we thus obtain for almost every η that

$$\left| \frac{d}{d\eta} \min_j (\tilde{\mathcal{P}}_j) \min_j (\tilde{\mathcal{U}}_j^+)(t,\eta) \right| = \left| \frac{d}{d\eta} d(t,\eta) \right|$$
$$\leqslant 2A^4 \big(\min_j (\tilde{\mathcal{P}}_j)^{1/2} + |\tilde{\mathcal{U}}_2| \big)(t,\eta).$$

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(ii) We have that

$$\begin{aligned}
&\big|\min_j(\tilde{\mathcal{P}}_j)(t,\eta) - \min_j(\tilde{\mathcal{P}}_j)(t,\tilde{\eta})\big| \\
&\quad \leqslant \max(|\tilde{\mathcal{P}}_1(t,\eta) - \tilde{\mathcal{P}}_1(t,\tilde{\eta})|, |\tilde{\mathcal{P}}_2(t,\eta) - \tilde{\mathcal{P}}_2(t,\tilde{\eta})|) \\
&\quad \leqslant \frac{A^4}{2}|\eta - \tilde{\eta}|.
\end{aligned}$$

Thus, for almost every η,

$$\begin{aligned}
\left|\frac{d}{d\eta}\big(\min_j(\tilde{\mathcal{P}}_j)\tilde{\mathcal{U}}_k\big)(t,\eta)\right| &= \left|\left(\frac{d}{d\eta}\min_j(\tilde{\mathcal{P}}_j)\right)\tilde{\mathcal{U}}_k(t,\eta) + \min_j(\tilde{\mathcal{P}}_j)\tilde{\mathcal{U}}_{k,\eta}(t,\eta)\right| \\
&\leqslant \frac{A^4}{2}\|\tilde{\mathcal{U}}_k\|_\infty + \|\tilde{\mathcal{P}}_k\tilde{\mathcal{U}}_{k,\eta}\|_\infty \\
&\leqslant \frac{1}{\sqrt{2}}A^6. \qquad \square
\end{aligned}$$

LEMMA A.4. (i) *The function* $\eta \mapsto \min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)(t,\eta)$ *is Lipschitz continuous with a uniformly bounded Lipschitz constant and thus differentiable almost everywhere with*

$$\begin{aligned}
&\left|\frac{d}{d\eta}\big(\min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\big)(t,\eta)\right| \\
&\quad \leqslant \mathcal{O}(1)\sqrt{A}A^4(\min_j(\tilde{\mathcal{D}}_j)^{1/2} + |\tilde{\mathcal{U}}_k|)(t,\eta), \quad k = 1, 2.
\end{aligned}$$

(ii) *The function* $\eta \mapsto \min_j(\tilde{\mathcal{D}}_j)\tilde{\mathcal{U}}_k(t,\eta)$, $k = 1, 2$, *is Lipschitz continuous with a uniformly bounded Lipschitz constant and thus differentiable almost everywhere with*

$$\left|\frac{d}{d\eta}\big(\min_j(\tilde{\mathcal{D}}_j)\tilde{\mathcal{U}}_k\big)(t,\eta)\right| \leqslant \mathcal{O}(1)A^7.$$

Proof. (i) We only present the proof for the case $k = 2$, since the case $k = 1$ is similar.

Given $0 \leqslant \eta_1 < \eta_2 \leqslant 1$, we assume without loss of generality that

$$0 \leqslant \min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)(t,\eta_2) - \min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)(t,\eta_1).$$

To ease the notation, we introduce the function

$$\bar{d}(t,\eta) = \min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)(t,\eta).$$

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We will distinguish several cases:

(1): If $\bar{d}(t,\eta_2)=0$, then $\bar{d}(t,\eta_1)=0$ and one has

$$\begin{aligned}0 &\leqslant \min_j(\tilde{\mathcal{U}}_j^+)(t,\eta_2)\big|\min_j(\tilde{\mathcal{D}}_j)(t,\eta_2)-\min_j(\tilde{\mathcal{D}}_j)(t,\eta_1)\big|\\&\leqslant \min_j(\tilde{\mathcal{U}}_j^+)(t,\eta_2)|(|\tilde{\mathcal{D}}_1(t,\eta_2)-\tilde{\mathcal{D}}_1(t,\eta_1)|+|\tilde{\mathcal{D}}_2(t,\eta_2)-\tilde{\mathcal{D}}_2(t,\eta_1)|)\\&\leqslant \mathcal{O}(1)A^5\min_j(\tilde{\mathcal{U}}_j^+)(t,\eta_2)|\eta_2-\eta_1|.\end{aligned}$$

(2): If $\bar{d}(t,\eta_2)>0$, then $\min_j(\tilde{\mathcal{D}}_j)(t,\eta_2)>0$ and $\min_j(\tilde{\mathcal{U}}_j^+)(t,\eta_2)>0$ or equivalently

$$\tilde{\mathcal{D}}_1(t,\eta_2)>0,\quad \tilde{\mathcal{D}}_2(t,\eta_2)>0,\quad \tilde{\mathcal{U}}_1(t,\eta_2)>0,\quad\text{and}\quad \tilde{\mathcal{U}}_2(t,\eta_2)>0.$$

(2a): Assume that $\bar{d}(t,\eta_1)=0$ and $\min_j(\tilde{\mathcal{D}}_j)(t,\eta_1)=0$. Then

$$\begin{aligned}0 &\leqslant \bar{d}(t,\eta_2)-\bar{d}(t,\eta_1)\\&\leqslant \big(\min_j(\tilde{\mathcal{D}}_j)(t,\eta_2)-\min_j(\tilde{\mathcal{D}}_j)(t,\eta_1)\big)\min_j(\tilde{\mathcal{U}}_j^+)(t,\eta_2)\\&\leqslant \min_j(\tilde{\mathcal{U}}_j^+)(t,\eta_2)(|\tilde{\mathcal{D}}_1(t,\eta_2)-\tilde{\mathcal{D}}_1(t,\eta_1)|+|\tilde{\mathcal{D}}_2(t,\eta_2)-\tilde{\mathcal{D}}_2(t,\eta_1)|)\\&\leqslant \mathcal{O}(1)A^5\min_j(\tilde{\mathcal{U}}_j^+)(t,\eta_2)|\eta_2-\eta_1|.\end{aligned}$$

(2b): Assume that $\bar{d}(t,\eta_1)=0$ and $\min_j(\tilde{\mathcal{U}}_j^+)(t,\eta_1)=0$. Then one either has that

(i): $\tilde{\mathcal{U}}_2(t,\eta_1)\leqslant\tilde{\mathcal{U}}_2^+(t,\eta_1)=\min_j(\tilde{\mathcal{U}}_j^+)(t,\eta_1)$ and we can write

$$\begin{aligned}0 &\leqslant d(t,\eta_2)-d(t,\eta_1)\\&\leqslant \tilde{\mathcal{U}}_2\min_j(\tilde{\mathcal{D}}_j)(t,\eta_2)-\tilde{\mathcal{U}}_2\min_j(\tilde{\mathcal{D}}_j)(t,\eta_1)\\&\leqslant \int_{\eta_1}^{\eta_2}\tilde{\mathcal{U}}_{2,\eta}\min_j(\tilde{\mathcal{D}}_j)(t,s)+\tilde{\mathcal{U}}_2\frac{d}{d\theta}\min_j(\tilde{\mathcal{D}}_j)(t,s)\,ds\\&\leqslant \int_{\eta_1}^{\eta_2}|\tilde{\mathcal{U}}_{2,\eta}|\min_j(\tilde{\mathcal{D}}_j)(t,s)\,ds\\&\quad+\int_{\eta_1}^{\eta_2}|\tilde{\mathcal{U}}_2|\bigg(\bigg|(\tilde{\mathcal{U}}_1^2-\tilde{\mathcal{P}}_1)\tilde{\mathcal{Y}}_{1,\eta}-\frac{1}{A_1}\tilde{\mathcal{D}}_1\tilde{\mathcal{Y}}_{1,\eta}+\frac{1}{2}A_1^5\bigg|\\&\quad+\bigg|(\tilde{\mathcal{U}}_2^2-\tilde{\mathcal{P}}_2)\tilde{\mathcal{Y}}_{2,\eta}-\frac{1}{A_2}\tilde{\mathcal{D}}_2\tilde{\mathcal{Y}}_{2,\eta}+\frac{1}{2}A_2^5\bigg|\bigg)(t,s)\,ds\end{aligned}$$

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$$\leqslant \sqrt{A}A^4 \int_{\eta_1}^{\eta_2} \min_j(\tilde{\mathcal{D}}_j)^{1/2}(t,s)\,ds + \mathcal{O}(1)A^5 \int_{\eta_1}^{\eta_2} |\tilde{\mathcal{U}}_2|(t,s)\,ds$$
$$\leqslant \mathcal{O}(1)A^7|\eta_2-\eta_1|$$

or

(ii): $\min_j(\tilde{\mathcal{U}}_j^+)(t,\eta_1) = \tilde{\mathcal{U}}_1^+(t,\eta_1)$. Then there exists a maximal interval $[\eta_1, a]$ such that $\tilde{\mathcal{U}}_1^+(t,s) < \tilde{\mathcal{U}}_2^+(t,s)$ for all $s \in [\eta_1, a)$ and $\tilde{\mathcal{U}}_1^+(t,a) = \tilde{\mathcal{U}}_2^+(t,a)$. Moreover, there exists a maximal interval $[b,a] \subset [\eta_1,a]$ such that $\tilde{\mathcal{U}}_1^+(t,s) > 0$ for all $s \in (b,a]$ and $\tilde{\mathcal{U}}_1^+(t,b) = 0$. Hence we can write

$$\begin{aligned}
0 &\leqslant \bar{d}(t,\eta_2) - \bar{d}(t,\eta_1)\\
&\leqslant \bar{d}(t,\eta_2) - \bar{d}(t,a) + \bar{d}(t,a) - \bar{d}(t,b)\\
&\leqslant \tilde{\mathcal{U}}_2 \min_j(\tilde{\mathcal{D}}_j)(t,\eta_2) - \tilde{\mathcal{U}}_2 \min_j(\tilde{\mathcal{D}}_j)(t,a)\\
&\quad + \tilde{\mathcal{U}}_1 \min_j(\tilde{\mathcal{D}}_j)(t,a) - \tilde{\mathcal{U}}_1 \min_j(\tilde{\mathcal{D}}_j)(t,b)\\
&\leqslant \int_a^{\eta_2} \Big(\tilde{\mathcal{U}}_{2,\eta} \min_j(\tilde{\mathcal{D}}_j) + \tilde{\mathcal{U}}_2 \frac{d}{d\eta} \min_j(\tilde{\mathcal{D}}_j)\Big)(t,s)\,ds\\
&\quad + \int_b^{a} \Big(\tilde{\mathcal{U}}_{1,\eta} \min_j(\tilde{\mathcal{D}}_j) + \tilde{\mathcal{U}}_1 \frac{d}{d\eta} \min_j(\tilde{\mathcal{D}}_j)\Big)(t,s)\,ds\\
&\leqslant \int_a^{\eta_2} (\sqrt{A}A^4 \min_j(\tilde{\mathcal{D}}_j)^{1/2} + \mathcal{O}(1)A^5|\tilde{\mathcal{U}}_2|)(t,s)\,ds\\
&\quad + \int_b^{a} (\sqrt{A}A^4 \min_j(\tilde{\mathcal{D}}_j)^{1/2} + \mathcal{O}(1)A^5|\tilde{\mathcal{U}}_1|)(t,s)\,ds\\
&\leqslant \sqrt{A}A^4 \int_b^{\eta_2} \min_j(\tilde{\mathcal{D}}_j)^{1/2}(t,s)\,ds + \mathcal{O}(1)A^5 \int_b^{\eta_2} |\tilde{\mathcal{U}}_2|(t,s)\,ds\\
&\leqslant \sqrt{A}A^4 \int_{\eta_1}^{\eta_2} \min_j(\tilde{\mathcal{D}}_j)^{1/2}(t,s)\,ds + \mathcal{O}(1)A^5 \int_{\eta_1}^{\eta_2} |\tilde{\mathcal{U}}_2|(t,s)\,ds\\
&\leqslant \mathcal{O}(1)A^7|\eta_2-\eta_1|.
\end{aligned}$$

Note, in the case that $\tilde{\mathcal{U}}_1^+(t,s) < \tilde{\mathcal{U}}_2^+(t,s)$ for all $s \in [\eta_1,\eta_2]$, the estimate starts with

$$0 \leqslant d(t,\eta_2) - d(t,\eta_1) \leqslant \tilde{\mathcal{U}}_1 \min_j(\tilde{\mathcal{D}}_j)(t,\eta_2) - \tilde{\mathcal{U}}_1 \min_j(\tilde{\mathcal{D}}_j)(t,b),$$

where $(b,\eta_2]$ denotes the maximal interval such that $\tilde{\mathcal{U}}_1^+(t,s) > 0$ for all $s \in (b,\eta_2]$.

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(2c): Assume that $\bar d(t,\eta_1)>0$; then $\min_j(\tilde{\mathcal{D}}_j)(t,\eta_1)>0$ and $\min_j(\tilde{\mathcal{U}}_j^+)(t,\eta_1)=\min_j(\tilde{\mathcal{U}}_j)(t,\eta_1)>0$. Then one either has that

(i): $\min_j(\tilde{\mathcal{U}}_j)(t,\eta_1)=\tilde{\mathcal{U}}_2(t,\eta_1)$, and we have (as before)

$$\begin{aligned}
0&\leqslant \bar d(t,\eta_2)-\bar d(t,\eta_1)\\
&\leqslant \tilde{\mathcal{U}}_2\min_j(\tilde{\mathcal{D}}_j)(t,\eta_2)-\tilde{\mathcal{U}}_2\min_j(\tilde{\mathcal{D}}_j)(t,\eta_1)\\
&\leqslant \int_{\eta_1}^{\eta_2}\Big(\frac{1}{2}\sqrt{A}A^6\min_j(\tilde{\mathcal{D}}_j)^{1/2}+\mathcal{O}(1)A^5|\tilde{\mathcal{U}}_2|\Big)(t,s)\,ds\\
&\leqslant \mathcal{O}(1)A^7|\eta_2-\eta_1|,
\end{aligned}$$

or

(ii): $\min_j(\tilde{\mathcal{U}}_j)(t,\eta_1)=\tilde{\mathcal{U}}_1(t,\eta_1)$. Then there exists a maximal interval $[\eta_1,a]$ such that $\tilde{\mathcal{U}}_1^+(t,s)<\tilde{\mathcal{U}}_2^+(t,s)$ for all $s\in[\eta_1,a)$ and $\tilde{\mathcal{U}}_1^+(t,a)=\tilde{\mathcal{U}}_2^+(t,a)$. Moreover, there exists a maximal interval $[b,a]\subset[\eta_1,a]$ such that $\tilde{\mathcal{U}}_1^+(t,s)>0$ for all $s\in(b,a]$ and $\tilde{\mathcal{U}}_1^+(t,b)=0$. Hence we can write

$$\begin{aligned}
0&\leqslant \bar d(t,\eta_2)-\bar d(t,\eta_1)\\
&\leqslant \bar d(t,\eta_2)-\bar d(t,a)+\bar d(t,a)-\bar d(t,b)\\
&\leqslant \tilde{\mathcal{U}}_2\min_j(\tilde{\mathcal{D}}_j)(t,\eta_2)-\tilde{\mathcal{U}}_2\min_j(\tilde{\mathcal{D}}_j)(t,a)\\
&\quad+\tilde{\mathcal{U}}_1\min_j(\tilde{\mathcal{D}}_j)(t,a)-\tilde{\mathcal{U}}_1\min_j(\tilde{\mathcal{D}}_j)(t,b)\\
&\leqslant \int_a^{\eta_2}\Big(\tilde{\mathcal{U}}_{2,\eta}\min_j(\tilde{\mathcal{D}}_j)+\tilde{\mathcal{U}}_2\frac{d}{d\eta}\min_j(\tilde{\mathcal{D}}_j)\Big)(t,s)\,ds\\
&\quad+\int_b^{a}\Big(\tilde{\mathcal{U}}_{1,\eta}\min_j(\tilde{\mathcal{D}}_j)+\tilde{\mathcal{U}}_1\frac{d}{d\eta}\min_j(\tilde{\mathcal{D}}_j)\Big)(t,s)\,ds\\
&\leqslant \int_a^{\eta_2}\Big(\frac{1}{2}\sqrt{A}A^6\min_j(\tilde{\mathcal{D}}_j)^{1/2}+\mathcal{O}(1)A^5|\tilde{\mathcal{U}}_2|\Big)(t,s)\,ds\\
&\quad+\int_b^{a}\Big(\frac{1}{2}\sqrt{A}A^6\min_j(\tilde{\mathcal{D}}_j)^{1/2}+\mathcal{O}(1)A^5|\tilde{\mathcal{U}}_1|\Big)(t,s)\,ds\\
&\leqslant \sqrt{A}A^4\int_b^{\eta_2}\min_j(\tilde{\mathcal{D}}_j)^{1/2}(t,s)\,ds+\mathcal{O}(1)A^5\int_b^{\eta_2}|\tilde{\mathcal{U}}_2|(t,s)\,ds\\
&\leqslant \sqrt{A}A^4\int_{\eta_1}^{\eta_2}\min_j(\tilde{\mathcal{D}}_j)^{1/2}(t,s)\,ds+\mathcal{O}(1)A^5\int_{\eta_1}^{\eta_2}|\tilde{\mathcal{U}}_2|(t,s)\,ds\\
&\leqslant \mathcal{O}(1)A^7|\eta_2-\eta_1|.
\end{aligned}$$

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Note, in the case that $\tilde{\mathcal{U}}_1^+(t,s) < \tilde{\mathcal{U}}_2^+(t,s)$ for all $s \in [\eta_1, \eta_2]$, the estimate starts with

$$0 \leqslant d(t,\eta_2) - d(t,\eta_1) \leqslant \tilde{\mathcal{U}}_1 \min_j(\tilde{\mathcal{D}}_j)(t,\eta_2) - \tilde{\mathcal{U}}_1 \min_j(\tilde{\mathcal{D}}_j)(t,b),$$

where $(b, \eta_2]$ denotes the maximal interval such that $\tilde{\mathcal{U}}_1^+(t,s) > 0$ for all $s \in (b, \eta_2]$.

