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Local Points of Motives in Semistable Reduction

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Abstract. In this paper we study – for a semistable scheme – a comparison map between its log-syntomic cohomology and the p-adic étale cohomology of its generic fiber. The image can be described in terms of what Bloch and Kato call the local points of the underlying motive. The results extend a proven conjecture of Schneider which treats the good reduction case. The proof uses the theory of logarithmic schemes, some crystalline cohomology theories defined on them and various techniques in p-adic Hodge theory, in particular the Fontaine–Jannsen conjecture proven by Kato and Tsuji as well as Fontaine's rings of p-adic periods and their properties. The comparison result may become useful with respect to cycle class maps.

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0. Introduction

For a prime p let K be a finite extension of $\mathbb{Q}_p$ and let X be a smooth projective variety over K. Let $\overline{X} := X \times_K \overline{K}$, where $\overline{K}$ is an algebraic closure of K. Consider the motive $H^i(X)(r)$, where i and r are integers and (r) denotes the r-fold Tate-twist. We define V to be the p-adic étale cohomology $H^i(\overline{X}, \mathbb{Q}_p(r))$ together with its $\mathrm{Gal}(\overline{K}/K) =: G_K$-action. When $p \geqslant 3$ and X has good reduction, i.e., if there exists a smooth proper model $\mathcal{X}$ of X over the ring of integers O_K, then Fontaine and Messing and Kato were able to construct a canonical map $H^{i+1}_{\mathrm{syn}}(\mathcal{X}, s_{\mathbb{Q}_p}(r)) \to H^{i+1}_{\mathrm{et}}(X, \mathbb{Q}_p(r))$ from the syntomic cohomology to p-adic étale cohomology.

Assuming that $\mathcal{X}$ is smooth and projective we have, for $i \neq 2r-1$, the isomorphism $g\colon H^{i+1}_{\mathrm{et}}(X, \mathbb{Q}_p(r)) \xrightarrow{\cong} H^1(G, V)$ by a weight argument and the Hochschild–Serre spectral sequence.

Let $H^1_f(G_K, V) := \ker(H^1(G_K, V) \to H^1(G_K, B_{\mathrm{crys}} \otimes V))$. Here B_{crys} is the ring of p-adic periods constructed by Fontaine. If $i \neq 2r-1, 2r$, Schneider's p-adic points conjecture predicts the following commutative diagram with vertical isomorphisms.

$$\begin{array}{ccc} H^{i+1}_{\mathrm{syn}}(\mathcal{X}, s_{\mathbb{Q}_p}(r)) & \hookrightarrow & H^{i+1}_{\mathrm{et}}(X, \mathbb{Q}_p(r)) \\ \cong\downarrow & & \downarrow g \cong \\ H^1_f(G_K, V) & \hookrightarrow & H^1(G_K, V) \end{array}$$

This provides a very nice alternative description of the group H^1_f which can be interpreted as the group of p-adic points of the motive $H^i(X)(r)$ (compare [BK] and [S], Sect. 1).

The conjecture has been proven in the following cases:

- $K/\mathbb{Q}_p$ unramified and $i \leqslant r < p-1$ (joint work by S. Saito and the author [L-S], Sect. 6),
- more generally for $p > 2$, and no restriction on the ramification by Jan Nekovář [Ne].

The purpose of this paper is to study a semistable analogue of this conjecture. This was suggested to me by K. Kato and S. Saito. Now let X have semistable reduction, i.e., there exists a regular proper model $\mathcal{X}$ of X such that the closed fiber Y of $\mathcal{X}$ is a reduced divisor with normal crossings in $\mathcal{X}$. Equivalently, $\mathcal{X}$ is étale locally given by Spec $O_K[T_1, \dots, T_d]/(T_1 \dots T_r - \pi)$, where π is a uniformizing element of O_K. Let $s_n^{\log'}(r)$ be the log-syntomic complex on $(\mathcal{X})_{\mathrm{et}}$ as defined by Kato ([Ka4], Sect. 5) and Tsuji ([Tsu], Sect. 2.1), using the construction '$1 - p^{-r}\varphi$' for $r < p$, where φ is the Frobenius and extended by Tsuji to all r. Tsuji defines a second log-syntomic complex $\widetilde{s_n^{\log}(r)}$ via the map $p^r - \varphi$ ([Tsu], Sect. 2.1). The definition of the complexes will be given in Section 1. Kato and Tsuji define canonical maps

$$\widetilde{s_n^{\log}(r)} \xrightarrow{\varepsilon_1} s_n^{\log'}(r) \xrightarrow{\varepsilon_2} i_* i^* R j_* \mathbb{Z}/p^n(r),$$

where $i\colon Y \hookrightarrow \mathcal{X}$, $j\colon X \hookrightarrow \mathcal{X}$ are the canonical inclusions. By ([Tsu] (2.1.2)), the kernel and cokernel of ε_1 is killed by p^r, thus ε_1 induces a canonical isomorphism

$$H^j_{\mathrm{naive}}(\mathcal{X}, \widetilde{s^{\log}_{\mathbb{Q}_p}}(r)) \xrightarrow{\cong} H^j_{\mathrm{naive}}(\mathcal{X}, s^{\log'}_{\mathbb{Q}_p}(r)),$$

where the naive cohomology is obtained by taking inverse limits and tensoring with $\mathbb{Q}_p$. Due to an exactness problem that occurs when passing to $\mathbb{Q}_p$-coefficients in various long exact cohomology sequences, we have to consider continuous log-syntomic cohomology

$$H^j_{\mathrm{cont}}(\mathcal{X}, \widetilde{s^{\log}_{\mathbb{Q}_p}(r)}) := H^j_{\mathrm{cont}}(\mathcal{X}, \widetilde{s^{\log}_{\mathbb{Z}_p}(r)}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p,$$

where $\widetilde{s^{\log}_{\mathbb{Z}_p}(r)}$ is considered – via the projective system $(\widetilde{s_n^{\log}(r)})_n$ – as a complex of $\mathbb{Z}_p$-sheaves that live in a suitable derived category of $\mathbb{Z}_p$-sheaves. For details, we refer the reader to ([E]). Important for us is the fact that the continuous cohomology sits in a short exact sequence

$$0 \to \varprojlim_n{}^1 H^{j-1}(\mathfrak{X}, \widetilde{s_n^{\log}(r)}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to H^j_{\mathrm{cont}}(\mathfrak{X}, \widetilde{s_{\mathbb{Q}_p}^{\log}(r)})$$

$$\to H^j_{\mathrm{naive}}(\mathfrak{X}, \widetilde{s_{\mathbb{Q}_p}^{\log}(r)}) \to 0.$$

Of course, the same construction works for $s_n^{\log'}(r)$ and ε_1 induces an isomorphism $H^j_{\mathrm{cont}}(\mathfrak{X}, \widetilde{s_{\mathbb{Q}_p}^{\log}(r)}) \cong H^j_{\mathrm{cont}}(\mathfrak{X}, s_{\mathbb{Q}_p}^{\log'}(r))$. In the following, we will just write $H^j(\mathfrak{X}, s_{\mathbb{Q}_p}^{\log}(r)) := H^j_{\mathrm{cont}}(\mathfrak{X}, s_{\mathbb{Q}_p}^{\log}(r))$ to denote one of the two $\mathbb{Q}_p$-vector spaces. Lemma 1.5 and the proof of the Theorems will show that both cohomology groups will work. ε_2 induces a canonical map

$$H^{i+1}(\mathfrak{X}, s_{\mathbb{Q}_p}^{\log}(r)) \to H^{i+1}_{\mathrm{naive}}(\mathfrak{X}, s_{\mathbb{Q}_p}^{\log}(r)) \xrightarrow{\varepsilon_2} H^{i+1}_{\mathrm{et}}(X, \mathbb{Q}_p(r)).$$

Assuming that $H^{i+1}(\overline{X}, \mathbb{Q}_p(r))^{G_K} = 0$, the Hochschild–Serre spectral sequence provides a map $\alpha\colon H^{i+1}(\mathfrak{X}, s_{\mathbb{Q}_p}^{\log}(r)) \to H^1(G_K, V)$ and one may ask what is the image of this map. Kato and Saito suggested that a good candidate for $\mathrm{Im}\,\alpha$ is $H^1_g(K, V)$ in the sense of Bloch–Kato [B-K], i.e.,

$$H^1_g(K, V) = \ker(H^1(K, V) \to H^1(K, B_{dR} \otimes V).$$

After an analysis of the semistable situation it turns out that a good formulation of a semistable analogue of the p-adic points conjecture is given as follows

CONJECTURE. *Let $\mathfrak{X}$ and α be above and assume that $H^{i+1}(\overline{X}, \mathbb{Q}_\ell(r))^{G_K} = 0 = H^i(\overline{X}, \mathbb{Q}_\ell(r))^{G_K}$ for all primes ℓ. Then* $\mathrm{Im}\,\alpha = H^1_g(K, V)$.

Let $D_i = H^i_{\log\,\mathrm{crys}}((Y, M)/(W(k), W(L)), O^{\mathrm{crys}}) \otimes_W K_0$ be the ith log-crystalline cohomology ($\otimes K_0$) of the closed fiber Y, equipped with an action of the monodromy operator N and the Frobenius φ. (K_0 is the quotient field of the Witt ring W.) We will also consider the log-crystalline cohomology $H^j((Y, M)/\mathrm{Spec}\, W_n\langle t\rangle, \mathcal{L}), O_n^{\mathrm{crys}})$ which is a module over $W_n\langle t\rangle$, the DP-envelope of the closed immersion $\mathrm{Spec}\, W_n \hookrightarrow \mathrm{Spec}\, W_n[t]$. We will give the precise definition in Section 2 and only note here that it is important in the construction of the monodromy operator and in the context of semistable Fontaine–Laffaille theory that was studied by C. Breuil ([Br]). Throughout the paper we will deal with the following technical assumption $(*)$ that is crucial for the existence of a certain exact cohomology sequence (2.5) and Lemma (2.6).

The projective system $(H^j((Y, M)/\mathrm{Spec}\, W_n\langle t\rangle, \mathcal{L}), O_n^{\mathrm{crys}}))_n$ satisfies the Mittag $-$ Leffler condition. $(*)$

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For $j \leqslant p-2$ and $K/\mathbb{Q}_p$ unramified this follows from the works of C. Breuil. More precisely, he showed that the cohomology groups

$$H^j((Y,M)/\operatorname{Spec} W_n\langle t\rangle, \mathcal{L}), O_n^{\mathrm{crys}})$$

are $W_n\langle t\rangle$-modules of finite length for $j \leqslant p-2$, hence $(*)$ holds (compare ([Br]), Corollary (2.2.3.3) and Theorem (2.3.2.1)). Once the restrictive assumptions in Fontaine–Laffaille theory can be removed one can expect $(*)$ to hold in general. As the main result of this paper, we will prove the following theorem:

THEOREM 0.1. *Let p be a prime and K a finite unramified extension of $\mathbb{Q}_p$. Assume that $i \leqslant p-3$ or that the condition $(*)$ holds for $j = i, i+1$. If $(D_i)^{N=0}_{\varphi=p^r} = 0 = (D_{i+1})^{N=0}_{\varphi=p^r}$, then $H^{i+1}(\overline{X}, \mathbb{Q}_p(r))^{G_K} = 0$ and $\mathrm{Im}(\alpha) = H^1_g(K, V)$.*

Here $(D_i)^{N=0}_{\varphi=p^r}$ means the eigenspace where the Frobenius acts as multiplication by p^r in the kernel of the monodromy operator.

Remarks. (1) The B_{st}-comparison isomorphism between the log-crystalline cohomology and p-adic étale cohomology (proven by Kato [Ka4], Sect. 6 and Tsuji ([Tsu], Theorem 4.4) implies that $H^{i+1}(\overline{X}, \mathbb{Q}_p(r))^{G_K}$ is contained in $(D_{i+1})^{N=0}_{\varphi=p^r}$. Therefore $H^{i+1}(\overline{X}, \mathbb{Q}_p(r))^{G_K} = 0$ is zero by our assumption and the map α is well-defined.

(2) The p-adic and ℓ-adic monodromy conjecture imply that the vanishing assumptions in the above conjecture are satisfied when $r < 0$ or $r > (i/2)+1$. A nontrivial example where these assumptions also hold is the motive $H^2(X)(2)$ where X is the self-product of a Tate-elliptic curve (so $i = r = 2$). Furthermore Mokrane's Conjecture about the coincidence of the weight and monodromy filtrations on log-crystalline cohomology ([Mo], Conj. 3.27, proven for curves and surfaces [Mo], Sect. 5 and 6) that I consider as part of the p-adic monodromy conjecture imply that the Hasse–Weil zeta function can be computed using log-crystalline cohomology, compare ([Mo], Thm. 6.3.3). This implies that the conditions

(i) $(D_i)^{N=0}_{\varphi=p^r} = 0$;
(ii) $H^i(\overline{X}, \mathbb{Q}_\ell(r))^{G_K} = 0$ for all primes ℓ;

are equivalent (compare the discussion in [J1], p. 348).

So we obtain

COROLLARY 0.1.1. *The p-adic and ℓ-adic monodromy conjecture imply the semistable p-adic points conjecture, if K is unramified over $\mathbb{Q}_p$ and $i \leqslant p-3$.*

COROLLARY 0.1.2. *The conjecture holds if $\dim X \leqslant 2$, $p > 5$ and $K/\mathbb{Q}_p$ is unramified.*

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Using methods that are similar to those developed by Perrin-Riou ([PR], Sect. 2.2) in her study of the Iwasawa-theory of local Galois representations we will also compute the kernel of α.

