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GENERALISED BERNOULLI POLYNOMIALS AND SERIES

CLÉMENT FRAPPIER

We present several results related to the recently introduced generalised Bernoulli polynomials. Some recurrence relations are given, which permit us to compute efficiently the polynomials in question. The sums $\sum_{k=1}^{\infty} 1/j_k^{2r}, r = 1, 2, 3, \ldots$, where $j_k = j_k(\alpha)$ are the zeros of the Bessel function of the first kind of order α, are evaluated in terms of these polynomials. We also study a generalisation of the series appearing in the Euler-MacLaurin summation formula.

1. INTRODUCTION

Let $g_\alpha(z) := 2^\alpha \Gamma(\alpha+1) J_\alpha(z)/z^\alpha$, where

$$J_\alpha(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{\alpha+2k}}{2^{\alpha+2k} k! \Gamma(\alpha+k+1)}$$

is the Bessel function of the first kind, of order α. The zeros $j_k = j_k(\alpha)$, $k = \pm 1, \pm 2, \ldots$ are ordered such that $j_{-k} = -j_k$ and $0 < |j_1| < |j_2| < \cdots$. They are known [**7**, p.482] to be real for $\alpha > -1$. The α-Bernoulli polynomials $B_{n,\alpha}(x)$ are defined [**3**] by the generating function

$$\frac{e^{(x-1/2)z}}{g_\alpha(iz/2)} = \sum_{n=0}^{\infty} B_{n,\alpha}(x)(z^n/n!), \quad |z| < 2|j_1|. \tag{1}$$

The $B_{n,\alpha}(x)$ are rational functions of α. We have $B_{n,(1/2)}(x) = B_n(x)$, the classical Bernoulli polynomials, and $B_{n,-(1/2)}(x) = E_n(x)$, the Euler polynomials. The α-Bernoulli numbers are $B_{n,\alpha} := B_{n,\alpha}(0)$. Some basic properties of these polynomials are contained in the following [**3**]

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Proposition. *For any complex number α (not a negative integer) we have*

$$B'_{n,\alpha}(x) = nB_{n-1,\alpha}(x), \quad n = 1, 2, 3, \dots \tag{2}$$

$$B_{n,\alpha}(1-x) = (-1)^n B_{n,\alpha}(x), \quad n = 0, 1, 2, \dots \tag{3}$$

(In particular, $B_{n,\alpha}(1) = (-1)^n B_{n,\alpha}$ and $B_{2m+1,\alpha}(1/2) = 0, m = 0, 1, 2, \dots$)

$$B_{n,\alpha}(x+y) = \sum_{j=0}^{n} \binom{n}{j} B_{j,\alpha}(y) x^{n-j}, \quad n = 0, 1, 2, \dots . \tag{4}$$

(In particular, $B_{n,\alpha}(x) = \sum_{j=0}^{n} \binom{n}{j} B_{j,\alpha} x^{n-j}$.)

We note in [**3**] that the limiting case $\alpha \to \infty$ of (4) is nothing but the binomial formula

$$(A+B)^n = \sum_{j=0}^{n} \binom{n}{j} A^j B^{n-j}, \tag{5}$$

where $A = y - 1/2$ and $B = x$. This is a consequence of

$$\lim_{\alpha\to\infty} B_{n,\alpha}(x) = \left(x - \frac{1}{2}\right)^n . \tag{6}$$

In this paper we study in more detail these generalised Bernoulli polynomials. In the next section we present some recurrence relations which permit us to compute easily the new polynomials. In Section 3, we show how sums of the form $\sum_{k=1}^{\infty} 1/j_k^{2r}$, $r = 1, 2, 3, \dots$, can be written in terms of the $B_{n,\alpha}(x)$. Finally we examine, in Section 4, a generalisation of the classical Bernoulli series appearing in the Euler-MacLaurin summation formula.

2. Reccurence Relations

Formula (4), in conjunction with the recurrence relation

$$\sum_{k=0}^{m} \frac{2^{4k} B_{2k,\alpha}(1/2)}{(2k)!(m-k)!\Gamma(\alpha+m-k+1)} = 0, \quad m = 1, 2, \dots , \tag{7}$$

allows us to compute the α-Bernoulli polynomials (see [**3**, Section 4.1]). The following result gives a more direct approach. The notation $[a]$ signifies the integer part of the real number a.

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THEOREM 1. *Let α be a complex number, not a negative integer. We have the identity*

$$\sum_{k=0}^{[n/2]} \frac{n!\Gamma(\alpha+1)B_{n-2k,\alpha}(x)}{2^{4k}k!(n-2k)!\Gamma(\alpha+k+1)} = \left(x-\frac{1}{2}\right)^n. \tag{8}$$

PROOF: We write (1) in the form

$$e^{(x-1/2)z} = g_\alpha\left(\frac{iz}{2}\right)\sum_{l=0}^{\infty} B_{l,\alpha}(x)\frac{z^l}{l!} = \left(\sum_{m=0}^{\infty} c_m z^m\right)\left(\sum_{l=0}^{\infty} B_{l,\alpha}(x)\frac{z^l}{l!}\right), \tag{9}$$

where

$$c_m := \begin{cases} \dfrac{\Gamma(\alpha+1)}{2^{4k}k!\Gamma(\alpha+k+1)} & \text{if} \quad m=2k \\ 0 & \text{if} \quad m=2k+1. \end{cases}$$