Thus we showed that

$$|\bar{d}(t,\eta_2) - \bar{d}(t,\eta_1)| \leqslant \mathcal{O}(1)A^7|\eta_2 - \eta_1|,$$

or, in other words, $\bar{d}(t,\cdot) = \min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)(t,\cdot)$ is Lipschitz continuous with Lipschitz constant $\mathcal{O}(1)A^7$, which is independent of time, and thus differentiable almost everywhere. Moreover, a close look reveals that

$$\begin{aligned}
&|\bar{d}(t,\eta_2) - \bar{d}(t,\eta_1)| \\
&\quad \leqslant \mathcal{O}(1)A^5 \min_j(\tilde{\mathcal{U}}_j^+)(t,\eta_2)|\eta_2 - \eta_1| \\
&\qquad + \sqrt{A}A^4\left|\int_{\eta_1}^{\eta_2} \min_j(\tilde{\mathcal{D}}_j)^{1/2}(t,s)\,ds\right| + \mathcal{O}(1)A^5\left|\int_{\eta_1}^{\eta_2} |\tilde{\mathcal{U}}_2|(t,s)\,ds\right|.
\end{aligned}$$

Since both $|\tilde{\mathcal{U}}_2|(t,\cdot)$ and $\min_j(\tilde{\mathcal{D}}_j)^{1/2}(t,\cdot)$ are continuous, the fundamental theorem of calculus implies that

$$\begin{aligned}
\left|\frac{\bar{d}(t,\eta_2) - \bar{d}(t,\eta_1)}{\eta_2 - \eta_1}\right| &\leqslant \mathcal{O}(1)A^5 \min_j(\tilde{\mathcal{U}}_j^+)(t,\eta_2) \\
&\quad + \sqrt{A}A^4 \min_j(\tilde{\mathcal{D}}_j)^{1/2}(t,\tilde{\eta}) + \mathcal{O}(1)A^5|\tilde{\mathcal{U}}_2|(t,\tilde{\eta})
\end{aligned}$$

for some $\tilde{\eta}$ between η_1 and η_2. Letting $\eta_2 \to \eta_1$ we thus obtain for almost every η that

$$\begin{aligned}
\left|\frac{d}{d\eta}\min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)(t,\eta)\right| &= \left|\frac{d}{d\eta}\bar{d}(t,\eta)\right| \\
&\leqslant \mathcal{O}(1)\sqrt{A}A^4\big(\min_j(\tilde{\mathcal{D}}_j)^{1/2} + |\tilde{\mathcal{U}}_2|\big)(t,\eta).
\end{aligned}$$

(ii) We have that

$$\begin{aligned}
&\big|\min_j(\tilde{\mathcal{D}}_j)(t,\eta) - \min_j(\tilde{\mathcal{D}}_j)(t,\tilde{\eta})\big| \\
&\quad \leqslant \max(|\tilde{\mathcal{D}}_1(t,\eta) - \tilde{\mathcal{D}}_1(t,\tilde{\eta})|, |\tilde{\mathcal{D}}_2(t,\eta) - \tilde{\mathcal{D}}_2(t,\tilde{\eta})|) \\
&\quad \leqslant \mathcal{O}(1)A^5|\eta - \tilde{\eta}|,
\end{aligned}$$

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and hence, for almost every η,

$$\left|\frac{d}{d\eta}\min_j(\tilde{\mathcal{D}}_j)(t,\eta)\right| \leqslant \mathcal{O}(1)A^5.$$

This implies, for almost every η,

$$\begin{aligned}\left|\frac{d}{d\eta}\big(\min_j(\tilde{\mathcal{D}}_j)\tilde{\mathcal{U}}_k\big)(t,\eta)\right| &= \left|\left(\frac{d}{d\eta}\min_j(\tilde{\mathcal{D}}_j)\right)\tilde{\mathcal{U}}_k(t,\eta) + \min_j(\tilde{\mathcal{D}}_j)\tilde{\mathcal{U}}_{k,\eta}(t,\eta)\right|\\ &\leqslant \mathcal{O}(1)A^7 + 2A\|\tilde{\mathcal{P}}_k|\tilde{\mathcal{U}}_{k,\eta}|\|\\ &\leqslant \mathcal{O}(1)A^7. \end{aligned}$$

□

LEMMA A.5. *The function*

$$\eta \mapsto a\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{U}}_j^+)(t,\theta)\,d\theta$$

is Lipschitz continuous with a uniformly bounded Lipschitz constant and thus differentiable almost everywhere with

$$\left|\frac{d}{d\eta}\left(a\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{U}}_j^+)(t,\theta)\,d\theta\right)\right| \leqslant \mathcal{O}(1)A^2.$$

Proof. To prove the existence and boundedness of the derivative, we will prove Lipschitz continuity. Let $0 \leqslant \eta_1 < \eta_2 \leqslant 1$; then

$$\begin{aligned} a\bigg|&\int_0^{\eta_1} \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_1)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{U}}_j^+)(t,\theta)\,d\theta\\ &\quad-\int_0^{\eta_2} \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_2)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{U}}_j^+)(t,\theta)\,d\theta\bigg|\\ &\leqslant a\int_{\eta_1}^{\eta_2} \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_2)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{U}}_j^+)(t,\theta)\,d\theta\\ &\quad+a\int_0^{\eta_1}\big(\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_1)-\tilde{\mathcal{Y}}_j(t,\theta))})\\ &\quad-\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_2)-\tilde{\mathcal{Y}}_j(t,\theta))})\big)\min_j(\tilde{\mathcal{U}}_j^+)(t,\theta)\,d\theta\\ &\leqslant \frac{aA^2}{\sqrt{2}}|\eta_2-\eta_1|\\ &\quad+a\int_0^{\eta_1}(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta_1)-\tilde{\mathcal{Y}}_1(t,\theta))}-e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\eta_2)-\tilde{\mathcal{Y}}_1(t,\theta))})\min_j(\tilde{\mathcal{U}}_j^+)(t,\theta)\,d\theta \end{aligned}$$

$$
\begin{aligned}
&+a\int_0^{\eta_1}(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta_1)-\tilde{\mathcal{Y}}_2(t,\theta))}-e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\eta_2)-\tilde{\mathcal{Y}}_2(t,\theta))})\min_j(\tilde{\mathcal{U}}_j^+)(t,\theta)\,d\theta\\
&\leqslant \frac{aA^2}{\sqrt{2}}|\eta_2-\eta_1|\\
&\quad+\int_0^{\eta_1}\int_{\eta_1}^{\eta_2} e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,s)-\tilde{\mathcal{Y}}_1(t,\theta))}\tilde{\mathcal{Y}}_{1,\eta}(t,s)\,ds\min_j(\tilde{\mathcal{U}}_j^+)(t,\theta)\,d\theta\\
&\quad+\int_0^{\eta_1}\int_{\eta_1}^{\eta_2} e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,s)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{Y}}_{2,\eta}(t,s)\,ds\min_j(\tilde{\mathcal{U}}_j^+)(t,\theta)\,d\theta\\
&\leqslant \frac{aA^2}{\sqrt{2}}|\eta_2-\eta_1|\\
&\quad+\int_{\eta_1}^{\eta_2}\left(\int_0^s e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,s)-\tilde{\mathcal{Y}}_1(t,\theta))}\tilde{\mathcal{U}}_1^+(t,\theta)\,d\theta\right)\tilde{\mathcal{Y}}_{1,\eta}(t,s)\,ds\\
&\quad+\int_{\eta_1}^{\eta_2}\left(\int_0^s e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,s)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^+(t,\theta)\,d\theta\right)\tilde{\mathcal{Y}}_{2,\eta}(t,s)\,ds\\
&\leqslant \frac{aA^2}{\sqrt{2}}|\eta_2-\eta_1|+\frac{1}{A_1^5}\int_{\eta_1}^{\eta_2}\left(\int_0^s e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,s)-\tilde{\mathcal{Y}}_1(t,\theta))}\right.\\
&\times\left.\left(\frac{1}{A_1}\tilde{\mathcal{P}}_1^2\tilde{\mathcal{Y}}_{1,\eta}+A_1\tilde{\mathcal{U}}_1^2\tilde{\mathcal{Y}}_{1,\eta}+\tilde{\mathcal{U}}_1^+\tilde{\mathcal{H}}_{1,\eta}\right)(t,\theta)\,d\theta\right)\tilde{\mathcal{Y}}_{1,\eta}(t,s)\,ds\\
&\quad+\frac{1}{A_2^5}\int_{\eta_1}^{\eta_2}\left(\int_0^s e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,s)-\tilde{\mathcal{Y}}_2(t,\theta))}\right.\\
&\quad\times\left.\left(\frac{1}{A_2}\tilde{\mathcal{P}}_2^2\tilde{\mathcal{Y}}_{2,\eta}+A_2\tilde{\mathcal{U}}_2^2\tilde{\mathcal{Y}}_{2,\eta}+\tilde{\mathcal{U}}_2^+\tilde{\mathcal{H}}_{2,\eta}\right)(t,\theta)\,d\theta\right)\tilde{\mathcal{Y}}_{2,\eta}(t,s)\,ds\\
&\leqslant \frac{aA^2}{\sqrt{2}}|\eta_2-\eta_1|\\
&\quad+\mathcal{O}(1)\frac{1}{A_1^3}\int_{\eta_1}^{\eta_2}\tilde{\mathcal{P}}_1\tilde{\mathcal{Y}}_{1,\eta}(t,s)\,ds\\
&\quad+\mathcal{O}(1)\frac{1}{A_2^3}\int_{\eta_1}^{\eta_2}\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,s)\,ds\\
&\leqslant \mathcal{O}(1)A^2|\eta_2-\eta_1|. \qquad \square
\end{aligned}
$$

LEMMA A.6. *The function*

$$
\eta\mapsto\min_k\left(\int_0^{\eta}\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{k,\eta}(t,\theta)\,d\theta\right)
$$

is Lipschitz continuous with a uniformly bounded Lipschitz constant and thus differentiable almost everywhere. The derivative satisfies,

$$\left|\frac{d}{d\eta}\min_k\left(\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{k,\eta}(t,\theta)\,d\theta\right)\right| \leqslant \mathcal{O}(1)A^7. \tag{A.9}$$

Proof. Introduce

$$\tilde{a}(t,\eta) = \min_k\left(\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{k,\eta}(t,\theta)\,d\theta\right)$$
$$\tilde{b}(t,\eta) = \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta$$
$$\tilde{c}(t,\eta) = \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)d\theta.$$

Thus

$$\tilde{a}(t,\eta) = \min(\tilde{b},\tilde{c})(t,\eta).$$

Then we have to show that $\tilde{a}(t,\cdot)$ is Lipschitz continuous with a Lipschitz constant, which only depends on A. Clearly, we have that

$$|\tilde{a}(t,\eta_1)-\tilde{a}(t,\eta_2)| \leqslant \max(|\tilde{b}(t,\eta_1)-\tilde{b}(t,\eta_2)|, |\tilde{c}(t,\eta_1)-\tilde{c}(t,\eta_2)|),$$

and it suffices to show that both $\tilde{b}(t,\cdot)$ and $\tilde{c}(t,\cdot)$ are Lipschitz continuous with a Lipschitz constant, which only depends on A. We are only going to establish the Lipschitz continuity for $\tilde{b}$, since the argument for $\tilde{c}$ follows the same lines.

Let $0 \leqslant \eta_1 < \eta_2 \leqslant 1$. Then we have to consider two cases:

$$0 \leqslant \tilde{b}(t,\eta_1)-\tilde{b}(t,\eta_2) \quad \text{and} \quad 0 \leqslant \tilde{b}(t,\eta_2)-\tilde{b}(t,\eta_1).$$

(i): $0 \leqslant \tilde{b}(t,\eta_2)-\tilde{b}(t,\eta_1)$: By definition, we have

$$\begin{aligned}\tilde{b}(t,\eta_2)-\tilde{b}(t,\eta_1) &= \int_0^{\eta_2} \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_2)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{P}}_j)\\ &\quad\times \min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\\ &\quad- \int_0^{\eta_1} \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_1)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{P}}_j)\\ &\quad\times \min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\\ &= \int_{\eta_1}^{\eta_2} \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_2)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{P}}_j)\end{aligned}$$

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$$
\begin{aligned}
&\times \min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\\
&+\int_0^{\eta_1}\Big(\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_2)-\tilde{\mathcal{Y}}_j(t,\theta))})\\
&-\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_1)-\tilde{\mathcal{Y}}_j(t,\theta))})\Big)\\
&\times \min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\\
\leqslant{}& \int_{\eta_1}^{\eta_2}\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_2)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{P}}_j)\\
&\times \min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta,
\end{aligned}
$$

where we used in the last step that $0 \leqslant \min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{1,\eta}(t,\eta)$ and that $\tilde{\mathcal{Y}}_1(t,\eta)$ is increasing, which implies

$$
\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_1)-\tilde{\mathcal{Y}}_j(t,\theta))}) \geqslant \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_2)-\tilde{\mathcal{Y}}_j(t,\theta))}).
$$

Moreover, note that $\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_2)-\tilde{\mathcal{Y}}_j(t,\theta))}) \leqslant 1$, for $0 \leqslant \eta_1 \leqslant \theta \leqslant \eta_2$, and that

$$
0 \leqslant 2\sqrt{2}\min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta) \leqslant A_1^7 \leqslant A^7,
$$

by (4.15b) and (4.15e). Thus

$$
\begin{aligned}
0 &\leqslant \tilde{b}(t,\eta_2)-\tilde{b}(t,\eta_1)\\
&\leqslant \int_{\eta_1}^{\eta_2}\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_2)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\\
&\leqslant \frac{A^7}{2\sqrt{2}}|\eta_2-\eta_1|.
\end{aligned}
$$

(ii): $0 \leqslant \tilde{b}(t,\eta_1)-\tilde{b}(t,\eta_2)$: By definition, we have

$$
\begin{aligned}
0 &\leqslant \tilde{b}(t,\eta_1)-\tilde{b}(t,\eta_2)\\
&= \int_0^{\eta_1}\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_1)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\\
&\quad-\int_0^{\eta_2}\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_2)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\\
&= \int_0^{\eta_1}(\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_1)-\tilde{\mathcal{Y}}_j(t,\theta))})-\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_2)-\tilde{\mathcal{Y}}_j(t,\theta))}))
\end{aligned}
$$

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$$
\begin{aligned}
&\times \min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\\
&-\int_{\eta_1}^{\eta_2}\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_2)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\\
\leqslant&\int_0^{\eta_1}(\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_1)-\tilde{\mathcal{Y}}_j(t,\theta))})-\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_2)-\tilde{\mathcal{Y}}_j(t,\theta))}))\\
&\times \min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta.
\end{aligned}
$$

Now we have to be much more careful than before. Namely, we have (as before)

$$
\begin{aligned}
0&\leqslant \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_1)-\tilde{\mathcal{Y}}_j(t,\theta))})-\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_2)-\tilde{\mathcal{Y}}_j(t,\theta))})\\
&\leqslant \frac{1}{a}\int_{\eta_1}^{\eta_2}(e^{\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\theta)-\tilde{\mathcal{Y}}_1(t,s))}\tilde{\mathcal{Y}}_{1,\eta}(t,s)+e^{\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,s))}\tilde{\mathcal{Y}}_{2,\eta}(t,s))\,ds.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\tilde{b}(t,\eta_1)-\tilde{b}(t,\eta_2)\\
&\quad\leqslant \int_0^{\eta_1}\Big(\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_1)-\tilde{\mathcal{Y}}_j(t,\theta))})-\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_2)-\tilde{\mathcal{Y}}_j(t,\theta))})\Big)\\
&\quad\times \min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\\
&\quad\leqslant \frac{1}{a}\int_0^{\eta_1}\int_{\eta_1}^{\eta_2}e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,s)-\tilde{\mathcal{Y}}_1(t,\theta))}\tilde{\mathcal{Y}}_{1,\eta}(t,s)\,ds\min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\\
&\qquad+\frac{1}{a}\int_0^{\eta_1}\int_{\eta_1}^{\eta_2}e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,s)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{Y}}_{2,\eta}(t,s)\,ds\min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\\
&\quad=\frac{1}{a}\int_{\eta_1}^{\eta_2}\tilde{\mathcal{Y}}_{1,\eta}(t,s)\bigg(\int_0^{\eta_1}e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,s)-\tilde{\mathcal{Y}}_1(t,\theta))}\min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\bigg)\,ds\\
&\qquad+\frac{1}{a}\int_{\eta_1}^{\eta_2}\tilde{\mathcal{Y}}_{2,\eta}(t,s)\bigg(\int_0^{\eta_1}e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,s)-\tilde{\mathcal{Y}}_2(t,\theta))}\min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\bigg)\,ds\\
&\quad\leqslant \frac{1}{a}\int_{\eta_1}^{\eta_2}\tilde{\mathcal{Y}}_{1,\eta}(t,s)\bigg(\int_0^{s}e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,s)-\tilde{\mathcal{Y}}_1(t,\theta))}\min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\bigg)\,ds\\
&\qquad+\frac{1}{a}\int_{\eta_1}^{\eta_2}\tilde{\mathcal{Y}}_{2,\eta}(t,s)\bigg(\int_0^{s}e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,s)-\tilde{\mathcal{Y}}_2(t,\theta))}\min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\bigg)\,ds\\
&\quad=\int_{\eta_1}^{\eta_2}\tilde{B}_1(t,s)\,ds+\int_{\eta_1}^{\eta_2}\tilde{B}_2(t,s)\,ds.
\end{aligned}
$$

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As far as $\tilde{B}_1(t,s)$ is concerned, we have

$$\begin{aligned}
&\int_{\eta_1}^{\eta_2} \tilde{B}_1(t,s)\,ds \\
&\quad = \frac{1}{a}\int_{\eta_1}^{\eta_2}\bigg(\int_0^s e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,s)-\tilde{\mathcal{Y}}_1(t,\theta))}\min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\bigg)\tilde{\mathcal{Y}}_{1,\eta}(t,s)\,ds \\
&\quad \leqslant \frac{1}{\sqrt{2}}\int_{\eta_1}^{\eta_2}\bigg(\int_0^s e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,s)-\tilde{\mathcal{Y}}_1(t,\theta))}\tilde{\mathcal{P}}_1^{3/4}\tilde{\mathcal{U}}_1^+\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\bigg)\tilde{\mathcal{Y}}_{1,\eta}(t,s)\,ds \\
&\quad \leqslant \frac{1}{\sqrt{2}}\int_{\eta_1}^{\eta_2}\bigg(\int_0^s e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,s)-\tilde{\mathcal{Y}}_1(t,\theta))}\tilde{\mathcal{P}}_1^{3/2}\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\bigg)^{1/2} \\
&\qquad \times\bigg(\int_0^s e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,s)-\tilde{\mathcal{Y}}_1(t,\theta))}\tilde{\mathcal{U}}_1^2\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\bigg)^{1/2}\tilde{\mathcal{Y}}_{1,\eta}(t,s)\,ds \\
&\quad \leqslant 3A^2\int_{\eta_1}^{\eta_2}\tilde{\mathcal{P}}_1\tilde{\mathcal{Y}}_{1,\eta}(t,s)\,ds \\
&\quad \leqslant \frac{3}{2}A^7|\eta_2-\eta_1|,
\end{aligned}$$

using (4.16k).