THEOREM 0.2. *Let the assumptions be as in Theorem* (0.1). *Then* $\ker\alpha$ *is canonically isomorphic to the cokernel of the map* $p^{r-1} - \varphi$, *acting on* $\operatorname{coker}(N\colon D_{i-1} \to D_{i-1})$. *Furthermore there is a canonical surjection* $\eta\colon \ker\alpha \twoheadrightarrow H^2(G_K, H^{i-1}(\overline{X}, \mathbb{Q}_p(r)))$. *If the reduction of* $\mathcal{X}$ *is ordinary in the sense of* ([H1], 1.9) *then* η *is an isomorphism.*

The paper is organized as follows:

After a review of some notations and definitions of logarithmic algebraic geometry introduced by Kato, Fontaine and Illusie we will prove Theorem (0.1) in the second paragraph. An important step in the proof is a relation between the crystalline cohomology $H^i((\mathcal{X}, M)/W, O_{\mathbb{Q}_p}^{\mathrm{crys}})$ that appears in the definition of the log-syntomic cohomology (here W is endowed with the trivial log-structure) and the log-crystalline cohomology D_i (more precisely $D_i \otimes \widehat{K_0\langle t\rangle}$, where $\widehat{K_0\langle t\rangle} = K_0 \otimes_W \widehat{W\langle t\rangle}$ and $\widehat{W\langle t\rangle}$ is the p-adic completion of the DP-envelope of $W[t]$). Even though this crystalline cohomology is 'big' compared to D_i, its eigenspaces under the Frobenius are 'small', i.e., finite-dimensional K_0-vector spaces. Another tool is some Galois descent arguments applied to certain exact sequences in terms of the rings of p-adic periods constructed by Fontaine and the B_{st}-comparison isomorphism between log-crystalline cohomology and p-adic étale cohomology (proven by Kato and Tsuji). The assumption on D_i is needed to assure that the exponential map of Bloch–Kato maps onto H^1_f. This will imply that H^1_f is actually contained in the image of α. The assumption on D_{i+1} is used to show that the image of α is contained in H^1_g. To prove the surjectivity onto H^1_g we combine Bloch and Kato's local Tate duality with Poincaré duality on log-crystalline cohomology (proven by Hyodo). Then Theorem (0.2) will be proven in the third paragraph. Finally in the last paragraph we will consider the special case $i + 1 = 2r$, which we had to exclude in Theorem (0.1) and (0.2) for weight arguments. Of course, it is of special interest when we want to look at cycle class maps.

1. In this section we recall some basic definitions of logarithmic algebraic geometry that can be found in the papers of Kato [Ka3], [Ka4] and Hyodo and Kato [H-K] and will be needed later.

Let Z be a scheme. All subsequent sheaves are taken with respect to the étale topology. A pre-logarithmic structure on Z is a sheaf of monoids M on Z together with a multiplicative morphism $a\colon M \to O_X$, sending 1 to 1 (M, a) is a logarithmic structure when a induces an isomorphism $a^{-1}(O_X^*) \cong O_X^*$. Then (Z, M) is called a logarithmic scheme. There are natural notions of morphisms between logarithmic schemes, log-structures associated to pre-logarithmic structures, direct and inverse images of log-structures with respect to a morphism $f\colon Z_1 \to Z_2$. The following examples are very important.

(1.1a) $M_Z = O_Z^*$ with the canonical inclusion $O_Z^* \subset O_Z$. This is called the trivial log-structure on Z.

(1.1b) Let $\mathcal{X}$ be a semistable scheme over O_K as defined in the introduction. Let $M = M_{\mathcal{X}} = O_{\mathcal{X}} \cap j_* O_X^* \hookrightarrow O_{\mathcal{X}}$, where $j\colon X \hookrightarrow \mathcal{X}$ is the inclusion of the generic fiber. This is called the canonical log-structure on $\mathcal{X}$. This notion generalizes to reduced normal crossing divisors on Noetherian regular schemes. We can apply this definition to Spec O_K itself and get the canonical log-structure N on Spec O_K.

(1.1c) Let P be a monoid together with a homomorphism $n\colon P \to \Gamma(Z, O_Z)$. Then we can consider the log-structure, associated to the pre-log-structure $P_Z \to O_Z$, induced by n (where P_Z is the constant sheaf with values in P on Z). Example (1.1b) is locally of this type: If étale locally $\mathcal{X} = \operatorname{Spec} O_K[T_1, \dots, T_d]/ (T_1, \dots, T_r - \pi)$, then consider the pre-log-structure defined by $\mathbb{N}^r \to \Gamma(\mathcal{X}, O_X)$, $(n_i) \to \prod_i T_i^{n_i}$, $M = M_{\mathcal{X}}$, defined in (1.1b) is the associated log-structure on $\mathcal{X}$. The log-structure N on Spec O_K is induced by $\mathbb{N} \to O_K$, $1 \mapsto \pi$.

A log-structure M on Z is called fine, if étale locally there is a finitely generated monoid P and a homomorphism $h\colon P_Z \to O_Z$, such that M is isomorphic to the log-structure associated to the pre-log-structure (P_Z, h). Of course Example (1.1b) is of this type.

In the following all morphisms $f\colon (Z_1, M_1) \to (Z_2, M_2)$ will be morphisms between schemes with fine log-structures. There are notions of f to be

- a closed immersion ([H-K], (2.8));
- an exact closed immersion ([H-K], (2.8));
- a (log-)étale morphism ([H-K], (2.9));
- a (log-)smooth morphism ([H-K], (2.9));
- an integral morphism ([H-K], (2.10));
- a (log-)syntomic morphism ([Ka4], (2.5)).

We can provide examples for these morphisms by using Example (1.1b): The morphism $(\mathcal{X}, M) \to (\operatorname{Spec} O_K, N)$ is log-smooth and integral ([H-K], 2.13.2). It induces a log-smooth morphism in the closed fibers $(Y, M_1) \to (\operatorname{Spec} k, N_1)$ where M_1, N_1 are the inverse images with respect to the closed immersion $Y \hookrightarrow \mathcal{X}$, Spec $k \hookrightarrow \operatorname{Spec} O_K$. If Spec O_K is equipped with the trivial log-structure, then the semistable scheme $\mathcal{X}$ induces a log-syntomic morphism $(\mathcal{X}, M) \to \operatorname{Spec} O_K$. In particular $(\mathcal{X}, M)$ is (log-)syntomic over $W = W(k)$, the Wittring of the residue field of O_K.

Let (T, L) be a scheme with a fine log-structure such that O_T is killed by some positive integer and assume that T is endowed with a PD (= divided power) ideal.

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For a scheme with an integral log-structure (X, M) over (T, L), we have the crystalline site $((X, M)/(T, L))_{\text{crys}}$ ([H-K], 2.15) and can consider its cohomology with respect to the structure sheaf O_X/T. Let W_n be the ring of Witt vectors of length n and denote by $W_n(L)$ the log-structure associated to $1 \mapsto 0$ as a morphism of monoids $\mathbb{N} \to W_n$ on W_n. We have closed immersions $\operatorname{Spec} k \to \operatorname{Spec} W_n$ and $(\operatorname{Spec} k, N_1) \mapsto (\operatorname{Spec} W_n, W_n(L))$, where in the first immersion we consider trivial log-structures. Let $\mathcal{X}$ be a semistable scheme with closed fiber Y as in Example (1.1b) and let $\mathcal{X}_n = \mathcal{X} \otimes \mathbb{Z}/p^n$. Then we have crystalline sites $((\mathcal{X}_n, M_n)/\operatorname{Spec} W_n)_{\text{crys}}$, $((Y, M_1)/\operatorname{Spec} W_n)_{\text{crys}}$, $((Y, M_1)/(\operatorname{Spec} W_n, W_n(L)))_{\text{crys}}$, where M_n is the inverse image of M and in the first two sites, Spec W_n is endowed with the trivial log-structure. In particular we recover the log-crystalline cohomology

$$D_i := \lim_{\substack{\longleftarrow \\ n}} H^i((Y, M_1)/(\operatorname{Spec} W_n, W_n(L)), O^{\text{crys}}) \otimes_W K_0$$

in the introduction, where K_0 is the quotient field of $W(k)$. Let U be the open subscheme of Y which is smooth over k, $u \colon U \hookrightarrow Y$. We recall the notion of a de Rham–Witt complex $W_n\omega^{\cdot}_Y$ on $(Y)_{\text{et}}$ which is a certain subcomplex of the graded differential algebra $u_* W_n \Omega^{\cdot}_U$ where $W_n \Omega^{\cdot}_U$ is the usual de Rham–Witt complex, compare ([H-K], 1.1). By Proposition (1.5) in [H-K] we have an exact sequence of complexes

$$\begin{gathered} 0 \longrightarrow W_n\omega^{\cdot}_Y[-1] \longrightarrow W_n\tilde{\omega}^{\cdot}_Y \longrightarrow W_n\omega^{\cdot}_Y \longrightarrow 0 \\ a \longmapsto a\theta, \quad \theta \longmapsto 0, \end{gathered} \tag{1.2}$$

where $W_n\tilde{\omega}^{\cdot}_Y$ is a modified de Rham–Witt complex, defined as a certain $W_n(O_Y)$-subalgebra of the graded diff. algebra $u_*(W_n\Omega^{\cdot}_U)[\theta]/\theta^2$, where θ is an indeterminate in degree one satisfying $\theta a = (-1)^a \theta$ for $a \in u_* W_n \Omega^a_U$ and $\mathrm{d}\theta = 0$.

The connecting homomorphism of (1.2) defines the monodromy operator N in the log-crystalline cohomology, using Theorem 4.19 in [H-K]. In the proof of the Theorem we will also use an alternative description of the exact sequence (1.2) via crystalline complexes.

For a logarithmic scheme (Z, M) that is syntomic over W, let $Z_n = Z \otimes \mathbb{Z}/p^n$ and M_n the log-structure on Z_n induced by M. In [Ka4] Kato defines a complex $s^{\log'}_{n,z}(r)$ in $\mathcal{D}_{\text{et}}(Z_n)$ via embedding systems into schemes (Z^i, M^i) that are log-smooth over W and have an action of the Frobenius. We do not give this definition here but refer to an alternative description of $s^{\log'}_{n,z}(r)$ that Kato uses in his paper ([Ka4], p. 286) and that is based on Theorem 1.7 in [Ka2]:

DEFINITION 1.3. Let $r < p$. Then $s^{\log'}_{n,z}(r)$ is defined to be the mapping fiber of

$$Ru_{(Z_n, M_n)/W_n *}(J^{[r]}_{Z_n/W_n}) \xrightarrow{1 - p^{-r}\varphi} Ru_{(Z_n, M_n)/W_n *}(O_{Z_n/W_n}),$$

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where $u\colon ((Z_n, M_n)/W_n)^{\sim}_{\text{crys}} \to (Z_n)^{\sim}_{\text{et}}$ is the canonical morphism of topoi and $J^{[r]}_{Z_n/W_n}$ is the rth divided power ideal sheaf of $J_{Z_n/W_n} = \ker(O_{Z_n/W_n} \to O_{Z_n})$.

Tsuji generalizes this definition to the case $r \geqslant p$. It is more complicated and we refer the reader to [Tsu], Section 2.1.

DEFINITION 1.4 [Tsu]. For $r \geqslant 0$ $s^{\log^{\sim}}_{n,z}(r)$ is the mapping fiber of

$$Ru_{(Z_n,M_n)/W_n*}(J^{[r]}_{Z_n/W_n}) \xrightarrow{\varphi - p^r} Ru_{(Z_n,M_n)/W_n*}(O_{Z_n/W_n}).$$

Remark. Definition (1.3) is the natural log-syntomic analogue of the syntomic complexes $Rv_*S^{[r]}_n$, that were studied in [Ka1]. Here $S^{[r]}_n$ is the syntomic sheaf of Fontaine and Messing [F-M] and $v\colon (Z)^{\sim}_{\text{syn}} \to (Z)^{\sim}_{\text{et}}$. According to Kato ([Ka4], p. 286), ‘adding log poles’ defines a canonical morphism $Rv_*S^{[r]}_n \to s^{\log'}_{n,z}(r)$.

Let now

$$H^j((Z, M/W, O^{\text{crys}}_{\mathbb{Q}_p}) := H^j_{\text{cont}}((Z, M)/W, O^{\text{crys}}_{\mathbb{Z}_p}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p,$$

where, $H^j_{\text{cont}}((Z, M)/W, O^{\text{crys}}_{\mathbb{Z}_p})$ is the continuous cohomology of the projective system $(O^{\text{crys}}_n)_n$ which is defined similarly as continuous étale cohomology in ([J3]). It sits in an exact sequence

$$0 \to \varprojlim_n{}^1 H^{j-1}((Z, M)/W, O^{\text{crys}}_n) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to H^j((Z, M)/W, O^{\text{crys}}_{\mathbb{Q}_p})$$
$$\to \varprojlim_n H^j((Z, M)/W, O^{\text{crys}}_n) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to 0.$$

Similarly one defines continuous cohomology

$$H^j((Z, M)/W, J^{[r]}_{\mathbb{Q}_p}), H^j((Z, M)/W, O^{\text{crys}}_{\mathbb{Q}_p}/J^{[r]}_{\mathbb{Q}_p}) \quad \text{and} \quad H^j((Z, M)/W, s^{\log}_{\mathbb{Q}_p}(r))$$

(compare the Introduction). In the following cohomology with $\mathbb{Q}_p$-coefficients will consistently mean continuous cohomology, if not stated otherwise. The naive cohomology $\lim_{\longleftarrow n} H^j(\,) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ will be denoted by $H^j_{\text{naive}}(\,)$. The following Lemma is now clear.

