# A PACKING PROBLEM

R.G. Stanton, H.C. Williams, C.R. Zarnke

(received August 27, 1966)

1. Introduction. Consider three externally tangent circles A, B, C, with radii a, b, c, respectively. Let D be the circle which is externally tangent to all three and enclosed by them. Pack circles so that $Z_1 = D$ and $Z_i$ is externally tangent to A, B, $Z_{i-1}$. Similarly $Y_1 = D$; $Y_i$ is externally tangent to A, C, $Y_{i-1}$. Also $X_1 = D$; $X_i$ is externally tangent to B, C, $X_{i-1}$. [see Fig. 1]. We use $x_i, y_i, z_i, d$, for the radii of $X_i, Y_i, Z_i$, D, respectively.

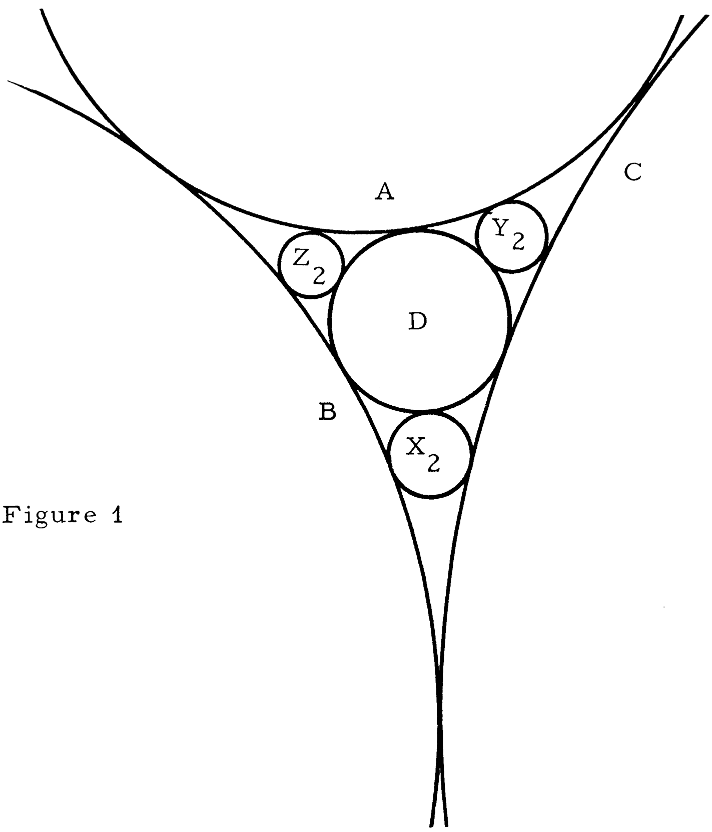


Figure 1



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We obtained and tabulated the ratio, $R(a, b, c)$, of the packed area, namely, $D + \sum_{2}^{\infty} (X_i + Y_i + Z_i)$ to the total area of the circular triangle bounded by $A, B, C$.

2. Method. It is well known (see, for example, [1], p. 16) that

$$d = \frac{abc}{bc+ca+ab+2\Delta},$$

where $\Delta^2 = abc(a+b+c)$. Hence

$$x_0 = a, \quad x_{i+1} = \frac{x_i bc}{bc+x_i b+x_i c+2\sqrt{x_i bc(x_i+b+c)}}.$$

This difference equation may be put into the following form:

$$\frac{1}{x_{i+1}} = \frac{1}{x_i} + \frac{b+c}{bc} + 2\sqrt{\frac{1}{bc} + \left(\frac{b+c}{bc}\right)\frac{1}{x_i}}.$$

Set $\frac{1}{x_i} = \frac{b+c}{bc} w_i^2 - \frac{1}{b+c}$; we obtain $w_{i+1}^2 = (1 + w_i)^2$. Since $w_i > 0$ for all $i$, we obtain

$$w_{i+1} = 1 + w_i, \quad w_i = i + K,$$

where $K$ is a constant determined by the initial conditions. Putting $i = 0$, we obtain

$$w_0^2 = K^2 = \frac{bc}{b+c}\left(\frac{1}{a} + \frac{1}{b+c}\right).$$

Hence $$x_i = \frac{1}{\frac{b+c}{bc} i^2 + 2i\sqrt{\frac{a+b+c}{abc}} + \frac{1}{a}}.$$