As far as $\tilde{B}_2(t,s)$ is concerned, we have to be more careful. Therefore recall that we have (cf. (4.13)) that

$$A_2^5 = 2\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)-\tilde{\mathcal{U}}_2^2\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)+\tilde{\mathcal{H}}_{2,\eta}(t,\eta)\leqslant 2\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)+\tilde{\mathcal{H}}_{2,\eta}(t,\eta).$$

Therefore we can write

$$\begin{aligned}
&\frac{1}{a}\int_0^s e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,s)-\tilde{\mathcal{Y}}_2(t,\theta))}\min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta \\
&\quad \leqslant \frac{\sqrt{2}}{a}\int_0^s e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,s)-\tilde{\mathcal{Y}}_2(t,\theta))}\min_j(\tilde{\mathcal{P}}_j)^{3/2}\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta \\
&\quad \leqslant \int_0^s e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,s)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2^{1/4}\tilde{\mathcal{P}}_1\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta \\
&\quad \leqslant \frac{A_1^5}{2A_2^5}\int_0^s e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,s)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2^{1/4}(2\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}+\tilde{\mathcal{H}}_{2,\eta})(t,\theta)\,d\theta \\
&\quad \leqslant \frac{A_1^5}{2A_2^5}\int_0^s e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,s)-\tilde{\mathcal{Y}}_2(t,\theta))}\bigg(2\tilde{\mathcal{P}}_2^{5/4}\tilde{\mathcal{Y}}_{2,\eta}+\frac{A_2}{\sqrt{2}}\tilde{\mathcal{H}}_{2,\eta}\bigg)(t,\theta)\,d\theta \\
&\quad \leqslant \frac{A_1^5}{2A_2^5}\bigg(2\sqrt{2}A_2^2\tilde{\mathcal{P}}_2(t,s)+2\int_0^s e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,s)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2^{5/4}\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\bigg).
\end{aligned}$$

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Note that the integral term can be bounded by $\mathcal{O}(1)\tilde{\mathcal{P}}_2(t,s)$ since

$$\int_0^s e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,s)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2^{5/4}\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta \leqslant \frac{15}{\sqrt{2}}A_2^2\tilde{\mathcal{P}}_2(t,s)$$

by (4.16k). We end up with

$$\begin{aligned}\int_{\eta_1}^{\eta_2}\tilde{B}_2(t,s)\,ds &= \frac{1}{a}\int_{\eta_1}^{\eta_2}\tilde{\mathcal{Y}}_{2,\eta}(t,s)\\ &\quad\times\left(\int_0^s e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,s)-\tilde{\mathcal{Y}}_2(t,\theta))}\min_j(\tilde{\mathcal{P}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\right)ds\\ &\leqslant \frac{17}{\sqrt{2}}\frac{A_1^5}{A_2^3}\int_{\eta_1}^{\eta_2}\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,s)\,ds\\ &\leqslant \frac{17}{2\sqrt{2}}A^7|\eta_2-\eta_1|.\end{aligned}$$

Moreover,

$$\begin{aligned}\tilde{b}(t,\eta_1)-\tilde{b}(t,\eta_2) &\leqslant \int_{\eta_1}^{\eta_2}\tilde{B}_1(t,s)\,ds+\int_{\eta_1}^{\eta_2}\tilde{B}_2(t,s)\,ds\\ &\leqslant \mathcal{O}(1)A^7|\eta_2-\eta_1|,\end{aligned}$$

where $\mathcal{O}(1)$ denotes some constant, which only depends on A and which remains bounded as $A\to 0$.

Finally combining both cases yields that there exists a constant $\mathcal{O}(1)$, which only depends on A and which remains bounded as $A\to 0$, such that

$$|\tilde{b}(t,\eta_2)-\tilde{b}(t,\eta_1)| \leqslant \mathcal{O}(1)A^7|\eta_2-\eta_1|$$

and subsequently

$$|\tilde{a}(t,\eta_2)-\tilde{a}(t,\eta_1)| \leqslant \mathcal{O}(1)A^7|\eta_2-\eta_1|.$$

This proves that the derivative exists for almost every η and is bounded by (A.9). □

LEMMA A.7. *The function*

$$\eta\mapsto\min_k\left(\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{k,\eta}(t,\theta)\,d\theta\right)$$

is Lipschitz continuous with uniformly bounded Lipschitz constant and thus differentiable almost everywhere. The derivative satisfies,

$$\left|\frac{d}{d\eta}\min_k\left(\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{k,\eta}(t,\theta)\,d\theta\right)\right| \leqslant \mathcal{O}(1)A^8. \tag{A.10}$$

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Proof. We present the following argument. Introduce

$$\begin{aligned}
\bar{a}(t,\eta) &= \min_k\left(\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{k,\eta}(t,\theta)\,d\theta\right),\\
\bar{b}(t,\eta) &= \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta,\\
\bar{c}(t,\eta) &= \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta.
\end{aligned}$$

Thus

$$\bar{a}(t,\eta) = \min(\bar{b},\bar{c})(t,\eta).$$

We have to show that $\bar{a}(t,\,\cdot\,)$ is Lipschitz continuous with a Lipschitz constant, which only depends on A. Clearly, we have that

$$|\bar{a}(t,\eta_1)-\bar{a}(t,\eta_2)| \leqslant \max(|\bar{b}(t,\eta_1)-\bar{b}(t,\eta_2)|, |\bar{c}(t,\eta_1)-\bar{c}(t,\eta_2)|),$$

and it suffices to show that both $\bar{b}(t,\,\cdot\,)$ and $\bar{c}(t,\,\cdot\,)$ are Lipschitz continuous with a Lipschitz constant, which only depends on A. We are only going to establish the Lipschitz continuity for $\bar{b}$ since the argument for $\bar{c}$ follows the same lines. Let $0 \leqslant \eta_1 < \eta_2 \leqslant 1$. Then we have to consider two cases:

$$0 \leqslant \bar{b}(t,\eta_1)-\bar{b}(t,\eta_2) \quad \text{and} \quad 0 \leqslant \bar{b}(t,\eta_2)-\bar{b}(t,\eta_1).$$

(i): $0 \leqslant \bar{b}(t,\eta_2)-\bar{b}(t,\eta_1)$: By definition, we have

$$\begin{aligned}
&\bar{b}(t,\eta_2)-\bar{b}(t,\eta_1)\\
&\quad= \int_0^{\eta_2} \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_2)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\\
&\qquad - \int_0^{\eta_1} \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_1)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\\
&\quad= \int_{\eta_1}^{\eta_2} \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_2)-\tilde{\mathcal{Y}}_j(t,\theta))})\\
&\qquad \times \min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\\
&\qquad + \int_0^{\eta_1} (\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_2)-\tilde{\mathcal{Y}}_j(t,\theta))})\\
&\qquad - \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_1)-\tilde{\mathcal{Y}}_j(t,\theta))}))\\
&\qquad \times \min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta
\end{aligned}$$

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$$\leqslant \int_{\eta_1}^{\eta_2} \min_j (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_2)-\tilde{\mathcal{Y}}_j(t,\theta))}) \times \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\, d\theta,$$

where we used in the last step that $0 \leqslant \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{1,\eta}(t,\eta)$ and that $\tilde{\mathcal{Y}}_i(t,\eta)$ is increasing, which implies

$$\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_1)-\tilde{\mathcal{Y}}_j(t,\theta))}) \geqslant \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_2)-\tilde{\mathcal{Y}}_j(t,\theta))}).$$

Moreover, note that $\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_2)-\tilde{\mathcal{Y}}_j(t,\theta))}) \leqslant 1$ for $0 \leqslant \eta_1 \leqslant \theta \leqslant \eta_2$ and that

$$0 \leqslant \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta) \leqslant \frac{1}{\sqrt{2}}A_1^8 \leqslant \frac{1}{\sqrt{2}}A^8$$

by (4.15b), (4.15e), and (4.15n). Thus

$$0 \leqslant \bar{b}(t,\eta_2) - \bar{b}(t,\eta_1) \leqslant \frac{1}{\sqrt{2}}A^8|\eta_2-\eta_1|.$$

(ii) $0 \leqslant \bar{b}(t,\eta_1) - \bar{b}(t,\eta_2)$: By definition, we have

$$\begin{aligned}
0 &\leqslant \bar{b}(t,\eta_1) - \bar{b}(t,\eta_2)\\
&= \int_0^{\eta_1} \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_1)-\tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\, d\theta\\
&\quad - \int_0^{\eta_2} \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_2)-\tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\, d\theta\\
&= \int_0^{\eta_1} \big(\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_1)-\tilde{\mathcal{Y}}_j(t,\theta))}) - \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_2)-\tilde{\mathcal{Y}}_j(t,\theta))})\big)\\
&\quad \times \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\, d\theta\\
&\quad - \int_{\eta_1}^{\eta_2} \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_2)-\tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\, d\theta\\
&\leqslant \int_0^{\eta_1} \big(\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_1)-\tilde{\mathcal{Y}}_j(t,\theta))}) - \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_2)-\tilde{\mathcal{Y}}_j(t,\theta))})\big)\\
&\quad \times \min_j(\tilde{\mathcal{D}}_j) \min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\, d\theta.
\end{aligned}$$

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Now we have to be much more careful than before. Namely, we have (as before)

$$
\begin{aligned}
0 &\leqslant \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_1)-\tilde{\mathcal{Y}}_j(t,\theta))}) - \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_2)-\tilde{\mathcal{Y}}_j(t,\theta))}) \\
&\leqslant \frac{1}{a}\int_{\eta_1}^{\eta_2} (e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,s)-\tilde{\mathcal{Y}}_1(t,\theta))}\tilde{\mathcal{Y}}_{1,\eta}(t,s) + e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,s)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{Y}}_{2,\eta}(t,s))\,ds.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\bar{b}(t,\eta_1) - \bar{b}(t,\eta_2) \\
&\quad\leqslant \int_0^{\eta_1} \big(\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_1)-\tilde{\mathcal{Y}}_j(t,\theta))}) - \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_2)-\tilde{\mathcal{Y}}_j(t,\theta))})\big) \\
&\qquad \times \min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta \\
&\quad\leqslant \frac{1}{a}\int_0^{\eta_1}\int_{\eta_1}^{\eta_2} e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,s)-\tilde{\mathcal{Y}}_1(t,\theta))}\tilde{\mathcal{Y}}_{1,\eta}(t,s)\,ds\,\min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta \\
&\qquad + \frac{1}{a}\int_0^{\eta_1}\int_{\eta_1}^{\eta_2} e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,s)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{Y}}_{2,\eta}(t,s)\,ds \\
&\qquad \times \min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta \\
&\quad= \frac{1}{a}\int_{\eta_1}^{\eta_2}\tilde{\mathcal{Y}}_{1,\eta}(t,s) \\
&\qquad \times \bigg(\int_0^{\eta_1} e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,s)-\tilde{\mathcal{Y}}_1(t,\theta))}\min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\bigg)\,ds \\
&\qquad + \frac{1}{a}\int_{\eta_1}^{\eta_2}\tilde{\mathcal{Y}}_{2,\eta}(t,s) \\
&\qquad \times \bigg(\int_0^{\eta_1} e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,s)-\tilde{\mathcal{Y}}_2(t,\theta))}\min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\bigg)\,ds \\
&\quad\leqslant \frac{1}{a}\int_{\eta_1}^{\eta_2}\tilde{\mathcal{Y}}_{1,\eta}(t,s) \\
&\qquad \times \bigg(\int_0^{s} e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,s)-\tilde{\mathcal{Y}}_1(t,\theta))}\min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\bigg)\,ds \\
&\qquad + \frac{1}{a}\int_{\eta_1}^{\eta_2}\tilde{\mathcal{Y}}_{2,\eta}(t,s) \\
&\qquad \times \bigg(\int_0^{s} e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,s)-\tilde{\mathcal{Y}}_2(t,\theta))}\min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\bigg)\,ds \\
&\quad= \int_{\eta_1}^{\eta_2}\bar{B}_1(t,s)\,ds + \int_{\eta_1}^{\eta_2}\bar{B}_2(t,s)\,ds.
\end{aligned}
$$

As far as $\bar{B}_1(t,s)$ is concerned, we have

$$\begin{aligned}
&\int_{\eta_1}^{\eta_2} \bar{B}_1(t,s)\,ds \\
&\quad = \frac{1}{a}\int_{\eta_1}^{\eta_2}\bigg(\int_0^s e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,s)-\tilde{\mathcal{Y}}_1(t,\theta))}\min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\bigg)\tilde{\mathcal{Y}}_{1,\eta}(t,s)\,ds \\
&\quad \leqslant \sqrt{2}A\int_{\eta_1}^{\eta_2}\bigg(\int_0^s e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,s)-\tilde{\mathcal{Y}}_1(t,\theta))}\tilde{\mathcal{P}}_1^{3/4}\tilde{\mathcal{U}}_1^+\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\bigg)\tilde{\mathcal{Y}}_{1,\eta}(t,s)\,ds \\
&\quad \leqslant \sqrt{2}A\int_{\eta_1}^{\eta_2}\bigg(\int_0^s e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,s)-\tilde{\mathcal{Y}}_1(t,\theta))}\tilde{\mathcal{P}}_1^{3/2}\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\bigg)^{1/2} \\
&\qquad \times\bigg(\int_0^s e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,s)-\tilde{\mathcal{Y}}_1(t,\theta))}\tilde{\mathcal{U}}_1^2\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\bigg)^{1/2}\tilde{\mathcal{Y}}_{1,\eta}(t,s)\,ds \\
&\quad \leqslant 6A^3\int_{\eta_1}^{\eta_2}\tilde{\mathcal{P}}_1\tilde{\mathcal{Y}}_{1,\eta}(t,s)\,ds \\
&\quad \leqslant 3A^8|\eta_2-\eta_1|,
\end{aligned}$$

where we used (4.15n).

As far as $\bar{B}_2(t,s)$ is concerned, we have to be more careful. Therefore recall that we have (cf. (4.13)) that

$$A_2^5 = 2\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)-\tilde{\mathcal{U}}_2^2\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)+\tilde{\mathcal{H}}_{2,\eta}(t,\eta)\leqslant 2\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)+\tilde{\mathcal{H}}_{2,\eta}(t,\eta).$$

Therefore we can write

$$\begin{aligned}
&\frac{1}{a}\int_0^s \mathrm{e}^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,s)-\tilde{\mathcal{Y}}_2(t,\theta))}\min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta \\
&\quad \leqslant \frac{2\sqrt{2}A}{a}\int_0^s e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,s)-\tilde{\mathcal{Y}}_2(t,\theta))}\min_j(\tilde{\mathcal{P}}_j)^{3/2}\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta \\
&\quad \leqslant 2A\int_0^s e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,s)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2^{1/4}\tilde{\mathcal{P}}_1\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta \\
&\quad \leqslant A\frac{A_1^5}{A_2^5}\int_0^s e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,s)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2^{1/4}(2\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}+\tilde{\mathcal{H}}_{2,\eta})(t,\theta)\,d\theta \\
&\quad \leqslant A\frac{A_1^5}{A_2^5}\int_0^s e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,s)-\tilde{\mathcal{Y}}_2(t,\theta))}\bigg(2\tilde{\mathcal{P}}_2^{5/4}\tilde{\mathcal{Y}}_{2,\eta}+\frac{A_2}{\sqrt{2}}\tilde{\mathcal{H}}_{2,\eta}\bigg)(t,\theta)\,d\theta \\
&\quad \leqslant A\frac{A_1^5}{A_2^5}\bigg(2\sqrt{2}A_2^2\tilde{\mathcal{P}}_2(t,s)+2\int_0^s e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,s)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2^{5/4}\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\bigg).
\end{aligned}$$

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Note that the integral term can be bounded by $\mathcal{O}(1)\tilde{\mathcal{P}}_2(t,s)$ since

$$\int_0^s e^{-\frac{1}{a}(\hat{\mathcal{Y}}_2(t,s)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2^{5/4}\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta \leqslant \frac{15}{\sqrt{2}}A_2^2\tilde{\mathcal{P}}_2(t,s)$$

by (4.16k). We end up with

$$\begin{aligned}
&\int_{\eta_1}^{\eta_2}\bar{B}_2(t,s)\,ds\\
&\quad=\frac{1}{a}\int_{\eta_1}^{\eta_2}\tilde{\mathcal{Y}}_{2,\eta}(t,s)\left(\int_0^s e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,s)-\tilde{\mathcal{Y}}_2(t,\theta))}\min_j(\tilde{\mathcal{D}}_j)\min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\right)ds\\
&\quad\leqslant\frac{17\sqrt{2}A^6}{A_2^3}\int_{\eta_1}^{\eta_2}\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,s)\,ds\\
&\quad\leqslant\frac{17A^8}{\sqrt{2}}|\eta_2-\eta_1|.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\bar{b}(t,\eta_1)-\bar{b}(t,\eta_2)&\leqslant\int_{\eta_1}^{\eta_2}\bar{B}_1(t,s)\,ds+\int_{\eta_1}^{\eta_2}\bar{B}_2(t,s)\,ds\\
&\leqslant\mathcal{O}(1)A^8|\eta_2-\eta_1|,
\end{aligned}$$

where $\mathcal{O}(1)$ denotes some constant, which only depends on A and which remains bounded as $A\to 0$.