Performing the Cauchy product in (9), we obtain

$$e^{(x-1/2)z} = \sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{c_m B_{n-m,\alpha}(x)}{(n-m)!}z^n = \sum_{n=0}^{\infty}\frac{(x-1/2)^n}{n!}z^n, \tag{10}$$

whence

$$\frac{(x-1/2)^n}{n!} = \sum_{m=0}^{n}\frac{c_m B_{n-m,\alpha}(x)}{(n-m)!}, \tag{11}$$

which readily gives (8). ∎

We may write formula (8) as

$$B_{n,\alpha}(x) = \left(x-\frac{1}{2}\right)^n - \sum_{k=1}^{n}\frac{n!\Gamma(\alpha+1)B_{n-2k,\alpha}(x)}{(n-2k)!k!2^{4k}\Gamma(\alpha+k+1)}. \tag{12}$$

Starting with $B_{0,\alpha}(x) = 1$, $B_{1,\alpha}(x) = x - 1/2$, we are then able to compute efficiently $B_{n,\alpha}(x)$ for $n \geq 2$. If we let $\alpha = 1/2$ in (8) and use the relation [4, p. 26]

$$\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}(2m)!}{2^{2m}m!}, \qquad m = 0, 1, 2, \dots, \tag{13}$$

then we get

$$\sum_{k=0}^{[n/2]}\frac{n!B_{n-2k}(x)}{(n-2k)!2^{2k}(2k+1)!} = \left(x-\frac{1}{2}\right)^n. \tag{14}$$

Differentiation of both sides of (14), in conjunction with (2) and (4), leads us to the well-known [**5**, p. 231] relation

$$B_n(x+1) - B_n(x) = nx^{n-1}, \qquad n = 1, 2, 3, \dots \tag{15}$$

(The notation in [5] is somewhat different of that generally used). We note also that formula (8), wherein $n = 2m$ and $x = 1/2$, becomes (7).

A small modification of the approach adopted in the proof of Theorem 1 leads us to a relation which has a slightly different character.

Theorem 1′. *Let α be a complex number, not a negative integer. We have the identity*

$$\sum_{k=0}^{n} \frac{\binom{n}{k}\Gamma(\alpha+1)\Gamma\left(\alpha+k+\frac{1}{2}\right) B_{n-k,\alpha}(x)}{2^k\Gamma\left(\alpha+\frac{k}{2}+1\right)\Gamma\left(\alpha+\frac{k+1}{2}\right)} = x^n. \tag{16}$$

Proof: We write (1) in the form

$$e^{xz} = e^{z/2}g_\alpha\left(\frac{iz}{2}\right)\sum_{l=0}^{\infty} B_{l,\alpha}(x)\frac{z^l}{l!} = \left(\sum_{m=0}^{\infty}\frac{d_m z^m}{2^m}\right)\left(\sum_{l=0}^{\infty} B_{l,\alpha}(x)\frac{z^l}{l!}\right), \tag{17}$$

where

$$e^z g_\alpha(iz) =: \sum_{m=0}^{\infty} d_m z^m. \tag{18}$$

Performing the Cauchy product in (17) yields

$$e^{xz} = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{d_k B_{n-k,\alpha}(x)}{2^k(n-k)!}z^n = \sum_{n=0}^{\infty}\frac{x^n}{n!}z^n, \tag{19}$$

whence

$$\sum_{k=0}^{n}\frac{d_k B_{n-k,\alpha}(x)}{2^k(n-k)!} = \frac{x^n}{n!}. \tag{20}$$

In order to evaluate the coefficients d_m, in (18), we use the representation [**7**, p. 48]

$$\frac{J_\alpha(z)}{z^\alpha} = \frac{1}{\sqrt{\pi}2^\alpha\Gamma(\alpha+1/2)}\int_{-1}^{1} e^{izt}\left(1-t^2\right)^{\alpha-1/2}dt, \tag{21}$$

for $Re(\alpha) > -1/2$ (obviously this restriction can be dropped in (16)). Replacing z by iz we obtain

$$\begin{aligned}
\left(e^z g_\alpha(iz)\right)^{(m)}(z=0) &= \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+1/2)} \int_{-1}^{1} (1-t)^m \left(1-t^2\right)^{\alpha-1/2} dt \\
&= \frac{2^{2\alpha+m}\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+1/2)} \int_0^1 u^{\alpha-1/2}(1-u)^{m+\alpha-1/2} du \\
&= \frac{2^{2\alpha+m}\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+1/2)} B\left(m+\alpha+\frac{1}{2}, \alpha+\frac{1}{2}\right) \\
&= \frac{2^{2\alpha+m}\Gamma(\alpha+1)\Gamma(m+\alpha+1/2)}{\sqrt{\pi}\Gamma(m+2\alpha+1)} \\
&= \frac{\Gamma(\alpha+1)\Gamma(m+\alpha+1/2)}{\Gamma\left(\alpha+\frac{m}{2}+1\right)\Gamma\left(\alpha+\frac{m+1}{2}\right)},
\end{aligned}$$

where the last step uses the duplication formula [**4**, p.29]

$$\sqrt{\pi}\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma(z+1/2). \tag{22}$$

So,

$$d_m = \frac{\Gamma(\alpha+1)\Gamma(\alpha+m+1/2)}{m!\Gamma\left(\alpha+\frac{m}{2}+1\right)\Gamma\left(\alpha+\frac{m+1}{2}\right)}, \tag{23}$$

and the result follows from (20). ∎

REMARK 1. We can also evaluate the coefficients d_m by performing the Cauchy product (see (9))

$$e^{z/2} g_\alpha\left(\frac{iz}{2}\right) = \left(\sum_{m=0}^{\infty} c_m z^m\right)\left(\sum_{l=0}^{\infty} \frac{z^l}{2^l l!}\right) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{c_k z^n}{(n-k)!2^{n-k}}.$$

