10

Ancillary Results

In this chapter, we discuss various results that were used earlier and for which good references are scarce or scattered. We work over an arbitrary base scheme, whenever possible.

10.1 S_2 Sheaves

Definition 10.1 Let F be a quasi-coherent sheaf on a scheme X. Its *annihilator,* denoted by $\operatorname{Ann}(F)$, is the largest ideal sheaf $I \subset \mathscr{O}_X$ such that $I \cdot F = 0$. The *support* of F is the zero set $Z(I) \subset X$, denoted by $\operatorname{Supp} F$.

The *dimension* of F at a point x, denoted by $\dim_x F$, is the dimension of its support at x. The dimension of F is $\dim F := \dim \operatorname{Supp} F$.

The set of all associated points (or primes) of a quasi-coherent sheaf F is denoted by $\operatorname{Ass}(F)$. An associated point of F is called *embedded* if it is contained in the closure of another associated point of F. Let $\operatorname{emb}(F) \subset F$ denote the largest subsheaf whose associated points are all embedded points of F. Thus $F/\operatorname{emb}(F)$ has no embedded points, hence it is S_1 (10.5). Informally speaking, $F \mapsto F/\operatorname{emb}(F)$ is the best way to associate an S_1 sheaf to another sheaf.

If F is coherent then it has only finitely many associated points and $\operatorname{Supp} F$ is the union of their closures.

Let $Z \subset X$ be a closed subscheme. Then $\operatorname{tors}_Z(F) \subset F$ denotes the Z-*torsion* subsheaf, consisting of all local sections whose support is contained in Z. There is a natural isomorphism $\operatorname{tors}_Z(F) \simeq \mathcal{H}^0_Z(X, F)$.

If X has a dimension function (see the Assumptions on p.347), then we use $\operatorname{tors}(F) \subset F$ to denote the *torsion* subsheaf, consisting of all local sections whose support has dimension $< \dim \operatorname{Supp} F$. A coherent sheaf F is called *pure*

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(of dimension n) if (the closure of) every associated point of F has dimension n. Thus $\mathrm{pure}(F) := F/\mathrm{tors}(F)$ is the maximal *pure quotient* of F. A scheme is pure iff its structure sheaf is.

If $\mathrm{Supp}\, F$ is pure dimensional, then $\mathrm{emb}(F) = \mathrm{tors}(F)$.

Let $f: X \to S$ be of finite type and F a coherent sheaf on X such that F_s is pure for every $s \in S$. Then the same holds after any base change $S' \to S$.

Warning. If X is pure dimensional, F is coherent and $\dim F = \dim X$, then our terminology agrees with every usage of "torsion" that we know of. However, the above distinction between $\mathrm{emb}(F)$ and $\mathrm{tors}(F)$ is not standard.

10.2 (Regular sequences and depth) Let A be a ring and M an A-module. Recall that $x \in A$ is *M-regular* if it is not a zero divisor on M, that is, if $m \in M$ and $xm = 0$ implies that $m = 0$. Equivalently, if x is not contained in any of the associated primes of M.

A sequence $x_1, \dots, x_r \in A$ is an *M-regular sequence* if x_1 is not a zero divisor on M and x_i is not a zero divisor on $M/(x_1, \dots, x_{i-1})M$ for all i.

Let $\mathrm{rad}\, A$ denote the *radical* (or Jacobson radical) of A, that is, the intersection of all maximal ideals. Let $I \subset \mathrm{rad}\, A$ be an ideal. The *depth* of M along I is the maximum length of an M-regular sequence $x_1, \dots, x_r \in I$. It is denoted by $\mathrm{depth}_I M$. If A is Noetherian, M is finite over A and $I \subset \mathrm{rad}\, A$, then all maximal M-regular sequences $x_1, \dots, x_r \in I$ have the same length; see Matsumura (1986, p.127) or Eisenbud (1995, sec.17).

Warning The literature is not fully consistent on the depth if $M = 0$ or if $I \not\subset \mathrm{rad}\, A$. While the definition of depth makes sense for arbitrary rings and ideals, it can give unexpected results.

10.3 (Comments on depth and S_m) Let F be a coherent sheaf on X. The *depth of F at x*, denoted by $\mathrm{depth}_x F$, is defined as the depth of its localization F_x along $m_{x,X}$ (as an $\mathscr{O}_{x,X}$-module). For a closed subscheme $Z \subset X$ we set

$$\mathrm{depth}_Z F := \inf\{\mathrm{depth}_z F : z \in Z\}. \tag{10.3.1}$$

If $X = \mathrm{Spec}\, A$ is affine, $Z = V(I)$ for some ideal $I \subset \mathrm{rad}\, A$ and $M = H^0(X, F)$ then $\mathrm{depth}_Z F = \mathrm{depth}_I M$. (This definition is for coherent sheaves only. See Grothendieck (1968, exp.III) for quasi-coherent sheaves.)

A coherent sheaf F on a scheme X satisfies *Serre's condition S_m* if

$$\mathrm{depth}_x F \geq \min\{m, \mathrm{codim}(x, \mathrm{Supp}\, F)\} \quad \text{for every } x \in X; \tag{10.3.2}$$

see Stacks (2022, tag 033P) for details.

It is important to note that over a local scheme (x, X), being S_m is *not* the same as $\mathrm{depth}_x F \geq m$; neither implies the other.

Definition 10.4 F is *Cohen–Macaulay* or *CM* if

$$\operatorname{depth}_x F = \dim_x F \quad \text{for every } x \in X. \tag{10.4.1}$$

It is easy to see that if F is CM then the local rings of $\operatorname{Supp} F$ are pure dimensional (Stacks, 2022, tag 00N2). In the literature, the definition of CM frequently includes the assumption that $\operatorname{Supp} F$ be pure dimensional; we will most likely lapse into this habit too.

In contrast with the S_m situation (10.3), if (10.4.1) holds at closed points, then it holds at every point of $\operatorname{Supp} F$; see Matsumura (1986, 17.4).

Condition S_1 can be described in terms of embedded points.

Lemma 10.5 *Let F be a coherent sheaf on a scheme X and $Z \subset X$ a closed subscheme. Then* $\operatorname{depth}_Z F \geq 1$ *iff none of the associated points of F is contained in Z. In particular, F is S_1 iff it has no embedded associated points.* □

The following lemma gives several characterizations of S_2 sheaves.

Lemma 10.6 *Let F be a coherent sheaf and $Z \subset$* $\operatorname{Supp} F$ *a nowhere dense subscheme. The following are equivalent.*

(10.6.1) $\operatorname{depth}_Z F \geq 2$.

(10.6.2) $\operatorname{depth}_Z F \geq 1$ *and* $\operatorname{depth}_Z(F|_D) \geq 1$ *whenever D is a Cartier divisor in an open subset of X that does not contain any associated prime of F.*

(10.6.3) $\operatorname{tors}_Z(F) = 0$ *and* $\operatorname{tors}_Z(F|_D) = 0$ *whenever D is as above.*

(10.6.4) *An exact sequence* $0 \to F \to F' \to Q \to 0$ *splits if* $\operatorname{Supp} Q \subset Z$.

(10.6.5) $\operatorname{depth}_Z F \geq 1$ *and for any exact sequence* $0 \to F \to F' \to Q \to 0$ *such that* $\emptyset \neq \operatorname{Supp} Q \subset Z$*, F' has an associated point in* $\operatorname{Supp} Q$.

(10.6.6) $F = j_*(F|_{X \setminus Z})$ *where $j\colon X \setminus Z \hookrightarrow X$ is the natural injection.*

(10.6.7) $\mathscr{H}^0_Z(X, F) = \mathscr{H}^1_Z(X, F) = 0$.

(10.6.8) *Let $z \in Z$ be any point. Then* $H^0_z(X_z, F_z) = H^1_z(X_z, F_z) = 0$.

Proof All but (4) are clearly local conditions on X. By assumption, $\operatorname{tors}_Z(F) = 0$. Thus, if in (4) there is a splitting locally then the unique splitting is given by $\operatorname{tors}_Z(F') \subset F'$. Thus (4) is also local, so we can assume that X is affine.

Conditions (2) and (3) are just restatements of the inductive definition of depth. Assume (1) and consider an extension $0 \to F \to F' \to Q \to 0$ where $\operatorname{Supp} Q \subset Z$. If $\operatorname{tors}_Z(F') \to Q$ is surjective then it gives a splitting. If not, then after quotienting out by $\operatorname{tors}_Z(F')$ and taking a coherent subsheaf $F'' \subset F'/\operatorname{tors}_Z(F')$, we get an extension $0 \to F \to F'' \to Q'' \to 0$ where

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$\mathrm{tors}_Z(F'') = 0$. Pick $s \in I_Z$ that is not a zero divisor on F and F'', but $s \cdot (F''/F) = 0$. Then sF'' is a nonzero submodule of F/sF supported on Z. This proves (1) ⇒ (4).

Assuming (4), we claim that $\mathrm{tors}_Z(F) = 0$. After localizing at a generic point of $\mathrm{tors}_Z(F)$, we may assume that $\mathrm{tors}_Z(F)$ is supported at $z \in Z$. Since the injective hull of $k(z)$ over $\mathcal{O}_X$ has infinite length, there is a nonsplit extension $j\colon \mathrm{tors}_Z(F) \hookrightarrow G$. Then the cokernel of $(1, j)\colon \mathrm{tors}_Z(F) \to F + G$ gives a non-split extension of F. The rest of (5) is clear.

If $\mathrm{depth}_Z F \geq 1$, then the natural map $F \to j_*(F|_{X\setminus Z})$ is an injection. The quotient is supported on Z, thus (5) ⇒ (6).

Assume (6). Then $F \to j_*(F|_{X\setminus Z})$ is an injection, so $\mathrm{depth}_Z F \geq 1$. If $\mathrm{depth}_Z F < 2$, then we can pick $s \in I_Z$ such that F/sF has a subsheaf Q supported on Z. Let $F' \subset F$ be the preimage of Q. Then $s^{-1}F' \subset j_*(F|_{X\setminus Z})$ shows that (6) ⇒ (1). We discuss (7) and (8) in (10.29). □

Corollary 10.7 *Let F be a coherent, S_2 sheaf and $G \subset F$ a subsheaf. Then G is S_2 iff every associated point of F/G has codimension ≤ 1 in* Supp F.

Proof Let $Z \subset \mathrm{Supp}\, F$ be a closed subset of codimension ≥ 2 and $j\colon U := X \setminus Z \hookrightarrow X$ the injection. Then $j_*(G|_U) \subset F$ and $\mathrm{depth}_Z G < 2 \Leftrightarrow G \neq j_*(G|_U)$ $\Leftrightarrow j_*(G|_U)/G \subset F/G$ is a nonzero subsheaf supported on Z. □

Corollary 10.8 *Let F be a coherent, S_2 sheaf and G any coherent sheaf. Then $\mathcal{H}om_X(G, F)$ is also S_2.*

Proof It is clear that every irreducible component of $\mathrm{Supp}\, \mathcal{H}om_X(G, F)$ is also an irreducible component of $\mathrm{Supp}\, F$.

Let $Z \subset \mathrm{Supp}\, F$ be a closed subset of codimension ≥ 2 and $j\colon X \setminus Z \hookrightarrow X$ the injection. Any homomorphism $\phi\colon G|_{X\setminus Z} \to F|_{X\setminus Z}$ uniquely extends to $j_*\phi\colon j_*(G|_{X\setminus Z}) \to j_*(F|_{X\setminus Z})$. Since F is S_2, the target equals F. We have a natural map $G \to j_*(G|_{X\setminus Z})$, whose kernel is $\mathrm{tors}_Z(G)$. Thus $\mathcal{H}om_X(G, F) = j_*(\mathcal{H}om_X(G, F)|_{X\setminus Z})$, hence $\mathcal{H}om_X(G, F)$ is S_2. □

An important property of S_2 sheaves is the following, which can be obtained by combining Hartshorne (1977, III.7.3 and III.12.11).