LEMMA 1.5. *There is a canonical isomorphism*

$$\varepsilon_1\colon H^j((Z, M)/W, s^{\log^{\sim}}_{\mathbb{Q}_p}(r)) \xrightarrow{\cong} H^j((Z, M)/W, s^{\log'}_{\mathbb{Q}_p}(r))$$

that fits into a commutative diagram of exact sequences of continuous cohomology

$$\begin{array}{ccccccc}
\to H^{j-1}((Z, M)/W, O^{\text{crys}}_{\mathbb{Q}_p}) & \to & H^j((Z, M)/W, s^{\log^{\sim}}_{\mathbb{Q}_p}(r)) & \to & H^j((Z, M)/W, J^{[r]}_{\mathbb{Q}_p}) & \xrightarrow{\varphi - p^r} \\
\downarrow = & & \varepsilon_1 \downarrow \cong & & \cong \downarrow \cdot p^r & \\
\to H^{j-1}((Z, M)/W, O^{\text{crys}}_{\mathbb{Q}_p}) & \to & H^j((Z, M)/W, s^{\log'}_{\mathbb{Q}_p}(r)) & \to & H^j((Z, M)/W, J^{[r]}_{\mathbb{Q}_p}) & \xrightarrow{1 - p^{-r}\cdot\varphi}
\end{array}$$

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Indeed, the exactness of the horizontal sequences is clear because we work with continuous cohomology, the commutativity follows from the explicit construction of the map ε_1 in ([Tsu] (2.1.2)).

Remark 1.6. We recall that for the closed fiber Y/k of a log-smooth morphism $(\mathfrak{X}, M) \to (\mathrm{Spec}\, O_K, N)$ the 'naive' Hyodo–Kato cohomology

$$D_i = \varprojlim_n H^i((Y, M_1)/(\mathrm{Spec}\, W_n(k), W_n(L)), O_n^{\mathrm{crys}}) \otimes_W K_0$$

coincides with continuous cohomology because

$$H^i((Y, M_1)/(\mathrm{Spec}\, W_n(k), W_n(L)), O_n^{\mathrm{crys}})$$

is of finite length over $W_n(k)$ (compare ([H-K]), (3.2))). In the next paragraph we will see that in two other important cases continuous cohomology coincides with naive cohomology. This will enable us to relate log-syntomic cohomology to Hyodo–Kato cohomology and to exploit the well-known properties of the latter one.

2. In this section we will prove Theorem (0.1). Recall that we assume that $K/\mathbb{Q}_p$ is unramified.

LEMMA 2.1. *There is a canonical isomorphism*

$$H^i((Y, M_1)/W_n)), O_n^{\mathrm{crys}}) \xrightarrow[\cong]{} H^i(((\mathfrak{X}_n, M_n)/W_n), O_n^{\mathrm{crys}}) \quad \textit{for all } i \textit{ and all } n.$$

Here we take M to be the canonical log-structure on $\mathfrak{X}$ and M_1 the inverse image of M on the closed fiber Y. Spec W_n is endowed with the trivial log-structure.

Proof. The classical rigidity property of crystalline cohomology says that given a PD-scheme $S = (S, I, \gamma)$ and a closed subscheme $S_0 \hookrightarrow S$ defined by a sub-PD-ideal of I and a S-scheme X with $X_0 = X \times_S S_0$ we have

$$H^i((X/S)_{\mathrm{crys}}, O_{X/S}^{\mathrm{crys}}) \cong H^i((X_0/S)_{\mathrm{crys}}, O_{X_0/S}^{\mathrm{crys}})$$

(compare [B-O], Theorem (5.17)). Now it is easy to see that the proof of this rigidity theorem also works if X is equipped with a log-structure M and M_{X_0} is the induced log-structure on X_0. Apply this to $(X, M) = (X_n, M_n)$ over $S = (\mathrm{Spec}\, W_n, (p))$ and $S_0 = \mathrm{Spec}\, k$. Since $K/\mathbb{Q}_p$ is unramified we have $X_0 = Y$ and Lemma (2.1) follows.

Recall the exact sequence of the de Rham–Witt complexes (1.2)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & W_n\omega^{\cdot}_Y[-1] & \longrightarrow & W_n\tilde{\omega}^{\cdot}_Y & \longrightarrow & W_n\omega^{\cdot}_Y & \longrightarrow & 0 \\ & & a & \longmapsto & a\theta & & & & \\ & & & & \theta & \longrightarrow & 0 & & \end{array}$$

In the following we will also work with an alternative construction of the monodromy-operator via crystalline complexes, given in ([H-K], 3.6).

Consider the exact closed immersion $(W_n, W_n(L)) \to (\mathrm{Spec}\, W_n[t], \mathcal{L})$, where $\mathcal{L}$ is the log-structure associated to $\mathbb{N} \to W_n[t]$, $1 \mapsto t$ and the morphism is defined by $W_n[t] \to W_n$, $t \to 0$, $\mathcal{L} \to W_n(L)$, $1 \in \mathbb{N} \to 1 \in \mathbb{N}$. Take an embedding system $(Y^{\cdot}, M^{\cdot}) \to (Z^{\cdot}, N^{\cdot})$ of $(Y, M_1) \to (\mathrm{Spec}\, W_n[t], \mathcal{L})$ in the sense of ([H-K], 2.18). Note that (Z^i, N^i) is also log-smooth over Spec W_n endowed with the trivial log-structure. This uses the fact that $(\mathrm{Spec}\, W_n[t], \mathcal{L})$ is log-smooth over Spec $W_n(\Gamma(\mathrm{Spec}\, W_n[t], \omega^1_{(\mathrm{Spec}\, W_n[t], \mathcal{L})/W_n})$ is a free $W_n[t]$-module of rank one with base $d\log(t)$), compare ([Ka4] (3.2)). Let C_{Y/W_n} be the crystalline complex associated to the embedding system $(Y^{\cdot}, M^{\cdot}) \to (Z^{\cdot}, N^{\cdot})$ (over Spec W_n, triv.) and $C_{Y/\mathrm{Spec}\, W_n\langle t\rangle}$ ($W_n\langle t\rangle$ is the PD-polynomial ring over W_n in one variable t and Spec $W_n\langle t\rangle$ is endowed with the inverse image of $\mathcal{L}$) be the crystalline complex associated to the embedding system $(Y^{\cdot}, M^{\cdot}) \to (Z^{\cdot} \times_{\mathrm{Spec}\, W_n[t]} \mathrm{Spec}\, W_n\langle t\rangle, (N^{\cdot})')$, where $(N^{\cdot})'$ s the inverse image of $N^{\cdot}$. For the notion of crystalline complexes we refer to ([H-K], Definition 2.19). According to ([H-K], 3.6) we have an exact sequence of complexes

$$0 \longrightarrow C_{Y/\mathrm{Spec}\, W_n\langle t\rangle}[-1] \longrightarrow C_{Y/W_n} \longrightarrow C_{Y/\mathrm{Spec}\, W_n\langle t\rangle} \longrightarrow 0$$

$$a \longmapsto a \wedge d\log t. \tag{2.2}$$

Let $\theta\colon (Y^{\cdot})^{\sim}_{\mathrm{et}} \to (Y)^{\sim}_{\mathrm{et}}$ be the obvious morphism of topoi. Applying $R\theta_*$ to the exact sequence (2.2) and using ([H-K], Prop. 2.20), we get the following exact sequence in $D(Y)_{\mathrm{et}}$

$$0 \to Ru_{(Y,M_1)/(\mathrm{Spec}\, W_n\langle t\rangle, \mathcal{L})}(O^{\mathrm{crys}}))[-1] \to Ru_{(Y,M_1)/W_n}(O^{\mathrm{crys}})$$
$$\to Ru_{(Y,M_1)/(\mathrm{Spec}\, W_n\langle t\rangle, \mathcal{L})}(O^{\mathrm{crys}}) \to, \tag{2.3}$$

where O^{crys} denotes here the structure sheaf on $((Y, M_1)/\mathrm{Spec}\, W_n\langle t\rangle, \mathcal{L})$ resp. $((Y, M_1)/W_n)$.

On the other hand, if one tensors (2.2) with W_n (with respect to $W_n\langle t\rangle \to W_n$, $t^{[i]} \to 0$, $i \geqslant 1$) one gets the exact sequence

$$0 \longrightarrow C_{Y/(W_n, W_n(L))}[-1] \longrightarrow W_n \otimes_{W_n\langle t\rangle} C_{Y/W_n}$$
$$\longrightarrow C_{Y/(W_n, W_n(L))} \longrightarrow 0 \tag{2.4}$$

where $C_{Y/(W_n, W_n(L))}$ is the crystalline complex with respect to the embedding-system $(Y^{\cdot}, M^{\cdot}) \to (Z^{\cdot} \times_{\mathrm{Spec}\, W_n[t]} W_n, (N^{\cdot})'')$, where $(N^{\cdot})''$ is the inverse image of N'. Applying $R\theta_*$ to (2.4) we get an exact sequence of complexes on $D(Y)_{\mathrm{et}}$ that is quasiisomorphic to (1.2) (this is shown in [H-K] (4.20)).

By our assumption $(*)$ the projective system

$$(H^j((Y, M_1)/(\mathrm{Spec}\, W_n\langle t\rangle, \mathcal{L}), O_n^{\mathrm{crys}}))_n$$

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satisfies the Mittag–Leffler condition and we have an isomorphism

$$H^j_{\mathrm{cont}}((Y, M_1)/(\mathrm{Spec}\, W\langle t\rangle, \mathcal{L}), \mathcal{O}^{\mathrm{crys}}_{\mathbb{Q}_p}) \cong H^j_{\mathrm{naive}}((Y, M_1)/(\mathrm{Spec}\, W\langle t\rangle, \mathcal{L}), \mathcal{O}^{\mathrm{crys}}_{\mathbb{Q}_p}).$$

Applying the crucial base change argument ([H-K], Prop. (4.13)) to

$$H^j_{\mathrm{naive}}((Y, M_1)/(\mathrm{Spec}\, W\langle t\rangle, \mathcal{L}), \mathcal{O}^{\mathrm{crys}}_{\mathbb{Q}_p})$$

and taking continuous cohomology of (2.3) we get a long exact sequence

$$\begin{aligned} &\longrightarrow H^{i-1}((Y, M_1)/(\mathrm{Spec}\, W, W(L)), \mathcal{O}^{\mathrm{crys}}_{\mathbb{Q}_p}) \otimes_K \widehat{K\langle t\rangle} \\ &\xrightarrow[\wedge d\log t]{} H^i((Y, M_1)/W, \mathcal{O}^{\mathrm{crys}}_{\mathbb{Q}_p}), \\ &\longrightarrow H^i((Y, M_1)/(\mathrm{Spec}\, W, W(L)), \mathcal{O}^{\mathrm{crys}}_{\mathbb{Q}_p}) \otimes_K \widehat{K\langle t\rangle} \\ &\xrightarrow[N_{\widehat{K\langle t\rangle}}]{} H^i((Y, M_1)/(\mathrm{Spec}\, W, W(L)), \mathcal{O}^{\mathrm{crys}}_{\mathbb{Q}_p}) \otimes_K \widehat{K\langle t\rangle} \longrightarrow \end{aligned} \tag{2.5}$$

where $\widehat{K\langle t\rangle} = K \otimes_W \widehat{W\langle t\rangle}$, $\widehat{W\langle t\rangle}$ is the p-adic completion of the DP-envelope of $W[t]$ and $N_{\widehat{K\langle t\rangle}}$ is induced by the connecting homomorphism of (2.2), such that $N_{\widehat{K\langle t\rangle}} \otimes K$ is the usual monodromy operator N that is defined via the connecting homomorphism of (2.4).

(2.5) and (2.4) induce a commutative diagram

$$\begin{array}{ccc} H^{i-1}((Y, M_1)/(\mathrm{Spec}\, W, W(L)), \mathcal{O}^{\mathrm{crys}}_{\mathbb{Q}_p}) \otimes_K \widehat{K\langle t\rangle} & \xrightarrow[t\to 0]{} & H^{i-1}((Y, M_1)/(\mathrm{Spec}\, W, W(L)), \mathcal{O}^{\mathrm{crys}}_{\mathbb{Q}_p}) \\ \downarrow{\scriptstyle \wedge d\log t} & & \downarrow \\ H^i((Y, M_1)/W, \mathcal{O}^{\mathrm{crys}}_{\mathbb{Q}_p}) & \xrightarrow[t\to 0]{} & H^i_{\mathrm{et}}(Y, W\tilde{\omega}_Y)_{\mathbb{Q}_p} \\ \downarrow & & \downarrow \\ H^i((Y, M_1)/(\mathrm{Spec}\, W, W(L)), \mathcal{O}^{\mathrm{crys}}_{\mathbb{Q}_p}) \otimes_K \widehat{K\langle t\rangle} & \xrightarrow[t\to 0]{} & H^i((Y, M_1)/(\mathrm{Spec}\, W, W(L)), \mathcal{O}^{\mathrm{crys}}_{\mathbb{Q}_p}) \\ \downarrow & & \downarrow \end{array}$$

where the vertical exact sequence on the right is the long exact cohomology sequence associated to (1.2) and using ([H-K], Theorem 4.19). Let $D_j := H^j((Y, M_1)/(\mathrm{Spec}\, W, W(L)), \mathcal{O}^{\mathrm{crys}}_{\mathbb{Q}_p})$ denote the log-crystalline cohomology as in the Introduction. Since the Frobenius φ acts on t via $\varphi(t) = t^p$, any eigenspace of φ acting on $D_j \otimes_K \widehat{K\langle t\rangle}$ is already contained in D_j. The monodromy operator $N_{\widehat{K\langle t\rangle}}$ is equal to $N \otimes 1 + 1 \otimes N$ where $N(t^{[n]}) = nt^{[n]}$ on $\widehat{K\langle t\rangle}$. Now we show

LEMMA 2.6. (i) coker $(N_{\widehat{K\langle t\rangle}})_{\varphi=p^{r-1}} = \mathrm{coker}(N)_{\varphi=p^{r-1}}$.