Finally, combining both cases yields that there exists a constant $\mathcal{O}(1)$, which only depends on A and which remains bounded as $A\to 0$, such that

$$|\bar{b}(t,\eta_2)-\bar{b}(t,\eta_1)|\leqslant\mathcal{O}(1)A^8|\eta_2-\eta_1|$$

and subsequently

$$|\bar{a}(t,\eta_2)-\bar{a}(t,\eta_1)|\leqslant\mathcal{O}(1)A^8|\eta_2-\eta_1|.$$

This proves that the derivative exists for almost every η and is bounded by (A.10). □

LEMMA A.8. *The function*

$$\eta\mapsto\min_k\left[\int_0^\eta\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{U}}_j^+)^3\tilde{\mathcal{Y}}_{k,\eta}(t,\theta)\,d\theta\right]$$

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is Lipschitz continuous with a uniformly bounded Lipschitz constant and thus differentiable almost everywhere. The derivative satisfies

$$\left|\frac{d}{d\eta}\min_k\left[\int_0^\eta \min_j(e^{-(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{U}}_j^+)^3\tilde{\mathcal{Y}}_{k,\eta}(t,\theta)\,d\theta\right]\right| \leqslant \mathcal{O}(1)A^6, \tag{A.11}$$

where $\mathcal{O}(1)$ denotes a constant, which only depends on A and which remains bounded as $A \to 0$.

Proof. To that end, we present the following argument. Introduce

$$\begin{aligned} a(t,\eta) &= \min_k\left[\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{U}}_j^+)^3\tilde{\mathcal{Y}}_{k,\eta}(t,\theta)\,d\theta\right],\\ b(t,\eta) &= \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{U}}_j^+)^3\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta,\\ c(t,\eta) &= \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{U}}_j^+)^3\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta. \end{aligned}$$

Thus

$$a(t,\eta) = \min(b,c)(t,\eta).$$

Then we have to show that $a(t,\cdot)$ is Lipschitz continuous with a Lipschitz constant, which only depends on A and hence on C. Clearly, we have that

$$|a(t,\eta_1)-a(t,\eta_2)| \leqslant \max(|b(t,\eta_1)-b(t,\eta_2)|,|c(t,\eta_1)-c(t,\eta_2)|),$$

and it suffices to show that both $b(t,\cdot)$ and $c(t,\cdot)$ are Lipschitz continuous with a Lipschitz constant, which only depends on A. We are only going to establish the Lipschitz continuity for b, since the argument for c follows the same lines. Let $0 \leqslant \eta_1 < \eta_2 \leqslant 1$. Then we have to consider two cases:

$$0 \leqslant b(t,\eta_1)-b(t,\eta_2) \quad \text{and} \quad 0 \leqslant b(t,\eta_2)-b(t,\eta_1).$$

(i): $0 \leqslant b(t,\eta_2)-b(t,\eta_1)$: By definition, we have

$$\begin{aligned} &b(t,\eta_2)-b(t,\eta_1)\\ &\quad= \int_0^{\eta_2} \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_2)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{U}}_j^+)^3\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\\ &\qquad- \int_0^{\eta_1} \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_1)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{U}}_j^+)^3\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\\ &\quad= \int_{\eta_1}^{\eta_2} \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_2)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{U}}_j^+)^3\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta \end{aligned}$$

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$$+\int_0^{\eta_1}\Big(\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_2)-\tilde{\mathcal{Y}}_j(t,\theta))})$$
$$-\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_1)-\tilde{\mathcal{Y}}_j(t,\theta))})\Big)$$
$$\times \min_j(\tilde{\mathcal{U}}_j^+)^3\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta$$
$$\leqslant \int_{\eta_1}^{\eta_2}\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_2)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{U}}_j^+)^3\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta,$$

where we used in the last step that $0\leqslant \min_j(\tilde{\mathcal{U}}_j^+)^3\tilde{\mathcal{Y}}_{1,\eta}(t,\eta)$ and that $\tilde{\mathcal{Y}}_1(t,\,\cdot\,)$ is increasing, which implies

$$\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_1)-\tilde{\mathcal{Y}}_j(t,\theta))})$$
$$=\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_1)-\tilde{\mathcal{Y}}_j(t,\eta_2)+\tilde{\mathcal{Y}}_j(t,\eta_2)-\tilde{\mathcal{Y}}_j(t,\theta))})$$
$$\geqslant \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_2)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_1)-\tilde{\mathcal{Y}}_j(t,\eta_2))})$$
$$\geqslant \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_2)-\tilde{\mathcal{Y}}_j(t,\theta))}).$$

Moreover, note that $\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_2)-\tilde{\mathcal{Y}}_j(t,\theta))})\leqslant 1$ since $\eta_1\leqslant\theta\leqslant\eta_2$, and that

$$0\leqslant \min_j(\tilde{\mathcal{U}}_j^+)^3\tilde{\mathcal{Y}}_{1,\eta}(t,\eta)\leqslant |\tilde{\mathcal{U}}_1^3|\tilde{\mathcal{Y}}_{1,\eta}(t,\eta)\leqslant \frac{1}{\sqrt{2}}A_1^7\leqslant\frac{1}{\sqrt{2}}A^7$$

by (4.15b) and (4.15g). Thus

$$0\leqslant b(t,\eta_2)-b(t,\eta_1)$$
$$\leqslant \int_{\eta_1}^{\eta_2}\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_2)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{U}}_j^+)^3\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta$$
$$\leqslant \frac{1}{\sqrt{2}}A^7|\eta_2-\eta_1|.$$

(ii): $0\leqslant b(t,\eta_1)-b(t,\eta_2)$: By definition, we have

$$b(t,\eta_1)-b(t,\eta_2)$$
$$=\int_0^{\eta_1}\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_1)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{U}}_j^+)^3\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta$$
$$-\int_0^{\eta_2}\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_2)-\tilde{\mathcal{Y}}_j(t,\theta))})\min_j(\tilde{\mathcal{U}}_j^+)^3\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta$$

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$$
\begin{aligned}
&= \int_0^{\eta_1} \Big(\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_1)-\tilde{\mathcal{Y}}_j(t,\theta))}) \\
&\quad - \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_2)-\tilde{\mathcal{Y}}_j(t,\theta))})\Big) \\
&\quad \times \min_j(\tilde{\mathcal{U}}_j^+)^3\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\, d\theta \\
&\quad - \int_{\eta_1}^{\eta_2} \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_2)-\tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{U}}_j^+)^3\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\, d\theta \\
&\leqslant \int_0^{\eta_1} \Big(\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_1)-\tilde{\mathcal{Y}}_j(t,\theta))}) \\
&\quad - \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_2)-\tilde{\mathcal{Y}}_j(t,\theta))})\Big) \\
&\quad \times \min_j(\tilde{\mathcal{U}}_j^+)^3\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\, d\theta.
\end{aligned}
$$

Now we have to be much more careful than before. Namely, if $\min_j(\tilde{\mathcal{Y}}_j(t,\theta) - \tilde{\mathcal{Y}}_j(t,\eta_2)) = \tilde{\mathcal{Y}}_i(t,\theta) - \tilde{\mathcal{Y}}_i(t,\eta_2)$, we have

$$
\begin{aligned}
0 &\leqslant \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_1)-\tilde{\mathcal{Y}}_j(t,\theta))}) - \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_2)-\tilde{\mathcal{Y}}_j(t,\theta))}) \\
&= \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_1)-\tilde{\mathcal{Y}}_j(t,\theta))}) - e^{\frac{1}{a}(\tilde{\mathcal{Y}}_i(t,\theta)-\tilde{\mathcal{Y}}_i(t,\eta_2))} \\
&\leqslant e^{\frac{1}{a}(\tilde{\mathcal{Y}}_i(t,\theta)-\tilde{\mathcal{Y}}_i(t,\eta_1))} - e^{\frac{1}{a}(\tilde{\mathcal{Y}}_i(t,\theta)-\tilde{\mathcal{Y}}_i(t,\eta_2))} \\
&= \frac{1}{a}\int_{\eta_1}^{\eta_2} e^{\frac{1}{a}(\tilde{\mathcal{Y}}_i(t,\theta)-\tilde{\mathcal{Y}}_i(t,s))}\tilde{\mathcal{Y}}_{i,\eta}(t,s)\, ds \\
&\leqslant \frac{1}{a}\int_{\eta_1}^{\eta_2} (e^{\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,\theta)-\tilde{\mathcal{Y}}_1(t,s))}\tilde{\mathcal{Y}}_{1,\eta}(t,s) + e^{\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,\theta)-\tilde{\mathcal{Y}}_2(t,s))}\tilde{\mathcal{Y}}_{2,\eta}(t,s))\, ds.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&b(t,\eta_1) - b(t,\eta_2) \\
&\quad \leqslant \int_0^{\eta_1} \Big(\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_1)-\tilde{\mathcal{Y}}_j(t,\theta))}) \\
&\qquad - \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta_2)-\tilde{\mathcal{Y}}_j(t,\theta))})\Big) \\
&\qquad \times \min_j(\tilde{\mathcal{U}}_j^+)^3\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\, d\theta \\
&\quad \leqslant \frac{1}{a}\int_0^{\eta_1}\int_{\eta_1}^{\eta_2} e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,s)-\tilde{\mathcal{Y}}_1(t,\theta))}\tilde{\mathcal{Y}}_{1,\eta}(t,s)\, ds \min_j(\tilde{\mathcal{U}}_j^+)^3\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\, d\theta \\
&\qquad + \frac{1}{a}\int_0^{\eta_1}\int_{\eta_1}^{\eta_2} e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,s)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{Y}}_{2,\eta}(t,s)\, ds \min_j(\tilde{\mathcal{U}}_j^+)^3\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\, d\theta
\end{aligned}
$$

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$$
\begin{aligned}
&= \frac{1}{a}\int_{\eta_1}^{\eta_2} \tilde{\mathcal{Y}}_{1,\eta}(t,s)\bigg(\int_0^{\eta_1} e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,s)-\tilde{\mathcal{Y}}_1(t,\theta))} \min_j(\tilde{\mathcal{U}}_j^+)^3\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\bigg)\,ds \\
&\quad + \frac{1}{a}\int_{\eta_1}^{\eta_2} \tilde{\mathcal{Y}}_{2,\eta}(t,s)\bigg(\int_0^{\eta_1} e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,s)-\tilde{\mathcal{Y}}_2(t,\theta))} \min_j(\tilde{\mathcal{U}}_j^+)^3\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\bigg)\,ds \\
&\leqslant \frac{1}{a}\int_{\eta_1}^{\eta_2} \tilde{\mathcal{Y}}_{1,\eta}(t,s)\bigg(\int_0^{s} e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,s)-\tilde{\mathcal{Y}}_1(t,\theta))} \min_j(\tilde{\mathcal{U}}_j^+)^3\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\bigg)\,ds \\
&\quad + \frac{1}{a}\int_{\eta_1}^{\eta_2} \tilde{\mathcal{Y}}_{2,\eta}(t,s)\bigg(\int_0^{s} e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,s)-\tilde{\mathcal{Y}}_2(t,\theta))} \min_j(\tilde{\mathcal{U}}_j^+)^3\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\bigg)\,ds \\
&= \int_{\eta_1}^{\eta_2} B_1(t,s)\,ds + \int_{\eta_1}^{\eta_2} B_2(t,s)\,ds.
\end{aligned}
$$

As far as $B_1(t,s)$ is concerned, recall that

$$
\begin{aligned}
\tilde{\mathcal{P}}_i(t,s) &= \frac{1}{4A_i}\int_0^1 e^{-\frac{1}{A_i}|\tilde{\mathcal{Y}}_i(t,s)-\tilde{\mathcal{Y}}_i(t,\theta)|}(2(\tilde{\mathcal{U}}_i^2-\tilde{\mathcal{P}}_i)\tilde{\mathcal{Y}}_{i,\eta}(t,\theta)+A_i^5)\,d\theta \\
&= \frac{1}{4A_i}\int_0^1 e^{-\frac{1}{A_i}|\tilde{\mathcal{Y}}_i(t,s)-\tilde{\mathcal{Y}}_i(t,\theta)|}(\tilde{\mathcal{U}}_i^2\tilde{\mathcal{Y}}_{i,\eta}+\tilde{\mathcal{H}}_{i,\eta})(t,\theta)\,d\theta,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\int_{\eta_1}^{\eta_2} B_1(t,s)\,ds \\
&\quad= \frac{1}{a}\int_{\eta_1}^{\eta_2} \tilde{\mathcal{Y}}_{1,\eta}(t,s)\bigg(\int_0^{s} e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,s)-\tilde{\mathcal{Y}}_1(t,\theta))} \min_j(\tilde{\mathcal{U}}_j^+)^3\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\bigg)\,ds \\
&\quad\leqslant \frac{1}{a}\int_{\eta_1}^{\eta_2} \tilde{\mathcal{Y}}_{1,\eta}(t,s)\bigg(\int_0^{s} e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,s)-\tilde{\mathcal{Y}}_1(t,\theta))} \min_j(\tilde{\mathcal{U}}_j^+)\tilde{\mathcal{U}}_1^2\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\bigg)\,ds \\
&\quad\leqslant \frac{a}{\sqrt{2}}\int_{\eta_1}^{\eta_2} \tilde{\mathcal{Y}}_{1,\eta}(t,s)\bigg(\int_0^{s} e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_1(t,s)-\tilde{\mathcal{Y}}_1(t,\theta))}\tilde{\mathcal{U}}_1^2\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\bigg)\,ds \\
&\quad\leqslant \frac{4}{\sqrt{2}}A^2\int_{\eta_1}^{\eta_2} \tilde{\mathcal{P}}_1\tilde{\mathcal{Y}}_{1,\eta}(t,s)\,ds \\
&\quad\leqslant \sqrt{2}A^7|\eta_2-\eta_1|.
\end{aligned}
$$

As far as $B_2(t,s)$ is concerned, we have to be more careful. Therefore recall that we have

$$
A_2^5 = 2\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)-\tilde{\mathcal{U}}_2^2\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)+\tilde{\mathcal{H}}_{2,\eta}(t,\eta) \leqslant 2\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\eta)+\tilde{\mathcal{H}}_{2,\eta}(t,\eta).
$$

Therefore we can write

$$
\frac{1}{a}\int_0^{s} e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,s)-\tilde{\mathcal{Y}}_2(t,\theta))} \min_j(\tilde{\mathcal{U}}_j^+)^3\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta
$$

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$$
\begin{aligned}
&\leqslant \frac{1}{a}\int_0^s e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,s)-\tilde{\mathcal{Y}}_2(t,\theta))}|\tilde{\mathcal{U}}_2|\tilde{\mathcal{U}}_1^2\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\\
&\leqslant \frac{A_1^5}{aA_2^5}\int_0^s e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,s)-\tilde{\mathcal{Y}}_2(t,\theta))}|\tilde{\mathcal{U}}_2|(2\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}+\tilde{\mathcal{H}}_{2,\eta})(t,\theta)\,d\theta\\
&\leqslant \frac{A_1^5}{aA_2^5}\int_0^s e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,s)-\tilde{\mathcal{Y}}_2(t,\theta))}\left(A_2\tilde{\mathcal{U}}_2^2\tilde{\mathcal{Y}}_{2,\eta}+\frac{1}{A_2}\tilde{\mathcal{P}}_2^2\tilde{\mathcal{Y}}_{2,\eta}+A_2^2\tilde{\mathcal{H}}_{2,\eta}\right)(t,\theta)\,d\theta\\
&\leqslant \frac{A_1^5}{aA_2^5}\int_0^1 e^{-\frac{1}{A_2}|\tilde{\mathcal{Y}}_2(t,s)-\tilde{\mathcal{Y}}_2(t,\theta)|}\left(A_2\tilde{\mathcal{U}}_2^2\tilde{\mathcal{Y}}_{2,\eta}+A_2^2\tilde{\mathcal{H}}_{2,\eta}+\frac{1}{A_2}\tilde{\mathcal{P}}_2^2\tilde{\mathcal{Y}}_{2,\eta}\right)(t,\theta)\,d\theta\\
&\leqslant \frac{A_1^5}{aA_2^5}\Bigg(4A_2^2\tilde{\mathcal{P}}_2(t,s)+4A_2^3\tilde{\mathcal{P}}_2(t,s)\\
&\quad+\frac{1}{A_2}\int_0^1 e^{-\frac{1}{A_2}|\tilde{\mathcal{Y}}_2(t,s)-\tilde{\mathcal{Y}}_2(t,\theta)|}\tilde{\mathcal{P}}_2^2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\Bigg).
\end{aligned}
$$

Note that the integral term can be bounded by $\mathcal{O}(1)\tilde{\mathcal{P}}_2(t,s)$ since

$$
\begin{aligned}
&\int_0^1 e^{-\frac{1}{A_2}|\tilde{\mathcal{Y}}_2(t,s)-\tilde{\mathcal{Y}}_2(t,\theta)|}\tilde{\mathcal{P}}_2^2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\\
&\quad= A_2^7\int_0^1 e^{-|\mathcal{Y}_2(t,A_2^2s)-\mathcal{Y}_2(t,A_2^2\theta)|}\mathcal{P}_2^2\mathcal{Y}_{2,\eta}(t,A_2^2\theta)\,d\theta\\
&\quad= A_2^5\int_0^{A_2^2} e^{-|\mathcal{Y}_2(t,A_2^2s)-\mathcal{Y}_2(t,\theta)|}\mathcal{P}_2^2\mathcal{Y}_{2,\eta}(t,\theta)\,d\theta\\
&\quad\leqslant \mathcal{O}(1)A_2^5\mathcal{P}_2(t,A_2^2s)=\mathcal{O}(1)A_2^3\tilde{\mathcal{P}}_2(t,s)
\end{aligned}
$$

by (3.34). We end up with

$$
\begin{aligned}
&\int_{\eta_1}^{\eta_2} B_2(t,s)\,ds\\
&\quad=\frac{1}{a}\int_{\eta_1}^{\eta_2}\tilde{\mathcal{Y}}_{2,\eta}(t,s)\left(\int_0^s e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_2(t,s)-\tilde{\mathcal{Y}}_2(t,\theta))}\min_j(\tilde{\mathcal{U}}_j^+)^3\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\right)ds\\
&\quad\leqslant \frac{A_1^5}{aA_2^5}\int_{\eta_1}^{\eta_2}\tilde{\mathcal{Y}}_{2,\eta}(t,s)\\
&\qquad\times\left(A_2^2\mathcal{O}(A)\tilde{\mathcal{P}}_2(t,s)+\frac{1}{A_2}\int_0^1 e^{-\frac{1}{A_2}|\tilde{\mathcal{Y}}_2(t,s)-\tilde{\mathcal{Y}}_2(t,\theta)|}\tilde{\mathcal{P}}_2^2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\right)ds\\
&\quad\leqslant \frac{\mathcal{O}(1)A_1^5A_2^2}{aA_2^5}\int_{\eta_1}^{\eta_2}\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,s)\,ds\\
&\quad\leqslant \mathcal{O}(1)A^6|\eta_2-\eta_1|.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
b(t,\eta_1)-b(t,\eta_2) &\leqslant \int_{\eta_1}^{\eta_2} B_1(t,s)\,ds + \int_{\eta_1}^{\eta_2} B_2(t,s)\,ds\\
&\leqslant \mathcal{O}(1)A^6|\eta_2-\eta_1|,
\end{aligned}
$$

where $\mathcal{O}(1)$ only depends on A and hence on $C=\frac{A^2}{2}$, and remains bounded as $A\to 0$.