Proposition 10.9 (Enriques–Severi–Zariski lemma) *Let $f\colon X \to S$ be a projective morphism and F a coherent sheaf on X that is flat over S, with S_2 fibers of pure dimension ≥ 2. Then $f_*F(-m) = R^1 f_*F(-m) = 0$ for $m \gg 1$.*

Therefore, if $H \in |\mathcal{O}_X(m)|$ does not contain any of the associated points of F, then the restriction map $f_*F \to (f|_H)_*(F|_H)$ is an isomorphism. □

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10.10 (Depth and flatness) Let $p\colon Y \to X$ be a morphism and G a coherent sheaf on Y that is flat over X. It is easy to see that for any point $y \in Y$ we have

$$\operatorname{depth}_y G = \operatorname{depth}_{p(y)} X + \operatorname{depth}_y G_{p(y)}. \tag{10.10.1}$$

Similarly, if $p\colon Y \to X$ is flat and F is a coherent sheaf on X, then

$$\operatorname{depth}_y p^* F = \operatorname{depth}_{p(y)} F + \operatorname{depth}_y Y_{p(y)}. \tag{10.10.2}$$

In particular, if $p\colon Y \to X$ is flat with S_m fibers and F is a quasi-coherent S_m sheaf on X then p^*F is also S_m. The converse also holds if p is faithfully flat.

The assumption on the fibers is necessary and a flat pull-back of an S_m sheaf need not be S_m; not even for products. Let X_1, X_2 be k-schemes. Then $X_1 \times X_2$ is S_m iff both of the X_i are S_m.

10.2 Flat Families of S_m Sheaves

We consider how the S_m property (2.72) varies in flat families.

Theorem 10.11 (Grothendieck, 1960, IV.12.1.6) *Let $\pi\colon X \to S$ be a morphism of finite type and F a coherent sheaf on X that is flat over S. Fix $m \in \mathbb{N}$. Then the set of points $\{x \in X\colon F_{\pi(x)}$ is pure and S_m at $x\}$ is open in X.*

This immediately implies the following variant for proper morphisms.

Corollary 10.12 *5 Let $\pi\colon X \to S$ be a proper morphism and F a coherent sheaf on X that is flat over S. Fix $m \in \mathbb{N}$. Then the set of points $\{s \in S : F_s$ is pure and $S_m\}$ is open in S.* □

For nonproper morphisms we get the following.

Corollary 10.13 *Let S be an integral scheme, $\pi\colon X \to S$ a morphism of finite type, and F a coherent sheaf on X. Assume that F is pure and S_m. Then there is a dense open subset $S^\circ \subset S$ such that F_s is pure and S_m for every $s \in S^\circ$.*

Proof Let $Z \subset X$ denote the set of points $x \in X$ such that either F is not flat at x or $F_{\pi(x)}$ is not pure and S_m at x. Note that Z is closed in X by (10.11) and by generic flatness (Eisenbud, 1995, 14.4).

The local rings of the generic fiber of π are also local rings of X, hence the restriction of F to the generic fiber is pure and S_m. Thus Z is disjoint from the generic fiber of π. Therefore $\pi(Z) \subset S$ is a constructible subset that does

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not contain the generic point, hence $S \setminus \pi(Z)$ contains a dense open subset $S^\circ \subset S$. □

10.14 (Nagata's openness criterion) In many cases, one can check openness of a subset of a scheme using the following easy to prove test, which is sometimes called the *Nagata openness criterion.*

Let X be a Noetherian topological space and $U \subset X$ an arbitrary subset. Then U is open iff the following conditions are satisfied.

(10.14.1) If $x_1 \in \bar{x}_2$ and $x_1 \in U$ then $x_2 \in U$.

(10.14.2) If $x \in U$ then there is a nonempty open $V \subset \bar{x}$ such that $V \subset U$.

Assume now that we want to use this to check openness of a fiber-wise property $\mathcal{P}$ for a morphism $\pi\colon X \to S$.

We start with condition (10.14.1). Pick points $x_1, x_2 \in X$ such that $x_1 \in \bar{x}_2$.

Let T be the spectrum of a DVR with closed point $0 \in T$, generic point $t_g \in T$, and $q\colon T \to X$ a morphism such that $q(0) = x_1$ and $q(t_g) = x_2$. After base change using $\pi \circ q$ we get $Y \to T$. Usually one cannot guarantee that the residue fields are unchanged under q. However, if property $\mathcal{P}$ is invariant under field extensions, then it is enough to check (10.14.1) for $Y \to T$. Thus we may assume that S is the spectrum of a DVR.

As for (10.14.2), we can replace S by the closure of $\pi(x)$. Then $\pi(x)$ is the generic point of S and then we may assume that S is regular.

We can summarize these considerations in the following form.

Proposition 10.15 (Openness criterion) *Let $\mathcal{P}$ be a property defined for coherent sheaves on schemes over fields. Assume that $\mathcal{P}$ is invariant under base field extensions. The following are equivalent.*

(10.15.1) *Let $\pi\colon X \to S$ be a morphism of finite type and F a coherent sheaf on X that is flat over S. Then $\{x \in X\colon F_{\pi(x)}$ satisfies property $\mathcal{P}$ at $x\}$ is open in X.*

(10.15.2) *The following hold, where $\sigma\colon S \to X$ denotes a section.*

(a) If S is the spectrum of a DVR with closed point 0*, generic point g and $\mathcal{P}$ holds for $\sigma(0) \in X_0$, then $\mathcal{P}$ holds for $\sigma(g) \in X_g$.*

(b) If S is the spectrum of a regular ring with generic point g and $\mathcal{P}$ holds for $\sigma(g) \in X_g$, then $\mathcal{P}$ holds in a nonempty open $U \subset \sigma(S)$. □

10.16 (Proof of 10.11) By (10.15), we may assume that S is affine and regular. We may also assume that π is affine and $X = \operatorname{Supp} F$.

First, we check (10.15.2.a) for $m = 1$. (Note that pure and S_1 is equivalent to pure (10.1).) Let $W \subset X$ be the closure of an associated prime of F. Then the irreducible components of $W \cap X_0$ are associated primes of F_0 by (10.22).

Since F_0 is pure, $W \cap X_0$ is an irreducible component of Supp F_0. Hence W is an irreducible component of Supp F. Thus F_g is also pure.

Next we check (10.15.2.a) for $m > 1$. Since S_m implies S_1, we already know that every fiber of F is pure. By (10.17) there is a subset $Z \subset X$ of codimension ≥ 2 such that F is CM over $X \setminus Z$. Let $Z \subset H \subset X$ be a Cartier divisor that does not contain any of the associated primes of F_0. Then $F|_H$ is flat over S and $(F|_H)_0 = F_0|_H$ is pure and S_{m-1}. Thus, by induction, $F|_H$ is pure and S_{m-1} on the generic fiber, hence F_{s_g} is pure and S_m along H. It is even CM on $X \setminus H$, hence F_{s_g} is pure and S_m.

For (10.15.2.b) we start with $m = 1$. We may assume that F_{s_g} is pure. By Noether normalization, after passing to some open subset of S, there is a finite surjection $p\colon X \to \mathbb{A}^n_S$ for some n. Note that p_*F is flat over S and it is pure on the generic fiber by (9.2), hence torsion-free. Using (9.2) in the reverse direction for the other fibers, we are reduced to the case when $X = \mathbb{A}^n_S$ and F is torsion-free at $x := \sigma(g)$ on the generic fiber. Thus there is an injection of the localizations $F_x \hookrightarrow \mathscr{O}^m_{x,X}$. By generic flatness (Eisenbud, 1995, 14.4), the quotient $\mathscr{O}^m_{x,X}/F_x$ is flat over an open, dense subset $S^\circ \subset S$. Thus if $s \in S^\circ$ then we have an injection $F|_U \hookrightarrow \mathscr{O}^m_U$. Thus every fiber F_s is torsion-free over $U \cap \pi^{-1}(S^\circ)$. For $m > 1$, we follow the same argument as above using $Z \subset H \subset X$ and induction. □

Lemma 10.17 *Let $\pi\colon X \to S$ be a morphism of finite type and F a coherent sheaf on X that is flat over S. Assume that* Supp F *is pure-dimensional over S. As in (7.26), let* $\mathrm{FlatCM}_S(X, F) \subset X$ *be the set of points x such that $F_{\pi(x)}$ is CM at x. Then, for every $s \in S$,*

(10.17.1) Supp $F_s \cap \mathrm{FlatCM}_S(X, F)$ *is dense in* Supp F_s, *and,*

(10.17.2) *if F_s is pure, then its complement has codimension ≥ 2 in* Supp F_s.

Proof We may assume that π is affine and $X = \mathrm{Supp}\, F$. By (10.49), after replacing X with an étale neighborhood of x, there is a finite surjection $g\colon X \to Y$ where $\tau\colon Y \to S$ is smooth.

Since g_*F is flat over S, it is locally free at a point $y \in Y$ iff the restriction of g_*F to the fiber $Y_{\tau(y)}$ is locally free at y. The latter holds outside a codimension ≥ 1 subset of each fiber Y_s. If F is pure then g_*F is torsion-free on each fiber, so local freeness holds outside a subset of codimension ≥ 2. □

Let F be a coherent, S_m sheaf on $\mathbb{P}^n$. If a hyperplane $H \subset \mathbb{P}^n$ does not contain any of the irreducible components of Supp F then $F|_H$ is S_{m-1}, essentially by definition. The following result says that $F|_H$ is even S_m for

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general hyperplanes, though we cannot be very explicit about the meaning of "general."

Corollary 10.18 (Bertini theorem for S_m) *Let F be a coherent, pure, S_m sheaf on a finite type k-scheme and $|V|$ a base point free linear system on X. Then there is a dense, open $U \subset |V|$ such that $F|_H$ is also pure and S_m for $H \in U$.*

Proof Let $Y \subset X \times |V|$ be the incidence correspondence (that is, the set of pairs (point $\in H$) with projections π and $\check{\pi}$). Note that π is a $\mathbb{P}^{n-1}$-bundle for $n = \dim |V|$, thus $\pi^* F$ is also pure and S_m by (10.10).

By (10.13) there is a dense open subset $U \subset |V|$ such that $F|_H$ is also pure and S_m for $H \in U$. For a divisor H, the restriction $F|_H$ is isomorphic to the restriction of $\pi^* F$ to the fiber of $\check{\pi}$ over $H \in |V|$. □

Corollary 10.19 (Bertini theorem for hulls) *Let $|V|$ be a base point free linear system on a finite type k-scheme X. Let F be a coherent sheaf on X with hull $q\colon F \to F^{[**]}$. Then there is a dense, open subset $U \subset |V|$ such that*

$$(F^{[**]})|_H = (F|_H)^{[**]} \quad \textit{for } H \in U.$$

Proof By definition we have an exact sequence

$$0 \to K \to F \to F^{[**]} \to Q \to 0,$$

where $\dim K \leq n-1$ and $\dim Q \leq n-2$. If $H \in |V|$ is general, then the restriction stays exact

$$0 \to K|_H \to F|_H \to (F^{[**]})|_H \to Q|_H \to 0,$$

$\dim K|_H \leq n-2$ and $\dim Q|_H \leq n-3$. Thus $(F^{[**]})|_H = (F|_H)^{[**]})$. □

Corollary 10.20 (Bertini theorem for S_m in families) *Let T be the spectrum of a local ring, $X \subset \mathbb{P}^n_T$ a quasi-projective scheme and F a coherent sheaf on X that is flat over T with pure, S_m fibers.*

Assume that either X is projective over T or $\dim T \leq 1$. *Then $F|_{H\cap X}$ is also flat over T with pure and S_m fibers for a general hyperplane $H \subset \mathbb{P}^n_T$.*

Proof The hyperplanes correspond to sections of $\check{\mathbb{P}}^n_T \to T$. If X is projective over T then we use (10.18) for the special fiber X_0 and conclude using (10.12).