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(ii) ker $N_{\widehat{K\langle t\rangle}} = \ker N$ (*N considered on D_i*).
(iii) *There is an exact sequence*

$$0 \longrightarrow \operatorname{coker}(N\colon D_i \to D_i)_{\varphi=p^{r-1}} \longrightarrow H^{i+1}((Y, M_1)/W, O^{\text{crys}}_{\mathbb{Q}_p})_{\varphi=p^r}$$
$$\longrightarrow (D_{i+1})^{N=0}_{\varphi=p^r}.$$

Proof. (ii) is clear from the formula $N_{\widehat{K\langle t\rangle}} = N\otimes 1 + 1\otimes N$, (iii) follows from (i) and (ii), so it remains to show (i). The canonical map $D_i \otimes_K \widehat{K\langle t\rangle} \to D_i (t^{[n]} \to 0$, for $n \geqslant 1)$ induces a surjection

$$\operatorname{coker}(N_{\widehat{K\langle t\rangle}})_{\varphi=p^{r-1}} \twoheadrightarrow \operatorname{coker}(N)_{\varphi=p^{r-1}}.$$

Let $x \in \ker(D_i \otimes_K \widehat{K\langle t\rangle} \to D_i)$ such that $(1-p^{1-r}\varphi)x = N_{\widehat{K\langle t\rangle}}(y)$ for some $y \in \ker(D_i \otimes_K \widehat{K\langle t\rangle} \to D_i)$. Then

$$x = (1-p^{1-r}\varphi)^{-1} N_{\widehat{K\langle t\rangle}}(y) = \sum_{n\geqslant 0}(p^{1-r}\varphi)^n N_{\widehat{K\langle t\rangle}}(y)$$
$$= N_{\widehat{K\langle t\rangle}}\left(\sum_{n\geqslant 0}(p^{-r}\varphi)^n y\right) = N_{\widehat{K\langle t\rangle}}(1-p^{-r}\varphi)^{-1}(y)$$

lies in the image of $N_{\widehat{K\langle t\rangle}}$, as required (we have used the formula $N_{\widehat{K\langle t\rangle}}\varphi = p\varphi N_{\widehat{K\langle t\rangle}}$; that $\sum_{n\geqslant 0}(p^{-s}\varphi)^n(y)$ converges – for $s \geqslant 0$ and $y = t^{[k]}\otimes m$, $m \in D_i$ – uniformly in k, will be shown in the proof of Proposition 3.1).

Lemma 2.6 will play a crucial role in the proof of Theorem 0.1 as well as the following

LEMMA 2.7. *There is a canonical isomorphism*

$$\psi\colon H^i_{\text{naive}}((\mathcal{X}, M)/W, O^{\text{crys}}_{\mathbb{Q}_p}/J^{[r]}_{\mathbb{Q}_p}) \xrightarrow{\cong} H^i_{DR}(X)/\text{Fil}^r.$$

Here

$$H^i_{\text{naive}}((\mathcal{X}, M)/W, O^{\text{crys}}_{\mathbb{Q}_p}/J^{[r]}_{\mathbb{Q}_p}) := \varprojlim_n H^i((\mathcal{X}_n, M_n)/W_n, O^{\text{crys}}_n/J^{[r]}_n)\otimes\mathbb{Q}_p$$

and Fil^r *is the* rth *step in the Hodge filtration on* $H^i_{DR}(X)$. *Moreover* $H^i_{\text{naive}}((\mathcal{X}, M)/W, O^{\text{crys}}_{\mathbb{Q}_p}/J^{[r]}_{\mathbb{Q}_p})$ *is isomorphic to the continuous cohomology* $H^i((\mathcal{X}, M)/W, O^{\text{crys}}_{\mathbb{Q}_p}/J^{[r]}_{\mathbb{Q}_p})$.

Proof. We closely follow the proof of ([K-M], Lemma (4.5)). Take an exact closed immersion $(\mathcal{X}_n, M_n) \to (Z_n, N_n)$ such that the ideal I of $\mathcal{X}_n$ in Z_n is generated at each point of $\mathcal{X}_n$ by a regular sequence and such that (Z_n, N_n) is log-smooth over W_n. Let D_n be the PD-envelope of $(\mathcal{X}_n, M_n)$ in (Z_n, N_n) and $I^{[r]}_n$ the

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rth divided power of $\ker(O_{D_n} \to O_{\mathcal{X}_n})$. Note that D_n coincides as a scheme with the classical PD-envelope of the closed immersion $\mathcal{X}_n \to Z_n$ ([Ka 3], (5.5.1)). Then $Ru^{\log}_{(\mathcal{X}_n,M_n)/W_n^*}(J_n^{[r]}/J_n^{[r+1]})$ is represented by the complex

$$\begin{aligned} I_n^{[r]}/I_n^{[r+1]} &\xrightarrow{d} I_n^{[r-1]}/I_n^{[r]} \otimes \omega^1_{(Z_n,N_n)/W_n} \longrightarrow \cdots \\ &\longrightarrow I_n^{[0]}/I_n^{[1]} \otimes \omega^r_{(Z_n,N_n)/W_n} \longrightarrow \end{aligned} \tag{2.7.1}$$

Since $\mathcal{X}_n$ is syntomic over W_n the direct sum $\oplus_{r\in\mathbb{Z}} I_n^{[r]}/I_n^{[r+1]}$ is isomorphic to the divides power-polynomial ring on the locally free sheaf I/I^2. In particular $I_n^{[r]}/I_n^{[r+1]}$ is isomorphic to the degree r part $(I/I^2)^{[r]}$ which is again locally free by ([B-O], Prop. A2). Now all entries of the complex (2.7.1) are locally free sheaves of finite rank on $(\mathcal{X}_n)_{\mathrm{et}}$. Since $\mathcal{X}_n$ is proper the cohomology of these sheaves is a finite W_n-module. Now the hypercohomology spectral sequence associated to (2.7.1) shows that $H^i((\mathcal{X}_n/M_n)/W_n, J_n^{[r]}/J_n^{[r+1]})$ is a finite W_n-module. An easy induction argument shows that the same property holds for $H^i((\mathcal{X}_n/M_n)/W_n, O_n^{\mathrm{crys}}/J_n^{[r]})$. In particular we get the isomorphism between continuous and naive cohomology for $O^{\mathrm{crys}}/J^{[r]}$. The canonical map $I_n^{[0]}/I_n^{[1]} \otimes \omega^r_{(Z_n,N_n)/W_n} \to \omega^r_{(\mathcal{X}_n,M_n)/W_n}$ defines a map

$$Ru^{\log}_{(\mathcal{X}_n,M_n)/W_n^*}(J_n^{[r]}/J_n^{[r+1]}) \longrightarrow \omega^r_{(\mathcal{X}_n,M_n)/W_n}[-r].$$

Since the generic fiber X of $\mathcal{X}$ is smooth, this map induces an isomorphism

$$Ru^{\log}_{X/K^*} J^{[r]}_{X/K}/J^{[r+1]}_{X/K} \xrightarrow[\cong]{} \omega^r_{X/K}[-r] \text{ in } \mathbb{Q} \otimes \varprojlim_n D(\mathcal{X}_n, O_{\mathcal{X}_n})$$

(for the notation compare [K-M]). To finish the proof of Lemma (2.7) one just follows the argument in ([K-M, Lemma 4.5).

Remark 2.8. This Lemma is implicitly used by Kato in his proof of the B_{st}-comparison isomorphism ([Ka 4], (6.4)). It actually holds more generally for schemes that are log-syntomic over W and such that the generic fiber is smooth and it provides a log-syntomic analogue for Lemma (4.5) in [K-M].

We are now ready to prove the following proposition:

PROPOSITION 2.9. *The following diagram is commutative:*

$$\begin{array}{ccccc} H^i((\mathcal{X}, M)/W, O^{\mathrm{crys}}_{\mathbb{Q}_p}) & \longrightarrow & H^{i+1}(\mathcal{X}, s^{\log}_{\mathbb{Q}_p}(r)) & \twoheadrightarrow & H^{i+1}((\mathcal{X}, M)/W, J^{[r]}_{\mathbb{Q}_p})_{\varphi=p^r} \\ \downarrow \lambda & & \downarrow \alpha & & \downarrow \overline{\alpha} \\ H^i_{DR}(X)/\mathrm{Fil}^r & \xrightarrow{\exp} & H^1(K, V) & \longrightarrow & H^1(K, B_{\mathrm{crys}} \otimes_{\mathbb{Q}_p} V) \end{array}$$

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Here $H^{i+1}((\mathcal{X}, M)/W, J^{[r]}_{\mathbb{Q}_p})_{\varphi=p^r}$ is a shorthand for

$$\ker(H^{i+1}((\mathcal{X}, M)/W, J^{[r]}_{\mathbb{Q}_p}) \xrightarrow{\varphi-p^r} H^{i+1}((\mathcal{X}, M)/W, O^{\mathrm{crys}}_{\mathbb{Q}_p}))$$

and λ is equal to the composition

$$H^i((\mathcal{X}, M)/W, O^{\mathrm{crys}}_{\mathbb{Q}_p}) \to \ker(N_{\widehat{K\langle t\rangle}})$$

$$= D_i^{N=0} \underset{\cong}{\xrightarrow{(p^{-r}\varphi-1)^{-1}}} D_i^{N=0} \hookrightarrow D_i \to D_i/\mathrm{Fil}^r.$$

The assumption $(D_i)^{N=0}_{\varphi=p^r} = 0$ implies that $p^{-r}\varphi - 1$ is invertible on $D_i^{N=0}$. The upper horizontal exact sequence is obtained from the diagram in Lemma (1.5). The lower horizontal sequence is exact, because $\mathrm{Im}(\exp) = H^1_f(K, V)$, where as in the introduction $V = H^i(\overline{X}, \mathbb{Q}_p(r))$. (Note that by using our assumption $(D_i)^{N=0}_{\varphi=p^r} = 0$ and the B_{st}-comparison-isomorphism we have $H^1_f(K, V)/H^1_e(K, V) = 0$, compare ([B-K], Sect. 3.8)) $\overline{\alpha}$ is defined via a composite map

$$H^{i+1}((\mathcal{X}, M)/W, J^{[r]}_{\mathbb{Q}_p})_{\varphi=p^r}$$

$$\to H^{i+1}((\mathcal{X}, M)/W, O^{\mathrm{crys}}_{\mathbb{Q}_p})_{\varphi=p^r} \xrightarrow{\tilde{\alpha}} H^1(K, B_{\mathrm{crys}} \otimes V)$$

where $\tilde{\alpha}$ is given as follows:

There is a canonical map

$$H^{i+1}((\mathcal{X}, M)/W, O^{\mathrm{crys}}_{\mathbb{Q}_p}) \to H^{i+1}_{\mathrm{naive}}((\overline{\mathcal{X}}, \overline{M})/W, O^{\mathrm{crys}}_{\mathbb{Q}_p}) \xrightarrow{\cong} (B^+_{\mathrm{st}} \otimes_K D_{i+1})^{N=0}$$

by ([Ka4], Theorem 4.1) which maps the $\varphi = p^r$-eigenspace into

$$((B^+_{st} \otimes_K D_{i+1})^{N=0})^{G_K}_{\varphi=p^r} = (D_{i+1})^{N=0}_{\varphi=p^r} \quad \text{and} \quad (D_{i+1})^{N=0}_{\varphi=p^r} = 0$$

by our assumption. So the Hochschild–Serre spectral sequence associated to $R\Gamma(G_K;\ ) \circ RH^\circ((\overline{\mathcal{X}}, \overline{M})/W, O^{\mathrm{crys}}_{\mathbb{Q}_p})$ defines $\tilde{\alpha}$ as a boundary map, where we use the Fontaine–Jannsen Conjecture (proven by Kato and Tsuji) that provides us with a map

$$\begin{aligned} H^i_{\mathrm{naive}}((\overline{\mathcal{X}}, \overline{M})/W, O^{\mathrm{crys}}_{\mathbb{Q}_p}) &\xrightarrow{\cong} (B^+_{\mathrm{st}} \otimes_K D_i)^{N=0} \\ &\hookrightarrow (B_{\mathrm{st}} \otimes_K D_i)^{N=0} \\ &\cong (B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V)^{N=0} \cong B_{\mathrm{crys}} \otimes_{\mathbb{Q}_p} V. \end{aligned}$$

Proof of Proposition 2.9. We first show the commutativity of the right-hand side. Tsuji has proven that there is a canonical isomorphism between the cohomology $\mathcal{H}^r(i^* s_n^{\log}(r))$ and the sheaf $M_n^r = i^* R^r j_* \mathbb{Z}/^n(r)$ of p-adic vanishing cycles