Finally, combining both cases yields that

$$|b(t,\eta_2)-b(t,\eta_1)| \leqslant \mathcal{O}(1)A^6|\eta_2-\eta_1|$$

and subsequently

$$|a(t,\eta_2)-a(t,\eta_1)| \leqslant \mathcal{O}(1)A^6|\eta_2-\eta_1|,$$

where $\mathcal{O}(1)$ remains bounded as $A\to 0$. This proves that the derivative exists for almost every η and is bounded by (A.11). □

We need to estimate the pointwise difference between two functions $\tilde{\mathcal{D}}_j$, $j=1$, 2. This is the content of the following lemma.

LEMMA A.9. *We have that*

$$
\begin{aligned}
|\tilde{\mathcal{D}}_1(t,\eta)-\tilde{\mathcal{D}}_2(t,\eta)| &\leqslant 2A^{3/2}\max_j(\tilde{\mathcal{D}}_j^{1/2})(t,\eta)|\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_2(t,\eta)|\\
&\quad + 2\sqrt{2}A^{3/2}\max_j(\tilde{\mathcal{D}}_j^{1/2})(t,\eta)\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|\\
&\quad + 4A^3\left(\int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)^2(t,\theta)\,d\theta\right)^{1/2}\\
&\quad + 2\sqrt{2}A^3\left(\int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})^2(t,\theta)\,d\theta\right)^{1/2}\\
&\quad + \frac{3A^4}{\sqrt{2}}\left(\int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,\theta)\,d\theta\right)^{1/2}\\
&\quad + \frac{12\sqrt{2}A^4}{\sqrt{3}e}\left(\int_0^\eta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}\,d\theta\right)^{1/2}|A_1-A_2|\\
&\quad + \tilde{\mathcal{U}}_j^2|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|(t,\eta)\\
&\quad + \tilde{\mathcal{P}}_j|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|(t,\eta)\\
&\quad + \frac{3A^4}{2}\int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|(t,\theta)\,d\theta
\end{aligned}
$$

$$+ 6A^4 \int_0^\eta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}\, d\theta |A_1 - A_2|,$$

for any value of $j = 1, 2$.

Proof. Direct calculations yield

$$\begin{aligned}
&|\tilde{\mathcal{D}}_1(t,\eta) - \tilde{\mathcal{D}}_2(t,\eta)| \\
&= \bigg| \int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))} \Big((\tilde{\mathcal{U}}_1^2 - \tilde{\mathcal{P}}_1)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta) + \frac{1}{2}A_1^5 \Big)\, d\theta \\
&\quad - \int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))} \Big((\tilde{\mathcal{U}}_2^2 - \tilde{\mathcal{P}}_2)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta) + \frac{1}{2}A_2^5 \Big)\, d\theta \bigg| \\
&\leqslant \bigg| \int_0^\eta (e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))} - e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}) \\
&\quad \times \Big((\tilde{\mathcal{U}}_1^2 - \tilde{\mathcal{P}}_1)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta) + \frac{1}{2}A_1^5 \Big) \mathbb{1}_{B(\eta)^c}(t,\theta)\, d\theta \bigg| \\
&\quad + \bigg| \int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))} - e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}) \\
&\quad \times \Big((\tilde{\mathcal{U}}_2^2 - \tilde{\mathcal{P}}_2)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta) + \frac{1}{2}A_2^5 \Big) \mathbb{1}_{B(\eta)}(t,\theta)\, d\theta \bigg| \\
&+ \bigg| \int_0^\eta \min_j (e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}) \Big((\tilde{\mathcal{U}}_1^2 - \tilde{\mathcal{P}}_1)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta) + \frac{1}{2}A_1^5 \Big)\, d\theta \\
&\quad - \int_0^\eta \min_j (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}) (\tilde{\mathcal{U}}_2^2 - \tilde{\mathcal{P}}_2)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta) + \frac{1}{2}A_2^5)\, d\theta \bigg| \\
&= \bar{d}_{11}(t,\eta) + \bar{d}_{12}(t,\eta) + \bar{d}_{13}(t,\eta),
\end{aligned}$$

where $B(\eta)$ is defined in (5.11).

For $\bar{d}_{11}(t,\eta)$ we immediately obtain

$$\begin{aligned}
\bar{d}_{11}(t,\eta) &\leqslant \frac{1}{A_1} \int_0^\eta (|\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_2(t,\eta)| + |\tilde{\mathcal{Y}}_1(t,\theta) - \tilde{\mathcal{Y}}_2(t,\theta)|) \\
&\quad \times e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))} \Big((\tilde{\mathcal{U}}_1^2 - \tilde{\mathcal{P}}_1)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta) + \frac{1}{2}A_1^5 \Big)\, d\theta \\
&\leqslant \frac{\tilde{\mathcal{D}}_1(t,\eta)}{A_1} |\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_2(t,\eta)| + \sqrt{2}A_1^{3/2}\tilde{\mathcal{D}}_1^{1/2}(t,\eta)\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\| \\
&\leqslant A_1^{3/2}\tilde{\mathcal{D}}_1^{1/2}(t,\eta)|\tilde{\mathcal{Y}}_1(t,\eta) - \tilde{\mathcal{Y}}_2(t,\eta)| + \sqrt{2}A_1^{3/2}\tilde{\mathcal{D}}_1^{1/2}(t,\eta)\|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2\|.
\end{aligned} \tag{A.12}$$

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Following the same lines, we end up with

$$\begin{aligned}\bar{d}_{12}(t,\eta) &\leqslant \frac{1}{A_2}\int_0^\eta (|\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_2(t,\eta)| + |\tilde{\mathcal{Y}}_1(t,\theta)-\tilde{\mathcal{Y}}_2(t,\theta)|) \\ &\quad\times e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\left|(\tilde{\mathcal{U}}_2^2-\tilde{\mathcal{P}}_2)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)+\frac{1}{2}A_2^5\right| d\theta \\ &\leqslant A_2^{3/2}\tilde{\mathcal{D}}_2^{1/2}(t,\eta)|\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_2(t,\eta)| + \sqrt{2}A_2^{3/2}\tilde{\mathcal{D}}_2^{1/2}(t,\eta)\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|. \end{aligned} \tag{A.13}$$

The last term $\bar{d}_{13}(t,\eta)$ needs a bit more work. Indeed,

$$\begin{aligned}\bar{d}_{13}(t,\eta) &= \left|\int_0^\eta \min_j(e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\left((\tilde{\mathcal{U}}_1^2-\tilde{\mathcal{P}}_1)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)+\frac{1}{2}A_1^5\right) d\theta\right. \\ &\quad \left. - \int_0^\eta \min_j(e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\left((\tilde{\mathcal{U}}_2^2-\tilde{\mathcal{P}}_2)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)+\frac{1}{2}A_2^5\right) d\theta\right| \\ &\leqslant \left|\int_0^\eta \Big(\min_j(e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\tilde{\mathcal{U}}_1^2\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\right. \\ &\quad \left. - \min_j(e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\tilde{\mathcal{U}}_2^2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\Big) d\theta\right| \\ &\quad + \left|\int_0^\eta \Big(\min_j(e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\tilde{\mathcal{P}}_1\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\right. \\ &\quad \left. - \min_j(e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\Big) d\theta\right| \\ &\quad + \left|\int_0^\eta \Big(\min_j(e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\frac{1}{2}A_1^5\right. \\ &\quad \left. - \min_j(e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\frac{1}{2}A_2^5\Big) d\theta\right| \\ &= \bar{T}_1(t,\eta)+\bar{T}_2(t,\eta)+\bar{T}_3(t,\eta). \end{aligned}$$

To estimate $\bar{T}_1(t,\eta)$, recall (3.40), (A.4), (A.5), and Lemma A.1 (ii), which imply

$$\begin{aligned}\bar{T}_1(t,\eta) &= \left|\int_0^\eta \Big(\min_j(\mathrm{e}^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\tilde{\mathcal{U}}_1^2\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\right. \\ &\quad \left. - \min_j(e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\tilde{\mathcal{U}}_2^2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\Big) d\theta\right| \\ &\leqslant \left|\int_0^\eta \min_j(e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})(\tilde{\mathcal{U}}_1^2-\tilde{\mathcal{U}}_2^2)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\mathbb{1}_{\tilde{\mathcal{U}}_2^2\leqslant\tilde{\mathcal{U}}_1^2}(t,\theta)\, d\theta\right| \end{aligned}$$

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$$
\begin{aligned}
&+ \left| \int_0^\eta \min_j(e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})(\tilde{\mathcal{U}}_1^2-\tilde{\mathcal{U}}_2^2)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\mathbb{1}_{\tilde{\mathcal{U}}_1^2<\tilde{\mathcal{U}}_2^2}(t,\theta)\,d\theta \right| \\
&+ \mathbb{1}_{A_1\leqslant A_2}\left| \int_0^\eta \Big(\min_j(e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}) \right. \\
&\left. - \min_j(e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\Big) \min_j(\tilde{\mathcal{U}}_j^2)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta \right| \\
&+ \mathbb{1}_{A_2< A_1}\left| \int_0^\eta \Big(\min_j(e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}) \right. \\
&\left. - \min_j(e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\Big) \min_j(\tilde{\mathcal{U}}_j^2)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta \right| \\
&+ \left| \int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{U}}_j^2)(\tilde{\mathcal{Y}}_{1,\eta}-\tilde{\mathcal{Y}}_{2,\eta})(t,\theta)\,d\theta \right| \\
\leqslant{}& 2\int_0^\eta \min_j(e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})|\tilde{\mathcal{U}}_1|\tilde{\mathcal{Y}}_{1,\eta}|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2|(t,\theta)\,d\theta \\
&+ 2\int_0^\eta \min_j(e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})|\tilde{\mathcal{U}}_2|\tilde{\mathcal{Y}}_{2,\eta}|\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2|(t,\theta)\,d\theta \\
&+ \mathbb{1}_{A_1\leqslant A_2}\frac{4}{ae}\left| \int_0^\eta \min_j(e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{U}}_j^2)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta \right| |A_1-A_2| \\
&+ \mathbb{1}_{A_2< A_1}\frac{4}{ae}\left| \int_0^\eta \min_j(e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{U}}_j^2)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta \right| |A_1-A_2| \\
&+ \left| \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{U}}_j^2)(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)\Big|_{\theta=0}^{\eta} \right. \\
&\left. - \int_0^\eta \frac{d}{d\theta}\Big(\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{U}}_j^2)\Big)(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)(t,\theta)\,d\theta \right| \\
\leqslant{}& 2\Big(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\tilde{\mathcal{U}}_1^2\tilde{\mathcal{Y}}_{1,\eta}^2(t,\theta)\,d\theta\Big)^{1/2} \\
&\times \Big(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)^2(t,\theta)\,d\theta\Big)^{1/2} \\
&+ 2\Big(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^2\tilde{\mathcal{Y}}_{2,\eta}^2(t,\theta)\,d\theta\Big)^{1/2} \\
&\times \Big(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)^2(t,\theta)\,d\theta\Big)^{1/2} \\
&+ \frac{2\sqrt{2}A}{e}\Big(\int_0^\eta e^{-\frac{3}{4A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{U}}_2^2\tilde{\mathcal{Y}}_{2,\eta}^2(t,\theta)\,d\theta\Big)^{1/2} \\
&\times \Big(\int_0^\eta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}\,d\theta\Big)^{1/2}|A_1-A_2|
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{2\sqrt{2}A}{e}\bigg(\int_0^\eta e^{-\frac{3}{4A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\tilde{\mathcal{U}}_1^2\tilde{\mathcal{Y}}_{1,\eta}^2(t,\theta)\,d\theta\bigg)^{1/2}\\
&\times\bigg(\int_0^\eta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}\,d\theta\bigg)^{1/2}|A_1-A_2|\\
&+ \tilde{\mathcal{U}}_j^2|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|(t,\eta)\\
&+ \frac{1}{a}\int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}\min_j(\tilde{\mathcal{U}}_j^2)(\tilde{\mathcal{Y}}_{1,\eta}+\tilde{\mathcal{Y}}_{2,\eta})|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|(t,\theta)\,d\theta\\
&+ A^4\int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|(t,\theta)\,d\theta\\
\leqslant{}& 4A^3\bigg(\int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)^2(t,\theta)\,d\theta\bigg)^{1/2}\\
&+ \frac{8\sqrt{2}A^4}{\sqrt{3}e}\bigg(\int_0^\eta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}\,d\theta\bigg)^{1/2}|A_1-A_2|\\
&+ \tilde{\mathcal{U}}_j^2|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|(t,\eta)\\
&+ \sqrt{2}A^4\bigg(\int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,\theta)\,d\theta\bigg)^{1/2}\\
&+ A^4\int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|(t,\theta)\,d\theta. \qquad \text{(A.14)}
\end{aligned}
$$

Following the same lines, one has

$$
\begin{aligned}
\bar{T}_2(t,\eta) ={}& \bigg|\int_0^\eta\Big(\min_j(\mathrm{e}^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\tilde{\mathcal{P}}_1\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\\
&- \min_j(e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\Big)\,d\theta\bigg|\\
\leqslant{}& \bigg|\int_0^\eta \min_j(e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})(\tilde{\mathcal{P}}_1-\tilde{\mathcal{P}}_2)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\mathbb{1}_{\tilde{\mathcal{P}}_2\leqslant\tilde{\mathcal{P}}_1}(t,\theta)\,d\theta\bigg|\\
&+ \bigg|\int_0^\eta \min_j(e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})(\tilde{\mathcal{P}}_1-\tilde{\mathcal{P}}_2)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\mathbb{1}_{\tilde{\mathcal{P}}_1<\tilde{\mathcal{P}}_2}(t,\theta)\,d\theta\bigg|\\
&+ \mathbb{1}_{A_1\leqslant A_2}\bigg|\int_0^\eta\Big(\min_j(e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\\
&- \min_j(e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\Big)\min_j(\tilde{\mathcal{P}}_j)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\,d\theta\bigg|\\
&+ \mathbb{1}_{A_2<A_1}\bigg|\int_0^\eta\Big(\min_j(e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\\
&- \min_j(e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\Big)\min_j(\tilde{\mathcal{P}}_j)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\,d\theta\bigg|
\end{aligned}
$$

$$
\begin{aligned}
&+ \left|\int_0^\eta \min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{P}}_j)(\tilde{\mathcal{Y}}_{1,\eta} - \tilde{\mathcal{Y}}_{2,\eta})(t,\theta)\, d\theta\right| \\
\leqslant\ & 2\int_0^\eta \min_j(e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})|\tilde{\mathcal{P}}_1^{1/2}|\tilde{\mathcal{Y}}_{1,\eta}|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t,\theta)\, d\theta \\
&+ 2\int_0^\eta \min_j(e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})|\tilde{\mathcal{P}}_2^{1/2}|\tilde{\mathcal{Y}}_{2,\eta}|\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2}|(t,\theta)\, d\theta \\
&+ \mathbb{1}_{A_1\leqslant A_2}\frac{4}{ae}\left|\int_0^\eta \min_j(e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{P}}_j)\tilde{\mathcal{Y}}_{2,\eta}(t,\theta)\, d\theta\right||A_1 - A_2| \\
&+ \mathbb{1}_{A_2< A_1}\frac{4}{ae}\left|\int_0^\eta \min_j(e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{P}}_j)\tilde{\mathcal{Y}}_{1,\eta}(t,\theta)\, d\theta\right||A_1 - A_2| \\
&+ \Bigg|\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{P}}_j)(\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)\Big|_{\theta=0}^{\eta} \\
&- \int_0^\eta \frac{d}{d\theta}\Big(\min_j(e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}) \min_j(\tilde{\mathcal{P}}_j)\Big)(\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2)(t,\theta)\, d\theta\Bigg| \\
\leqslant\ & 2\left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\tilde{\mathcal{P}}_1\tilde{\mathcal{Y}}_{1,\eta}^2(t,\theta)\, d\theta\right)^{1/2} \\
&\times \left(\int_0^\eta e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}(\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2(t,\theta)\, d\theta\right)^{1/2} \\
&+ 2\left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}^2(t,\theta)\, d\theta\right)^{1/2} \\
&\times \left(\int_0^\eta e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}(\tilde{\mathcal{P}}_1^{1/2} - \tilde{\mathcal{P}}_2^{1/2})^2(t,\theta)\, d\theta\right)^{1/2} \\
&+ \frac{2A}{e}\left(\int_0^\eta e^{-\frac{3}{4A_2}(\tilde{\mathcal{Y}}_2(t,\eta)-\tilde{\mathcal{Y}}_2(t,\theta))}\tilde{\mathcal{P}}_2\tilde{\mathcal{Y}}_{2,\eta}^2(t,\theta)\, d\theta\right)^{1/2} \\
&\times \left(\int_0^\eta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}\, d\theta\right)^{1/2}|A_1 - A_2| \\
&+ \frac{2A}{e}\left(\int_0^\eta e^{-\frac{3}{4A_1}(\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_1(t,\theta))}\tilde{\mathcal{P}}_1\tilde{\mathcal{Y}}_{1,\eta}^2(t,\theta)\, d\theta\right)^{2} \\
&\times \left(\int_0^\eta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}\, d\theta\right)^{1/2}|A_1 - A_2| \\
&+ \tilde{\mathcal{P}}_j|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t,\eta) \\
&+ \frac{1}{a}\int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))} \min_j(\tilde{\mathcal{P}}_j)(\tilde{\mathcal{Y}}_{1,\eta} + \tilde{\mathcal{Y}}_{2,\eta})|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t,\theta)\, d\theta \\
&+ \frac{A^4}{2}\int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}|\tilde{\mathcal{Y}}_1 - \tilde{\mathcal{Y}}_2|(t,\theta)\, d\theta
\end{aligned}
$$

$$\leqslant 2\sqrt{2}A^3\Big(\int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})^2(t,\theta)\,d\theta\Big)^{1/2}$$
$$+\frac{4\sqrt{2}A^4}{\sqrt{3e}}\Big(\int_0^\eta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}\,d\theta\Big)^{1/2}|A_1-A_2|$$
$$+\tilde{\mathcal{P}}_j|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|(t,\eta)$$
$$+\frac{A^4}{\sqrt{2}}\Big(\int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,\theta)\,d\theta\Big)^{1/2}$$
$$+\frac{A^4}{2}\int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|(t,\theta)\,d\theta. \tag{A.15}$$