If $\dim T = 1$ then we use (10.18) both for the special fiber X_0 and the generic fibers X_{g_i}. We get open subsets $U_0 \subset \check{\mathbb{P}}^n_0$ and $U_{g_i} \subset \check{\mathbb{P}}^n_{g_i}$. Let $W_i \subset \check{\mathbb{P}}^n_T$ denote the closure of $\check{\mathbb{P}}^n_{g_i} \setminus U_{g_i}$. For dimension reasons, W_i does not contain $\check{\mathbb{P}}^n_0$. Thus any hyperplane corresponding to a section through a point of $U_0 \backslash (\cup_i W_i)$ works. □

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Example 10.21 If $\dim T \geq 2$ then (10.20) does not hold for nonproper maps. Here is a similar example for the classical Bertini theorem on smoothness. Set

$$X := (x^2 + y^2 + z^2 = s) \setminus (x = y = z = s = 0) \subset \mathbb{A}^3_{xyz} \times \mathbb{A}^2_{st}$$

with smooth second projection $f\colon X \to \mathbb{A}^2_{st}$. Over the origin we start with the hyperplane $H_{00} := (x = 0)$, it is a typical member of the base point free linear system $|ax + by + cz = 0|$.

A general deformation of it is given by $H_{st} := x+b(s,t)y+c(s,t)z = d(s,t)$. It is easy to compute that the intersection $H_{st} \cap X_{st}$ is singular iff $s(1+b^2+c^2) = d^2$. This equation describes a curve in $\mathbb{A}^2_{st}$ that passes through the origin.

10.22 (Associated points of restrictions) Let X be a scheme, $D \subset X$ a Cartier divisor and F a coherent sheaf on X. We aim to compare $\mathrm{Ass}(F)$ and $\mathrm{Ass}(F|_D)$. If D does not contain any of the associated points of G then $\mathrm{Tor}^1(G, \mathscr{O}_D) = 0$. Thus if $0 = F_0 \subset \cdots \subset F_r = F$ is a filtration of F by subsheaves and D does not contain any of the associated points of F_i/F_{i-1} then $0 = F_0|_D \subset \cdots \subset F_r|_D = F|_D$ is a filtration of $F|_D$ and $F_i|_D/F_{i-1}|_D \simeq (F_i/F_{i-1})|_D$. We can also choose any of the associated points of F to be an associated point of F_1, proving the following.

Claim 10.22.1 If D does not contain any of the associated points of F, then
- (a) $\mathrm{Ass}(F|_D) \subset \cup_i \mathrm{Ass}((F_i/F_{i-1})|_D)$ and
- (b) for every $x \in \mathrm{Ass}(F)$, every generic point of $D \cap \bar{x}$ is in $\mathrm{Ass}(F|_D)$. □

By (10.25), we can choose the F_i such that $\mathrm{Ass}(F_i/F_{i-1})$ is a single associated point of F for every i. Thus it remains to understand $\mathrm{Ass}(G|_D)$ when G is pure. Let $G^{[**]} \supset G$ denote the hull of G and set $Q := G^{[**]}/G$. As we have noted, if D does not contain any of the associated points of Q then $G^{[**]}|_D \supset G|_D$, thus $\mathrm{Ass}(G^{[**]}|_D) = \mathrm{Ass}(G|_D)$. Finally, since $G^{[**]}$ is S_2, the restriction $G^{[**]}|_D$ is S_1, hence its associated points are exactly the generic points of $D \cap \mathrm{Supp}\, G$. We have thus proved the following.

Claim 10.22.2 Let $D \subset X$ be a Cartier divisor that contains neither an associated point of F nor an associated point of $(F_i/F_{i-1})^{[**]}/(F_i/F_{i-1})$. Then
- (a) the associated points of $F|_D$ are exactly the generic points of $D \cap \bar{x}$ for all $x \in \mathrm{Ass}(F)$, and
- (b) $(F/\operatorname{emb}(F))|_D \simeq (F|_D)/(\operatorname{emb}(F|_D))$. □

Note that the associated points of $(F_i/F_{i-1})^{[**]}/(F_i/F_{i-1})$ depend on the choice of the F_i, they are not determined by F. For the Claim to hold, it is enough to take the intersection of all possible sets. This set is still hard to determine, but

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in many applications the key point is that, as long as X is excellent, we need D to avoid only a finite set of points.

The next result describes how the associated points of fibers of a flat sheaf fit together. The proof is a refinement of the arguments used in (10.16).

Theorem 10.23 *Let $f\colon X \to S$ be a morphism of finite type and F a coherent sheaf on X. Then the following hold.*

(10.23.1) *There are finitely many locally closed $W_i \subset X$ such that* $\operatorname{Ass}(F_s)$ *equals the set of generic points of the $(W_i)_s$ for every $s \in S$.*

(10.23.2) *If F is flat over S then we can choose the W_i to be closed and such that each $f|_{W_i}\colon W_i \to f(W_i)$ is equidimensional.*

Proof Using Noetherian induction it is enough to prove that (1) holds over a non-empty open subset of $\operatorname{red} S$. We may thus assume that S is integral with generic point $g \in S$.

Assume first that X is integral and F is torsion-free. By Noether normalization, after again passing to some non-empty open subset of S there is a finite surjection $p\colon X \to \mathbb{A}^m_S$. Then p_*F is torsion-free of generic rank say r, hence there is an injection $j\colon p_*F \hookrightarrow \mathscr{O}^r_{\mathbb{A}^m_S}$. After again passing to some non-empty open subset we may assume that $\operatorname{coker}(j)$ is flat over S, thus

$$j_s\colon p_*(F_s) = (p_*F)_s \hookrightarrow \mathscr{O}^r_{\mathbb{A}^m_s}$$

is an injection for every $s \in S$. Thus each F_s is torsion-free and its associated points are exactly the generic points of the fiber X_s.

In general, we use (10.25) for the generic fiber and then extend the resulting filtration to X. Thus, after replacing S by a nonempty open subset if necessary, we may assume that there is a filtration $0 = F^0 \subset \cdots \subset F^n = F$ such that each F^{m+1}/F^m is a coherent, torsion-free sheaf over some integral subscheme $W_m \subset X$ and $W_{m_1} \not\subset W_{m_2}$ for $m_1 > m_2$. As we proved, we may assume that the associated points of each $\left(F^{m+1}/F^m\right)_s$ are exactly the generic points of the fiber $(W_m)_s$. Using generic flatness, we may also assume that each F^{m+1}/F^m is flat over S and, after further shrinking S, none of the generic points of $(W_{m_1})_s$ are contained in $(W_{m_2})_s$ for $m_1 > m_2$. Then the associated points of each F_s are exactly the generic points of the fibers $(W_m)_s$ for every m. This proves (1).

In order to see (2), consider first the case when the base $(0 \in T)$ is the spectrum of a DVR. The filtration given by (10.25) for the generic fiber extends to a filtration $0 = F^0 \subset \cdots \subset F^n = F$ over X giving closed integral subschemes $W_m \subset X$. Since T is the spectrum of a DVR, the F^{m+1}/F^m are flat over T, hence the associated points of F_0 are exactly the generic points of the fibers $(W_m)_0$ for every m.

To prove (2) in general, we take the $W_i \subset X$ obtained in (1) and replace them by their closures. A possible problem arises if $f|_{W_i}\colon W_i \to f(W_i)$ is not equidimensional. Assume that $W_i \to f(W_i)$ has generic fiber dimension d and let $(W_i)_s$ be a special fiber. Pick any closed point $x \in (W_i)_s$ and the spectrum of a DVR $(0 \in T)$ mapping to W_i such that the special point of T maps to $f(x)$ and the generic point of T to the generic point of $f(W_i)$. After base change to T we see that F_s has a d-dimensional associated subscheme containing x. Thus $(W_i)_s$ is covered by d-dimensional associated subschemes of F_s. Since F_s is coherent, this is only possible if $\dim(W_i)_s = d$ and every generic point of the $(W_i)_s$ is an associated point of F_s. □

10.24 (Semicontinuity and depth) Let X be a scheme and F a coherent sheaf on X. As we noted in (10.3), the function $x \mapsto \operatorname{depth}_x F$ is not lower semicontinuous. This is, however, caused by the non-closed points. A quick way to see this is the following.

Assume that X is regular and let $0 \in X$ be a closed point. By the Auslander–Buchsbaum formula as in Eisenbud (1995, 19.9), F_0 has a projective resolution of length $\dim X - \operatorname{depth}_0 F$. Thus there is an open subset $0 \in U \subset X$ such that $F|_U$ has a projective resolution of length $\dim X - \operatorname{depth}_0 F$. This shows that

$$\operatorname{depth}_x F \geq \operatorname{depth}_0 F - \dim \bar{x} \quad \forall x \in U. \tag{10.24.1}$$

That is, $x \mapsto \operatorname{depth}_x F$ is lower semicontinuous for closed points. In general, we have the following analog of (10.11).

Proposition 10.24.2 Let $\pi\colon X \to S$ be a morphism of finite type and F a coherent sheaf on X that is flat over S with pure fibers. Let $0 \in X$ be a closed point. Then there is an open subset $0 \in U \subset X$ such that

$$\operatorname{depth}_x F_{\pi(x)} \geq \operatorname{depth}_0 F_{\pi(0)} - \operatorname{tr-deg}_{k(\pi(x))} k(x) \quad \forall x \in U,$$

where $F_{\pi(x)}$ is the restriction of F to the fiber $X_{\pi(x)}$ and tr-deg denotes the transcendence degree. Hence $x \mapsto \operatorname{depth}_x F_{\pi(x)}$ is lower semicontinuous on closed points.

Proof Using Noether normalization and (10.17.1) as in (10.16), we can reduce to the case when $X = \mathbb{A}^n_S$ for some n. Next we take a projective resolution of the fiber $F_{\pi(0)}$ and lift it to a suitable neighborhood $0 \in U \subset X$ using the flatness of F. □

Dévissage is a method that writes a coherent sheaf as an extension of simpler coherent sheaves and uses these to prove various theorems. There are many ways to do this, and different ones are useful in different contexts; see Stacks (2022, tag 07UN) for some of them.

Recall that $\operatorname{Ass}(F)$ denotes the set of associated points of a sheaf F (10.1) and that a sheaf is S_1 iff it has no embedded points (10.5). As in (10.1), $\operatorname{tors}_Z(F) \subset F$ is the largest subsheaf whose support is contained in Z.

Lemma 10.25 (Dévissage) *Let X be a Noetherian scheme, F a coherent sheaf on X. Write $\operatorname{Ass}(F) = \{w_i : i = 1, \dots, m\}$ in some fixed order and let W_i be the closure of w_i. Assume that $W_j \not\subset W_i$ for $i < j$. Then the following hold.*

(10.25.1) *There is a unique filtration $0 = G_0 \subset G_1 \subset \cdots \subset G_m = F$ such that each G_i/G_{i-1} is a torsion-free sheaf supported on W_i. Moreover, the natural map $\operatorname{tors}_{W_i}(F) \to G_i/G_{i-1}$ is an isomorphism at w_i.*

(10.25.2) *There is a non-unique refinement $G_i = G_{i,0} \subset G_{i,1} \subset \cdots \subset G_{i,r_i} = G_{i+1}$ such that each $G_{i,j+1}/G_{i,j}$ is a rank 1, torsion-free sheaf over $\operatorname{red} W_i$.*

Proof It is easy to see that we must set $G_1 = \operatorname{tors}_{W_1}(F)$. Then pass to F/G_1 and use induction on the number of associated points to get (1).