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([Tsu], Thm. 3.2). His proof relies on a filtration $\mathrm{Fil}^{\cdot}$ on M_n^r that was defined by Hyodo ([H1], (1.4)) and is induced by a symbol map on Milnor K-theory. Hyodo has shown ([H1], Thm. 1.6) that the highest graded quotient $\mathrm{gr}^0 M_n^r$ sits in an extension

$$0 \longrightarrow W_n\omega_{Y,\log}^{r-1} \longrightarrow \mathrm{gr}^0 M_n^r \longrightarrow W_n\omega_{Y,\log}^{r} \longrightarrow 0,$$

where $W_n\omega_{Y,\log}^{i}$ are the modified logarithmic Hodge–Witt sheaves ([H1] (1.5)). On the other hand Hyodo and Kato ([H-K], Prop. 1.5) constructed an exact sequence of Hodge–Witt sheaves $0 \longrightarrow W_n\omega_Y^{r-1} \longrightarrow W_n\tilde{\omega}_Y^r \longrightarrow W_n\omega_Y^r \longrightarrow 0$ and used the connecting homomorphism on the level of cohomology to define the monodromy operator on log-crystalline cohomology. It follows from the work of Tsuji ([Tsu], Sect. 2.4) that there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & W_n\omega_{Y,\log}^{r-1} & \longrightarrow & \mathrm{gr}^0 M_n^r & \longrightarrow & W_n\omega_{Y,\log}^{r} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & W_n\omega_Y^{r-1} & \longrightarrow & W_n\tilde{\omega}_Y^r & \longrightarrow & W_n\omega_Y^r & \longrightarrow & 0 \end{array}$$

such that the upper exact sequence is obtained by taking the kernel of $1 - F$ acting on the lower exact sequence, where F is the Frobenius. The canonical map of complexes $W_n\tilde{\omega}_Y^r[-r] \longrightarrow W_n\tilde{\omega}_Y^{\cdot}$ yields a canonical map on the eigenspaces of the Frobenius $H^{i+1-r}(Y, W_n\tilde{\omega}_Y^r)^{F=1} \longrightarrow H^{i+1}(Y, W_n\tilde{\omega}_Y^{\cdot})_{\varphi=p^r}$. From (2.6) we get the canonical isomorphism $H^{i+1}(\mathcal{X}, M)/W, \mathcal{O}_{\mathbb{Q}_p}^{\mathrm{crys}})_{\varphi=p^r} \cong (H^{i+1}(Y, W\tilde{\omega}_Y^{\cdot})_{\mathbb{Q}_p})_{\varphi=p^r}$ and therefore a commutative diagram

$$\begin{array}{ccccc} H_{\mathrm{et}}^{i+1}(\mathcal{X}, s_{\mathbb{Q}_p}^{\log}(r)) & & \longrightarrow & & H^{i+1}(\mathcal{X}, M)/W, J_{\mathbb{Q}_p}^{[r]})_{\varphi=p^r} \\ \downarrow & & & & \downarrow \\ H_{\mathrm{et}}^{i+1}(\mathcal{X}, \tau_{\leqslant r} Rj_*\mathbb{Q}_p(r)) & \longrightarrow & H^{i+1-r}(Y, \mathrm{gr}^0 M_{\mathbb{Q}_p}^r) & \longrightarrow & H^{i+1}(\mathcal{X}, M)/W, \mathcal{O}_{\mathbb{Q}_p}^{\mathrm{crys}})_{\varphi=p^r} \end{array}$$

In order to see that the map $H_{\mathrm{et}}^{i+1}(\mathcal{X}, s_{\mathbb{Q}_p}^{\log}(r)) \to H_{\mathrm{et}}^{i+1}(X, \mathbb{Q}_p(r))$ actually factors through $H_{\mathrm{et}}^{i+1}(\mathcal{X}, \tau_{\leqslant r} Rj_*\mathbb{Q}_p(r))$ we need an additional argument. In ([Ka1], Theorem (3.6)) Kato shows that for a syntomic scheme Z/W and $r < p$ his complexes $s_n^{\log}(r)$ that are defined in the same way as Tsuji's complexes $s_n^{\log'}(r)$ but without log-structures satisfy the vanishing property $\mathcal{H}^q(s_n^{\log}(r)) = 0$ for $q > r$. Now the proof is the same when we consider a log-scheme (Z, M) that is syntomic over W and work with the complexes $s_n^{\log'}(r)$ for $r < p$. By using Tsuji's extended definition of $s_n^{\log}(r)$ ([Tsu], Sect. 2.1) for $r \geqslant p$ it is then easy to see $\mathcal{H}^q(s_n^{\log'}(r)) = 0$ for $q > r$ and no restriction on r by the same arguments as in Kato's proof. Therefore the left vertical map in the above diagram is well-defined. Finally, it

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follows from the construction of the B_{st}-comparison isomorphism ([Ka4], Sect. 6) and the functoriality of the Hochschild–Serre spectral sequence that the diagram

$$\begin{array}{ccc} H^{i+1}_{\mathrm{et}}(\mathcal{X}, \tau_{\leqslant r} Rj_*\mathbb{Q}_p(r)) & \longrightarrow & H^{i+1}((\mathcal{X}, M)/W, \mathcal{O}^{\mathrm{crys}}_{\mathbb{Q}_p})_{\varphi=p^r} \\ \downarrow & & \downarrow \tilde{\alpha} \\ H^1(K, V) & \longrightarrow & H^1(K, B_{\mathrm{crys}} \otimes V) \end{array}$$

commutes. From the above explanations the commutativity on the right-hand side of the diagram in (2.9) is clear. Furthermore, ([Ka4] 6.4) implies that $\overline{\alpha}$ factors through

$$H^{i+1}((\mathcal{X}, M)/W, J^{[r]}_{\mathbb{Q}_p})_{\varphi=p^r} \xrightarrow{\hat{\alpha}} H^1(K, \mathrm{Fil}^r(B_{\mathrm{crys}} \otimes H^i(\overline{X}, \mathbb{Q}_p)))$$
$$\longrightarrow H^1(K, B_{\mathrm{crys}} \otimes V).$$

Now consider the following commutative diagram

$$\begin{array}{ccc} H^i((\mathcal{X}, M)/W, J^{[r]}_{\mathbb{Q}_p}) & \xrightarrow{1-p^{-r}\varphi} & H^i((\mathcal{X}, M)/W, \mathcal{O}^{\mathrm{crys}}_{\mathbb{Q}_p}) \\ \downarrow & & \downarrow \\ 0 \to V \to \mathrm{Fil}^r(B_{\mathrm{st}} \otimes_K D_i)^{N=0} & \xrightarrow{1-p^{-r}\varphi} & (B_{\mathrm{st}} \otimes_K D_i)^{N=0} \to 0. \end{array}$$

Here the lower exact sequence is derived from tensoring the exact sequence

$$0 \to \mathbb{Q}_p(r) \to \mathrm{Fil}^r(B_{\mathrm{crys}}) \xrightarrow{1-p^{-r}\varphi} B_{\mathrm{crys}} \to 0 \qquad \text{([Fo1], 5.3.7)}$$

with $H^i(\overline{X}, \mathbb{Q}_p)$ and then applying the B_{st}-comparison-isomorphism. Taking Galois-invariants yields a commutative diagram

$$\begin{array}{ccc} H^i((\mathcal{X}, M)/W, \mathcal{O}^{\mathrm{crys}}_{\mathbb{Q}_p}) & \longrightarrow & H^{i+1}_{\mathrm{et}}(\mathcal{X}, s^{\log}_{\mathbb{Q}_p}(r)) \\ \downarrow & & \downarrow \alpha \\ H^i_{DR}(X)^{N=0}/(1-p^{-r}\varphi)(\mathrm{Fil}^r)^{N=0} & \hookrightarrow & H^1(K, V) \end{array}$$

using Lemma 9.5 in [J2] and the isomorphism $D_i \cong H^i_{DR}(X)$ ([H-K], Thm. 5.1). By our assumption $(\mathrm{Fil}^r)^{N=0}_{\varphi=p^r} = 0$, so the inverse of $(1-p^{-r}\varphi)$: $D_i^{N=0} \to D_i^{N=0}$ induces an isomorphism

$$H^i_{DR}(X)^{N=0}/(1-p^{-r}\varphi)(\mathrm{Fil}^r)^{N=0} \xrightarrow{\cong} H^i_{DR}(X)^{N=0}/(\mathrm{Fil}^r)^{N=0}$$

and we get by composition a commutative diagram

$$\begin{array}{ccc} H^i((\mathcal{X}, M)/W, \mathcal{O}^{\mathrm{crys}}_{\mathbb{Q}_p}) & \longrightarrow & H^{i+1}_{\mathrm{et}}(\mathcal{X}, s^{\log}_{\mathbb{Q}_p}(r)) \\ \downarrow & & \downarrow \alpha \\ H^i_{DR}(X)^{N=0}/(\mathrm{Fil}^r)^{N=0} & \xhookrightarrow{\exp} & H^1(K, V) \end{array}$$

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It remains to show that the lower horizontal map is the restriction of the exponential map in the sense of Bloch–Kato. (Note that the map λ that appears in the diagram (2.9) coincides with the left vertical arrow in the above diagram composed with the canonical map $D_i^{N=0}/(\mathrm{Fil}^r)^{N=0} \hookrightarrow D_i/\mathrm{Fil}^r$.) Consider the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Q}_p & \longrightarrow & t^{-r}\mathrm{Fil}^r B_{\mathrm{crys}}^+ & \xrightarrow{1-\varphi} & t^{-r}B_{\mathrm{crys}}^+ & \longrightarrow & 0 \\ & & \Big\downarrow = & & \Big\downarrow \theta_1 & & \Big\downarrow \theta_2 & & \\ 0 & \longrightarrow & \mathbb{Q}_p & \longrightarrow & B_{\mathrm{crys}}^+ \oplus B_{dR}^+ & \longrightarrow & B_{\mathrm{crys}} \oplus B_{dR} & \longrightarrow & 0. \end{array}$$

In this diagram (and only here, so there is no confusion with the element t defined earlier) t denotes the element defined via the inclusion $\mathbb{Q}_p(1) \hookrightarrow B_{\mathrm{crys}}^+$ in the sense of Fontaine. The upper exact sequence is a version of ([Fo1], 5.3.7), the lower exct sequence is the one derived by Bloch and Kato ([B-K], Sect. 1), θ_1 and θ_2 are given as follows: $\theta_1(x) = (x, \varphi(x))$, $\theta_2(x) = (x, x)$. (Note that $t^{-r}\,\mathrm{Fil}^r B_{\mathrm{crys}}^+$ is contained in B_{dR}^+, so the definition makes sense.) After tensoring the above diagram with $V = H^i(\overline{X}, \mathbb{Q}_p(r))$, using the formula $B_{st}^{N=0} = B_{\mathrm{crys}}$ and applying the B_{st}-, resp. B_{dR}-comparison isomorphism it is easy to see that $H_{DR}^i(X)^{N=0}/(\mathrm{Fil}^r)^{N=0} \xrightarrow{\exp} H^1(K, V)$ is the restriction of the exponential map $DR(V)/DR^0(V) \xrightarrow{\exp} H^1(K, V)$. Another proof for the commutativity of the first square in Proposition 2.9 has been pointed out to me by the referee and is given as follows:

– The map $H^i((\mathcal{X}, M)/W, \mathcal{O}_{\mathbb{Q}_p}^{\mathrm{crys}}) \to H_{\mathrm{et}}^{i+1}(\mathcal{X}, s_{\mathbb{Q}_p}^{\log}(r) \xrightarrow{\alpha} H^1(K, V)$ is equal to

$$H^i((\mathcal{X}, M)/W, \mathcal{O}_{\mathbb{Q}_p}^{\mathrm{crys}}) \xrightarrow{\ker} (N_{\widehat{K\langle t\rangle}})$$

$$= D_i^{N=0} \xrightarrow{\gamma} \mathrm{Coker}(\delta) = H_f^1(K, V) \subset H^1(K, V),$$

where $\delta\colon D_i^{N=0} \to D_i^{N=0} \oplus D_i/\mathrm{Fil}^r$ is given by $\delta(y) = ((1 - p^{-r}\varphi)(y), y)$ and $\gamma(x) =$ the class of $(x, 0)$.

– The exponential map of [B-K] exp: $D_i/\mathrm{Fil}^r \to \mathrm{Coker}(\delta) = H_f^1(K, V)$ is given by $\exp(x) =$ the class of $(0, x)$.

– The following commutative diagram is commutative:

$$\begin{array}{ccc} H^i((\mathcal{X}, M)/W, \mathcal{O}_{\mathbb{Q}_p}^{\mathrm{crys}}) & \longrightarrow & H^i((\mathcal{X}, M)/W, \mathcal{O}_{\mathbb{Q}_p}^{\mathrm{crys}}/J_{\mathbb{Q}_p}^{[r]}) \\ \Big\downarrow & & \Big\downarrow \psi \\ D_i^{N=0} & \longrightarrow & D_i/\mathrm{Fil}^r \end{array}$$

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Consequently, if $x \in D_i^{N=0}$ and $x = (p^{-r}\varphi - 1)(y)$ with $y \in D_i^{N=0}$, then the class of $(x, 0)$ in Coker(δ) is equal to the class of $(x, 0) + \delta(y) = (0, y)$, i.e., $\exp(y)$, as claimed.

This finishes the proof of the Proposition.

We study the right-hand side of the diagram in Proposition 2.9 in more detail. Using (2.6), Lemmas 2.1 and 2.7 and our assumption $(D_{i+1})^{N=0}_{\varphi=p^r} = 0$ we can draw the following conclusions:

(i) The map $H^{i+1}((\mathcal{X}, M)/W, J^{[r]}_{\mathbb{Q}_p})_{\varphi=p^r} \twoheadrightarrow H^{i+1}((\mathcal{X}, M)/W, \mathcal{O}^{\mathrm{crys}}_{\mathbb{Q}_p})_{\varphi=p^r}$ is surjective.

(ii) $H^{i+1}((\mathcal{X}, M)/W, \mathcal{O}^{\mathrm{crys}}_{\mathbb{Q}_p})_{\varphi=p^r}$ is contained in

$$\Sigma := \mathrm{image}\left(D_i \hookrightarrow D_i \underset{K}{\otimes} \widehat{K\langle t\rangle} \xrightarrow[\wedge d\log t]{} H^{i+1}((Y, M_1)/W, \mathcal{O}^{\mathrm{crys}}_{\mathbb{Q}_p})\right).$$

Now using Fontaine's exact sequence ([Fo] 3.2.3)

$$0 \longrightarrow B_{\mathrm{crys}} \longrightarrow B_{\mathrm{st}} \xrightarrow{N} B_{\mathrm{st}} \longrightarrow 0,$$

tensoring it with V, applying the B_{st}-comparison-isomorphism and taking G_K-invariants we get a commutative diagram

$$\begin{array}{ccc} \mathrm{coker}(H^i_{DR}(X) \xrightarrow{N} H^i_{DR}(X)) & \xrightarrow{\cong} & \Sigma \\ & \searrow\hookrightarrow & \downarrow \tilde{\alpha} \\ & & H^1(K, B_{\mathrm{crys}} \otimes V). \end{array} \qquad (2.10)$$

Here the commutativity follows again from ([J2], Lemma 9.5). As a result we obtain

LEMMA 2.11. *$\tilde{\alpha}$ is injective and*

$$\ker \overline{\alpha} = \ker(H^{i+1}((\mathcal{X}, M)/W, J^{[r]}_{\mathbb{Q}_p}) \longrightarrow H^{i+1}((\mathcal{X}, M)/W, \mathcal{O}^{\mathrm{crys}}_{\mathbb{Q}_p}).$$

The above diagram (2.10) also shows that the composite map

$$\Sigma \xrightarrow[\tilde{\alpha}]{} H^1(K, B_{\mathrm{crys}} \otimes V) \longrightarrow H^1(K, B_{\mathrm{st}} \otimes V)$$

is zero.