We find that

$$d_n = \sum_{k=0}^{[n/2]} \frac{\Gamma(\alpha+1)}{2^{2k}k!(n-2k)!\Gamma(\alpha+k+1)} \tag{24}$$

and the comparison of (23) and (24) gives the identity

$$\sum_{k=0}^{[n/2]} \frac{n!}{k!(n-2k)!2^{2k}\Gamma(\alpha+k+1)} = \frac{\Gamma(\alpha+n+1/2)}{\Gamma\left(\alpha+\frac{n}{2}+1\right)\Gamma\left(\alpha+\frac{n+1}{2}\right)}. \tag{25}$$

The difference between Theorem 1 and Theorem 1′ can at once be seen if we let $\alpha \to \infty$. In (8), all the terms of the summation tend to zero, except the one corresponding to $k = 0$, and we get (6). In (16), it is possible to use Stirling's formula and we readily obtain the binomial formula (5) with $A = 1/2$, $B = x - 1/2$.

When $\alpha = m$ is a non-negative integer formula (16) may be written, with the help of (13), as

$$\sum_{k=0}^{n} \frac{\binom{n}{k}\binom{2m+2k}{k}}{2^{2k}\binom{m+k}{k}} B_{n-k,m}(x) = x^n. \tag{26}$$

When $\alpha = m + 1/2$, where m is again a non-negative integer, we find

$$\sum_{k=0}^{n} \frac{\binom{n}{k}}{\binom{2m+k+1}{m+1}} B_{n-k,m+1/2}(x) = \frac{x^n}{\binom{2m+1}{m+1}}. \tag{27}$$

The special case $m = 0\,(\alpha = 1/2)$ of (27) gives the known [**5**, p.249] formula

$$\sum_{k=0}^{n} \frac{B_{n-k}(x)}{(n-k)!(k+1)!} = \frac{x^n}{n!}. \tag{28}$$

If we take $m = 1\,(\alpha = 3/2)$ in (27) and differentiate three times then we obtain

$$\sum_{k=2}^{n} (k-2)\binom{n}{k} B_{n-k,3/2}(x) = \binom{n}{3} x^{n-3}. \tag{29}$$

On the other hand, differentiation of (4) with respect to y, where x has been first replaced by y and y by x, yields

$$\sum_{k=1}^{n} k\binom{n}{k} B_{n-k,3/2}(x) y^{k-1} = nB_{n-1,3/2}(x+y). \tag{30}$$

Using the last relation with $y = 1$, and (29), we get

$$\begin{aligned} \text{(31)}\quad nB_{n-1,3/2}(x+1) - 2B_{n,3/2}(x+1) + nB_{n-1,3/2}(x) + 2B_{n,3/2}(x) \\ = \binom{n}{3} x^{n-3}, \qquad n \geq 3. \end{aligned}$$

Formula (31) is for $\alpha = 3/2$ what (15) is for $\alpha = 1/2$. It reduces to

$$2B_{2N+1,3/2} = -(2N+1)B_{2N,3/2}, \qquad N \geq 2, \tag{32}$$

for $x = 0$ and $n = 2N + 1$. The relation (32) has been obtained by another method in [**3**, Section 4.3].

It may appear useless to give separately the relations of Theorem 1 and 1′ since the basic idea of the proofs is essentially the same. The relation (1) can indeed be written as

$$e^{(x-a)z} = e^{bz} g_\alpha\left(\frac{iz}{2}\right) \sum_{l=0}^{\infty} B_{l,\alpha}(x) \frac{z^l}{l!}, \tag{33}$$

with $a + b = 1/2$; the formulas (9) and (17) correspond respectively to the cases $a = 1/2$, $b = 0$ and $a = 0$, $b = 1/2$. Proceeding as in the remark following the proof of Theorem 1′, we find that

$$e^{bz} g_\alpha\left(\frac{iz}{2}\right) = \sum_{m=0}^{\infty} \sum_{p=0}^{m} \frac{c_p b^{m-p} z^m}{(m-p)!}, \tag{34}$$

which leads us to the apparently more general formula

$$\sum_{m=0}^{n} \sum_{k=0}^{[m/2]} \frac{n!\Gamma(\alpha+1) b^{m-2k} B_{n-m,\alpha}(x)}{2^{4k} k!(m-2k)!(n-m)!\Gamma(\alpha+k+1)} = (x-a)^n, \qquad n = 0, 1, 2, \dots, \tag{35}$$

with $a + b = 1/2$. However, permuting the order of summation in (35), and using the relation (4) in the form $\sum\limits_{j=0}^{n-2m} \binom{n-2m}{j} B_{j,\alpha}(x) b^{n-2m-j} = B_{n-2m,\alpha}(x+b)$, we obtain

$$\sum_{m=0}^{[n/2]} \frac{n!\Gamma(\alpha+1) B_{n-2m,\alpha}(x+b)}{2^{4m} m!(n-2m)!\Gamma(\alpha+m+1)} = (x-a)^n, \qquad a + b = \frac{1}{2}, \tag{36}$$

which is exactly formula (8) where x is replaced by $x+b$. Formula (35) is thus equivalent to Theorem 1. It is also indirectly equivalent to Theorem 1′ via the relation (25).