For (2), any filtration of $(G_{i+1}/G_i)_{w_i}$ whose successive quotients are $k(w_i)$ extends uniquely to the required $G_{i,j}$. □

10.3 Cohomology over Non-proper Schemes

The cohomology theory of coherent sheaves is trivial over affine schemes and well understood over proper schemes. If X is a scheme and $j\colon U \hookrightarrow X$ is an open subscheme then one can study the cohomology theory of coherent sheaves on U by understanding the cohomology theory of quasi-coherent sheaves on X and the higher direct image functors $R^i j_*$. The key results are (10.26) and (10.30); see Grothendieck (1960, IV.5.11.1).

Proposition 10.26 *Let X be an excellent scheme, $Z \subset X$ a closed subscheme and $U := X \setminus Z$ with injection $j\colon U \hookrightarrow X$. Let G be a coherent sheaf on U. Then j_*G is coherent iff $\operatorname{codim}_W(Z \cap W) \geq 2$, whenever $W \subset X$ is the closure of an associated point w of G.*

The case of arbitrary Noetherian schemes is discussed in Kollár (2017).

Proof This is a local question, hence we may assume that X is affine. By (10.25) G has a filtration $0 = G_0 \subset \cdots \subset G_r = G$ such that each G_{m+1}/G_m is isomorphic to a subsheaf of some $\mathscr{O}_{W\cap U}$ where w is an associated prime of G. Since j_* is left exact, it is enough to show that each $j_*\mathscr{O}_{W\cap U}$ is coherent.

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Let $p\colon V \to W$ be the normalization. Since X is excellent, p is finite. $\mathscr{O}_V$ is S_2 (by Serre's criterion) and so is $p_*\mathscr{O}_V$ by (9.2). Thus

$$j_*\mathscr{O}_{W\cap U} \subset j_*(p_*\mathscr{O}_V|_U) = p_*\mathscr{O}_V,$$

where the equality follows from (10.6) using $\operatorname{codim}_W(Z \cap W) \geq 2$. Thus $j_*\mathscr{O}_{W\cap U}$ is coherent. □

It is frequently quite useful to know that coherent sheaves are "nice" over large open subsets. For finite type schemes this was established in (10.17).

Proposition 10.27 *Let X be a Noetherian scheme. Assume that every integral subscheme $W \subset X$ has an open dense subscheme $W^\circ \subset W$ that is regular, or at least CM. (For example, X is excellent.) Let F be a coherent sheaf on X.*

(10.27.1) *There is a closed subset $Z_1 \subset \operatorname{Supp} F$ of codimension ≥ 1 such that F is CM on $X \setminus Z_1$.*

(10.27.2) *If F is S_1 then there is a closed subset $Z_2 \subset \operatorname{Supp} F$ of codimension ≥ 2 such that F is CM on $X \setminus Z_2$.*

Proof We put the intersections of different irreducible components of $\operatorname{Supp} F$ into Z_1. Since (1) is a local question, we may thus assume that $\operatorname{Supp} F$ is irreducible. Since an extension of CM sheaves of the same dimensional support is CM (10.28), using (10.25) we may assume that F is torsion-free over an integral subscheme $W \subset X$. Then F is locally free over a dense open subset $W^\circ \subset W$ and we can take $Z_1 := W \setminus W^*$, where W^* is the regular locus of W°.

In order to prove (2), we may assume that X is affine. Let $s = 0$ be a local equation of Z_1. We apply the first part to F/sF to obtain a closed subset $Z_2 \subset \operatorname{Supp}(F/sF)$ of codimension ≥ 1 such that F/sF is CM on $X \setminus Z_2$. Thus F is CM on $X \setminus Z_2$. □

The next lemma is quite straightforward; see Kollár (2013b, 2.60).

Lemma 10.28 *Let X be a scheme and $0 \to F' \to F \to F'' \to 0$ a sequence of coherent sheaves on X that is exact at $x \in X$.*

(10.28.1) *If* $\operatorname{depth}_x F \geq r$ *and* $\operatorname{depth}_x F'' \geq r-1$ *then* $\operatorname{depth}_x F' \geq r$.

(10.28.2) *If* $\operatorname{depth}_x F \geq r$ *and* $\operatorname{depth}_x F' \geq r-1$ *then* $\operatorname{depth}_x F'' \geq r-1$. □

10.29 (Cohomology over quasi-affine schemes) Grothendieck (1967)

Let X be an affine scheme, $Z \subset X$ a closed subscheme and $U := X \setminus Z$. Here our primary interest is in the case when $Z = \{x\}$ is a closed point.

For a quasi-coherent sheaf F on X, let $H^0_Z(X, F)$ denote the space of global sections whose support is in Z. There is a natural exact sequence

$$0 \to H^0_Z(X, F) \to H^0(X, F) \to H^0(U, F|_U).$$

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This induces a long exact sequence of the corresponding higher cohomology groups. Since X is affine, $H^i(X, F) = 0$ for $i > 0$, hence the long exact sequence breaks up into a shorter exact sequence

$$0 \to H^0_Z(X, F) \to H^0(X, F) \to H^0(U, F|_U) \to H^1_Z(X, F) \to 0 \qquad (10.29.1)$$

and a collection of isomorphisms

$$H^i(U, F|_U) \simeq H^{i+1}_Z(X, F) \quad \text{for } i \geq 1. \qquad (10.29.2)$$

The vanishing of the local cohomology groups is closely related to the depth of the sheaf F. Two instances of this follow from already established results. First, for coherent sheaves (10.5) can be restated as

$$H^0_Z(X, F) = 0 \iff \operatorname{depth}_Z F \geq 1. \qquad (10.29.3)$$

Second, (10.6) tells us when the map $H^0(X, F) \to H^0(U, F|_U)$ in (10.29.1) is an isomorphism. This implies that, for coherent sheaves,

$$H^0_Z(X, F) = H^1_Z(X, F) = 0 \iff \operatorname{depth}_Z F \geq 2. \qquad (10.29.4)$$

More generally, Grothendieck's vanishing theorem (see Grothendieck (1967, sec.3) or Bruns and Herzog (1993, 3.5.7)) says that

$$\operatorname{depth}_Z F = \min\{i : H^i_Z(X, F) \neq 0\}. \qquad (10.29.5)$$

Combined with (10.29.2–3), this shows that

$$H^i(U, F|_U) = 0 \quad \text{for } 1 \leq i \leq \operatorname{depth}_Z F - 2. \qquad (10.29.6)$$

All these cohomology groups are naturally modules over $H^0(X, \mathscr{O}_X)$ and we need to understand when they are finitely generated.

More generally, let G be a coherent sheaf on U. When is the group $H^i(U, G)$ a finite $H^0(X, \mathscr{O}_X)$-module? Since X is affine, $H^i(U, G) = H^0(X, R^i j_* G)$, where $j \colon U \hookrightarrow X$ denotes the natural open embedding. Thus $H^i(U, G)$ is a finite $H^0(X, \mathscr{O}_X)$-module iff $R^i j_* G$ is a coherent sheaf. For $i \geq 1$, the sheaves $R^i j_* G$ are supported on Z, which implies the following.

Claim 10.29.7 Assume that $i \geq 1$. Then every associated prime of $H^i(U, G)$ (viewed as an $H^0(X, \mathscr{O}_X)$-module) is contained in Z, and, if $Z = \{x\}$, then $H^i(U, G)$ is a finite $H^0(X, \mathscr{O}_X)$-module iff $H^i(U, G)$ has finite length. □

The general finiteness condition is stated in (10.30); but first we work out the special cases that we use. We start with $H^0(U, G)$; here we have the following restatement of (10.26).

Claim 10.29.8 Let X be an excellent, affine scheme, $Z \subset X$ a closed subscheme, $U := X \setminus Z$, and G a coherent sheaf on U. Assume, in addition, that $Z \cap \bar{W}_i$

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has codimension ≥ 2 in $\bar{W}_i$ for every associated prime $W_i \subset U$ of G. Then $H^0(U,G)$ is a finite $H^0(X,\mathscr{O}_X)$-module. □

It is considerably harder to understand finiteness for $H^1(U,G)$. The following special case is used in Section 5.4.

Claim 10.29.9 Let X be an excellent scheme, $Z \subset X$ a closed subscheme, $U := X \setminus Z$, and G a coherent sheaf on U. Assume in addition that G is S_2, there is a coherent CM sheaf F on X and an injection $G \hookrightarrow F|_U$, and Z has codimension ≥ 3 in $\operatorname{Supp} F$. Then $R^1 j_* G$ is coherent.

Proof Set $Q = F|_U/G$. Since G is S_2, it has no extensions with a sheaf whose support has codimension ≥ 2 by (10.6), thus every associated prime of Q has codimension ≤ 1 in $\operatorname{Supp} F$. Thus Q satisfies the assumptions of (10.26) and so j_*Q is coherent. By (10.29.4) $R^1 j_*(F|_U) = 0$, hence the exact sequence

$$0 \to j_*G \to j_*(F|_U) \to j_*Q \to R^1 j_*G \to R^1 j_*(F|_U) = 0$$

shows that $R^1 j_*G$ is coherent. □

Not every S_2-sheaf can be realized as a subsheaf of a CM sheaf, but this can be arranged in some important cases.

Claim 10.29.10 Assume in addition that X is embeddable into a regular, affine scheme R as a closed subscheme, $\operatorname{Supp} G$ has pure dimension $n \geq 3$, $Z = \{x\}$ is a closed point, and G is S_2.

Then $H^1(U,G)$ has finite length. Thus, if X is of finite type over a field k, then $H^1(U,G)$ is a finite dimensional k-vector space.

Outline of proof X plays essentially no role. Let $Y \subset R$ be a complete intersection subscheme defined by $\dim R - n$ elements of $\operatorname{Ann} G$. Then Y is Gorenstein, we can view G as a coherent sheaf on $Y \setminus \{x\}$, and $H^i(X \setminus \{x\}, G) = H^i(Y \setminus \{x\}, G)$. Thus it is enough to prove vanishing of the latter for $i = 1$. By (10.29.11) there is an embedding $G \hookrightarrow \mathscr{O}^m_{Y\setminus\{x\}}$, hence (10.29.9) applies. □

Claim 10.29.11 Let U be a quasi-affine scheme of pure dimension n and G a pure, coherent sheaf on U of dimension n. Assume that either U is reduced, or U is Gorenstein at its generic points.

Then G is isomorphic to a subsheaf of $\mathscr{O}^m_U$ for some m.

Outline of proof Assume that such an embedding exists at the generic points. Then we have an embedding $G \hookrightarrow \mathscr{O}^m_U$ over some dense open set $U^\circ \subset U$. Pick $s \in \mathscr{O}_U$ invertible at the generic points and vanishing along $U \setminus U^\circ$. Multiplying by s^r for $r \gg 1$ gives the embedding $G \hookrightarrow \mathscr{O}^m_U$.

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The remaining question is, what happens at the generic point. The existence of the embedding is clear if U is reduced.

In general, we are reduced to the following algebra question: given an Artinian ring A, is every finite A-module M a submodule of A^m for some m? Usually, the answer is no. However, local duality theory (see, for instance, Eisenbud (1995, secs.21.1–2)) shows that every finite A-module is a submodule of ω_A^m for some m. Finally, A is Gorenstein iff $A \simeq \omega_A$. □

Much of the next result can be proved using these methods, but local duality theory works better, as in Grothendieck (1968, VIII.2.3).

Theorem 10.30 *Let X be an excellent scheme, $Z \subset X$ a closed subscheme, $U := X \setminus Z$, and $j\colon U \hookrightarrow X$ the open embedding. Assume in addition that X is locally embeddable into a regular scheme. For a coherent sheaf G on U and $n \in \mathbb{N}$ the following are equivalent.*

(10.30.1) *$R^i j_* G$ is coherent for $i < n$.*

(10.30.2) $\operatorname{depth}_u G \geq n$ *for every point $u \in U$ such that* $\operatorname{codim}_{\bar{u}}(Z \cap \bar{u}) = 1$. □

10.4 Volumes and Intersection Numbers

We have used several general results that compare intersection numbers and volumes under birational morphisms.