Similarly, $y_i$ and $z_i$ may be written down by applying the cyclic permutation $(abc)$ once and twice respectively to $x_i$. Using these results, we end up with the formula

$$R(a, b, c) = T^{-1} \pi[d^2 + \sum_{i=2}^{\infty} (x_i^2 + y_i^2 + z_i^2)] ,$$

where

$$T = \Delta - a^2 \tan^{-1} \frac{\Delta}{a(a+b+c)} - b^2 \tan^{-1} \frac{\Delta}{b(a+b+c)} - c^2 \tan^{-1} \frac{\Delta}{c(a+b+c)} .$$

The referee has kindly pointed out that this result has just appeared in Melzak, Z.A., Infinite Packings of Disks, C.J.M. 18 (1966), 838-852.

3. Special Cases. Clearly

$$R(a, b, c) = R(Ka, Kb, Kc) = R(a, c, b) = R(c, a, b), \quad \text{etc.},$$

for $K$ any positive constant of similarity. Keeping this fact in mind, we may discuss five cases: (1) $R(1, \infty, \infty)$; (2) $R(1, 1, \infty)$; (3) $R(1, 1, 1)$; (4) $R(1, b, c)$; (5) $R(1, b, \infty)$ .

CASE 1. $R(1, \infty, \infty)$ .

We have a unit circle with two parallel tangents; we obtain $x_i = 1$ , $y_i = z_i = i^{-2}$ . Thus

$$F(1, \infty, \infty) = \lim_{n \to \infty} \frac{n\pi + 2 \sum_{i=2}^{\infty} i^{-4}}{2 - \frac{\pi}{2} + 4n} = \frac{\pi}{4} \approx .785398 .$$

This is of course clear geometrically from consideration of the ratio of the area of a disk to that of its circumscribed square.

CASE 2. $R(1, 1, \infty)$ .

We have

$$x_i = y_i = \frac{1}{i^2 + 2i + 1} = \frac{1}{(i+1)^2} , \quad z_i = \frac{1}{2i(i+1)} .$$

Since $T = 2 - \frac{\pi}{2}$ , we obtain

$$(2 - \frac{\pi}{2}) R(1, 1, \infty) = 2 \pi \sum_{i=2}^{\infty} \frac{1}{(i+1)^4} + \frac{\pi}{4} \sum_{i=1}^{\infty} (\frac{1}{i} - \frac{1}{i+1})^2$$



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$$= 2\pi\left(\frac{\pi^4}{90} - \frac{17}{16}\right) + \frac{\pi}{4}\sum_{i=1}^{\infty}\left\{\frac{1}{i^2} + \frac{1}{(i+1)^2} - \frac{2}{i(i+1)}\right\}$$

$$= 2\pi\left(\frac{\pi^4}{90} - \frac{17}{16}\right) + \frac{\pi}{4}\left\{\frac{\pi^2}{6} + \frac{\pi^2}{6} - 1 - 2\right\} .$$

Thus

$$R(1, 1, \infty) = \frac{\pi\left(\frac{\pi^4}{45} - \frac{17}{8}\right) + \frac{\pi}{4}\left(\frac{\pi^2 - 9}{3}\right)}{2 - \frac{\pi}{2}}$$

$$\approx .820624 .$$

CASE 3. $R(1, 1, 1)$ .

We set $\beta = (\sqrt{3} - 1) / 2$ and obtain

$$x_i = y_i = z_i = \frac{1}{2i^2 + 2\sqrt{3}i + 1} = \frac{1}{2}\left\{\frac{1}{i+\beta} - \frac{1}{i+\beta+1}\right\} .$$

Now

$$\pi\sum_{i=1}^{\infty} x_i^2 = \frac{\pi}{4}\sum_{i=1}^{\infty}\left[\frac{1}{(i+\beta)^2} + \frac{1}{(i+\beta+1)^2} - \frac{2}{(i+\beta)(i+\beta+1)}\right]$$

and

$$\sum_{i=1}^{\infty}\frac{1}{(i+\beta)(i+\beta+1)} = \frac{1}{1+\beta} ;$$

therefore,

$$\pi\sum_{i=1}^{\infty} x_i^2 = \frac{\pi}{4}\left\{2\sum_{i=1}^{\infty}\frac{1}{(i+\beta)^2} - \frac{1}{(1+\beta)^2} - \frac{2}{1+\beta}\right\}$$