3. Generalised Riemann-Zeta Sums

It is well-known [**5**, p 244] that the sums $\sum\limits_{k=1}^{\infty} 1/k^{2r}$, $r = 1, 2, 3, \dots$, can be expressed in terms of the Bernoulli numbers:

$$\sum_{k=1}^{\infty} \frac{1}{k^{2r}} = \frac{(-1)^{r+1}(2\pi)^{2r}}{2(2r)!} B_{2r}. \tag{37}$$

We consider the more general sums

$$\sigma_\alpha(r) := \sum_{k=1}^{\infty} \frac{1}{j_k^{2r}}, \tag{38}$$

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where the series converges for $r = 1, 2, 3, \ldots$ since [**7**, p.506] $j_k \sim k\pi$, $k \to \infty$. Values of $\sigma_\alpha(r)$ are given in [**7**, p.502] for $r = 1, 2, 3, 4, 5$ and $r = 8$.

In this section we present a recurrence relation, easy to implement, which permits us to express $\sigma_\alpha(r)$ in terms of the α-Bernoulli polynomials. For instances, apart the values given in [**7**], we find that

$$\sigma_\alpha(6) = \frac{21\alpha^3 + 181\alpha^2 + 513\alpha + 473}{2^{11}(\alpha+1)^6(\alpha+2)^3(\alpha+3)^2(\alpha+4)(\alpha+5)(\alpha+6)},$$
$$\sigma_\alpha(7) = \frac{33\alpha^3 + 329\alpha^2 + 1081\alpha + 1145}{2^{12}(\alpha+1)^7(\alpha+2)^3(\alpha+3)^2(\alpha+4)(\alpha+5)(\alpha+6)(\alpha+7)},$$
$$\sigma_\alpha(9) = \frac{715\alpha^6 + 16567\alpha^5 + 158568\alpha^4 + 798074\alpha^3 + 2217079\alpha^2 + 3212847\alpha + 1893046}{2^{17}(\alpha+1)^9(\alpha+2)^4(\alpha+3)^3(\alpha+4)^2(\alpha+5)(\alpha+6)(\alpha+7)(\alpha+8)(\alpha+9)},$$

and so on. The link between the sums $\sigma_\alpha(r)$ and the α-Bernoulli polynomials is given by the following result.

THEOREM 2. *Let α be a complex number, not a negative integer. For each $n = 1, 2, 3, \ldots$, we have the identity*

$$(39)\quad B_{n+1,\alpha}(x) - \left(x - \frac{1}{2}\right) B_{n,\alpha}(x) = \sum_{r=0}^{[n-1/2]} \frac{(-1)^{r+1}(2r+1)!}{2^{2r+1}} \binom{n}{2r+1} B_{n-2r-1,\alpha}(x)\sigma_\alpha(r+1).$$

PROOF: We start with the factorisation of $J_\alpha(z)/z^\alpha$ [**7**, p.498], which can be written

$$(40)\qquad g_\alpha\left(\frac{iz}{2}\right) = \prod_{k=1}^{\infty}\left(1 + \frac{z^2}{4j_k^2}\right).$$

Taking the logarithmic derivative on both sides yields

(41)
$$\frac{\frac{i}{2}g'_\alpha(iz/2)}{g_\alpha(iz/2)} = \sum_{k=1}^{\infty} \frac{z}{2j_k^2\left(1 + \frac{z^2}{4j_k^2}\right)} = \sum_{k=1}^{\infty}\sum_{r=0}^{\infty} \frac{(-1)^r z^{2r+1}}{2^{2r+1} j_k^{2r+2}} = \sum_{r=0}^{\infty} \frac{(-1)^r \sigma_\alpha(r+1) z^{2r+1}}{2^{2r+1}},$$

for $|z| < 2|j_1|$. On the other hand, differentiation of both sides of (1), with respect to z, gives

$$(42)\qquad \sum_{n=1}^{\infty} B_{n+1,\alpha}(x)\frac{z^n}{n!} = \frac{(x-1/2)\,e^{(x-1/2)z}}{g_\alpha(iz/2)} - \frac{\frac{i}{2}g'_\alpha(iz/2)\,e^{(x-1/2)z}}{(g_\alpha(iz/2))^2},$$

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for $|z| < 2|j_1|$. Replacing respectively, in (42), $(i/2)g'_\alpha(iz/2)/g_\alpha(iz/2)$ and $\left(e^{(x-1/2)z}\right)/g_\alpha(iz/2)$ by the right-hand member of (41) and (1), we obtain

(43)

$$\sum_{n=0}^{\infty} B_{n+1,\alpha}(x)\frac{z^n}{n!} = \sum_{n=0}^{\infty}\left(x-\frac{1}{2}\right)B_{n,\alpha}(x)\frac{z^n}{n!} - \sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\frac{(-1)^r B_{m,\alpha}(x)\sigma_\alpha(r+1)}{2^{2r+1}m!}z^{m+2r+1},$$

whence

$$(44)\quad \sum_{n=1}^{\infty}\left(B_{n+1,\alpha}(x)-\left(x-\frac{1}{2}\right)B_{n,\alpha}(x)\right)\frac{z^n}{n!} = -\sum_{n=1}^{\infty}\sum_{r=0}^{[(n-1/2)]}\frac{(-1)^r B_{n-2r-1,\alpha}(x)}{2^{2r+1}(n-2r-1)!}\sigma_\alpha(r+1)z^n,$$

from which (39) follows immediately. ∎

REMARK 2. Formula (39) shows that $C_{n-1,\alpha}(x) := B_{n+1,\alpha}(x) - (x-1/2)\,B_{n,\alpha}(x)$ is a polynomial *of degree* $(n-1)$ whose leading coefficient is $-n/8(\alpha+1)$ (since $\sigma_\alpha(1) = 1/4(\alpha+1)$). We also note that $C'_{n-1,\alpha}(x) = nC_{n-2,\alpha}(x)$.