For the last term $\bar{T}_3(t,\eta)$, we have

$$2\bar{T}_3(t,\eta)=\Big|\int_0^\eta \big(\min_j(e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})A_1^5-\big(\min_j(e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})A_2^5\big)\,d\theta\Big|$$
$$\leqslant \mathbb{1}_{A_1\leqslant A_2}\Big|\int_0^\eta (e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})(A_1^5-A_2^5)\,d\theta\Big|$$
$$+\mathbb{1}_{A_2<A_1}\Big|\int_0^\eta (e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})(A_1^5-A_2^5)\,d\theta\Big|$$
$$+a^5\int_0^\eta \big|\min_j(e^{-\frac{1}{A_1}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})-\min_j(e^{-\frac{1}{A_2}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))})\big|\,d\theta$$
$$\leqslant 10A^4\int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}\,d\theta|A_1-A_2|$$
$$+\frac{4a^4}{e}\int_0^\eta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}\,d\theta|A_1-A_2|$$
$$\leqslant 12A^4\int_0^\eta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}\,d\theta|A_1-A_2|. \tag{A.16}$$

Thus we end up with

$$|\tilde{\mathcal{D}}_1(t,\eta)-\tilde{\mathcal{D}}_2(t,\eta)|\leqslant 2A^{3/2}\max_j(\tilde{\mathcal{D}}_j^{1/2})(t,\eta)|\tilde{\mathcal{Y}}_1(t,\eta)-\tilde{\mathcal{Y}}_2(t,\eta)|$$
$$+2\sqrt{2}A^{3/2}\max_j(\tilde{\mathcal{D}}_j^{1/2})(t,\eta)\|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2\|$$
$$+4A^3\Big(\int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}(\tilde{\mathcal{U}}_1-\tilde{\mathcal{U}}_2)^2(t,\theta)\,d\theta\Big)^{1/2}$$
$$+2\sqrt{2}A^3\Big(\int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}(\tilde{\mathcal{P}}_1^{1/2}-\tilde{\mathcal{P}}_2^{1/2})^2(t,\theta)\,d\theta\Big)^{1/2}$$
$$+\frac{3A^4}{\sqrt{2}}\Big(\int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}(\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2)^2(t,\theta)\,d\theta\Big)^{1/2}$$

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$$
\begin{aligned}
&+\frac{12\sqrt{2}A^4}{\sqrt{3}e}\Big(\int_0^\eta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}\,d\theta\Big)^{1/2}|A_1-A_2|\\
&+\tilde{\mathcal{U}}_j^2|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|(t,\eta)\\
&+\tilde{\mathcal{P}}_j|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|(t,\eta)\\
&+\frac{3A^4}{2}\int_0^\eta e^{-\frac{1}{a}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}|\tilde{\mathcal{Y}}_1-\tilde{\mathcal{Y}}_2|(t,\theta)\,d\theta\\
&+6A^4\int_0^\eta e^{-\frac{3}{4A}(\tilde{\mathcal{Y}}_j(t,\eta)-\tilde{\mathcal{Y}}_j(t,\theta))}\,d\theta|A_1-A_2|,
\end{aligned}
$$

which proves the lemma. □

LEMMA A.10. *We have the following estimates:*

$$\int_0^\eta e^{-\frac{3}{2A}(\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta))}\tilde{\mathcal{P}}\tilde{\mathcal{Y}}_\eta(t,\theta)\,d\theta \leqslant 2A\tilde{\mathcal{P}}(t,\eta), \tag{A.17}$$

$$\int_0^\eta e^{-\frac{3}{2A}(\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta))}\tilde{\mathcal{H}}_\eta(t,\theta)\,d\theta \leqslant 4A\tilde{\mathcal{P}}(t,\eta), \tag{A.18}$$

$$\int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta))}\tilde{\mathcal{U}}^2(t,\theta)d\theta \leqslant 6\tilde{\mathcal{P}}(t,\eta), \tag{A.19}$$

$$\int_0^\eta e^{-\frac{5}{4A}(\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta))}\tilde{\mathcal{P}}\tilde{\mathcal{Y}}_\eta(t,\theta)\,d\theta \leqslant 4A\tilde{\mathcal{P}}(t,\eta), \tag{A.20}$$

$$\int_0^\eta e^{-\frac{3}{2A}(\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta))}\tilde{\mathcal{P}}(t,\theta)\,d\theta \leqslant 7\tilde{\mathcal{P}}(t,\eta), \tag{A.21}$$

$$\int_0^\eta e^{-\frac{1}{2A}(\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta))}\tilde{\mathcal{P}}^2\tilde{\mathcal{Y}}_\eta(t,\theta)\,d\theta \leqslant A\mathcal{O}(1)\tilde{\mathcal{P}}^{1/2}(t,\eta), \tag{A.22}$$

$$\int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta))}\tilde{\mathcal{P}}^{1+\beta}\tilde{\mathcal{Y}}_\eta(t,\theta)\,d\theta \leqslant 3\frac{1+\beta}{\beta}\frac{A^{1+4\beta}}{4^\beta}\tilde{\mathcal{P}}(t,\eta), \quad \beta>0. \tag{A.23}$$

Proof. The proof of (A.17) goes as follows:

$$
\begin{aligned}
&\int_0^\eta e^{-\frac{3}{2A}(\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta))}\tilde{\mathcal{P}}\tilde{\mathcal{Y}}_\eta(t,\theta)\,d\theta\\
&\quad=\frac{2}{3}A\tilde{\mathcal{P}}(t,\eta)-\frac{2}{3A}\int_0^\eta e^{-\frac{3}{2A}(\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta))}\tilde{\mathcal{Q}}\tilde{\mathcal{Y}}_\eta(t,\theta)\,d\theta\\
&\quad\leqslant\frac{2}{3}A\tilde{\mathcal{P}}(t,\eta)+\frac{2}{3}\int_0^\eta e^{-\frac{3}{2A}(\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta))}\tilde{\mathcal{P}}\tilde{\mathcal{Y}}_\eta(t,\theta)\,d\theta,
\end{aligned}
$$

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which implies

$$\int_0^\eta e^{-\frac{3}{2A}(\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta))}\tilde{\mathcal{P}}\tilde{\mathcal{Y}}_\eta(t,\theta)\,d\theta \leqslant 2A\tilde{\mathcal{P}}(t,\eta).$$

Next, we use that

$$\int_0^\eta e^{-\frac{3}{2A}(\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta))}\tilde{\mathcal{H}}_\eta(t,\theta)\,d\theta \leqslant \int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta))}\tilde{\mathcal{H}}_\eta(t,\theta)\,d\theta \leqslant 4A\tilde{\mathcal{P}}(t,\eta)$$

(see (4.16d)), showing (A.18).

For (A.19) we find

$$\begin{aligned}
&\int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta))}\tilde{\mathcal{U}}^2(t,\theta)\,d\theta \\
&\quad = \theta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta))}\tilde{\mathcal{U}}^2(t,\theta)\Big|_{\theta=0}^{\eta} \\
&\qquad - \int_0^\eta \theta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta))}\left(\frac{1}{A}\tilde{\mathcal{U}}^2\tilde{\mathcal{Y}}_\eta + 2\tilde{\mathcal{U}}\tilde{\mathcal{U}}_\eta\right)(t,\theta)\,d\theta \\
&\quad = \eta\tilde{\mathcal{U}}^2(t,\eta) - \int_0^\eta \theta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta))}\left(\frac{1}{A}\tilde{\mathcal{U}}^2\tilde{\mathcal{Y}}_\eta + 2\tilde{\mathcal{U}}\tilde{\mathcal{U}}_\eta\right)(t,\theta)\,d\theta \\
&\quad \leqslant \tilde{\mathcal{U}}^2(t,\eta) + \int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta))}\left(\frac{1}{A}\tilde{\mathcal{U}}^2\tilde{\mathcal{Y}}_\eta + 2|\tilde{\mathcal{U}}\tilde{\mathcal{U}}_\eta|\right)(t,\theta)\,d\theta \\
&\quad \leqslant \tilde{\mathcal{U}}^2(t,\eta) + 4\tilde{\mathcal{P}}(t,\eta) \\
&\quad \leqslant 6\tilde{\mathcal{P}}(t,\eta).
\end{aligned}$$

The proof of (A.20),

$$\int_0^\eta e^{-\frac{5}{4A}(\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta))}\tilde{\mathcal{P}}\tilde{\mathcal{Y}}_\eta(t,\theta)\,d\theta \leqslant 4A\tilde{\mathcal{P}}(t,\eta),$$

follows in the same manner as (A.17).

Furthermore, for (A.21) we find

$$\begin{aligned}
&\int_0^\eta e^{-\frac{3}{2A}(\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta))}\tilde{\mathcal{P}}(t,\theta)\,d\theta \\
&\quad = \theta e^{-\frac{3}{2A}(\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta))}\tilde{\mathcal{P}}(t,\theta)\Big|_{\theta=0}^{\eta} \\
&\qquad - \int_0^\eta \theta e^{-\frac{3}{2A}(\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta))}\left(\frac{3}{2A}\tilde{\mathcal{P}}\tilde{\mathcal{Y}}_\eta + \frac{1}{A^2}\tilde{\mathcal{Q}}\tilde{\mathcal{Y}}_\eta\right)(t,\theta)\,d\theta
\end{aligned}$$

$$\leqslant \eta\tilde{\mathcal{P}}(t,\eta) + \frac{3}{A}\int_0^\eta e^{-\frac{3}{2A}(\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta))}\tilde{\mathcal{P}}\tilde{\mathcal{Y}}_\eta(t,\theta)\,d\theta$$
$$\leqslant 7\tilde{\mathcal{P}}(t,\eta),$$

which follows from (A.17).

In order to prove (A.22), we do as follows:

$$\begin{aligned}
&\int_0^\eta e^{-\frac{1}{2A}(\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta))}\tilde{\mathcal{P}}^2\tilde{\mathcal{Y}}_\eta(t,\theta)\,d\theta\\
&\quad = 2A\tilde{\mathcal{P}}^2(t,\eta) - 8\tilde{\mathcal{P}}\tilde{\mathcal{Q}}(t,\eta)\\
&\qquad + 8\int_0^\eta e^{-\frac{1}{2A}(\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta))}\Big(\tilde{\mathcal{P}}^2 + \frac{1}{A^2}\tilde{\mathcal{Q}}^2\Big)\tilde{\mathcal{Y}}_\eta(t,\theta)\,d\theta\\
&\qquad + 8\int_0^\eta e^{-\frac{1}{2A}(\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta))}\Big((\tilde{\mathcal{P}}-\tilde{\mathcal{U}}^2)\tilde{\mathcal{Y}}_\eta(t,\theta) - \frac{1}{2}A^5\Big)\tilde{\mathcal{P}}(t,\theta)\,d\theta,
\end{aligned}$$

and hence

$$\begin{aligned}
&\int_0^\eta e^{-\frac{1}{2A}(\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta))}\tilde{\mathcal{P}}^2\tilde{\mathcal{Y}}_\eta(t,\theta)\,d\theta\\
&\quad \leqslant 8\tilde{\mathcal{P}}\tilde{\mathcal{Q}}(t,\eta) + 8\int_0^\eta e^{-\frac{1}{2A}(\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta))}\Big((\tilde{\mathcal{U}}^2-\tilde{\mathcal{P}})\tilde{\mathcal{Y}}_\eta(t,\theta) + \frac{1}{2}A^5\Big)\tilde{\mathcal{P}}(t,\theta)\,d\theta\\
&\quad \leqslant 8A\tilde{\mathcal{P}}^2(t,\eta) + 8\Big(\int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta))}\Big((\tilde{\mathcal{U}}^2-\tilde{\mathcal{P}})\tilde{\mathcal{Y}}_\eta(t,\theta) + \frac{1}{2}A^5\Big)\,d\theta\Big)^{1/2}\\
&\qquad \times\Big(\int_0^\eta\Big((\tilde{\mathcal{U}}^2-\tilde{\mathcal{P}})\tilde{\mathcal{Y}}_\eta(t,\theta) + \frac{1}{2}A^5\Big)\tilde{\mathcal{P}}^2(t,\theta)\,d\theta\Big)^{1/2}\\
&\quad \leqslant A\mathcal{O}(1)\tilde{\mathcal{P}}^{1/2}(t,\eta).
\end{aligned}$$

As for the proof of (A.23), we proceed as follows:

$$\begin{aligned}
&\int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta))}\tilde{\mathcal{P}}^{1+\beta}\tilde{\mathcal{Y}}_\eta(t,\theta)\,d\theta\\
&\quad = Ae^{-\frac{1}{A}(\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta))}\tilde{\mathcal{P}}^{1+\beta}(t,\theta)\big|_{\theta=0}^{\eta}\\
&\qquad - A(1+\beta)\int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta))}\tilde{\mathcal{P}}^{\beta}\tilde{\mathcal{P}}_\eta(t,\theta)\,d\theta\\
&\quad = A\tilde{\mathcal{P}}^{1+\beta}(t,\eta) - \frac{1}{A}(1+\beta)\int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta))}\tilde{\mathcal{P}}^{\beta}\tilde{\mathcal{Q}}\tilde{\mathcal{Y}}_\eta(t,\theta)\,d\theta\\
&\quad = A\tilde{\mathcal{P}}^{1+\beta}(t,\eta) - (1+\beta)e^{-\frac{1}{A}(\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta))}\tilde{\mathcal{P}}^{\beta}\tilde{\mathcal{Q}}(t,\theta)\big|_{\theta=0}^{\eta}\\
&\qquad + (1+\beta)\int_0^\eta e^{-\frac{1}{A}(\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta))}(\beta\tilde{\mathcal{P}}^{\beta-1}\tilde{\mathcal{P}}_\eta\tilde{\mathcal{Q}} + \tilde{\mathcal{P}}^{\beta}\tilde{\mathcal{Q}}_\eta)(t,\theta)\,d\theta
\end{aligned}$$

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$$= A\tilde{\mathcal{P}}^{1+\beta}(t,\eta) - (1+\beta)\tilde{\mathcal{P}}^{\beta}\tilde{\mathcal{Q}}(t,\eta) \\ + (1+\beta)\int_0^{\eta} e^{-\frac{1}{A}(\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta))}\Big(\beta\tilde{\mathcal{P}}^{\beta-1}\tilde{\mathcal{Q}}^2\tilde{\mathcal{Y}}_\eta\frac{1}{A^2} + \tilde{\mathcal{P}}^{\beta}\tilde{\mathcal{Q}}_\eta\Big)(t,\theta)\,d\theta. \tag{A.24}$$

Recall (4.6), (4.9), and (4.10), which together imply

$$\tilde{\mathcal{Q}}_\eta(t,\eta) = \tfrac{1}{2}(\tilde{\mathcal{E}}_\eta - \tilde{\mathcal{D}}_\eta)(t,\eta) = -(\tilde{\mathcal{U}}^2 - \tilde{\mathcal{P}})\tilde{\mathcal{Y}}_\eta(t,\eta) - \tfrac{1}{2}A^5 + \tilde{\mathcal{P}}\tilde{\mathcal{Y}}_\eta(t,\eta).$$

Thus

$$\beta\tilde{\mathcal{P}}^{\beta-1}\tilde{\mathcal{Q}}^2\tilde{\mathcal{Y}}_\eta\frac{1}{A^2} + \tilde{\mathcal{P}}^{\beta}\tilde{\mathcal{Q}}_\eta \\ = \frac{\beta}{A^2}\tilde{\mathcal{Q}}^2\tilde{\mathcal{P}}^{\beta-1}\tilde{\mathcal{Y}}_\eta + \tilde{\mathcal{P}}^{\beta+1}\tilde{\mathcal{Y}}_\eta + \tilde{\mathcal{P}}^{\beta}\Big(-(\tilde{\mathcal{U}}^2 - \tilde{\mathcal{P}})\tilde{\mathcal{Y}}_\eta - \frac{1}{2}A^5\Big).$$

Inserting this expression in (A.24) and re-ordering the terms, we find

$$\beta\int_0^{\eta} e^{-\frac{1}{A}(\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta))}\tilde{\mathcal{P}}^{1+\beta}\tilde{\mathcal{Y}}_\eta(t,\theta)\,d\theta \\ + \frac{1+\beta}{A^2}\int_0^{\eta} e^{-\frac{1}{A}(\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta))}\tilde{\mathcal{P}}^{\beta-1}\tilde{\mathcal{Q}}^2\tilde{\mathcal{Y}}_\eta(t,\theta)\,d\theta \\ = (1+\beta)\tilde{\mathcal{P}}^{\beta}\tilde{\mathcal{Q}}(t,\eta) - A\tilde{\mathcal{P}}^{1+\beta}(t,\eta) \\ + (1+\beta)\int_0^{\eta} e^{-\frac{1}{A}(\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta))}\tilde{\mathcal{P}}^{\beta}\Big((\tilde{\mathcal{U}}^2 - \tilde{\mathcal{P}})\tilde{\mathcal{Y}}_\eta + \frac{1}{2}A^5\Big)(t,\theta)\,d\theta.$$

Estimating this, using (4.5a), we find

$$\beta\int_0^{\eta} e^{-\frac{1}{A}(\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta))}\tilde{\mathcal{P}}^{1+\beta}\tilde{\mathcal{Y}}_\eta(t,\theta)\,d\theta \\ \leqslant (1+\beta)\tilde{\mathcal{P}}^{\beta}|\tilde{\mathcal{Q}}|(t,\eta) \\ + (1+\beta)\int_0^{\eta} e^{-\frac{1}{A}(\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta))}\tilde{\mathcal{P}}^{\beta}\Big((\tilde{\mathcal{U}}^2 - \tilde{\mathcal{P}})\tilde{\mathcal{Y}}_\eta + \frac{1}{2}A^5\Big)(t,\theta)\,d\theta \\ \leqslant (1+\beta)\|\tilde{\mathcal{P}}\|_\infty^{\beta}A\tilde{\mathcal{P}}(t,\eta) \\ + (1+\beta)\|\tilde{\mathcal{P}}\|_\infty^{\beta}\int_0^{\eta} e^{-\frac{1}{A}(\tilde{\mathcal{Y}}(t,\eta)-\tilde{\mathcal{Y}}(t,\theta))}\Big((\tilde{\mathcal{U}}^2 - \tilde{\mathcal{P}})\tilde{\mathcal{Y}}_\eta + \frac{1}{2}A^5\Big)(t,\theta)\,d\theta \\ \leqslant (1+\beta)\|\tilde{\mathcal{P}}\|_\infty^{\beta}A\tilde{\mathcal{P}}(t,\eta) + (1+\beta)\|\tilde{\mathcal{P}}\|_\infty^{\beta}\tilde{\mathcal{D}}(t,\eta) \\ \leqslant 3(1+\beta)\Big(\frac{A^4}{4}\Big)^{\beta}A\tilde{\mathcal{P}}(t,\eta),$$

where we used (4.8), (4.15a), and (4.15c). □

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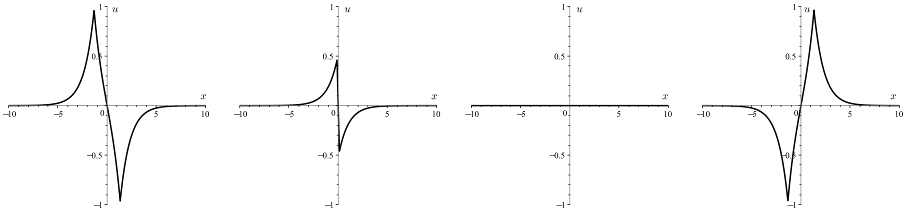

Figure B.3. Time evolution of $u(t, x)$ with $C = E^2 = 4$ and $t_0 = 2$ at $t = 0$, 1.5, 2, 4.