Definition 10.31 (Lazarsfeld, 2004, sec.2.2.C) Let X be a proper scheme of dimension n over a field and D a Mumford $\mathbb{R}$-divisor on X. Its *volume* is

$$\operatorname{vol}(D) := \lim_{m\to\infty} \frac{h^0(X, \mathscr{O}_X(\lfloor mD \rfloor))}{m^n/n!}.$$

Numerically equivalent divisors have the same volume, and, for $D = \sum d_i D_i$, the volume is a continuous function of the d_i; see Lazarsfeld (2004, 2.2.41–44). If D is nef then $\operatorname{vol}(D) = (D^n)$ (11.52).

Proposition 10.32 *Let $p\colon Y \to X$ be a birational morphism of normal, proper varieties of dimension n. Let D_Y be a p-nef $\mathbb{R}$-Cartier $\mathbb{R}$-divisor such that $D_X := p_*(D_Y)$ is also $\mathbb{R}$-Cartier. Then*

(10.32.1) $\operatorname{vol}(D_X) \geq \operatorname{vol}(D_Y)$*, and*

(10.32.2) *if D_X is ample then equality holds iff $D_Y \sim_{\mathbb{R}} p^* D_X$.*

Furthermore, let H be an ample divisor on X. Then

(10.32.3) *$I(H, D_X) \geq I(p^*H, D_Y)$ (with $I(*, *)$ as in (5.13)), and*

(10.32.4) *equality holds iff $D_Y \sim_{\mathbb{R}} p^* D_X$.*

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Proof Write $D_Y = p^*D_X - E$ where E is p-exceptional. By assumption $-E$ is p-nef, hence E is effective by (11.60). Thus $\operatorname{vol}(D_X) = \operatorname{vol}(p^*D_X) \geq \operatorname{vol}(D_Y)$, proving (1). Parts (2) and (4) are special cases of (10.39), but here is a more direct argument.

Set $r = \dim(p(\operatorname{Supp} E))$. For any $\mathbb{R}$-Cartier divisors A_i on X, the intersection number $(p^*A_1 \cdots p^*A_j \cdot E)$ vanishes whenever $j > r$. Thus, if $j > r$ then

$$(p^*H^j \cdot D_Y^{n-j}) = (p^*H^j \cdot (p^*D_X - E)^{n-j}) = (p^*H^j \cdot p^*D_X^{n-j}) = (H^j \cdot D_X^{n-j}),$$

and for $j = r$ we get that

$$(p^*H^r \cdot D_Y^{n-r}) = (H^r \cdot D_X^{n-r}) + (p^*H^r \cdot (-E)^{n-r}).$$

Thus we need to understand $(p^*H^r \cdot (-E)^{n-r})$. We may assume that H is very ample. Intersecting with p^*H is then equivalent to restricting to the preimage of a general member of $|H|$. Using this r-times (and normalizing if necessary), we get a birational morphism $p' \colon Y' \to X'$ between normal varieties of dimension $n - r$ and an effective, nonzero, p-exceptional $\mathbb{R}$-Cartier $\mathbb{R}$-divisor E' such that $-E'$ is p'-nef and $p'(E_i')$ is 0-dimensional. Thus, by (10.33), $(p^*H^r \cdot (-E)^{n-r}) = (-E')^{n-r} < 0$ which proves (3–4).

If D_X is ample then we can use this for $H := D_X$. Then $(H^r \cdot D_X^{n-r}) = (D_X^n)$ and we get (2). □

Lemma 10.33 *Let $p \colon Y \to X$ be a proper, birational morphism of normal schemes. Let E be an effective, nonzero, p-exceptional $\mathbb{R}$-Cartier $\mathbb{R}$-divisor such that $p(E)$ is 0-dimensional and $-E$ is p-nef. Set $n = \dim E$.*

Then $-(-E)^{n+1} = (-E|_E)^n > 0$.

Proof Assume that there is an effective, nonzero, p-exceptional $\mathbb{R}$-Cartier $\mathbb{R}$-divisor F such that $p(F) = p(E)$, $-F$ is p-nef and $-(-F)^{n+1} > 0$. Note that E, F have the same support, namely $p^{-1}(p(E))$, thus $E - \varepsilon F$ is effective for $0 < \varepsilon \ll 1$. Thus $-(-E)^n \geq -(-\varepsilon F)^n$ by (10.34) applied to $N_2 = -E, N_1 = -\varepsilon F$.

Such a divisor F exists on the normalization of the blow-up $B_{p(E)}X$. Let $Z \to X$ be a proper, birational morphism that dominates both Y and $B_{p(E)}X$. We can apply the above observation to the pull-backs of E and F to Z. □

Lemma 10.34 *Let N_1, N_2 be $\mathbb{R}$-Cartier divisors with proper support on an $n+1$-dimensional scheme. Assume that there exists an effective divisor with proper support D such that $D \sim_{\mathbb{R}} N_1 - N_2$ and the $N_i|_D$ are both nef. Then $(N_1^{n+1}) \geq (N_2^{n+1})$.*

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Proof $(N_1^{n+1}) - (N_2^{n+1}) = D \cdot \sum_{i=0}^{n} N_1^i N_2^{n-i} = \sum_{i=0}^{n} (N_1|_D)^i (N_2|_D)^{n-i}$. □

The next results compare the volumes of different perturbations of the canonical divisor.

Lemma 10.35 *Let X be a normal, proper variety of dimension n, and D an effective $\mathbb{R}$-divisor such that $K_X + D$ is $\mathbb{R}$-Cartier, nef and big. Let Y be a smooth, proper variety birational to X. Then*
(10.35.1) $\operatorname{vol}(K_Y) \le (K_X + D)^n$, *and*
(10.35.2) *equality holds iff $D = 0$ and X has canonical singularities.*

Proof Let Z be a normal, proper variety birational to X such that there are morphisms $q\colon Z \to Y$ and $p\colon Z \to X$. Write

$$K_Z \sim_{\mathbb{R}} q^* K_Y + E \quad \text{and} \quad K_Z \sim_{\mathbb{R}} p^*(K_X + D) - p_*^{-1} D + F, \tag{10.35.3}$$

where E is effective, q-exceptional and F is p-exceptional (not necessarily effective). Thus

$$q^* K_Y \sim_{\mathbb{R}} p^*(K_X + D) - p_*^{-1} D + F - E. \tag{10.35.4}$$

Write $F - E = G^+ - G^-$ where G^+, G^- are effective and without common irreducible components. Note that G^+ is p-exceptional, therefore

$$H^0(Z, \mathscr{O}_Z(\lfloor mp^*(K_X + D) + mG^+ \rfloor)) = H^0(Z, \mathscr{O}_Z(\lfloor mp^*(K_X + D) \rfloor)), \quad \text{so}$$
$$H^0(Z, \mathscr{O}_Z(\lfloor mp^*(K_X + D) - p_*^{-1}(mD) + mG^+ - mG^- \rfloor))$$
$$= H^0(Z, \mathscr{O}_Z(\lfloor mp^*(K_X + D) - p_*^{-1}(mD) - mG^- \rfloor)).$$

This implies that

$$\begin{aligned}\operatorname{vol}(K_Y) &= \operatorname{vol}(p^*(K_X + D) - p_*^{-1} D + G^+ - G^-) \\ &= \operatorname{vol}(p^*(K_X + D) - p_*^{-1} D - G^-) \\ &\le \operatorname{vol}(p^*(K_X + D)) = \operatorname{vol}(K_X + D) = (K_X + D)^n.\end{aligned}$$

Furthermore, by (10.39) equality holds iff $p_*^{-1} D + G^- = 0$, that is, when $D = 0$ and $G^- = 0$. In such a case (10.35.4) becomes $q^* K_Y \sim_{\mathbb{R}} p^* K_X + G^+$ and G^+ is effective. Thus $a(E, X) \ge a(E, Y)$ for every divisor E by (11.4.3), hence X has canonical singularities. □

A similar birational statement does not hold for pairs in general, but a variant holds if Y is a resolution of X. We can also add some other auxiliary divisors; these are needed in our applications.

Lemma 10.36 *Let X be a normal, proper variety of dimension n and Δ a reduced, effective $\mathbb{R}$-divisor on X. Let A be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor and D an effective $\mathbb{R}$-divisor such that $K_X + \Delta + A + D$ is $\mathbb{R}$-Cartier, nef and big. Let $p\colon Y \to X$ be any log resolution of (X, Δ). Then*
(10.36.1) $\operatorname{vol}(K_Y + p_*^{-1}\Delta + p^*A) \le (K_X + \Delta + A + D)^n$ *and*
(10.36.2) *equality holds iff $D = 0$ and (X, Δ) is canonical.*

Proof There are p-exceptional, effective divisors F_i such that

$$K_Y + p_*^{-1}\Delta \sim_{\mathbb{R}} p^*(K_X + \Delta + D) - p_*^{-1}D - F_1 + F_2. \tag{10.36.3}$$

As in (10.35), we get that

$$\begin{aligned} &H^0(Y, \mathscr{O}_Y(\lfloor mp^*(K_X + \Delta + A + D) - p_*^{-1}(mD) - mF_1 + mF_2\rfloor)) \\ &\qquad = H^0(Y, \mathscr{O}_Y(\lfloor mp^*(K_X + \Delta + A + D) - p_*^{-1}(mD) - mF_1\rfloor)), \quad \text{and} \end{aligned}$$

$$\begin{aligned} &\operatorname{vol}(K_Y + p_*^{-1}\Delta + p^*A) = \operatorname{vol}\big(p^*(K_X + \Delta + A + D) - p_*^{-1}D + F_2 - F_1\big) \\ &\quad = \operatorname{vol}\big(p^*(K_X + \Delta + A + D) - p_*^{-1}D - F_1\big) \le \operatorname{vol}\big(p^*(K_X + \Delta + A + D)\big) \\ &\qquad\qquad = \operatorname{vol}(K_X + \Delta + A + D) = (K_X + \Delta + A + D)^n. \end{aligned}$$

Furthermore, by (10.39), equality holds iff $p_*^{-1}D + F_1 = 0$, that is, when $D = 0$ and $F_1 = 0$. Thus (10.36.3) becomes $K_Z + p_*^{-1}\Delta \sim_{\mathbb{R}} p^*(K_X + \Delta) + F_2$, where F_2 is effective. This says that (X, Δ) is canonical. □

Essentially the same argument gives the following log canonical version.

Lemma 10.37 *Let X be a normal, proper variety of dimension n, Δ a reduced, effective $\mathbb{R}$-divisor on X and A an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. Let $q\colon \bar{X} \to X$ be a proper birational morphism, $\bar{E}$ the reduced q-exceptional divisor, $\bar{\Delta} := q_*^{-1}\Delta$, and $\bar{D}$ an effective $\mathbb{R}$-divisor on $\bar{X}$ such that $K_{\bar{X}} + \bar{\Delta} + \bar{E} + D + q^*A$ is $\mathbb{R}$-Cartier, nef and big. Let $p\colon Y \to X$ be any log resolution of singularities with reduced exceptional divisor E. Then*
(10.37.1) $\operatorname{vol}(K_Y + p_*^{-1}\Delta + E + p^*A) \le (K_{\bar{X}} + \bar{\Delta} + \bar{E} + \bar{D} + q^*A)^n$ *and*
(10.37.2) *equality holds iff $\bar{D} = 0$ and $(\bar{X}, \bar{\Delta} + \bar{E})$ is log canonical.* □

We have also used the following elementary estimate.

Lemma 10.38 *Let $p\colon Y \to X$ be a separable, generically finite morphism between smooth, proper varieties. Then* $\operatorname{vol}(K_Y) \ge \deg(Y/X) \cdot \operatorname{vol}(K_X)$.