A possible approach to evaluating the sums $\sigma_\alpha(r)$, $r = 1, 2, \dots$, consists in taking $n = 2m+1$ and $x = 1/2$ in (39). It results in the recurrence relation

$$(45)\quad \frac{(-1)^{m+1}(2m+1)!}{2^{2m+1}}\sigma_\alpha(m+1) = B_{2m+2,\alpha}\left(\frac{1}{2}\right) + \sum_{r=0}^{m-1}\frac{(-1)^r(2r+1)!}{2^{2r+1}}\binom{2m+1}{2r+1}B_{2m-2r,\alpha}\left(\frac{1}{2}\right)\sigma_\alpha(r+1).$$

The numbers $B_{2j,\alpha}(1/2)$ are easily calculated with the help of (7). Starting with the initial value $\sigma_\alpha(1) = 1/4(\alpha+1)$, we can compute $\sigma_\alpha(m+1)$ for $m = 1, 2, \dots$. For instance, we have $B_{2,\alpha}(1/2) = \left(-1/8(\alpha+1)\right)$, $B_{4,\alpha}(1/2) = 3(\alpha+3)/\left(64(\alpha+1)^2(\alpha+2)\right)$, and it follows from (45) that $\sigma_\alpha(2) = 1/\left(2^4(\alpha+1)^2(\alpha+2)\right)$.

Another possibility is to take $x = 0$ in (39). When $n = 2m+1$ is odd we obtain

$$(46)\quad \frac{(-1)^{m+1}(2m+1)!}{2^{2m+1}}\sigma_\alpha(m+1) = B_{2m+2,\alpha} + \frac{1}{2}B_{2m+1,\alpha} + \sum_{r=0}^{m-1}\frac{(-1)^r(2r+1)!}{2^{2r+1}}\binom{2m+1}{2r+1}B_{2m-2r,\alpha}\sigma_\alpha(r+1),$$

for $m = 1, 2, \ldots$. For instance, we have

$$B_{2,\alpha} = \frac{2\alpha+1}{8(\alpha+1)},$$
$$B_{4,\alpha} = \frac{(2\alpha+1)(2\alpha^2+\alpha-7)}{64(\alpha+1)^2(\alpha+2)},$$
$$B_{5,\alpha} = \frac{-(2\alpha-1)(2\alpha^2-\alpha-13)}{128(\alpha+1)^2(\alpha+2)},$$
$$B_{6,\alpha} = \frac{(2\alpha+1)(4\alpha^4-69\alpha^2-32\alpha+213)}{512(\alpha+1)^3(\alpha+2)(\alpha+3)},$$

and it follows from (46) that

$$\sigma_\alpha(3) = \frac{1}{2^5(\alpha+1)^3(\alpha+2)(\alpha+3)}.$$

When $n = 2m+2$ is even we obtain

$$(47)\qquad \frac{(-1)^m(2m+2)!}{2^{2m+2}}\sigma_\alpha(m+1) = B_{2m+3,\alpha} + \frac{1}{2}B_{2m+2,\alpha} + \sum_{r=0}^{m-1}\frac{(-1)^r(2r+1)!}{2^{2r+1}}\binom{2m+2}{2r+1}B_{2m-2r+1,\alpha}\sigma_\alpha(r+1),$$

for $m = 1, 2, \ldots$. Formula (47) reduces at once to (37) when $\alpha = 1/2$ since $j_k(1/2) = k\pi$, $k = \pm1, \pm2, \ldots$ and $B_{2j+1} = 0$, $j = 1, 2, \ldots$.

4. Generalised Bernoulli Series

We consider now the integrals

$$(48)\qquad I_{n,\alpha} := \int_0^1 B_{n,\alpha}(t) f^{(n+1)}(t)\,dt,$$

for $n = 0, 1, 2, \ldots$. Integrating by parts, and recalling that $B_{n,\alpha}(1) = (-1)^n B_{n,\alpha}$, $B'_{n,\alpha}(t) = nB_{n-1,\alpha}(t)$, yield immediately

$$(49)\qquad I_{n,\alpha} = B_{n,\alpha}\left((-1)^n f^{(n)}(1) - f^{(n)}(0)\right) - nI_{n-1,\alpha}.$$

We infer that

$$(50)\qquad \frac{(-1)^n}{n!}I_{n,\alpha} = \sum_{p=0}^{n}\frac{B_{p,\alpha}}{p!}\left(f^{(p)}(1) - (-1)^p f^{(p)}(0)\right).$$

For $\alpha = 1/2$ and $n = 2m$,

$$\frac{1}{(2m)!} I_{2m,\alpha} = f(1) - f(0) - \frac{1}{2}\left(f'(1) + f'(0)\right) + \sum_{p=1}^{m} \frac{B_{2p}}{(2p)!}\left(f^{(2p)}(1) - f^{(2p)}(0)\right)$$

is the basic series of the Euler-MacLaurin summation formula [**5,** p.260] or [**4,** p.455].

4.1. If f is a polynomial of degree $\le n$ then $I_{n,\alpha} = 0$; it follows from (50) that $\sum_{p=0}^{n} \left(B_{p,\alpha}/p!\right)\left(f^{(p)}(1) - (-1)^p f^{(p)}(0)\right) = 0$. In general the α-Bernoulli series $\sum_{p=0}^{\infty} (B_{p,\alpha}/p!)\ \left(f^{(p)}(1) - (-1)^p f^{(p)}(0)\right)$ may not converge. We prove the following result.