Let $H^1_{st}(K, V) := \ker(H^1(K, V) \to H^1(K, B_{\mathrm{st}} \otimes V)$. Then we have $H^1_{\mathrm{st}}(K, V) = H^1_g(K, V)$. This is an unpublished result due to Hyodo that is quoted in ([Fo2], 6.2.2) and has also been proven by Nekovář ([Ne], 1.2.4).

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We immediately get by combining this result with Propositions 2.9 and 2.10.

LEMMA 2.12. $\operatorname{Im}\alpha \subset H^1_g(K, V)$.

PROPOSITION 2.13. *The composite map*

$$H^{i+1}_{\text{et}}(\mathcal{X}, s^{\log}_{\mathbb{Q}_p}(r)) \xrightarrow{\alpha} H^1_g(K, V) \longrightarrow H^1_g(K, V)/H^1_f(K, V)$$

is surjective.

Proof. It remains to show that

$$\operatorname{coker}(H^i_{DR}(X) \xrightarrow{N} H^i_{DR}(X))_{\varphi=p^{r-1}} \cong H^{i+1}((\mathcal{X}, M)/W, O^{\text{crys}}_{\mathbb{Q}_p})_{\varphi=p^r} \quad (2.6)$$

is isomorphic to $H^1_g(K, V)/H^1_f(K, V) =: H^1_g/H^1_f$.

Now H^1_g/H^1_f is, by local Tate-Duality, dual to

$$H^1_f(K, H^{2d-i}(\overline{X}, \mathbb{Q}_p(d+1-r)))/H^1_e(K, \ldots)$$

and this vector space is, by the B_{st}-comparison-isomorphism isomorphic to $H^{2d-i}_{DR}(X)^{N=0}/1-f$. Here d is the dimension of X and f acts as $p^{-(d+1-r)} \cdot \varphi_p$, where φ_p is the Frobenius on log-crystalline cohomology. Using Poincaré duality for log-crystalline cohomology ([H2], (3.7)) we obtain an isomorphism

$$H^1_g/H^1_f \cong \operatorname{coker}(N\colon H^i_{DR}(X) \longrightarrow H^i_{DR}(X))_{f=1}$$

where f acts as $p^{-(r-1)}\varphi_p$. This finishes the proof of Proposition 2.13.

LEMMA 2.14. *In the commutative diagram in Proposition* 2.9 *the canonical map* $\ker\alpha \to \ker\overline{\alpha}$, *which is induced by the snake lemma, is the zero map.*

Before proving Lemma 2.14 we finish the proof of the main theorem.

COROLLARY 2.15. *The canonical map* $\ker\overline{\alpha} \longrightarrow \operatorname{coker}(\lambda)$, *that is induced by the snake lemma, is an isomorphism.*

Proof. The short exact sequence $0 \to J^{[r]}_n \to O^{\text{crys}}_n \to O^{[r]}_n/J^{[r]}_n \to 0$ in $(\mathcal{X}_n, M_n/W_n))^{\sim}_{\text{crys}}$ implies – in the associated long exact continuous cohomology sequence with $\mathbb{Q}_p$-coefficients – an isomorphism of finite-dimensional $\mathbb{Q}_p$-vector spaces (use Lemma 2.7 and 2.11)

$$\begin{aligned}\ker(\overline{\alpha}) &\cong \operatorname{coker}(H^i((\mathcal{X}, M)/W, O^{\text{crys}}_{\mathbb{Q}_p}) \to H^i((\mathcal{X}, M)/W, O^{\text{crys}}_{\mathbb{Q}_p}/J^{[r]}_{\mathbb{Q}_p})\\ &\cong \operatorname{coker}(D_i^{N=0} \to D_i/\operatorname{Fil}^r) \cong \operatorname{coker}(\lambda).\end{aligned}$$

On the other hand, the canonical map under consideration is injective by Lemma 2.14 and therefore has to be an isomorphism.

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COROLLARY 2.16. $H^1_f(K, V) \subset \operatorname{Im} \alpha$.

Proposition 2.9, Proposition 2.13 and Corollary 2.16 yield Theorem 0.1.

It remains to give a proof of Lemma 2.14. For this it suffices to show that the map

$$\begin{array}{ccc} \ker \overline{\alpha} & \longrightarrow & H^1(K, \mathrm{Fil}^r(B_{\mathrm{st}} \otimes_K D_i)^{N=0}) \\ & & \Big\downarrow \cong \\ & & H^1(K, \mathrm{Fil}^r(B_{\mathrm{crys}} \otimes_{\mathbb{Q}_p} H^i(\overline{X}, \mathbb{Q}_p)), \quad \text{induced by } \hat{\alpha}, \end{array}$$

is injective.

We consider the following commutative diagram

$$\begin{array}{ccccccccc} & & H^i((\mathcal{X}, M)/W, J^{[r]}_{\mathbb{Q}_p}) & \longrightarrow & H^i((\mathcal{X}, M)/W, O^{\mathrm{crys}}_{\mathbb{Q}_p}) & \longrightarrow & H^i((\mathcal{X}, M)/W, O^{\mathrm{crys}}_{\mathbb{Q}_p}/J^{[r]}_{\mathbb{Q}_p}) & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Fil}^r(B^+_{\mathrm{st}} \otimes_K D_i)^{N=0} & \longrightarrow & H^i_{\mathrm{naive}}((\overline{\mathcal{X}}, \overline{M})/W, O^{\mathrm{crys}}_{\mathbb{Q}_p}) & \longrightarrow & H^i_{\mathrm{naive}}((\overline{\mathcal{X}}, \overline{M}/W, O^{\mathrm{crys}}_{\mathbb{Q}_p}/J^{[r]}_{\mathbb{Q}_p}) & & \\ & & \Big\downarrow & & \downarrow \cong & & \downarrow \cong & & \\ & & \Big\downarrow & & (B^+_{\mathrm{st}} \otimes_K D_i)^{N=0} & \longrightarrow & (B^+_{DR} \otimes_K H^i_{DR}(X))/\mathrm{Fil}^r & & \\ & & \Big\downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Fil}^r(B_{\mathrm{st}} \otimes_K D_i)^{N=0} & \longrightarrow & (B_{\mathrm{st}} \otimes_K D_i)^{N=0} & \longrightarrow & (B_{dR} \otimes_K D_i)/\mathrm{Fil}^r & \longrightarrow & 0 \end{array}$$

Here we have used ([Ka4], (6.4)) for the isomorphism in the right sequence of vertical maps and the fact that $B_{\mathrm{crys}}/\mathrm{Fil}^r \cong B_{dR}/\mathrm{Fil}^r$ ([P-R] (1.4)) in order to obtain the lower horizontal exact sequence.

Taking G_K-invariants of the lower exact sequence and using Lemma 2.7 together with ([J2] Lemma 9.5) we get the commutative diagram

$$\begin{array}{ccc} H^i_{DR}(X)/\mathrm{Fil}^r & \twoheadrightarrow & \ker \overline{\alpha} \\ \Big\downarrow & \nearrow \cong & \Big\downarrow \hat{\alpha} \\ H^i_{DR}(X)/\langle H^i_{DR}(X)^{N=0}, \mathrm{Fil}^r \rangle & \hookrightarrow & H^1(K, \mathrm{Fil}^r(B_{\mathrm{st}} \otimes_K D_i)^{N=0}) \end{array}$$

Here the diagonal map exists and is an isomorphism because the composite map of the middle vertical arrows maps $H^i((\mathcal{X}, M)/W, O^{\mathrm{crys}}_{\mathbb{Q}_p})$ onto $(D_i)^{N=0}$. For this one uses the diagram constructed after (2.5). Therefore $\hat{\alpha}$ is injective and Lemma 2.14 follows.

3. In this section we compute the kernel of α, i.e., we prove Theorem 0.2. Let $\tilde{H}^i((\mathcal{X}, M)/W, J^{[r]}_{\mathbb{Q}_p})$ be the kernel of the composite map (compare Lemma 2.6)

$$H^i(\mathcal{X}, M)/W, O^{\mathrm{crys}}_{\mathbb{Q}_p}) \to \ker N_{\widehat{K\langle t\rangle}} = D_i^{N=0} \to D_i^{N=0}/(p^r - \varphi)(\mathrm{Fil}^r)^{N=0}.$$

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Then we have a commutative diagram

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
H^i((\mathcal{X},M)/W, J^{[r]}_{\mathbb{Q}_p}) & \xrightarrow{p^r-\varphi} & \tilde{H}^i((\mathcal{X},M)/W, J^{[r]}_{\mathbb{Q}_p}) \\
\downarrow{=} & & \downarrow \\
H^i((\mathcal{X},M)/W, J^{[r]}_{\mathbb{Q}_p}) & \xrightarrow{p^r-\varphi} & H^i((\mathcal{X},M)/W, \mathcal{O}^{\mathrm{crys}}_{\mathbb{Q}_p}) \\
\downarrow & & \downarrow \\
0 & \longrightarrow & D_i^{N=0}/(p^r-\varphi)(\mathrm{Fil}^r)^{N=0}
\end{array}$$

with vertical exact sequences. Taking cokernels of the above horizontal maps yields an exact sequence

$$0 \longrightarrow \mathcal{K}_1 \longrightarrow \mathcal{K}_2 \longrightarrow D_i^{N=0}/(p^r-\varphi)(\mathrm{Fil}^r)^{N=0}.$$

Proposition 2.9 and Lemma 2.14 imply that there is a canonical isomorphism $\mathcal{K}_1 \xrightarrow{\cong} \ker\alpha$. Now look at the following commutative diagram

$$\begin{array}{ccc}
 & & 0 \\
 & & \downarrow \\
H^i((\mathcal{X},M)/W, J^{[r]}_{\mathbb{Q}_p}) & \xrightarrow{p^r-\varphi} & \tilde{H}^i((\mathcal{X},M)/W, J^{[r]}_{\mathbb{Q}_p}) \\
\downarrow & & \downarrow \\
H^i((\mathcal{X},M)/W, \mathcal{O}^{\mathrm{crys}}_{\mathbb{Q}_p}) & \xrightarrow{p^r-\varphi} & H^i((\mathcal{X},M)/W, \mathcal{O}^{\mathrm{crys}}_{\mathbb{Q}_p}) \\
\downarrow & & \downarrow \\
H^i_{DR}(X)^{N=0}/(\mathrm{Fil}^r)^{N=0} & \xrightarrow{p^r-\varphi} & H^i_{DR}(X)^{N=0}/(p^r-\varphi)(\mathrm{Fil}^r)^{N=0} \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

By our assumption the lower horizontal map is an isomorphism and we get

$$\mathcal{K}_1 \cong \ker\alpha \cong \mathrm{coker}(H^i((\mathcal{X},M)/W, \mathcal{O}^{\mathrm{crys}}_{\mathbb{Q}_p}) \xrightarrow{p^r-\varphi} H^i((\mathcal{X},M)/W, \mathcal{O}^{\mathrm{crys}}_{\mathbb{Q}_p}).$$

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To compute this cokernel we use again the exact sequence (2.5) and get a commutative diagram (note that $\varphi(d\log t) = pd\log t$)

$$\begin{array}{ccc}
H^{i-1}_{DR}(X) \otimes_K \widehat{K\langle t\rangle} & \xrightarrow{p^r-\varphi} & H^{i-1}_{DR}(X) \otimes_K \widehat{K\langle t\rangle} \\
\downarrow{\scriptstyle N_{\widehat{K\langle t\rangle}}} & & \downarrow{\scriptstyle N_{\widehat{K\langle t\rangle}}} \\
H^{i-1}_{DR}(X) \otimes \widehat{K\langle t\rangle} & \xrightarrow{p^{r-1}-\varphi} & H^{i-1}_{DR}(X) \otimes \widehat{K\langle t\rangle} \\
\downarrow{\scriptstyle \wedge d\log t} & & \downarrow{\scriptstyle \wedge d\log t} \\
H^i((\mathfrak{X}, M)/W, \mathcal{O}^{\mathrm{crys}}_{\mathbb{Q}_p}) & \xrightarrow{p^r-\varphi} & H^i((\mathfrak{X}, M)/W, \mathcal{O}^{\mathrm{crys}}_{\mathbb{Q}_p}) \\
\downarrow & & \downarrow \\
H^i_{DR}(X)^{N=0} & \xrightarrow{p^r-\varphi} & H^i_{DR}(X)^{N=0} \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

Now we compute the cokernel of the upper (and lower) horizontal map by adapting certain methods developed by Perrin-Riou [PR].

PROPOSITION 3.1. *The cokernel of the map*

$$H^j_{DR}(X) \otimes \widehat{K\langle t\rangle} \xrightarrow{p^r-\varphi} H^j_{DR}(X) \otimes \widehat{K\langle t\rangle}$$

is isomorphic to $H^j_{DR}(X)/(p^r - \varphi)$.