Proof This is obvious if $\operatorname{vol}(K_X) = 0$, hence we may assume that K_X is big. Pulling back differential forms gives a natural map $p^*\omega_X \to \omega_Y$. This gives an

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injection $\omega_X^r \otimes p_*\omega_Y \hookrightarrow p_*(\omega_Y^{r+1})$. Since $p_*\omega_Y$ has rank $\deg(Y/X)$ and K_X is big, $H^0(X, \omega_X^r \otimes p_*\omega_Y)$ grows at least as fast as $\deg(Y/X) \cdot H^0(X, \omega_X^r)$. □

The following result describes the variation of the volume near a nef and big divisor. The assertions are special cases of Fulger et al. (2016, thms.A–B).

Theorem 10.39 *Let X be a proper variety, L a big $\mathbb{R}$-Cartier divisor, and E an effective divisor. The following are equivalent.*
(10.39.1) $\mathrm{vol}(L - E) = \mathrm{vol}(L)$*, and*
(10.39.2) $H^0(\mathscr{O}_X(\lfloor mL - mE \rfloor)) = H^0(\mathscr{O}_X(\lfloor mL \rfloor))$ *for every* $m \geq 0$.
If L is nef then these are further equivalent to
(10.39.3) $E = 0$.

Note that (3) ⇒ (2) ⇒ (1) are clear, but the converse is somewhat surprising. It says that although the volume measures only the asymptotic growth of the Hilbert function, one cannot change the Hilbert function without changing the volume. For proofs, see Fulger et al. (2016, thms.A–B).

10.5 Double Points

We used a variety of results about hypersurface double points. For the rest of the section, we work with rings R that contain $\frac{1}{2}$. In this case, all the definitions that we have seen are equivalent to the ones given below. If $\frac{1}{2} \notin R$, there are differing conventions, especially if $\operatorname{char} R = 2$.

The following results on normal forms, deformations, and resolutions of double points are well known, but not easy to find in one place.

Definition 10.40 A *quadratic form* over a field k is a degree 2 homogeneous polynomial $q(x_1, \dots, x_n) \in k[x_1, \dots, x_n]$. The *rank* of q is defined either as the dimension of the space spanned by the derivatives $\left\langle \frac{\partial q}{\partial x_1}, \dots, \frac{\partial q}{\partial x_n} \right\rangle$, or as the rank of $\operatorname{Hess}(q) := \left(\frac{\partial^2 q}{\partial x_i \partial x_j} \right)$, or as the number of variables in any diagonalized form $q = a_1 y_1^2 + \cdots + a_r y_r^2$ where $a_i \in k^\times$. More abstractly, if V is a k-vector space, we can think of q as an element of the symmetric square of its dual $S^2(V^*)$.

Definition 10.41 Let (S, m) be a regular local ring with residue field k such that $\operatorname{char} k \neq 2$. We can identify m^2/m^3 with $S^2(m/m^2)$. Thus, for any $g \in m^2$, we can view its image in m^2/m^3 as a quadratic form.

Let Y be a smooth variety over a field of characteristic $\neq 2$ and $X = (g = 0) \subset Y$ a hypersurface. Given a point $p \in X$, we let $\operatorname{rank}_p X$ denote the rank of the image of g in m_p^2/m_p^3.

We say that $p \in X$ is a *double point* if $\operatorname{rank}_p X \geq 1$, a *cA point* if $\operatorname{rank}_p X \geq 2$, and an *ordinary double point* if $\operatorname{rank}_p X = \dim_p X$. An ordinary double point is also called a *node,* especially if $\dim S = 2$.

If $y_1, \dots, y_n$ are étale coordinates on Y then $\operatorname{Hess}_{\mathbf{y}}(g) = \left(\frac{\partial^2 g}{\partial y_i \partial y_j}\right)$. Since the rank is lower semicontinuous, $\{p \in \operatorname{Sing} X\colon\ \operatorname{rank}_p X \geq r\}$ is open in $\operatorname{Sing} X$ for every r. For us the most interesting case is $r = 2$. The relative version is then the following.

Claim 10.41.1 Let $f\colon Y \to S$ be smooth and $X \subset Y$ a relative Cartier divisor. Then $\{p \in X\colon\ p \text{ is } cA \text{ (or smooth) on } X_{f(p)}\} \subset X$ is open. □

This implies that if $X \to S$ is proper and X_s has only cA-singularities (and smooth points) outside a closed subset $Z_s \subset X_s$ of codimension $\geq m$ for some $s \in S$ then the same holds in an open neighborhood $s \in S^\circ \subset S$.

Corollary 10.42 *Let $\pi\colon X \to S$ be a flat and pure dimensional morphism. Then the set of points $\{x\colon X_{\pi(x)}$ is demi-normal at $x\}$ is open in X.*

Proof Being S_2 is an open condition by (10.12). An S_1 scheme is geometrically reduced iff it is generically smooth and smoothness is an open condition. Thus being S_2 and geometrically reduced is an open condition.

It remains to show that having only nodes in codimension 1 is also an open condition. If all residue characteristics are $\neq 2$, this follows from (10.41.3) since having only cA-singularities in codimension 1 is an open condition.

See Kollár (2013b, 1.41) for nodes in characteristic 2. □

Let f be a function on $\mathbb{R}^n$ that has an ordinary critical point at the origin. The Morse lemma says that in suitable local coordinates $y_1, \dots, y_n$ we can write f as $\pm y_1^2 \pm \cdots \pm y_n^2$; see Milnor (1963, p.6) and Arnol′d et al. (1985, vol.I.sec.6.2) for differentiable and analytic versions. Algebraically, the best is to work with formal power series. We prove a form that also works if $\operatorname{char}(R/m) = 2$.

Lemma 10.43 (Formal Morse lemma with parameters) *Let (R, m) be a complete local ring and $G \in R[[x_1, \dots, x_n]]$. Assume that $G = q + H$, where q is a quadratic form with reduction modulo m denoted by $\bar{q}$ such that*

(10.43.1) $\dim\langle \partial\bar{q}/\partial x_1, \dots, \partial\bar{q}/\partial x_n \rangle = n$, *and*

(10.43.2) $H \in (x_1, \dots, x_n)^3 + mR[[x_1, \dots, x_n]]$.

Then there are local coordinates $y_1, \dots, y_n$ such that

(10.43.3) $y_i \equiv x_i \mod (x_1, \dots, x_n)^2 + mR[[x_1, \dots, x_n]]$, *and*

(10.43.4) $G = q(y_1, \dots, y_n) + b$ *for some $b \in m$.*

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Proof Let us start with the case when $R=k$ is a field. Set $x_{2,i} := x_i$. Assume inductively (starting with $r = 2$) that there are local coordinate systems $(x_{s,1}, \dots, x_{s,n})$ for $3 \le s \le r$ such that

$$\begin{aligned} x_{s,i} &\equiv x_{s-1,i} \mod (x_1, \dots, x_n)^{s-1} \quad \text{and} \\ G &\equiv q(x_{r,1}, \dots, x_{r,n}) \mod (x_1, \dots, x_n)^{r+1}. \end{aligned}$$

Next we choose $x_{r+1,i} := x_{r,i} + h_{r,i}$ for suitable $h_{r,i} \in (x_1, \dots, x_n)^r$. Note that

$$q(x_{r+1,1}, \dots, x_{r+1,n}) = q(x_{r,1}, \dots, x_{r,n}) + \textstyle\sum_i h_{r,i} \frac{\partial q}{\partial x_i} \mod (x_1, \dots, x_n)^{2r}.$$

(We use this only modulo $(x_1, \dots, x_n)^{r+2}$.) Since q is nondegenerate,

$$\textstyle\sum_i \frac{\partial q}{x_i} (x_1, \dots, x_n)^r = (x_1, \dots, x_n)^{r+1}.$$

Thus we can choose the $h_{r,i}$ such that

$$G - q(x_{r+1,1}, \dots, x_{r+1,n}) \in (x_1, \dots, x_n)^{r+2}.$$

In the limit we get $(x_{\infty,1}, \dots, x_{\infty,n})$ as required.

Applying this to $k = R/m$, we can assume from now on that

$$G - q(x_1, \dots, x_n) \in mR[[x_1, \dots, x_n]].$$

Working inductively (starting with $r = 1$), assume that there are local coordinate systems $(y_{s,1}, \dots, y_{s,n})$ for $3 \le s \le r$ such that

$$\begin{aligned} y_{s,i} &\equiv y_{s-1,i} \mod m^{s-1} R[[x_1, \dots, x_n]] \quad \text{and} \\ G &\equiv q(y_{r,1}, \dots, y_{r,n}) \mod m + m^r R[[x_1, \dots, x_n]]. \end{aligned}$$

Next we choose $y_{r+1,i} := y_{r,i} + c_{r,i}$ for suitable $c_{r,i} \in m^r R[[x_1, \dots, x_n]]$. Note that

$$q(y_{r+1,1}, \dots, y_{r+1,n}) = q(y_{r,1}, \dots, y_{r,n}) + \textstyle\sum_i c_{r,i} \frac{\partial q}{\partial x_i} \mod m^{2r} R[[x_1, \dots, x_n]].$$

(We use this only modulo $m^{r+1} R[[x_1, \dots, x_n]]$.) Since q is nondegenerate,

$$\textstyle\sum_i \frac{\partial q}{\partial x_i} m^r R[[x_1, \dots, x_n]] = (x_1, \dots, x_n) m^r R[[x_1, \dots, x_n]].$$

Thus we can choose the $c_{r,i}$ such that

$$G - q(y_{r+1,1}, \dots, y_{r+1,n}) \in m + m^{r+1} R[[x_1, \dots, x_n]].$$

In the limit we get $(y_{\infty,1}, \dots, y_{\infty,n})$ as required. □

In (1.27), we used various results on resolutions of double points of surfaces that contain a pair of lines and double points of 3–folds that contain a pair of planes. The normal forms can be obtained using the method of (10.43), but we did not follow how linear subvarieties transform under the (non-linear)

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coordinate changes used there. However, in the next examples, one can be quite explicit about the coordinate changes and the resolutions.

10.44 (Ordinary double points of surfaces) Let $S := (h(x_1, x_2, x_3) = 0) \subset \mathbb{A}^3$ be a surface with an ordinary double point at the origin that contains the pair of lines $(x_1x_2 = x_3 = 0)$. Then h can be written as

$$h = f(x_1, x_2, x_3)x_1x_2 - g(x_1, x_2, x_3)x_3.$$

Since the quadratic part has rank 3, then $f(0, 0, 0) \neq 0$ and we can write $g = x_1g_1 + x_2g_2 + x_3g_3$ for some polynomials g_i. Thus

$$h = f(x_1 - f^{-1}g_1x_3)(x_2 - f^{-1}g_2x_3) - (g_3 + f^{-1}g_1g_2)x_3^2.$$

Here $g_3 + f^{-1}g_1g_2$ is nonzero at $(0, 0, 0)$ and we can set

$$y_1 := x_1 - f^{-1}g_1x_3,\ y_2 := f(x_2 - f^{-1}g_2x_3)(g_3 + f^{-1}g_1g_2)^{-1} \quad \text{and} \quad y_3 := x_3$$

to bring the equation to the normal form $S = (y_1y_2 - y_3^2 = 0)$. The pair of lines is $(y_1y_2 = y_3 = 0)$.

Now we consider three ways of resolving the singularity of X. First, one can blow up the origin $0 \in \mathbb{A}^3$. We get $B_0\mathbb{A}^3 \subset \mathbb{A}^3_{\mathbf{y}} \times \mathbb{P}^2_{\mathbf{s}}$ defined by the equations $\{y_is_j = y_js_i \colon 1 \leq i, j \leq 3\}$. Besides these equations, B_0S is defined by $y_1y_2 - y_3^2 = s_1s_2 - s_3^2 = y_1s_2 - y_3s_3 = s_1y_2 - y_3s_3 = 0$.

One can also blow up (y_1, y_3). We get $B_{(y_1,y_3)}\mathbb{A}^3 \subset \mathbb{A}^3_{\mathbf{y}} \times \mathbb{P}^1_{u_1u_3}$ defined by the equation $y_1u_3 = y_3u_1$. Besides this equation, $B_{(y_1,y_3)}S$ is defined by $y_1y_2 - y_3^2 = u_1y_2 - u_3y_3 = 0$.