Theorem 3. *Let α be a complex number, not a negative integer. For each entire function f of exponential type $\tau < 2|j_1|$ which is bounded on the real axis, we have*

$$f(1) = f(0) - \sum_{p=1}^{\infty} \frac{B_{p,\alpha}}{p!}\left(f^{(p)}(1) - (-1)^p f^{(p)}(0)\right). \tag{51}$$

Proof: If $|f(x)| \le M$ for $-\infty < x < \infty$ then, by Bernstein's inequality [**1**],

$$\left|f^{(k)}(x)\right| \le M\tau^k, \quad -\infty < x < \infty, \quad k = 0, 1, 2, \ldots \tag{52}$$

The integral remainder $r_{n,\alpha} := \left((-1)^n/n!\right) I_{n,\alpha}$, in (50), can thus be bounded as follows. We have

(52)

$$|r_{n,\alpha}| \le \frac{1}{n!}\int_0^1 |B_{n,\alpha}(t)|\left|f^{(n+1)}(t)\right| dt \le M\frac{\tau^{n+1}}{n!}\int_0^1 |B_{n,\alpha}(t)|\, dt \le K^* \frac{\tau^n}{(2|j_1|)^n}, \quad n > n^*,$$

where the last step uses the uniform asymptotic relation [**3,** p.314]

$$B_{n,\alpha}(t) \sim \frac{-n!\left(e^{(2t-1)ij_1} + (-1)^n e^{-(2t-1)ij_1}\right)}{2^\alpha \Gamma(\alpha+1)(2i)^n j_1^{n-\alpha+1} J'_\alpha(j_1)}, \tag{53}$$

for $0 \le t \le 1$, as $n \to \infty$ (there are some exceptional values of t but it does not matter here). So,

$$\lim_{n\to\infty} r_{n,\alpha} = 0 \qquad \text{if} \quad \tau < 2|j_1|, \tag{54}$$

and the result follows. ∎

The particular case $\alpha = 1/2$ of Theorem 3 gives the expansion [**2,** formula (9.5)] (see also [**6,** p. 220])

$$f(1) = f(0) + \frac{1}{2}\left(f'(1) + f'(0)\right) - \sum_{p=1}^{\infty} \frac{B_{2p}}{(2p)!}\left(f^{(2p)}(1) - f^{(2p)}(0)\right), \tag{55}$$

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where f is any entire function of exponential type $< 2\pi$, bounded on the real axis. The limiting case $\alpha \to \infty$ of (51), namely

$$f(1) = f(0) + \sum_{p=1}^{\infty} \left(f^{(p)}(0) - (-1)^p f^{(p)}(1) \right) \frac{(1/2)^p}{p!}, \tag{56}$$

is clearly a consequence of Taylor's formula.

4.2. The remainder $r_{n,\alpha} = \left((-1)^n/n!\right) I_{n,\alpha}$, in (50), can be expressed in a different way in certain circumstances. When $n = 2m$, $m = 1, 2, \ldots$, formula (50) can be written in the form

$$\sum_{p=0}^{2m-1} \frac{B_{p,\alpha}}{p!} \left(f^{(p)}(1) - (-1)^p f^{(p)}(0) \right) = \frac{1}{(2m)!} \int_0^1 (B_{2m,\alpha}(t) - B_{2m,\alpha}) f^{(2m+1)}(t)\, dt. \tag{57}$$

If the polynomial $B_{2m,\alpha}(x) - B_{2m,\alpha}$ is real for real x and has no root for $0 < x < 1$ then we can apply the mean value theorem for integrals to get

$$\sum_{p=0}^{2m-1} \frac{B_{p,\alpha}}{p!} \left(f^{(p)}(1) - (-1)^p f^{(p)}(0) \right) = \frac{1}{(2m)!} \int_0^1 (B_{2m,\alpha}(t) - B_{2m,\alpha})\, dt \cdot f^{(2m+1)}(c), \tag{58}$$

for some c in the interval $(0, 1)$. Since

$$\int_0^1 B_{2m,\alpha}(t)\, dt = \frac{B_{2m+1,\alpha}(1) - B_{2m,\alpha}(0)}{2m+1} = \frac{-2}{2m+1} B_{2m+1,\alpha}, \tag{59}$$

we see that (58) becomes

$$\begin{aligned} \sum_{p=0}^{2m-1} \frac{B_{p,\alpha}}{p!} & \left(f^{(p)}(1) - (-1)^p f^{(p)}(0) \right) \\ & = - \left(\frac{2}{(2m+1)!} B_{2m+1,\alpha} + \frac{1}{(2m)!} B_{2m,\alpha} \right) f^{(2m+1)}(c), \end{aligned} \tag{60}$$

for some $c \in (0, 1)$. This formula is obtained under the crucial assumption that $B_{2m,\alpha}(x) - B_{2m,\alpha}$ has the same sign for $0 < x < 1$. The assumption is not true for all α as can be seen with numerical calculations. Note also that, in view of (32), $\left(2/(2m+1)!\right) B_{2m+1,3/2} + \left(1/(2m)!\right) B_{2m,3/2} = 0$, $m \geq 2$; thus the representation (60) cannot hold in general for $\alpha = 3/2$. Based on empirical evidence, we make the following

CONJECTURE. Let $m \geq 1$ be an integer. For each real number α such that $-1 < \alpha \leq 1/2$, we have $(-1)^m B_{2m-1,\alpha}(x) > 0$ for $0 < x < 1/2$.