Proof. We closely follow ([PR], Sect. 2.2). Perrin-Riou deals, instead of $\widehat{K\langle t\rangle}$ with the subring of $K[[T]]$ of all power series that are convergent on the open unit disc. Another important difference is the action of the Frobenius: in our situation we have $\varphi(t) = t^p$, Perrin-Riou defines the action of φ as $\varphi(T) = (1+T)^p - 1$.

It suffices to show that any element in $(\widehat{K\langle t\rangle} \cap t \cdot K[[t]]) \otimes H^j_{DR}(X)$ is in the image of the map $1 - p^{-r}\varphi$. Let M be a W-lattice in $D_j = H^j_{DR}(X)$, that is stable under φ. We consider elements of the form $t^{[k]} \otimes m$, $k \geqslant 1$, $m \in M$.

Claim. $z = \sum_{n\geqslant 0}(p^{-r}\varphi)^n(t^{[k]} \otimes m) \in \widehat{K\langle t\rangle} \otimes_W M$ and the infinite series defining z converges uniformly in k.

Indeed, we have the equality

$$(p^{-r}\varphi)^n(t^{[k]} \otimes m) = \frac{(kp^n)!}{k!p^{rn}} t^{[kp^n]} \otimes m', \quad m' \in M$$

and $\mathrm{ord}_p((kp^n)!/k!) \geqslant rn$ whenever $p^n \geqslant prn$ (for all $k \geqslant 1$). This shows that the coefficients $(kp^n)!/k!p^{rn}$ converge p-adically to zero. Now $(1 - p^{-r}\varphi)z = t^{[k]} \otimes m$ and therefore the claim and Proposition 3.1 follow.

Since $p^r - \varphi$ is an isomorphism on $H^i_{DR}(X)^{N=0}$ we get

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COROLLARY 3.2. $\ker\alpha$ *is isomorphic to the cokernel of the map* $p^{r-1}-\varphi$, *acting on* $\mathrm{coker}(N\colon H^{i-1}_{DR}(X)\to H^{i-1}_{DR}(X))$. By Poincaré duality for log-crystalline cohomology we see that $\ker\alpha$ is $\mathbb{Q}_p$-dual to $H^{2d-(i-1)}_{DR}(X)^{N=0}_{\varphi=p^{d-(r-1)}}$ and the B_{st}-comparison isomorphism implies a canonical injection

$$\eta^{\nu}\colon H^{2d-(i-1)}(\overline{X},\mathbb{Q}_p(d-r+1))^{G_K}\hookrightarrow H^{2d-(i-1)}_{DR}(X)^{N=0}_{\varphi=p^{d-(r-1)}}.$$

Local Tate-Duality therefore yields the desired surjection

$$\eta\colon \ker\alpha\twoheadrightarrow H^2(G_K,H^{i-1}(\overline{X},\mathbb{Q}_p(r))).$$

If the reduction of $\mathcal{X}$ is ordinary we know from ([Il], 2.6) that the $\varphi=p^j$-eigenspace in $H^s_{DR}(X)$ is contained in $\mathrm{Fil}^j H^s_{DR}(X)$ and in this case the above maps η^{ν} and η are isomorphisms. This finishes the proof of Theorem 0.2.

By the construction of the map η and by using again ([J2], Lemma 9.5) we get the following commutative diagram

$$\begin{array}{ccc}
0 & & 0\\
\downarrow & & \downarrow\\
\ker\alpha & \xrightarrow{\ \eta\ } & H^2(G_K,H^{i-1}(\overline{X},\mathbb{Q}_p(r)))\\
\downarrow & & \downarrow\\
H^{i+1}_{\mathrm{et}}(\mathcal{X},s^{\log}_{\mathbb{Q}_p}(r)) & \xrightarrow{\ \varepsilon_2\ } & H^{i+1}(X,\mathbb{Q}_p(r))\\
\downarrow{\scriptstyle\alpha} & & \downarrow\\
H^1_g(G_K,H^i(\overline{X},\mathbb{Q}_p(r))) & \hookrightarrow & H^1(G_K,H^i(\overline{X},\mathbb{Q}_p(r)))\\
\downarrow & & \downarrow\\
0 & & 0
\end{array}$$

The diagram yields a nice picture on the comparison of the log-syntomic cohomology of the semistable scheme $\mathcal{X}$ and the p-adic étale cohomology of its generic fiber X.

4. In this section we examine the special case $i+1=2r$ which is of particular interest when we want to study cycle class maps. We will finish the paper with some speculations on them. The notations are as before. If $\overline{\mathcal{X}}=\mathcal{X}\times_{O_K}O_{\overline{K}}$ we will consider the following subvectorspace of our log-syntomic cohomology

$$\tilde{H}^{2r}(\mathcal{X},s^{\log}_{\mathbb{Q}_p}(r)):=\ker(H^{2r}(\mathcal{X},s^{\log}_{\mathbb{Q}_p}(r))\longrightarrow H^{2r}(\overline{\mathcal{X}},s^{\log}_{\mathbb{Q}_p}(r))).$$

We have the canonical map constructed by Kato and Tsuji

$$\tilde{H}^{2r}(\mathcal{X},s^{\log}_{\mathbb{Q}_p}(r))\longrightarrow \ker(H^{2r}(X,\mathbb{Q}_p(r))\longrightarrow H^{2r}(\overline{X},\mathbb{Q}_p(r)))$$

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which together with the Hochschild–Serre spectral sequence yields a canonical map $\alpha\colon \tilde{H}^{2r}(\mathcal{X}, s^{\log}_{\mathbb{Q}_p}(r)) \longrightarrow H^1(\mathrm{Gal}(\overline{K}/K), V)$, where $V := H^{2r-1}(\overline{X}, \mathbb{Q}_p(r))$.

Under certain conditions related to the p-adic and ℓ-adic monodromy conjecture we will describe the image and the kernel of α. The results are in analogy to Theorems 0.1 and 0.2.

Let as before for an integer j $(D_j)^{N=0}_{\varphi=p^k}$ be the eigenspace where the Frobenius φ acts as multiplication by p^k in the kernel of the monodromy operator N. Throughout the paragraph we will consider the following hypothesis

$$(D_{2r-1})^{N=0}_{\varphi=p^r} = 0. \tag{H}$$

From the works of Mokrane [Mo] and Jannsen [J1] and the p-adic and ℓ-adic monodromy conjecture we conclude that the condition (H) is equivalent to the following

CONJECTURE. $H^{2r-1}(\overline{X}, \mathbb{Q}_\ell(r))^{G_K} = 0$ *for all primes* ℓ.

Even though this conjecture is not stated explicitly in [J1] evidence for it is given in ([J1] p. 349). Now we reformulate the main results in the case $i + 1 = 2r$.

THEOREM 4.1. *Under the assumptions of Theorem* 0.1 *and the condition* (H) *we have* $\mathrm{Im}\,\alpha = H^1_f(G_K, V) = H^1_g(G_K, V)$.

THEOREM 4.2. *Under the assumptions of Theorem* 4.1 *we have a canonical surjection* $\eta\colon \ker\alpha \twoheadrightarrow H^2(G_K, H^{2r-2}(\overline{X}, \mathbb{Q}_p(r)))$. *If* $\mathcal{X}$ *has ordinary semistable reduction, then* η *is an isomorphism.*

Theorem 4.2 will be an easy consequence of the analogous Theorem 0.2 proven in Section 3 whereas the proof of Theorem 4.1 requires a bit more work. But the methods are very similar to those developed in Section 2.

We start to prove Theorem 4.1. From the definition of $s^{\log}_n(r)$, we obtain a commutative diagram of exact sequences

$$\begin{array}{ccccccc} \longrightarrow H^{2r-1}((\mathcal{X}, M)/W, \mathcal{O}^{\mathrm{crys}}_{\mathbb{Q}_p}) & \longrightarrow & H^{2r}(\mathcal{X}, s^{\log}_{\mathbb{Q}_p}(r)) & \longrightarrow & H^{2r}((\mathcal{X}, M)/W, J^{[r]}_{\mathbb{Q}_p}) & \longrightarrow \\ \downarrow & & \downarrow & & \downarrow & \\ 0 & \longrightarrow & H^{2r}_{\mathrm{naive}}(\overline{\mathcal{X}}, s^{\log}_{\mathbb{Q}_p}(r)) & \longrightarrow & H^{2r}_{\mathrm{naive}}((\overline{\mathcal{X}}, \overline{M})/W, J^{[r]}_{\mathbb{Q}_p}) & \longrightarrow \end{array} \tag{4.3}$$

It follows from ([Ka4], Section 6) that the map

$$H^{2r}(\overline{\mathcal{X}}, s^{\log}_{\mathbb{Q}_p}(r)) \longrightarrow H^{2r}((\overline{\mathcal{X}}, \overline{M})/W, J^{[r]}_{\mathbb{Q}_p})$$

is injective and therefore the diagram is commutative.

Let as in Section 2 $H^{2r}((\mathcal{X}, M)/W, J^{[r]}_{\mathbb{Q}_p})_{\varphi=p^r}$ be the kernel of the map

$$H^{2r}((\mathcal{X}, M)/W, J^{[r]}_{\mathbb{Q}_p}) \xrightarrow[p^r-\phi]{} H^{2r}((\mathcal{X}, M)/W, \mathcal{O}^{\mathrm{crys}}_{\mathbb{Q}_p}).$$

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$H^{2r}((\overline{\mathcal{X}}, \overline{M})/W, J^{[r]}_{\mathbb{Q}_p})_{\varphi=p^r}$ is defined similarly. From (4.3) we get an exact sequence

$$H^{2r-1}((\mathcal{X}, M)/W, O^{\mathrm{crys}}_{\mathbb{Q}_p}) \longrightarrow \tilde{H}^{2r}(\mathcal{X}, s^{\log}_{\mathbb{Q}_p}(r)) \longrightarrow B \longrightarrow 0, \qquad (4.4)$$

where

$$B := \ker(H^{2r}((\mathcal{X}, M)/W, J^{[r]}_{\mathbb{Q}_p})_{\varphi=p^r} \longrightarrow H^{2r}((\overline{\mathcal{X}}, \overline{M})/W, J^{[r]}_{\mathbb{Q}_p})_{\varphi=p^r}.$$

PROPOSITION 4.5. *There is a commutative diagram of exact sequences*

$$\begin{array}{ccccc} H^{2r-1}((\mathcal{X}, M)/W, O^{\mathrm{crys}}_{\mathbb{Q}_p}) & \longrightarrow & \tilde{H}^{2r}(\mathcal{X}, s^{\log}_{\mathbb{Q}_p}(r)) & \longrightarrow & B \longrightarrow 0 \\ \downarrow \lambda & & \downarrow \alpha & & \downarrow \overline{\alpha} \\ D_{2r-1}/\mathrm{Fil}^r & \xhookrightarrow[\exp]{} & H^1(G_K, V) & \longrightarrow & H^1(G_K, B_{\mathrm{crys}} \otimes V). \end{array}$$

Here λ is the map given in Proposition 2.9.

The map $\overline{\alpha}$ is defined in the same way as in Proposition 2.9. Note that our vanishing assumption on $(D_{2r-1})^{N=0}_{\varphi=p^r}$ implies that 'exp' maps onto $H^1_f(G_K, V)$ and so the lower horizontal sequence is exact.

Proof. The Proposition is shown in the same way as Proposition 2.9.

Note that $\overline{\alpha}$ factors through

$$B \longrightarrow \ker(H^{2r}((\mathcal{X}, M)/W, O^{\mathrm{crys}}_{\mathbb{Q}_p})_{\varphi=p^r} \longrightarrow H^{2r}_{\mathrm{naive}}((\overline{\mathcal{X}}, \overline{M})/W, O^{\mathrm{crys}}_{\mathbb{Q}_p})_{\varphi=p^r}$$
$$\longrightarrow H^1(G_K, B_{\mathrm{crys}} \otimes V).$$

By using ([Ka4], Thm. 4.1) we have an isomorphism

$$H^{2r}_{\mathrm{naive}}((\overline{\mathcal{X}}, \overline{M})/W, O^{\mathrm{crys}}_{\mathbb{Q}_p}) \xrightarrow{\cong} (B^+_{st} \otimes D_{2r})^{N=0}$$

that fits into a commutative diagram

$$\begin{array}{ccc} H^{2r}((\mathcal{X}, M)/W, O^{\mathrm{crys}}_{\mathbb{Q}_p}) & \twoheadrightarrow & D^{N=0}_{2r} = [D_{2r} \otimes_K \widehat{K\langle t\rangle}]^{N=0} \\ \downarrow & & \downarrow v \mapsto 1 \otimes v \\ H^{2r}_{\mathrm{naive}}((\overline{\mathcal{X}}, \overline{M})/W, O^{\mathrm{crys}}_{\mathbb{Q}_p}) & \xrightarrow[\cong]{} & (B^+_{st} \otimes D_{2r})^{N=0} \end{array} \qquad (4.6)$$

where the upper surjective morphism is obtained from the exact sequence (2.5) and Lemma 2.6(ii). The commutativity follows from the explicit construction of

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the lower horizontal isomorphism in the proof of ([Ka4] Thm. 4.1) which is also based on the exact sequence of crystalline complexes (2.2) ([Ka4], Lemma 4.2). By Lemma 2.6, (iii) we have an exact sequence

$$0 \longrightarrow \operatorname{coker}(N\colon D_{2r-1} \to D_{2r-1})_{\varphi=p^{r-1}} \longrightarrow H^{2r}((\mathcal{X}, M)/W, \mathcal{O}^{\mathrm{crys}}_{\mathbb{Q}_p})_{\varphi=p^r}$$
$$\longrightarrow (D_{2r})^{N=0}_{\varphi=p^r} \longrightarrow . \tag{4.7}$$

LEMMA 4.8. *Under the condition* (H) *we have* $\operatorname{coker}(N\colon D_{2r-1} \to D_{2r-1})_{\varphi=p^{r-1}} = 0$.