These two blow-ups are actually isomorphic, as shown by the embedding

$$\mathbb{A}^3_{\mathbf{y}} \times \mathbb{P}^1_{u_1u_3} \hookrightarrow \mathbb{A}^3_{\mathbf{y}} \times \mathbb{P}^2_{\mathbf{s}} \quad : \quad ((y_1, y_2, y_3), (u_1{:}u_3)) \mapsto ((y_1, y_2, y_3), (u_1^2{:}u_3^2{:}u_1u_3))$$

restricted to $B_{(y_1,y_3)}S$. The same things happen if we blow up (y_2, y_3).

10.45 (Ordinary double points of 3-folds) Let $X := (h(x_1, \dots, x_4) = 0) \subset \mathbb{A}^4$ be a hypersurface with an ordinary double point at the origin that contains the pair of planes $(x_1x_2 = x_3 = 0)$. Then h can be written as

$$h = f(x_1, \dots, x_4)x_1x_2 - g(x_1, \dots, x_4)x_3.$$

The quadratic part has rank 4 iff $f(0, \dots, 0) \neq 0$ and x_4 appears in g with nonzero coefficient. In this case, we can set $y_i := x_i$ for $i = 1, 2, 3$ and $y_4 := f^{-1}g$ to bring the equation to the normal form $X = (y_1y_2 - y_3y_4 = 0)$. The original pair of planes is $(y_1y_2 = y_3 = 0)$.

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Now we consider three ways of resolving the singularity of X. First, one can blow up the origin $0 \in \mathbb{A}^4$. We get $B_0\mathbb{A}^4 \subset \mathbb{A}^4_{\mathbf{y}} \times \mathbb{P}^3_{\mathbf{s}}$, defined by the equations $\{y_i s_j = y_j s_i \colon 1 \le i, j \le 4\}$, and $p\colon B_0X \to X$ by the additional equations

$$y_1y_2 - y_3y_4 = s_1s_2 - s_3s_4 = y_i s_{3-i} - y_j s_{7-j} = 0 \,:\, i \in \{1,2\}, j \in \{3,4\}.$$

The exceptional set is the smooth quadric $(s_1s_2 = s_3s_4) \subset \mathbb{P}^3$ lying over the origin $0 \in \mathbb{A}^4$.

One can also blow up (y_1, y_3). Then $B_{(y_1,y_3)}\mathbb{A}^4 \subset \mathbb{A}^4_{\mathbf{y}} \times \mathbb{P}^1_{u_1u_3}$ is defined by the equation $y_1u_3 = y_3u_1$. Besides this equation, $B_{(y_1,y_3)}X$ is defined by $y_1y_2 - y_3y_4 = u_1y_2 - u_3y_4 = 0$. The exceptional set is the smooth rational curve $E \simeq \mathbb{P}^1_{u_1u_3}$ lying over the origin $0 \in \mathbb{A}^4$.

Note furthermore that the birational transform P^*_{24} of the plane $P_{24} := (y_2 = y_4 = 0)$ is the blown-up plane B_0P_{24}, but the birational transform P^*_{14} of the plane $P_{14} := (y_1 = y_4 = 0)$ is the plane $(y_1 = u_1 = 0)$. The latter intersects E at the point $(u_1 = 0) \in E$, thus $(P^*_{14} \cdot E) = 1$. Since $P^*_{14} + P^*_{24}$ is the pull-back of the Cartier divisor $(y_4 = 0)$, it has 0 intersection number with E. Thus $(P^*_{24} \cdot E) = -1$.

By direct computation, the rational map $p\colon \mathbb{A}^4_{\mathbf{y}} \times \mathbb{P}^3_{\mathbf{s}} \dashrightarrow \mathbb{A}^4_{\mathbf{y}} \times \mathbb{P}^1_{\mathbf{u}}$ given by $p_1\colon (y_1, \dots, y_4, s_1{:}\cdots{:}s_4) \mapsto (y_1, \dots, y_4, s_1{:}s_3)$ gives a morphism $p_1\colon B_0X \to B_{(y_1,y_3)}X$. Similarly, we obtain $p_2\colon B_0X \to B_{(y_2,y_3)}X$ and an isomorphism

$$p_1 \times p_2\colon B_0X \simeq B_{(y_1,y_3)}X \times_X B_{(y_2,y_3)}X.$$

Finally, set $S := (y_3 = y_4) \subset X$. By the computations of (10.44), the p_i restrict to isomorphisms $p_i\colon B_0S \simeq B_{(y_i,y_3)}S$. Thus $p^{-1}S = B_0S \cup E$ and B_0S is the graph of the isomorphism $p_2 \circ p_1^{-1}\colon B_{(y_1,y_3)}S \simeq B_{(y_2,y_3)}S$.

10.6 Noether Normalization

10.46 (Classical versions) Noether's normalization theorem says that if X is an affine (resp. projective) k-variety of dimension n then it admits a finite morphism to $\mathbb{A}^n_k$ (resp. $\mathbb{P}^n_k$).

We aim to generalize this to arbitrary morphisms. For the projective case, let $X \subset \mathbb{P}^N_S$ be projective over S and $n = \dim X_s$ for some $s \in S$. Choose a linear subspace $L_s \subset \mathbb{P}^N_s$ of dimension $N - n - 1$ that is disjoint from X_s. (This is always possible if $k(s)$ is infinite, otherwise we may need to take a high enough Veronese embedding first.) Lifting L_s to $\mathbb{P}^N_S$ and projecting from it gives the following.

Claim 10.46.1 Let $p\colon X \to S$ be a projective morphism and $n = \dim X_s$ for some $s \in S$. Then there is an open neighborhood $s \in S^\circ \subset S$ such that $p|_{X^\circ}$ can be factored as

$$p|_{X^\circ}: X^\circ \xrightarrow{\text{finite}} \mathbb{P}^n_{S^\circ} \longrightarrow S^\circ. \qquad \square$$

In general, we have the following weaker local version.

Claim 10.46.2 Let $p : X \to S$ be a finite type morphism and $x \in X$ a closed point. Then there is an open neighborhood $x \in X^\circ \subset X$ and an open embedding $X^\circ \hookrightarrow X^*$, where $p^* : X^* \to S$ is projective of relative dimension $\le \dim X_s$.

Proof Set $d := \dim X_s$ and pick $g_1, \dots, g_d \in \mathscr{O}_{x,X}$ that generate an $m_{x,X}$-primary ideal. They give a rational map $X \dashrightarrow \mathbb{A}^d_S \hookrightarrow \mathbb{P}^d_S$ that is quasi-finite on some $x \in X^\circ \subset X$. We then take $X^\circ \subset X^*$ such that $X^* \to \mathbb{P}^d_S$ is finite. $\square$

Next we give two examples showing that in (10.46.2) one cannot choose X° such that $X^\circ \to \mathbb{A}^d_S$ is finite, not even when S is local. After that we discuss an étale local version for finite type morphisms due to Raynaud and Gruson (1971). Arbitrary morphisms are discussed in (10.52); these results work best for morphisms of complete local schemes.

Example 10.47 We give an example of a morphism of pure relative dimension one $p: X \to S$ from an affine 3-fold X to a smooth, pointed surface $s \in S$ that cannot be factored as

$$p: X \xrightarrow{\text{finite}} \mathbb{A}^1 \times S \longrightarrow S,$$

not even over a formal neighborhood of s. Such examples are quite typical and there does not seem to be any affine version of Noether normalization over base schemes of dimension ≥ 2.

Let S denote the localization (or completion) of $\mathbb{A}^2_{st}$ at the origin and consider the affine scheme

$$X := ((x^3 + y^3 + 1)(1 + tx) + sy = 0) \subset \mathbb{A}^2_{xy} \times S.$$

Then $\pi: X \to S$ is a flat family of curves. We claim that there is no finite morphism of it onto $\mathbb{A}^1 \times S$.

Assume to the contrary that such a map $g: X \to \mathbb{A}^1 \times S$ exists. Then g can be extended to a finite morphism $\bar{g}: \bar{X} \to \mathbb{P}^1 \times S$.

Here $\bar{X}_{(0,0)}$ is a compactification of $X_{(0,0)}$, hence a curve of genus 1.

For $t \neq 0$, the line $(1 + tx = s = 0)$ gives an irreducible component of $\bar{X}_{(0,t)}$ that is a rational curve. As $t \to 0$, the limit of these rational curves is a union of rational, irreducible, geometric components of $\bar{X}_{(0,0)}$, a contradiction.

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Example 10.48 In $\mathbb{A}^4_{xyst}$ consider the surface $X := (x - sy^2 = y - tx^2 = 0)$. Projection to $\mathbb{A}^2_{xy}$ is birational with inverse $(x, y) \mapsto (s, t) = (x/y^2, y/x^2)$. The projection to $\mathbb{A}^2_{st}$ is quasi-finite.

Consider the projection $\pi\colon \mathbb{A}^4_{xyst} \to \mathbb{A}^3_{zst}$ given by $z = x + y$. We claim that the closure of its image contains the z-axis. Indeed, for any c, the curve

$$t \mapsto \left(t, c - t, \tfrac{t}{(c-t)^2}, \tfrac{c-t}{t^2}\right)$$

lies on X and its projection converges to $(c, 0, 0)$ as $t \to \infty$.

It is easy to see that the same happens for every perturbation of π. In fact, given $(x, y) \mapsto (a(s, t)x+b(s, t)y+c(s, t))$, the closure of the image of X contains the z-axis whenever $a(0, 0) \neq 0 \neq b(0, 0)$.

The next result of Raynaud and Gruson (1971) shows that Noether normalization works étale locally. The version given in Stacks (2022, tag 052D) states the first part, but following the proof gives the additional information about the choices.

Theorem 10.49 *Let $f\colon X \to S$ be a finite type morphism. Pick $s \in S$, a closed point $x \in X_s$ and set $n = \dim_x X_s$. Then there is an elementary étale neighborhood (2.18) $\pi\colon (x', X') \to (x, X)$ such that $f \circ \pi$ factors as*

$$(x', X') \xrightarrow{g} (y, Y) \xrightarrow{\tau} (s, S), \tag{10.49.1}$$

where g is finite, $g^{-1}(y) = \{x'\}$ (as sets), τ is smooth of relative dimension n, and $k(y) = k(s)$.

Moreover, pick $c \in \mathbb{N}$ and $x_1, \dots, x_n \in m_{x,X_s}$ that generate an m_{x,X_s}-primary ideal. Then we can choose (10.49.1) such that there are $y_1, \dots, y_n \in m_{y,Y_s}$ satisfying $g_s^* y_i \equiv \pi_s^* x_i \mod m^c_{x',X'_s}$ for every i. □

If X_s is generically geometrically reduced, then we can choose $y_{n+1} \in m_{x,X_s}$ with specified residue modulo m^c_{x',X'_s} and which embeds the generic fiber of $X_s \to Y_s$ into $\mathbb{A}^1_{k(Y_s)}$. Lifting it to X' and setting $Y' := \mathbb{A}^1_Y$ gives the following birational version of Noether normalization.

Corollary 10.50 *Let $f\colon X \to S$ be a finite type morphism. Pick $s \in S$ and a closed point $x \in X_s$. Assume that X_s is generically geometrically reduced and of pure dimension n. Then there is an elementary étale neighborhood $\pi\colon (x', X') \to (x, X)$ such that $f \circ \pi$ factors as*

$$(x', X') \xrightarrow{g'} (y', Y') \xrightarrow{\tau'} (s, S), \tag{10.50.1}$$

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where g' is finite, $(g')^{-1}(y') = \{x'\}$ (as sets), $g'_s\colon X'_s \to Y_s \times \mathbb{A}^1$ is birational, τ' is smooth of relative dimension $n+1$, and $k(y') = k(s)$.