Since $T = \sqrt{3} - \pi/2$ , we have

$$R(1, 1, 1) = \frac{\pi}{4T}\left\{6\sum_{i=1}^{\infty}\frac{1}{(i+\beta)^2} - \frac{3}{(1+\beta)^2} - \frac{6}{1+\beta} - \frac{2}{(1+\beta)^2(2+\beta)^2}\right\} .$$

$$\approx .822206 .$$



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CASES 4 AND 5. $R(1, b, c)$ , $R(1, b, \infty)$ .

Originally the results for Cases 1, 2, and 3 were worked out from geometric considerations before the general formula for $R(a, b, c)$ was obtained. The sequences of radii and of abscissae and ordinates of centres thus obtained are quite interesting in their own right. For Cases 4 and 5, values of $R$ were computed on an I.B.M. 7040 computer for all values of $b$ and $c$ $(b < c)$ at unit intervals from $b = c = 1$ up to $b = c = 150$ .

The series for $R(a, b, c)$ is not very rapidly convergent; indeed, round-off error from many terms tended to pile up. In order to obtain accurate results, the series was evaluated by adding the first 98 terms together and then employing a correction term (see, for example, [2], p.129) obtained from the Euler-Maclaurin Series.

For most of the entries in the table, the correction term began to have an effect in the third or fourth decimal place. The results of the calculation, to six figures of accuracy, are recorded in Table I for Case 4; the similar results for Case 5 are recorded in Table II.

As indicated above, the tables represent only a portion of the computations performed. We have tabulated the results for $b$ and $c$ at unit intervals from 1 to 10 and thereafter at intervals of 10 from 10 to 150.

It might be useful to record here a very useful alternative form of $x_i$ which has already been employed, namely,

$$x_i = \frac{\sqrt{bc}}{2} \left[\frac{1}{i+\alpha_1} - \frac{1}{i+\alpha_2}\right]$$

where

$$\alpha_1 = \frac{\Delta - a\sqrt{bc}}{a(b+c)} = \frac{\sqrt{p}}{a(\sqrt{s}+\sqrt{a})} , \quad \alpha_2 = \frac{\Delta + a\sqrt{bc}}{a(b+c)} = \frac{\sqrt{p}}{a(\sqrt{s}-\sqrt{a})} ,$$

with $p = abc$ , $s = a+b+c$ . The vaules of $y_i$ and $z_i$ are successively obtained by applying the cyclic permutation $(abc)$ .



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TABLE I

| b \ c | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | .822206 | | | | |
| 2 | .821959 | .821765 | | | |
| 3 | .821704 | .821430 | .820966 | | |
| 4 | .821522 | .821164 | .820586 | .820103 | |
| 5 | .821390 | .820960 | .820287 | .819718 | .819258 |
| 6 | .821291 | .820802 | .820051 | .819409 | .818885 |
| 7 | .821214 | .820676 | .819860 | .819157 | .818579 |
| 8 | .821153 | .810574 | .819704 | .818948 | .818323 |
| 9 | .821103 | .820489 | .819573 | .818773 | .818106 |
| 10 | .821062 | .820418 | .819462 | .818623 | .817921 |
| 20 | .820859 | .820058 | .818887 | .817830 | .816923 |
| 30 | .820785 | .819920 | .818661 | .817513 | .816517 |
| 40 | .820746 | .819848 | .818541 | .817342 | .816296 |
| 50 | .820723 | .819803 | .818467 | .817236 | .816157 |
| 60 | .820707 | .819773 | .818416 | .817163 | .816062 |
| 70 | .820695 | .819751 | .818379 | .817110 | .815993 |
| 80 | .820686 | .819734 | .818351 | .817070 | .815941 |
| 90 | .820680 | .819721 | .818329 | .817038 | .815899 |
| 100 | .820674 | .819702 | .818312 | .817013 | .815838 |
| 110 | .820670 | .819711 | .818297 | .816992 | .815838 |
| 120 | .820666 | .819695 | .818285 | .816974 | .815815 |
| 130 | .820663 | .819689 | .818275 | .816959 | .815795 |
| 140 | .820660 | .819683 | .818266 | .816946 | .815778 |
| 150 | .820658 | .819679 | .818258 | .816935 | .815764 |