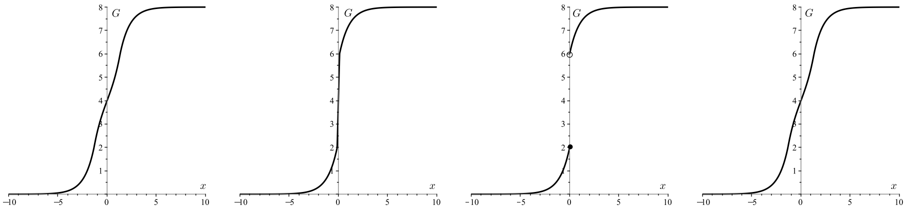

Figure B.4. Time evolution of $G(t, x)$ with $C = E^2 = 4$ and $t_0 = 2$ at $t = 0$, 1.5, 2, 4.

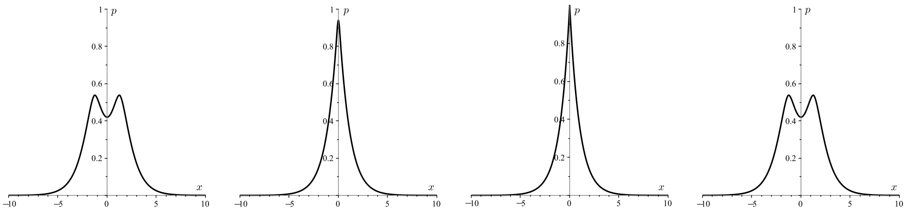

Figure B.5. Time evolution of $p(t, x)$ with $C = E^2 = 4$ and $t_0 = 2$ at $t = 0$, 1.5, 2, 4.

Appendix B. The antisymmetric peakon–antipeakon example

Fortunately, one can compute explicitly the quantities described in this paper in the important case of an antisymmetric peakon–antipeakon solution. The various functions are depicted on Figures B.3–B.14.

More precisely, consider the function [41]

$$u(t, x) = \begin{cases} \beta(t)\sinh(x), & |x| \leqslant \gamma(t), \\ \operatorname{sign}(x)\alpha(t)e^{-|x|}, & |x| \geqslant \gamma(t), \end{cases}$$

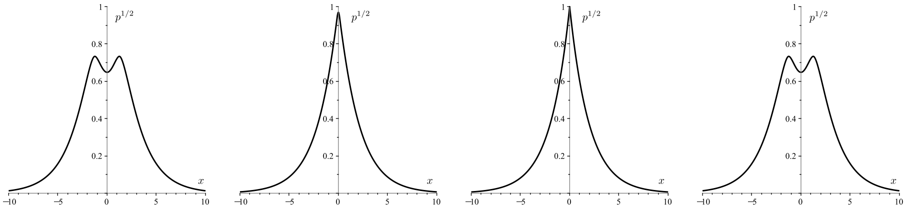

Figure B.6. Time evolution of $p^{1/2}(t,x)$ with $C = E^2 = 4$ and $t_0 = 2$ at $t = 0$, 1.5, 2, 4.

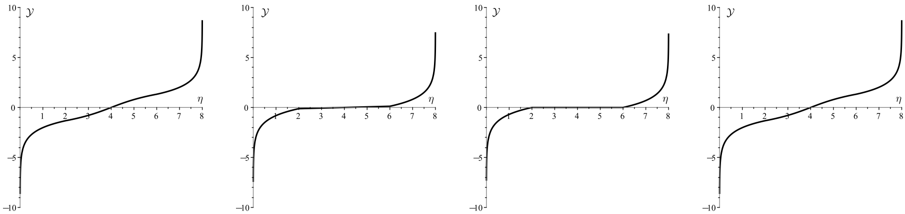

Figure B.7. Time evolution of $\mathcal{Y}(t,\eta)$ with $C = E^2 = 4$ and $t_0 = 2$ at $t = 0$, 1.5, 2, 4.

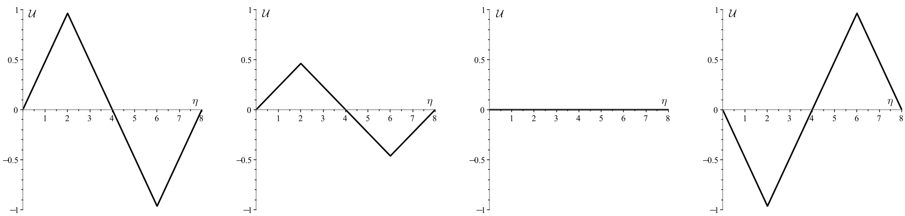

Figure B.8. Time evolution of $\mathcal{U}(t,\eta)$ with $C = E^2 = 4$ and $t_0 = 2$ at $t = 0$, 1.5, 2, 4.

$$= \begin{cases} -\alpha(t)e^{x}, & x \leqslant -\gamma(t), \\ \beta(t)\sinh(x), & -\gamma(t) \leqslant x \leqslant \gamma(t), \\ \alpha(t)e^{-x}, & \gamma(t) \leqslant x, \end{cases}$$

where

$$\alpha(t) = \frac{E}{2}\sinh\left(\frac{E}{2}(t-t_0)\right), \quad \beta(t) = E\frac{1}{\sinh(\frac{E}{2}(t-t_0))},$$
$$\gamma(t) = \ln\left(\cosh\left(\frac{E}{2}(t-t_0)\right)\right).$$

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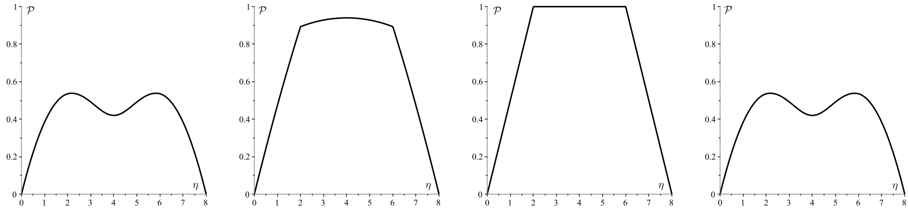

Figure B.9. Time evolution of $\mathcal{P}(t, \eta)$ with $C = E^2 = 4$ and $t_0 = 2$ at $t = 0$, 1.5, 2, 4.

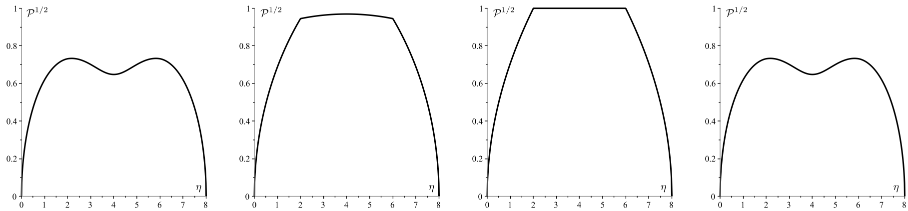

Figure B.10. Time evolution of $\mathcal{P}^{1/2}(t, \eta)$ with $C = E^2 = 4$ and $t_0 = 2$ at $t = 0$, 1.5, 2, 4.

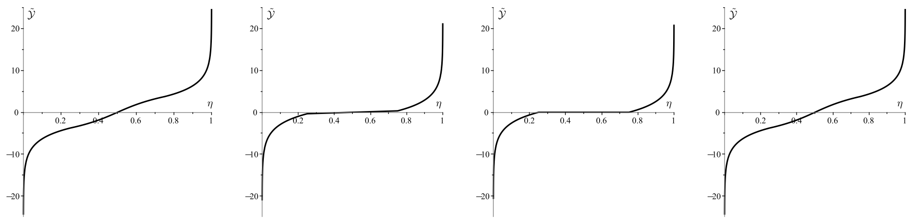

Figure B.11. Time evolution of $\tilde{\mathcal{Y}}(t, \eta)$ with $C = E^2 = 4$ and $t_0 = 2$ at $t = 0$, 1.5, 2, 4.

Here $E = \|u(t)\|_{H^1}$, $t \neq t_0$, denotes the total energy. The corresponding energy density is given by

$$\mu(t, x) = (u^2 + u_x^2)(t, x) = \begin{cases} 2\alpha^2(t)e^{2x}, & x \leqslant -\gamma(t), \\ \beta^2(t)\cosh(2x), & -\gamma(t) \leqslant x \leqslant \gamma(t), \qquad t \neq t_0, \\ 2\alpha^2(t)e^{-2x}, & \gamma(t) \leqslant x, \end{cases}$$

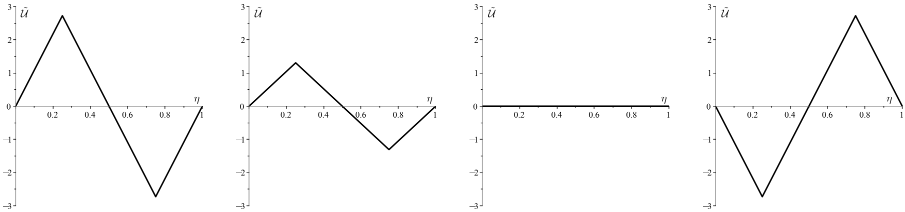

Figure B.12. Time evolution of $\tilde{\mathcal{U}}(t,\eta)$ with $C = E^2 = 4$ and $t_0 = 2$ at $t = 0$, 1.5, 2, 4.

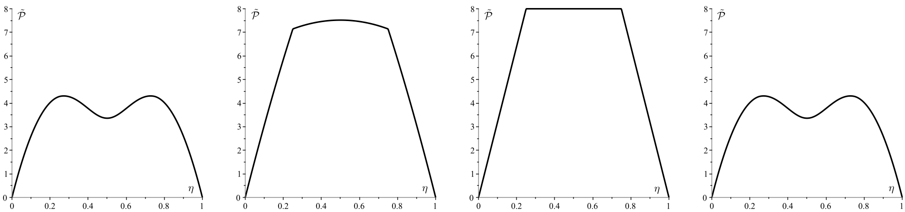

Figure B.13. Time evolution of $\tilde{\mathcal{P}}(t,\eta)$ with $C = E^2 = 4$ and $t_0 = 2$ at $t = 0$, 1.5, 2, 4.

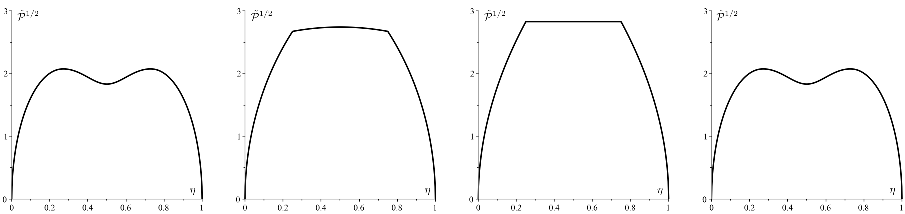

Figure B.14. Time evolution of $\tilde{\mathcal{P}}^{1/2}(t,\eta)$ with $C = E^2 = 4$ and $t_0 = 2$ at $t = 0$, 1.5, 2, 4.

with $\mu(t_0, x) = E^2\delta_0(x)$ for $t = t_0$. Hence $C = \mu(t,\mathbb{R}) = E^2$, and

$$F(t,x) = \begin{cases} \alpha(t)^2 e^{2x}, & x < -\gamma(t), \\ \dfrac{E^2}{4}\tanh^2\left(\dfrac{E}{2}(t-t_0)\right), & x = -\gamma(t), \\ \dfrac{1}{2}E^2 + \dfrac{1}{2}\beta(t)^2\sinh(2x), & -\gamma(t) < x < \gamma(t), \\ E^2 - \dfrac{E^2}{4}\tanh^2\left(\dfrac{E}{2}(t-t_0)\right), & x = \gamma(t), \\ E^2 - \alpha(t)^2 e^{-2x}, & \gamma(t) < x. \end{cases}$$

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In particular, this solution experiences wave breaking at time $t = t_0$, that is, $u_x(t, 0)$ tends to $-\infty$ as $t \to t_0-$ and

$$F(t_0-, x) = \begin{cases} 0, & x \leqslant 0, \\ E^2, & 0 < x. \end{cases}$$

The corresponding function $p_x(t, x)$, which can be computed using $p_x(t, x) = -u_t(t, x) - uu_x(t, x)$, is given by

$$p_x(t, x) = \begin{cases} \alpha'(t)e^x - \alpha(t)^2 e^{2x}, & x < -\gamma(t), \\ \dfrac{E^2}{4} - \dfrac{E^2}{4}\tanh^2\left(\dfrac{E}{2}(t - t_0)\right), & x = -\gamma(t), \\ -\beta'(t)\sinh(x) - \frac{1}{2}\beta(t)^2\sinh(2x), & -\gamma(t) < x < \gamma(t), \\ -\dfrac{E^2}{4} + \dfrac{E^2}{4}\tanh^2\left(\dfrac{E}{2}(t - t_0)\right), & \gamma(t) = x, \\ -\alpha'(t)e^{-x} + \alpha(t)^2 e^{-2x}, & \gamma(t) < x. \end{cases}$$

In particular, one obtains at wave breaking time $t = t_0$ that

$$p_x(t_0-, x) = \begin{cases} \dfrac{E^2}{4}e^x, & x < 0, \\ -\dfrac{E^2}{4}e^{-x}, & 0 < x. \end{cases}$$

Here it is important to note that $p_x(t_0-, x)$ has a (negative) jump of height $-\frac{E^2}{2}$ at $x = 0$ at time $t = t_0$. For all other points $x \in \mathbb{R}$, the function $p_x(t_0-, x)$ is continuous.

Thus the function $G(t, x) = 2p_x(t, x) + 2F(t, x)$ is given by

$$G(t, x) = \begin{cases} 2\alpha'(t)e^x, & x < -\gamma(t), \\ \dfrac{E^2}{2}, & x = -\gamma(t), \\ E^2 - 2\beta'(t)\sinh(x), & -\gamma(t) < x < \gamma(t), \\ \dfrac{3E^2}{2}, & x = \gamma(t), \\ 2E^2 - 2\alpha'(t)e^{-x}, & \gamma(t) < x. \end{cases}$$

In particular, one observes that $G_x(t,x) > 0$ for all $x \in \mathbb{R}$,

$$\lim_{x\to-\infty} G(t,x) = 0 \quad \text{and} \quad \lim_{x\to\infty} G(t,x) = 2E^2 = 2C.$$

Thus the limits at $\pm\infty$ are independent of time. Moreover, one has

$$G(t_0-,x) = \begin{cases} \dfrac{E^2}{2}e^x, & x \leqslant 0, \\ 2E^2 - \dfrac{E^2}{2}e^{-x}, & 0 < x. \end{cases}$$

Here it is important to note that $G(t_0-,x)$ has a jump of size E^2 due to the wave breaking at $t = t_0$, that is, $\mu(t_0,x) = E^2\delta_0(x)$.