A standard argument shows that the conjecture is true for $\alpha = 1/2$ (see [**5**, p. 239]). This argument is not directly applicable in general since $B_{2m-1,\alpha}$ is not always equal to zero for $m \geq 2$, so that Rolle's theorem is not a priori sufficient to infer the result. Here are two consequences of the conjecture, the first one being relevant in (60).

CONSEQUENCE 1. The polynomial $B_{2m,\alpha}(x) - B_{2m,\alpha}$ has no root for $0 < x < 1$.

CONSEQUENCE 2. The polynomial $B_{2m+1,\alpha}(x) + (2x-1)B_{2m+1,\alpha}$ has no root for $0 < x < 1/2$.

That the conjecture implies the two consequences can be seen with a simple adaptation of the corresponding argument for $\alpha = 1/2$. We note also that Theorem 2 implies, for $\alpha > -1$, that

$$(61)\quad (-1)^{m+1}\left(B_{2m+1,\alpha}(x) - (x-1/2)\,B_{2m,\alpha}(x)\right) > 0 \ \text{ if } (-1)^k B_{2k-1,\alpha}(x) > 0,\ k = 1,2,\dots,m.$$

4.3. Considering a remark made in Section 3, according to which the polynomial $C_{n-1,\alpha}(x) = B_{n+1,\alpha}(x) - (x-1/2)\,B_{n,\alpha}(x)$ satisfies the relation $C'_{n-1,\alpha}(x) = nC_{n-2,\alpha}(x)$, we are led to consider, as a point of departure, the integrals

$$(62)\qquad J_{n,\alpha} := \int_0^1 C_{n-1,\alpha}(t) f^{(n)}(t)\,dt,$$

rather that (48). The same kind of arguments lead us to the formula

$$(63)\qquad \frac{(-1)^{n+1}}{n!} J_{n,\alpha} = \sum_{p=1}^{n} \frac{1}{p!}\left(B_{p+1,\alpha} + \frac{1}{2}B_{p,\alpha}\right)\left(f^{(p-1)}(1) + (-1)^p f^{(p-1)}(0)\right),$$

from which we deduce the following variant of Theorem 3.

THEOREM 3′. *Let α be a complex number, not a negative integer. For each entire function f of exponential type $\tau < 2|j_1|$ which is bounded on the real axis, we have*

$$(64)\qquad \sum_{p=1}^{\infty} \frac{1}{p!}\left(B_{p+1,\alpha} + \frac{1}{2}B_{p,\alpha}\right)\left(f^{(p-1)}(1) + (-1)^p f^{(p-1)}(0)\right) = 0.$$

4.4. It is of course possible to follow the spirit of the Euler-MacLaurin summation formula and apply the standard procedure of that context. Given the sequence of numbers $x_i = x_1 + (i-1)h$, $i = 1,2,\dots,q$, we find that

$$(65)\quad h\sum_{i=1}^{q-1} f'(x_i) = f(x_q) - f(x_1) - \frac{h}{2}\left(f'(x_q) - f'(x_1)\right) + \sum_{p=2}^{n}\sum_{i=1}^{q-1} \frac{B_{p,\alpha}}{p!} h^p \left(f^{(p)}(x_{i+1}) - (-1)^p f^{(p)}(x_i)\right) - r_{n,\alpha}(q),$$

where

$$r_{n,\alpha}(q) = \frac{(-1)^n}{n!} h^n \sum_{i=1}^{q-1} \int_{x_i}^{x_{i+1}} B_{n,\alpha}\left(\frac{t-x_i}{h}\right) f^{(n+1)}(t)\,dt. \tag{66}$$

The special case $\alpha = 1/2$ is the classical Euler-MacLaurin formula; the limiting case $\alpha \to \infty$ gives the summation formula

$$\begin{aligned} h\sum_{i=1}^{q-1} f'(x_i) &= f(x_q) - f(x_1) - \frac{h}{2}(f'(x_q) - f'(x_1)) \\ &\quad + \sum_{p=2}^{n}\sum_{i=1}^{q-1} \frac{1}{p!}\left(\frac{h}{2}\right)^p \left((-1)^p f^{(p)}(x_{i+1}) - f^{(p)}(x_i)\right) - r_{n,\infty}(q), \end{aligned} \tag{67}$$

where

$$r_{n,\infty}(q) = \frac{(-1)^n}{n!} h^n \sum_{i=1}^{q-1} \int_{x_i}^{x_{i+1}} \left(\frac{t-x_i}{h} - \frac{1}{2}\right)^n f^{(n+1)}(t)\,dt. \tag{68}$$

If we take $n = 2m$ then (65) can be written in the form
(69)

$$2\sum_{p=1}^{m}\sum_{i=1}^{q-1} \frac{B_{2p-1,\alpha}}{(2p-1)!} h^{2p-1} f^{(2p-1)}(x_i) = -\sum_{p=0}^{2m} \frac{B_{p,\alpha}}{p!} h^p \left(f^{(p)}(x_q) - f^{(p)}(x_1)\right) + r_{2m,\alpha}(q).$$