Proof. It follows from Proposition 2.13 that this vectorspace is canonically isomorphic to $H^1_g(G_K, V)/H^1_f(G_K, V)$. By applying the functor $D_{\mathrm{st}} = (B_{\mathrm{st}} \otimes \cdot)^{G_K}$ to the Hard Lefschetz theorem for étale cohomology, we get an isomorphism $D^*_{2r-1} \xrightarrow{\cong} D_{2r-1}(2r-1)$, where $*$ denotes the Poincaré-dual of Hyodo–Kato cohomology. After (Tate-) twisting we have an isomorphism $(D_{2r-1}(r-1))^* \xrightarrow{\cong} D_{2r-1}(r)$. Under this isomorphism the vanishing statement of the Lemma is equivalent to $D^{N=0}_{2r-1}/(1-\varphi p^{-r}) = 0$ which is equivalent to the condition (H). The Lemma follows.

LEMMA 4.9. *Under the condition* (H) *we have the inclusion* $\operatorname{Im}\alpha \subset H^1_g(G_K, V) = H^1_f(G_K, V) = H^1_e(G_K, V)$.

Proof. This is shown in the same way as Lemma 2.12.

COROLLARY 4.10. *The map* $\overline{\alpha}$ *is the zero map.*

Proof. Combine the remark on $\overline{\alpha}$ after Proposition 4.5 with 4.6, 4.7 and Lemma 4.8.

LEMMA 4.11. *B coincides with* $\ker(H^{2r}((\mathcal{X}, M)/W, J^{[r]}_{\mathbb{Q}_p}) \longrightarrow H^{2r}((\mathcal{X}, M)/W, \mathcal{O}^{\mathrm{crys}}_{\mathbb{Q}_p}))$.

Proof. Combine the two facts that the canonical maps

$$H^{2r}_{\mathrm{naive}}((\overline{\mathcal{X}}, \overline{M})/W, J^{[r]}_{\mathbb{Q}_p}) \longrightarrow H^{2r}_{\mathrm{naive}}((\overline{\mathcal{X}}, \overline{M})/W, \mathcal{O}^{\mathrm{crys}}_{\mathbb{Q}_p})$$

and

$$H^{2r}((\mathcal{X}, M)/W, \mathcal{O}^{\mathrm{crys}}_{\mathbb{Q}_p})_{\varphi=p^r} \longrightarrow H^{2r}_{\mathrm{naive}}((\overline{\mathcal{X}}, \overline{M})/W, \mathcal{O}^{\mathrm{crys}}_{\mathbb{Q}_p})_{\varphi=p^r}$$

are both injective. To see this one uses ([Ka4], Sect. 6) and the commutative diagram in the proof of Lemma 2.14 for the first map and (4.6), (4.7) and Lemma 4.8 for the second one.

LEMMA 4.12. *The canonical map ker* $\alpha \to B$*, that is induced from the diagram in Proposition 4.5 is the zero map.*

Proof. By using Lemma 4.11 this is shown in the same way as Lemma 2.14.

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By Lemma 4.12, Proposition 4.5, the snake lemma and the dimension argument used in the proof of Corollary 2.15 we see that $\mathrm{Im}(\exp) = H^1_f(G_K, V)$ is contained in the image of α. This finishes the proof of Theorem 4.1.

Finally Lemma 4.12 and the same methods as developed in Section 3 immediately imply Theorem 4.2.

4.13. Let $\mathrm{CH}^r(\mathcal{X})$, resp. $\mathrm{CH}^r(X)$ be the Chow group of cycles of codimension r on $\mathcal{X}$, resp. on X modulo rational equivalence. Since $\mathcal{X}$ as a semistable scheme is syntomic over O_K we have well-defined syntomic cycle class maps $\mathrm{cl}_{\mathrm{syn}}\colon \mathrm{CH}^r(\mathcal{X}) \to H^{2r}_{\mathrm{syn}}(\mathcal{X}, S_n^{[r]})$, where $S_n^{[r]}$ is the sheaf of Fontaine–Messing. They are induced from Chern class maps $c_{r,\mathrm{syn}}$ on K_0 via Grothendieck's comparison isomorphism between higher Chow groups and K-groups and the formula $\mathrm{cl}_{\mathrm{syn}} = ((-1)^{r-1}/(r-1)!)c_{r,\mathrm{syn}}$. If $\pi\colon (\mathcal{X})^{\sim}_{\mathrm{syn}} \to (\mathcal{X})^{\sim}_{\mathrm{et}}$ denotes the evident morphism of topoi, one has a canonical map of complexes in $D_{\mathrm{et}}(\mathcal{X})$ $R\pi_* S_n^{[r]} \to s_n^{\log}(r)$ by adding log poles ([Ka4], p. 286).

By composing this map with $\mathrm{cl}_{\mathrm{syn}}$ and passing to $\mathbb{Q}_p$-coefficients we get a log-syntomic cycle class $\mathrm{cl}^{\log}_{\mathrm{syn}}\colon \mathrm{CH}^r(\mathcal{X}) \to H^{2r}(\mathcal{X}, s^{\log}_{\mathbb{Q}_p}(r))$. We also have the étale cycle class map $\mathrm{cl}_{\mathrm{et}}\colon \mathrm{CH}^r(X) \to H^{2r}_{\mathrm{et}}(X, \mathbb{Q}_p(r))$.

Now let

$$\mathrm{CH}^r(X)_0 := \ker(\mathrm{CH}^r(X) \to H^{2r}_{\mathrm{et}}(\overline{X}, \mathbb{Q}_p(r))$$

and

$$\mathrm{CH}^r(\mathcal{X})_0 := \ker(\mathrm{CH}^r(\mathcal{X}) \to \mathrm{CH}^r(X) \to H^{2r}(\overline{X}, \mathbb{Q}_p(r)).$$

The map is obtained by composing $\mathrm{cl}_{\mathrm{et}}$ with the canonical map $H^{2r}_{\mathrm{et}}(X, \mathbb{Q}_p(r)) \to H^{2r}(\overline{X}, \mathbb{Q}_p(r))$. From the Hochschild–Serre spectral sequence we get a map also denoted by $\mathrm{cl}_{\mathrm{et}}\colon \mathrm{CH}^r(X)_0 \to H^1(G_K, V)$. If the reduction of $\mathcal{X}$ is ordinary in the sense of ([H1] (1.9)) we know that the spectral sequence of p-adic vanishing cycles degenerates at E_2 up to bounded torsion by ([H1] Thm. 1.10). This implies that the canonical map ($\overline{\mathcal{X}} = \mathcal{X} \times_{O_K} O_{\overline{K}}$), $H^{2r}(\overline{\mathcal{X}}, \tau_{\leqslant r} Rj_* \mathbb{Q}_p(r)) \to H^{2r}(\overline{X}, \mathbb{Q}_p(r))$ is injective. By ([Tsu] Thm. 3.3.4) we have an isomorphism $H^{2r}(\overline{\mathcal{X}}, s^{\log}_{\mathbb{Q}_p}(r)) \cong H^{2r}(\overline{\mathcal{X}}, \tau_{\leqslant r} Rj_* \mathbb{Q}_p(r))$. Assuming the compatibility of the log-syntomic cycle class (on $\overline{\mathcal{X}}$) with the étale cycle class map (on $\overline{X}$) under Kato's and Tsuji's comparison map we see that the above map $\mathrm{cl}^{\log}_{\mathrm{syn}}$ is – at least in the ordinary case – expected to induce by restriction a map $\mathrm{cl}^{\log}_{\mathrm{sn}}\colon \mathrm{Ch}^r(\mathcal{X})_0 \to \tilde{H}^{2r}(\mathcal{X}, s^{\log}_{\mathbb{Q}_p}(r))$. Then the desired compatibility of the log-syntomic cycle class with the étale cycle class map can be formulated in the following

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CONJECTURE. *The following diagram commutes*

$$\begin{array}{ccc} \mathrm{CH}^r(\mathcal{X})_0 & \xrightarrow{\text{can}} & \mathrm{CH}^r(X)_0 \\ \Big\downarrow{\scriptstyle \mathrm{cl}_{\mathrm{syn}}^{\log}} & & \Big\downarrow{\scriptstyle \mathrm{cl}_{\mathrm{et}}} \\ \tilde{H}^{2r}(\mathcal{X}, s_{\mathbb{Q}_p}^{\log}(r)) & \xrightarrow{\alpha} & H^1(G_K, V) \end{array}$$

If $\mathcal{X}$ is smooth and projective over O_K, i.e., in the good reduction case, a similar compatibility has been recently shown by Niziol [Ni]. Perhaps her methods do apply here to prove this Conjecture, which holds in fact for $r = 1$ by [Ka4], (5.6.4.)). Therefore, Theorem 4.1 gives – under the assumption (H) – a modest evidence that the image of the étale cycle class map c_{et} is contained in $H_f^1(G_K, V) = H_g^1(G_K, V)$. This expected relation would give some local support to a 'global' Conjecture of Bloch–Kato ([B-K], Sect. 5) on the motivic cohomology of a smooth projective variety over an algebraic number field. Of course the above Conjecture can also be formulated for the higher algebraic K-Theory of $\mathcal{X}$, resp. X and the more general results on log-syntomic cohomology proven in Section 2 may become useful after proving the compatibility Conjecture in general. All this will be further discussed elsewhere.

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References

[B-K] Bloch, S. and Kato, K.: L-functions and Tamagawa numbers of motives, In: *The Grothendieck Festschrift*, Part I (1990), 533–600.

[B-O] Berthelot, P., and Ogus, A.: *Notes on Crystalline Cohomology*, Princeton University Press, Princeton, 1978.

[Br] Breuil, C.: Cohomologie étale de p-torsion et cohomologie crystalline en réduction semistable, Prépublication, Orsay, 1997.

[E] Ekedahl, T.: On the A-dic formalism, In: *The Grothendieck Festschrift*, Vol. II, *Progr. Math.* 87, Birkhäuser, Boston, 1990, pp. 197–218.

[F-M] Fontaine, J. M. and Messing, W.: p-adic periods and p-adic étale cohomology, *Contemp. Math.* **67** (1987), 179–207.

[Fo1] Fontaine, J. M.: Le corps des périodes p-adiques, In: *Périodes p-adiques, Astérisque* **223** (1994), 57–111.

[Fo2] Fontaine, J. M.: Représentations p-adiques semi-stables, In: *Périodes p-adiques, Astérisque* **223** (1994), 113–184.

[H1] Hyodo, O.: A note on p-adic étale cohomology in the semistable reduction case, *Invent. Math.* **91** (1988), 543–557.

[H2] Hyodo, O.: On the de Rham–Witt complex attached to a semistable family, *Compositio Math.* **78** (1991), 241–260.

[H-K] Hyodo, O. and Kato, K.: Semistable reduction and crystalline cohomology with logarithmic poles, In: *Périodes p-adiques, Astérisque* **223** (1994), 221–268.

[Il] Illusie, L.: Réduction semistable ordinaire, cohomologie étale p-adique et cohomologie de de Rham d'après Bloch–Kato et Hyodo, In: *Périodes p-adiques, Astérisque* **223** (1994), 209–220.

[J1] Jannsen, U.: On the ℓ-adic cohomology of varieties over number fields and its Galois cohomology, In: *Galois Groups of* $\mathbb{Q}$, Math. Sci. Res. Inst. Publ., Springer, New York, 1989, pp. 315–360.

[J2] Jannsen, U.: *Mixed Motives and Algebraic K-Theory*, Lecture Notes in Math. 1400, Springer, New York, 1990.

[J3] Jannsen, U.: Continuous étale cohomology, *Math. Annal.* **280** (1988), 207–245.

[Ka1] Kato, K.: On p-adic vanishing cycles (Application of ideals of Fontaine–Messing), In: *Algebraic Geometry, Sendai* (1985), Adv. Studies Pure Math. 10, Kinokuniya, Tokyo, 1987, pp. 207–251.

[Ka2] Kato, K.: The explicit reciprocity law and the cohomology of Fontaine–Messing, *Bull. Soc. Math. France* **119** (1991), 397–441.

[Ka3] Kato, K.: Logarithmic structures of Fontaine–Illusie, In: *Algebraic Analysis, Geometry and Number Theory*, The Johns Hopkins University Press, 1989, pp. 191–224.

[Ka4] Kato, K.: Semistable reduction and p-adic étale cohomology, In: *Périodes p-adiques, Astérisque* **223** (1994), 269–293.

[K-M] Kato, K. and Messing, W.: Syntomic cohomology and p-adic étale cohomology, *Tohoku Math. J.* **44** (1992), 1–9.

[L-S] Langer, A. and Saito, S.: Torsion zero-cycles on the self-product of a modular elliptic curve, In: *Duke Math. J.* **85**(2) (1996).

[Mo] Mokrane, A.: La suite spectrale des poids en cohomologie de Hyodo–Kato, *Duke Math. J.* **72** (1993), 301–337.

[Ne1] Nekovář, J.: On p-adic height-pairings, In: *Seminaire de Théorie de Nombres 1990–1991* (1993), 127–202.

[Ne2] Nekovář, Syntomic cohomology and p-adic regulateurs, In preparation.

[Ni] Niziol, W.: On the image of p-adic regulators, *Invent. Math.* **127** (1997), 375–400.

[PR] Perrin-Riou, B.: Théorie d'Iwasawa des répresentations p-adiques sur un corps local, *Invent. Math.* **115** (1994), 81–149.

[S] Schneider, P.: p-adic points of motives, In: U. Jannsen, S. Kleiman, J.-P. Serre (eds), *Motives*, Proc. Sympos. Pure Math., Amer. Math. Soc., Providence, 1994.

[Tsu] Tsuji, T.: p-Adic étale cohomology and crystalline cohomology in the semistable reduction case, Preprint, 1996.