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TABLE I continued

| b \ c | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|
| 1 | | | | | |
| 2 | | | | | |
| 3 | | | | | |
| 4 | | | | | |
| 5 | | | | | |
| 6 | .818457 | | | | |
| 7 | .818103 | .817706 | | | |
| 8 | .817805 | .817372 | .817004 | | |
| 9 | .817552 | .817086 | .816690 | .816349 | |
| 10 | .817334 | .816839 | .816317 | .816053 | .815736 |
| 20 | .816147 | .815479 | .814898 | .814388 | .813938 |
| 30 | .815655 | .814907 | .814250 | .813670 | .813154 |
| 40 | .815386 | .814592 | .813891 | .813270 | .812714 |
| 50 | .815217 | .814392 | .813663 | .813015 | .812433 |
| 60 | .815100 | .814254 | .813506 | .812837 | .812237 |
| 70 | .815015 | .814154 | .813390 | .812707 | .812093 |
| 80 | .814950 | .814077 | .813301 | .812608 | .811983 |
| 90 | .814899 | .814016 | .813232 | .812529 | .811896 |
| 100 | .814858 | .813967 | .813175 | .812465 | .811825 |
| 110 | .814824 | .813927 | .813129 | .812413 | .811767 |
| 120 | .814795 | .813893 | .813089 | .812368 | .811717 |
| 130 | .814771 | .813864 | .813056 | .812331 | .811675 |
| 140 | .814750 | .813839 | .813027 | .812298 | .811639 |
| 150 | .814731 | .813817 | .813002 | .812270 | .811608 |



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TABLE I continued

| b \ c | 20 | 30 | 40 | 50 | 60 |
|---|---|---|---|---|---|
| 20 | .811243 | | | | |
| 30 | .810074 | .808427 | | | |
| 40 | .809240 | .807498 | .806436 | | |
| 50 | .808755 | .806877 | .805717 | .804924 | |
| 60 | .808412 | .806431 | .805196 | .804345 | .803721 |
| 70 | .808155 | .806095 | .804801 | .803903 | .803241 |
| 80 | .807957 | .805833 | .804490 | .803555 | .802861 |
| 90 | .807799 | .805623 | .804240 | .803272 | .802552 |
| 100 | .807669 | .805450 | .804034 | .803038 | .802296 |
| 110 | .807562 | .805306 | .803860 | .802842 | .802080 |
| 120 | .807471 | .805184 | .803713 | .802674 | .801895 |
| 130 | .807393 | .805079 | .803586 | .802529 | .801735 |
| 140 | .807326 | .804988 | .803476 | .802403 | .801595 |
| 150 | .807267 | .804908 | .803379 | .802291 | .801472 |
| | | | | | |
| b \ c | 70 | 80 | 90 | 100 | 110 |
| 70 | .802731 | | | | |
| 80 | .802325 | .801897 | | | |
| 90 | .801994 | .801547 | .801181 | | |
| 100 | .801718 | .801255 | .800875 | .800556 | |
| 110 | .801485 | .801008 | .800615 | .800286 | .800005 |
| 120 | .801286 | .800796 | .800392 | .800053 | .799763 |
| 130 | .801113 | .800612 | .800198 | .799850 | .799553 |
| 140 | .800962 | .800450 | .800027 | .799671 | .799367 |
| 150 | .800828 | .800307 | .799876 | .799513 | .799203 |
| | | | | | |
| b \ c | 120 | 130 | 140 | 150 | |
| 120 | .799514 | | | | |
| 130 | .799296 | .799072 | | | |
| 140 | .799104 | .798874 | .798671 | | |
| 150 | .798934 | .798698 | .798491 | .798306 | |