For an arbitrary sequence of numbers x_i, $1 \le i \le q$, we find the more general formula

$$\begin{aligned} &2\sum_{p=1}^{m}\sum_{i=1}^{q-1} \frac{B_{2p-1,\alpha}}{(2p-1)!}(x_{i+1}-x_i)^{2p-1} f^{(2p-1)}(x_i) \\ &\quad + \sum_{p=0}^{2m}\sum_{i=1}^{q-1} \frac{B_{p,\alpha}}{p!}(x_{i+1}-x_i)^p \left(f^{(p)}(x_{i+1}) - f^{(p)}(x_i)\right) \\ &\quad = \sum_{i=1}^{q-1} \frac{(x_{i+1}-x_i)^{2m}}{(2m)!} \int_{x_i}^{x_{i+1}} B_{2m,\alpha}\left(\frac{t-x_i}{x_{i+1}-x_i}\right) f^{(2m+1)}(t)\,dt. \end{aligned} \tag{70}$$

5. Concluding Remarks

5.1. Some additional computations, using the generating function (1) and the Faa Di Bruno formula, lead us to the following explicit expression for the α-Bernoulli polynomials:

$$B_{n,\alpha}(x) = \left(x - \frac{1}{2}\right)^n + \sum_{s=1}^{[n/2]} \frac{(-1)^s (2s)! \binom{n}{2s} P_{N(s)}(\alpha)}{2^{4s} s! \prod_{\nu=1}^{s} (\alpha+\nu)^{[s/\nu]}} \left(x - \frac{1}{2}\right)^{n-2s}, \tag{71}$$

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where

(72)
$$P_{N(s)}(\alpha) := \prod_{\nu=1}^{s} (\alpha+\nu)^{[s/\nu]} \cdot \sum_{r=1}^{s} \sum_{\pi(s,r)} (-1)^{r+s} r!\ c(k_1,\dots,k_s) \cdot \prod_{\nu=1}^{s} \left(\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\nu+1)} \right)^{k_\nu}$$

is a polynomial in α of degree $N(1) := 0$ and $N(s) := \sum_{j=2}^{s} [s/j]$, $s = 2, 3, \dots$. The notation $\pi(s,r)$ means that the summation is extended over all non-negative integers $k_1, \dots, k_s$ such that $k_1 + 2k_2 + \dots + sk_s = s$, $k_1 + k_2 + \dots + k_s = r$, and $c(k_1,\dots,k_s) := \left(s!/k_1!\dots k_s!(1!)^{k_1}\dots(s!)^{k_s}\right)$. The polynomials $P_{N(s)}(\alpha)$ appear in the MacLaurin expansion of the function $1/\left(g_\alpha(z)\right)$, that is,

$$\frac{1}{g_\alpha(z)} = 1 + \sum_{s=1}^{\infty} \frac{P_{N(s)}(\alpha) z^{2s}}{2^{2s} s! \prod_{\nu=1}^{s} (\alpha+\nu)^{[s/\nu]}}. \tag{73}$$

For instances, $P_0(\alpha) = 1$, $P_1(\alpha) = \alpha + 3$, $P_2(\alpha) = \alpha^2 + 8\alpha + 19$ and $P_4(\alpha) = \alpha^4 + 17\alpha^3 + 117\alpha^2 + 379\alpha + 422$.

5.2. Formula (4), wherein $x = y = 1/2$, can be applied with $n = 2m$ and $n = 2m + 1$. We obtain respectively $B_{2m,\alpha} = \sum_{j=0}^{m} \binom{2m}{2j} B_{2j,\alpha}(1/2)(1/2)^{2m-2j}$ and $B_{2m+1,\alpha} = -\sum_{j=0}^{m} \binom{2m+1}{2j} B_{2j,\alpha}(1/2)(1/2)^{2m+1-2j}$. Since $\binom{2m+1}{2j} = \binom{2m}{2j} + \binom{2m}{2j-1}$ we see that the coefficients appearing in formula (63) can be written as

$$B_{2m+1,\alpha} + \frac{1}{2} B_{2m,\alpha} = -\sum_{j=1}^{m} \binom{2m}{2j-1} B_{2j,\alpha}\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^{2m+1-2j}. \tag{74}$$

5.3. We add another observation concerning the sign of our quantities. We have

$$(-1)^m B_{2m,\alpha}\left(\frac{1}{2}\right) > 0 \quad \text{if} \quad \alpha > -1. \tag{75}$$

This is a consequence of (45) via an easy mathematical induction.

5.4. Finally we mention two examples to illustrate formula (69) where, in both cases, $x_1 = h = 1$ that is, $x_i = i$, and also $q \to \infty$. Firstly we take $f(x) = \ln(x)$; this gives us, for $m = 2, 3, \dots$,

$$2\sum_{p=2}^{m} \frac{B_{2p-1,\alpha}}{(2p-1)} \zeta(2p-1) - \gamma = \sum_{p=1}^{2m} \frac{(-1)^{p-1}}{p} B_{p,\alpha} + \int_0^1 B_{2m,\alpha}(t) \sum_{i=1}^{\infty} \frac{1}{(t+i)^{2m+1}}\, dt, \tag{76}$$

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where γ is Euler constant. Secondly we take $f(x) = x^{-s}$, $Re(s) > 0$; we obtain, for $m = 1, 2, \ldots$,

$$(77)\quad 2\sum_{p=1}^{m} \frac{B_{2p-1,\alpha}}{(2p-1)!}\Gamma(s+2p-1)\zeta(s+2p-1) + \sum_{p=0}^{2m} \frac{(-1)^p}{p!} B_{p,\alpha}\Gamma(s+p) = \frac{\Gamma(s+2m+1)}{(2m)!}\int_0^1 B_{2m,\alpha}(t)\sum_{i=1}^{\infty}\frac{1}{(t+i)^{s+2m+1}}\,dt.$$

Formula (76) readily turns out to be the limiting case $s \to 0$ of (77).

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