Direct computations using $p = p_{xx} + \frac{1}{2}u^2 + \frac{1}{2}d\mu$ yield

$$p(t,x) = \begin{cases} \alpha'(t)e^x - \frac{1}{2}\alpha(t)^2e^{2x}, & x \leqslant -\gamma(t), \\ -\beta'(t)\cosh(x) - \frac{1}{2}\beta(t)^2\cosh^2(x), & -\gamma(t) \leqslant x \leqslant \gamma(t), \\ \alpha'(t)e^{-x} - \frac{1}{2}\alpha(t)^2e^{-2x}, & \gamma(t) \leqslant x. \end{cases}$$

In the new coordinates, using $G(t,\mathcal{Y}(t,\eta)) = \eta$, the solution reads as

$$\mathcal{Y}(t,\eta) = \begin{cases} \ln\left(\dfrac{\eta}{2\alpha'(t)}\right), & 0 < \eta \leqslant \dfrac{E^2}{2}, \\ \sinh^{-1}\left(\dfrac{E^2-\eta}{2\beta'(t)}\right), & \dfrac{E^2}{2} \leqslant \eta \leqslant \dfrac{3E^2}{2}, \\ \ln\left(\dfrac{2\alpha'(t)}{2E^2-\eta}\right), & \dfrac{3E^2}{2} \leqslant \eta < 2E^2, \end{cases}$$

and, applying $\mathcal{U}(t,\eta) = u(t,\mathcal{Y}(t,\eta))$,

$$\mathcal{U}(t,\eta) = \begin{cases} -\dfrac{\alpha(t)}{2\alpha'(t)}\eta, & 0 < \eta \leqslant \dfrac{E^2}{2}, \\ \dfrac{\beta(t)}{2\beta'(t)}(E^2-\eta), & \dfrac{E^2}{2} \leqslant \eta \leqslant \dfrac{3E^2}{2}, \\ \dfrac{\alpha(t)}{2\alpha'(t)}(2E^2-\eta), & \dfrac{3E^2}{2} \leqslant \eta < 2E^2, \end{cases}$$

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and, by invoking $\mathcal{P}(t,\eta) = p(t, \mathcal{Y}(t,\eta))$,

$$\mathcal{P}(t,\eta) = \begin{cases} \frac{1}{2}\eta - \frac{\alpha(t)^2}{8\alpha'(t)^2}\eta^2, & 0 < \eta \leqslant \frac{E^2}{2}, \\ -\beta'(t)\sqrt{1+\frac{(E^2-\eta)^2}{4\beta'(t)^2}} \\ \quad -\frac{1}{2}\beta(t)^2\left(1+\frac{(E^2-\eta)^2}{4\beta'(t)^2}\right), & \frac{E^2}{2} \leqslant \eta \leqslant \frac{3E^2}{2}, \\ \frac{1}{2}(2E^2-\eta) - \frac{\alpha(t)^2}{8\alpha'(t)^2}(2E^2-\eta)^2, & \frac{3E^2}{2} \leqslant \eta < 2E^2, \end{cases}$$

where we used the convention that $\sinh^{-1}(x)$ denotes the inverse of $\sinh(x)$.

For the scaled quantities we find, when we introduce $A = \sqrt{2C} = \sqrt{2}E$, that

$$\tilde{\mathcal{Y}}(t,\eta) = A\mathcal{Y}(t, A^2\eta) = \sqrt{2C}\begin{cases} \ln\left(\frac{C\eta}{\alpha'(t)}\right), & 0 < \eta \leqslant \frac{1}{4}, \\ \sinh^{-1}\left(\frac{C(1-2\eta)}{2\beta'(t)}\right), & \frac{1}{4} \leqslant \eta \leqslant \frac{3}{4}, \\ \ln\left(\frac{\alpha'(t)}{C(1-\eta)}\right), & \frac{3}{4} \leqslant \eta < 1, \end{cases}$$

$$\tilde{\mathcal{U}}(t,\eta) = A\mathcal{U}(t, A^2\eta) = \sqrt{2C}\begin{cases} -\frac{\alpha(t)}{\alpha'(t)}C\eta, & 0 < \eta \leqslant \frac{1}{4}, \\ \frac{\beta(t)}{2\beta'(t)}C(1-2\eta), & \frac{1}{4} \leqslant \eta \leqslant \frac{3}{4}, \\ \frac{\alpha(t)}{\alpha'(t)}C(1-\eta), & \frac{3}{4} \leqslant \eta < 1, \end{cases}$$

$$\begin{aligned}\tilde{\mathcal{P}}(t,\eta) &= A^2\mathcal{P}(t, A^2\eta) \\ &= 2C\begin{cases} C\eta - \frac{\alpha(t)^2}{2\alpha'(t)^2}C^2\eta^2, & 0 < \eta \leqslant \frac{1}{4}, \\ -\beta'(t)\sqrt{1+\frac{C^2(1-2\eta)^2}{4\beta'(t)^2}} \\ \quad -\frac{1}{2}\beta(t)^2\left(1+\frac{C^2(1-2\eta)^2}{4\beta'(t)^2}\right), & \frac{1}{4} \leqslant \eta \leqslant \frac{3}{4}, \\ C(1-\eta) - \frac{\alpha(t)^2}{2\alpha'(t)^2}C^2(1-\eta)^2, & \frac{3}{4} \leqslant \eta < 1. \end{cases}\end{aligned}$$

Conflict of Interest: None.

References

[1] L. Ambrosio, N. Gigli and G. Savaré, *Gradient Flows in Metric Spaces and in the Space of Probability Measures*, Lectures in Mathematics ETH Zürich (Birkhäuser Verlag, Basel, 2008).

[2] A. Blanchet, V. Calvez and J. A. Carrillo, 'Convergence of the mass-transport steepest descent scheme for the subcritical Patlak–Keller–Segel model', *SIAM J. Numer. Anal.* **46** (2008), 691–721.

[3] F. Bolley, J. A. Cañizo and J. A. Carrillo, 'Stochastic mean-field limit: non-Lipschitz forces and swarming', *Math. Models Methods Appl. Sci.* **21** (2011), 2179–2210.

[4] G. A. Bonaschi, J. A. Carrillo, M. Di Francesco and M. A. Peletier, 'Equivalence of gradient flows and entropy solutions for singular nonlocal interaction equations in 1D', *ESAIM Control Optim. Calc. Var.* **21** (2015), 414–441.

[5] W. Braun and K. Hepp, 'The Vlasov dynamics and its fluctuations in the $1/N$ limit of interacting classical particles', *Comm. Math. Phys.* **56** (1977), 101–113.

[6] A. Bressan and A. Constantin, 'Global conservative solutions of the Camassa–Holm equation', *Arch. Ration. Mech. Anal.* **183** (2007), 215–239.

[7] A. Bressan and A. Constantin, 'Global dissipative solutions of the Camassa–Holm equation', *Anal. Appl. (Singap.)* **5** (2007), 1–27.

[8] A. Bressan and M. Fonte, 'An optimal transportation metric for solutions of the Camassa–Holm equation', *Methods Appl. Anal.* **12** (2005), 191–219.

[9] A. Bressan, H. Holden and X. Raynaud, 'Lipschitz metric for the Hunter–Saxton equation', *J. Math. Pures Appl. (9)* **94** (2010), 68–92.

[10] R. Camassa and D. D. Holm, 'An integrable shallow water equation with peaked solutions', *Phys. Rev. Lett.* **71** (1993), 1661–1664.

[11] R. Camassa, D. D. Holm and J. Hyman, 'A new integrable shallow water equation', *Adv. Appl. Mech.* **31** (1994), 1–33.

[12] J. A. Cañizo, J. A. Carrillo and J. Rosado, 'A well-posedness theory in measures for some kinetic models of collective motion', *Math. Models Methods Appl. Sci.* **21** (2011), 515–539.

[13] J. A. Carrillo, Y.-P. Choi, M. Hauray and S. Salem, 'Mean-field limit for collective behavior models with sharp sensitivity regions', *J. Eur. Math. Soc. (JEMS)* **21** (2019), 121–161.

[14] J. A. Carrillo, M. Di Francesco, A. Figalli, T. Laurent and D. Slepčev, 'Global-in-time weak measure solutions and finite-time aggregation for nonlocal interaction equations', *Duke Math. J.* **156** (2011), 229–271.

[15] J. A. Carrillo, M. Di Francesco and G. Toscani, 'Strict contractivity of the 2-Wasserstein distance for the porous medium equation by mass-centering', *Proc. Amer. Math. Soc.* **135** (2007), 353–363.

[16] J. A. Carrillo, K. Grunert and H. Holden, 'A Lipschitz metric for the Hunter–Saxton equation', *Comm. Partial Differential Equations* **44** (2019), 309–334.

[17] J. A. Carrillo and J. S. Moll, 'Numerical simulation of diffusive and aggregation phenomena in nonlinear continuity equations by evolving diffeomorphisms', *SIAM J. Sci. Comput.* **31** (2009/10), 4305–4329.

[18] J. A. Carrillo, H. Ranetbauer and M.-T. Wolfram, 'Numerical simulation of nonlinear continuity equations by evolving diffeomorphisms', *J. Comput. Phys.* **327** (2016), 186–202.

[19] J. A. Carrillo and G. Toscani, 'Wasserstein metric and large–time asymptotics of non-linear diffusion equations', in *New Trends in Mathematical Physics* (World Sci. Publ., Hackensack, NJ, 2004), 234–244.

[20] J. A. Carrillo and G. Toscani, 'Contractive probability metrics and asymptotic behavior of dissipative kinetic equations', *Riv. Mat. Univ. Parma (7)* **6** (2007), 75–198.

[21] R. M. Chen and Y. Liu, 'Wave breaking and global existence for a generalized two-component Camassa–Holm system', *Int. Math. Res. Not. IMRN* **2011**(6) (2011), 1381–1416.

[22] M. Chen, S.-Q. Liu and Y. Zhang, 'A two-component generalization of the Camassa–Holm equation and its solutions', *Lett. Math. Phys.* **75** (2006), 1–15.

[23] A. Chertock, J.-G. Liu and T. Pendleton, 'Convergence of a particle method and global weak solutions for a family of evolutionary PDEs', *SIAM J. Numer. Anal.* **50** (2012), 1–21.

[24] G. M. Coclite, H. Holden and K. H. Karlsen, 'Global weak solutions to a generalized hyperelastic-rod wave equation', *SIAM J. Math. Anal.* **37** (2005), 1044–1069.

[25] A. Constantin and J. Escher, 'Wave breaking for nonlinear nonlocal shallow water equations', *Acta Math.* **181** (1998), 229–243.

[26] A. Constantin and B. Kolev, 'Geodesic flow on the diffeomorphism group of the circle', *Comment. Math. Helv.* **78** (2003), 787–804.

[27] A. Constantin and D. Lannes, 'The hydrodynamical relevance of the Camassa–Holm and Degasperis–Procesi equations', *Arch. Ration. Mech. Anal.* **192** (2009), 165–186.

[28] H.-H. Dai and Y. Huo, 'Solitary shock waves and other travelling waves in a general compressible hyperelastic rod', *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **456** (2000), 331–363.

[29] R. L. Dobrushin, 'Vlasov equations', *Funct. Anal. Appl.* **13** (1979), 115–123.

[30] B. Düring, D. Matthes and J. P. Milišić, 'A gradient flow scheme for nonlinear fourth order equations', *Discrete Contin. Dyn. Syst. Ser. B* **14** (2010), 935–959.

[31] J. Escher and B. Kolev, 'Geodesic completeness for Sobolev H^s-metrics on the diffeomorphism group of the circle', *J. Evol. Equ.* **14** (2014), 949–968.

[32] J. Escher, O. Lechtenfeld and Z. Yin, 'Well-posedness and blow-up phenomena for the 2-component Camassa–Holm equation', *Discrete Contin. Dyn. Syst.* **19** (2007), 493–513.

[33] Y. Fu and C. Qu, 'Well posedness and blow-up solution for a new coupled Camassa–Holm equations with peakons', *J. Math. Phys.* **50** (2009), 012906.

[34] T. Gallouët and F.-X. Vialard, 'The Camassa–Holm equation as an incompressible Euler equations: a geometric point of view', *J. Differential Equations* **264** (2018), 4199–4234.

[35] F. Golse, 'The mean-field limit for the dynamics of large particle systems', *J. Équ. aux Dérivées Partielles*, **9**, Univ. Nantes, Nantes, 2003, 47pp.

[36] F. Golse, 'On the dynamics of large particle systems in the mean-field limit', in *Macroscopic and Large Scale Phenomena: Coarse Graining, Mean Field Limits and Ergodicity*, Lecture Notes in Applied and Mathematics and Mechanics, 3 (Springer, Cham, 2016), 1–144.

[37] L. Gosse and G. Toscani, 'Lagrangian numerical approximations to one-dimensional convolution-diffusion equations', *SIAM J. Sci. Comput.* **28** (2006), 1203–1227.

[38] L. Gosse and G. Toscani, 'Identification of asymptotic decay to self-similarity for one-dimensional filtration equations', *SIAM J. Numer. Anal.* **43** (2006), 2590–2606.

[39] M. Grasmair, K. Grunert and H. Holden, 'On the equivalence of Eulerian and Lagrangian variables for the two-component Camassa–Holm system', in *Current Research in Nonlinear Analysis: In Honor of Haim Brezis and Louis Nirenberg*, (ed. Th. M. Rassias) Springer Optimization and Its Applications, 135 (Springer, Cham, Switzerland, 2018), 157–201.

https://doi.org/10.1017/fms.2020.22 Published online by Cambridge University Press

[40] K. Grunert, 'Solutions of the Camassa–Holm equation with accumulating breaking times', *Dyn. Partial Differ. Equ.* **13** (2016), 91–105.

[41] K. Grunert and H. Holden, 'The general peakon-antipeakon solution for the Camassa–Holm equation', *J. Hyperbolic Differ. Equ.* **13** (2016), 353–380.

[42] K. Grunert, H. Holden and X. Raynaud, 'Lipschitz metric for the periodic Camassa–Holm equation', *J. Differential Equations* **250** (2011), 1460–1492.

[43] K. Grunert, H. Holden and X. Raynaud, 'Lipschitz metric for the Camassa–Holm equation on the line', *Discrete Contin. Dyn. Syst.* **33** (2013), 2809–2827.

[44] K. Grunert, H. Holden and X. Raynaud, 'Global conservative solutions to the Camassa–Holm equation for initial data with nonvanishing asymptotics', *Discrete Contin. Dyn. Syst.* **32** (2012), 4209–4227.

[45] K. Grunert, H. Holden and X. Raynaud, 'Global dissipative solutions of the two-component Camassa–Holm system for initial data with nonvanishing asymptotics', *Nonlinear Anal. Real World Appl.* **17** (2014), 203–244.

[46] K. Grunert, H. Holden and X. Raynaud, 'Global solutions for the two-component Camassa–Holm system', *Comm. Partial Differential Equations* **37** (2012), 2245–2271.

[47] K. Grunert, H. Holden and X. Raynaud, 'A continuous interpolation between conservative and dissipative solutions for the two-component Camassa–Holm system', *Forum Math. Sigma* **3** (2014), e1, 73 pages.

[48] C. Guan, K. H. Karlsen and Z. Yin, 'Well-posedness and blow-up phenomenal for a modified two-component Camassa–Holm equation', in *Nonlinear Partial Differential Equations and Hyperbolic Wave Phenomena*, (eds. H. Holden and K. H. Karlsen) Contemporary Mathematics, 526 (American Mathematical Society, 2010), 199–220.

[49] C. Guan and Z. Yin, 'Global existence and blow-up phenomena for an integrable two-component Camassa–Holm shallow water system', *J. Differential Equations* **248** (2010), 2003–2014.

[50] C. Guan and Z. Yin, 'Global weak solutions for a two-component Camassa–Holm shallow water system', *J. Funct. Anal.* **260** (2011), 1132–1154.

[51] G. Gui and Y. Liu, 'On the global existence and wave breaking criteria for the two-component Camassa–Holm system', *J. Funct. Anal.* **258** (2010), 4251–4278.

[52] G. Gui and Y. Liu, 'On the Cauchy problem for the two-component Camassa–Holm system', *Math. Z.* **268** (2011), 45–66.

[53] Z. Guo and Y. Zhou, 'On solutions to a two-component generalized Camassa–Holm equation', *Stud. Appl. Math.* **124** (2010), 307–322.

[54] H. Holden and X. Raynaud, 'Global conservative solutions for the Camassa–Holm equation — a Lagrangian point of view', *Comm. Partial Differential Equations* **32** (2007), 1511–1549.

[55] H. Holden and X. Raynaud, 'Global conservative solutions of the generalized hyperelastic-rod wave equation', *J. Differential Equations* **233** (2007), 448–484.

[56] H. Holden and X. Raynaud, 'Dissipative solutions for the Camassa–Holm equation', *Discrete Contin. Dyn. Syst.* **24** (2009), 1047–1112.

[57] H. Holden and X. Raynaud, 'Global dissipative multipeakon solutions of the Camassa–Holm equation', *Comm. Partial Differential Equations* **33** (2008), 2040–2063.

[58] R. S. Johnson, 'Camassa–Holm, Korteweg–de Vries and related models for water waves', *J. Fluid Mech.* **455** (2002), 63–82.

[59] O. Junge, D. Matthes and H. Osberger, 'A fully discrete variational scheme for solving nonlinear Fokker–Planck equations in multiple space dimensions', *SIAM J. Numer. Anal.* **55** (2017), 419–443.

[60] S. Kouranbaeva, 'The Camassa–Holm equation as a geodesic flow on the diffeomorphism group', *J. Math. Phys.* **40** (1999), 857–868.

[61] H. Li and G. Toscani, 'Long-time asymptotics of kinetic models of granular flows', *Arch. Ration. Mech. Anal.* **172** (2004), 407–428.

[62] D. Matthes and H. Osberger, 'Convergence of a variational Lagrangian scheme for a nonlinear drift diffusion equation', *ESAIM Math. Model. Numer. Anal.* **48** (2014), 697–726.

[63] H. Neunzert, 'An introduction to the nonlinear Boltzmann–Vlasov equation', in *Kinetic Theories and the Boltzmann Equation (Montecatini, 1981)*, Lecture Notes in Mathematics, 1048 (Springer, Berlin, 1984), 60–110.

[64] H. Spohn, *Large Scale Dynamics of Interacting Particles*, Texts and Monographs in Physics (Springer, Berlin, 1991).

[65] A.-S. Sznitman, 'Topics in propagation of chaos', in *Ecole d'Eté de Probabilités de Saint-Flour XIX 1989*, Lecture Notes in Mathematics, 1464 (Springer, Berlin, 1991), 165–251.

[66] G. Toscani, 'One-dimensional kinetic models of granular flows', *M2AN Math. Model. Numer. Anal.* **34** (2000), 1277–1291.

[67] C. Villani, *Topics in Optimal Transportation*, Graduate Studies in Mathematics, 58 (American Mathematical Society, Providence, RI, 2003).

[68] M. Westdickenberg and J. Wilkening, 'Variational particle schemes for the porous medium equation and for the system of isentropic Euler equations', *M2AN Math. Model. Numer. Anal.* **44** (2010), 133–166.

https://doi.org/10.1017/fms.2020.22 Published online by Cambridge University Press