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TABLE II

| b \ c | |
|---|---|
| 1 | .820624 |
| 2 | .819614 |
| 3 | .818148 |
| 4 | .816775 |
| 5 | .815552 |
| 6 | .814469 |
| 7 | .813504 |
| 8 | .812639 |
| 9 | .811858 |
| 10 | .811148 |
| 20 | .806390 |
| 30 | .803691 |
| 40 | .801876 |
| 50 | .800540 |
| 60 | .799501 |
| 70 | .798661 |
| 80 | .797962 |
| 90 | .797368 |
| 100 | .796855 |
| 110 | .796406 |
| 120 | .796008 |
| 130 | .795652 |
| 140 | .795331 |
| 150 | .795040 |

## 4. Conjectures.

From the work included in this paper and the further tables computed by the authors, it seems likely that for $a$ and $b$ fixed and less than $c$, then $c_1 < c_2$ implies $R(a, b, c_1)$ greater than $R(a, b, c_2)$. The complexity of the expression for $R(a, b, c)$, however, has prevented us from giving a proof in the present paper.

The referee has drawn our attention to a stronger form of this conjecture, namely: if $a$ and $b$ are fixed, then $R(a, b, c)$ is a function of $c$ which attains its maximum for $c$ between



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a and b . For example, we read off from the tables the following values for R(1, 10, c) .

| c | 0 | 1/10 | 1/8 | 1/5 | 1/3 |
|---|---|---|---|---|---|
| R(1, 10, c) | .785398 | .811825 | .813301 | .816157 | .818661 |

Note that R(1, 10, 0) = R(1, ∞, ∞) , R(1, 10, 1/10) = R(1, 10, 100), R(1, 10, 1/8) = R(1, 8, 80) , etc.

| c | 1/2 | 1 | 2 | 5 | 10 |
|---|---|---|---|---|---|
| R(1, 10, c) | .820058 | .821062 | .820418 | .817921 | .815736 |

| c | 50 | 100 | 150 | ∞ |
|---|---|---|---|---|
| R(1, 10, c) | .812433 | .811825 | .811608 | .811148 |

If we tabulate R(1, 10, c) for values of c from .50 to 1.49, we can find the location of the c-value which makes R a maximum. The tabulation, in part, is given below.

| | | | |
|---|---|---|---|
| a = 1, b = 10, c = | .50 | R(1, 10, c) = | .8200576 |
| | .60 | | .8204936 |
| | .70 | | .8207663 |
| | .80 | | .8209314 |
| | .90 | | .8210228 |
| | 1.00 | | .8210623 |
| | 1.01 | | .8210640 |
| | 1.02 | | .8210653 |
| | 1.03 | | .8210663 |
| | 1.04 | | .8210669 |
| | 1.05 | | .8210673 |
| | 1.06 | | .8210673 |
| | 1.07 | | .8210670 |
| | 1.08 | | .8210664 |
| | 1.09 | | .8210655 |
| | 1.10 | | .8210644 |
| | 1.20 | | .8210393 |
| | 1.30 | | .8209940 |
| | 1.40 | | .8209338 |

We thus see that R(1, 10, c) attains a maximum for c = 1.055 .

In a similar way, we can tabulate R(1, 1, c) , R(1, 2, c) ,



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R(1, 3, c) , ... , R(1, 10, c) . We obtain the following results, which certainly support the conjecture.

| | | | | |
|---|---|---|---|---|
| a = 1, b = 1 | R(1, b, c) = | .8222063 | occurs for c = | 1.000 |
| 2 | | .8219811 | | 1.160 |
| 3 | | .8217257 | | 1.150 |
| 4 | | .8215391 | | 1.120 |
| 5 | | .8214033 | | 1.105 |
| 6 | | .8213016 | | 1.090 |
| 7 | | .8212228 | | 1.075 |
| 8 | | .8211603 | | 1.070 |
| 9 | | .8211094 | | 1.060 |
| 10 | | .8210673 | | 1.055 |

Another consequence of the conjecture would be that R(a, b, c) attains its absolute maximum when a = b = c . We notice that .822206 is in fact the largest entry in Table I .

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2. R.G. Stanton, Numerical Methods for Science and Engineering. Prentice-Hall, Englewood Cliffs, 1961.

University of Manitoba