Moreover, pick $c \in \mathbb{N}$ and $x_1, \dots, x_{n+1} \in m_{x,X_s}$ that generate an m_{x,X_s}-primary ideal. Then we can arrange that there are $y_1, \dots, y_{n+1} \in m_{y,Y_s}$ satisfying ${g'_s}^* y_i \equiv \pi_s^* x_i \mod m^c_{x',X'_s}$ for every i. □

Corollary 10.51 *Let $f\colon X \to S$ be a finite type morphism of pure relative dimension n. Pick $s \in S$ and a closed point $x \in X_s$ such that $k(x) = k(s)$. Assume that S is normal and f is flat at the generic points of X_s. Assume also that* $\operatorname{embdim}_x \operatorname{pure}(X_s) \le n+1$. *Then there is an elementary étale $\pi\colon (x', X') \to (x, X)$ such that (10.50.1) further factors as*

$$(x', X') \xrightarrow{g'} (y', D') \hookrightarrow (y', Y') \xrightarrow{\tau'} (s, S), \qquad (10.51.1)$$

where, $D' \subset Y'$ is a relative Cartier divisor, g' is birational, $g'_s\colon X'_s \to D'_s$ is birational and induces a local isomorphism $\operatorname{pure}(X'_s) \to D'_s$ *at x'.*

Proof Since $\operatorname{embdim}(\operatorname{pure}(X_s) \le n+1$, we can choose $x_1, \dots, x_{n+1} \in m_{x,X_s}$ that generate the ideal of $x \in \operatorname{pure}(X_s)$. Applying (10.50) with $c = 2$ guarantees that $\operatorname{pure}(X'_s) \to D'_s$ is a local isomorphism at x'.

D' is a relative Cartier divisor by (4.4) and then (10.54) implies that g' is a local isomorphism at the generic points of X'_s. Thus g' is birational. □

Informally speaking, (10.51) says that partial normalizations of flat deformations of hypersurfaces describe all deformations over normal base schemes. For double points this approach leads to a complete answer (10.68). More substantial applications are in de Jong and van Straten (1991).

Next we turn to local morphisms of Noetherian local schemes

10.52 (Noether normalization, local version) Let $f\colon (x, X) \to (s, S)$ be a morphism of local, Noetherian schemes. We would like to factor f as

$$f\colon (x, X) \xrightarrow{p} (s', S') \xrightarrow{q} (s, S), \qquad (10.52.1)$$

where p has "finiteness" properties and q has "smoothness" properties. Let us start with the case when $k(x) \supset k(s)$ is a finitely generated field extension. Pick any transcendence basis $\bar{y}_1, \dots, \bar{y}_n$ of $k(x)/k(s)$ and lift these back to $y_1, \dots, y_n \in \mathscr{O}_X$. We can then take S' to be the localization of $\mathbb{A}^n_S$ at the generic point of the fiber over $s \in S$. Thus we have proved the following.

Claim 10.52.2 Let $f\colon (x, X) \to (s, S)$ be a local morphism of local, Noetherian schemes such that $k(x) \supset k(s)$ is a finitely generated field extension. Then we can factor f as $f\colon (x, X) \xrightarrow{p} (s', S') \xrightarrow{q} (s, S)$, where $k(x)/k(s')$ is a finite field extension, q is the localization of a smooth morphism and $q^{-1}(s) = s'$ (as schemes). □

For Henselian schemes, we can do better. Pick $\bar{y} \in k(x)$ that is separable over $k(s')$ with separable, monic equation $\bar{g}(\bar{y}) = 0$. If $\mathcal{O}_X$ is Henselian then we can lift $\bar{y}$ to $y \in \mathcal{O}_X$ such that y satisfies a separable, monic equation $g(y) = 0$. We can now replace S' with the Henselization of $\mathcal{O}_{S'}[y]/(g(y))$ at the generic point of the central fiber, and obtain the following.

Claim 10.52.3 Let $f\colon (x, X) \to (s, S)$ be a local morphism of local, Henselian, Noetherian schemes such that $k(x)/k(s)$ is a finitely generated field extension. Then we can factor f as $f\colon (x, X) \xrightarrow{p} (s', S') \xrightarrow{q} (s, S)$, where p is finite, $k(x)/k(s')$ is a purely inseparable field extension, q is the localization of a smooth morphism and $q^{-1}(s) = s'$ (as schemes). □

Combining these with (10.53) gives the following.

Claim 10.52.4 Let $f\colon (x, X) \to (s, S)$ be a local morphism of local, complete, Noetherian schemes. Then we can factor f as $f\colon (x, X) \xrightarrow{p} (s', S') \xrightarrow{q} (s, S)$, where $k(x)/k(s')$ is a purely inseparable field extension, q is formally smooth, faithfully flat, regular and $q^{-1}(s) = s'$ (as schemes). □

Putting these together we get the following.

Claim 10.52.5 Let $f\colon (x, X) \to (s, S)$ be a local morphism of local, complete, Noetherian schemes such that $k(x)/k(s)$ is separable. Set $n := \dim X_s$.

Then we can factor f as

$$f\colon (x, X) \xrightarrow{p} ((s', 0), \hat{\mathbb{A}}^n_{S'}) \xrightarrow{\pi} (s', S') \xrightarrow{q} (s, S),$$

where p is finite, $k(x) = k(s', 0) = k(s')$, π is the coordinate projection, $q^{-1}(s) = s'$ (as schemes), q is the localization of a smooth morphism if $k(x)/k(s)$ is finitely generated and formally smooth, faithfully flat and regular in general.

Proof By (10.52.4), we have $q\colon (s', S') \to (s, S)$ such that $k(x) = k(s')$. Since $\mathcal{O}_{X_s}$ has dimension n, there are $\bar{t}_1, \dots, \bar{t}_n \in \mathcal{O}_{X_s}$ that generate an ideal that is primary to the maximal ideal. Lift these back to $t_1, \dots, t_n \in \mathcal{O}_X$. These define $p\colon (x, X) \to ((s', 0), \hat{\mathbb{A}}^n_{S'})$. By construction, $\mathcal{O}_X/(m_S, t_1, \dots, t_n) \simeq \mathcal{O}_{X_s}/(\bar{t}_1, \dots, \bar{t}_n)$ is finite over $k(s')$. Thus p is finite. □

Notation 10.52.6 Let R be a complete, local ring and $Y = \operatorname{Spec} R$. We write $\hat{\mathbb{A}}^n_Y := \operatorname{Spec} R[[x_1, \dots, x_n]]$. Note that $\hat{\mathbb{A}}^n_Y$ is *not* the product of $\hat{\mathbb{A}}^n$ with Y in any sense. If $X \to Y$ is a finite morphism then $\hat{\mathbb{A}}^n_X \simeq X \times_Y \hat{\mathbb{A}}^n_Y$.

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10.53 (Residue field extensions) Let (s, S) be a Noetherian, local scheme and $K/k(s)$ a field extension. By Grothendieck (1960, 0_{III}.10.3.1), there is a Noetherian, local scheme (x, X) and a flat morphism $g\colon (x, X) \to (s, S)$ such that $g^* m_{s,S} = m_{x,X}$ (that is, the scheme fiber $g^{-1}(s)$ is the reduced point $\{x\}$) and $k(x) \simeq K$.

If $K/k(s)$ is a finitely generated separable extension then we can choose $g\colon (x, X) \to (s, S)$ to be the localization of a smooth morphism. In particular, if S is normal then so is X.

Combining Grothendieck (1960, 0_{III}.10.3.1) and Stacks (2022, tag 07PK) shows that if $K/k(s)$ is an arbitrary separable extension, then we can choose $g\colon (x, X) \to (s, S)$ to be formally smooth. If S is complete then g is also regular. In particular, if S is normal then so is X.

Note that infinite inseparable extensions do cause problems in the above arguments. One difficulty is that they can lead to non-excellent schemes; see Nagata (1962, p.206).

10.54 (Openness for isomorphism) Let $g\colon (x, X) \to (s, S)$ be a local morphism of local, Noetherian schemes and $g\colon G \to F$ a map of coherent sheaves on X. Assume that F is flat over S. Then g is an isomorphism iff g_s is an isomorphism, and g_s is injective iff g is injective and $\operatorname{coker} g$ is flat over S. See Matsumura (1986, 22.5) or Kollár (1996, I.7.4.1) for proofs. Applying this to the structure sheaf of a scheme and its image, we get the following.

Claim 10.54.1 Let $\pi\colon X \to Y$ be a finite morphism of S-schemes. Assume that X is flat over S. Then π is an isomorphism (resp. closed embedding) in a neighborhood of a fiber X_s iff $\pi_s\colon X_s \to Y_s$ is an isomorphism (resp. closed embedding).

10.7 Flatness Criteria

Let $g\colon X \to S$ be a morphism and F a coherent sheaf on X. We are mainly interested in those cases when F is flat over S with pure fibers of dimension d for some d. In practice, we already know that $F|_U$ is flat for some dense open subset $U \subset X$ and we aim to find conditions that guarantee flatness.

Note that such a result is possible only if $F|_U$ determines F. Thus we at least need to assume that none of the associated point of F are contained in Z.

10.55 (Flatness and associated points) Let $f\colon X \to S$ be a morphism of Noetherian schemes and F a coherent sheaf on X.

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Claim 10.55.1 If F is flat over S then $f(\mathrm{Ass}(F)) \subset \mathrm{Ass}(S)$.

Proof Let $x \in X$ be an associated point of F and $s := f(x)$. Assume that s is not an associated point of S. Then there is an $r \in m_{s,S}$ such that $r\colon \mathscr{O}_S \to \mathscr{O}_S$ is injective near s. Tensoring with F shows that $r\colon F \to F$ is injective near X_s. Thus none of the points of X_s is in $\mathrm{Ass}(F)$. □

Claim 10.55.2 Assume that F is flat over S and $x \in \mathrm{Ass}(F)$. Then every generic point of $\mathrm{Supp}(\bar{x} \cap X_s)$ is an associated point of F_s. In particular, if F is flat with pure fibers then every $x \in \mathrm{Ass}(F)$ is a generic point of $\mathrm{Supp}(F_{f(x)})$.

Proof Let $G \subset F$ be the largest subsheaf supported on $\bar{x}$. After localizing at a generic point of $\mathrm{Supp}(\bar{x} \cap X_s)$, we have $\mathrm{Supp}(\bar{x} \cap X_s) = \{w\}$, a single closed point. There is an $n \geq 0$ such that $G \subset m_{s,S}^n F$, but $G \not\subset m_{s,S}^{n+1} F$. Thus $m_{s,S}^n F / m_{s,S}^{n+1} F \simeq (m_{s,S}^n / m_{s,S}^{n+1}) \otimes F_s$ has a nonzero subsheaf supported on w. □

Note that flatness is needed for (10.55.2) as illustrated by the restriction of either of the coordinate projections to the union of the axes $(xy = 0)$.

Claim 10.55.3 Assume f is of finite type, F is flat over S, and $x \in \mathrm{Ass}(F)$. Then every fiber of $\bar{x} \to f(\bar{x})$ has the same dimension.

Proof We may assume that $f(x)$ is a minimal associated point of S. Assume that we have $s \in f(\bar{x})$ such that $\dim(X_s \cap \bar{x})$ is larger than the expected dimension d. By restricting to a general relative Cartier divisor $H \subset X$, $F|_H$ is flat along H_s by (10.56) and $H_s \cap \bar{x}$ is a union of associated points of $F|_H$ by (10.22.1). Repeating this $d+1$ times we get Cartier divisors $H^1, \dots, H^{d+1} \subset X$ and a complete intersection $Z := H^1 \cap \cdots \cap H^{d+1}$ such that $F|_Z$ is flat along Z_s, the generic points of $Z \cap \bar{x}$ are associated points of $F|_Z$ yet they do not dominate $f(\bar{x})$. This is impossible by (10.55.1). □

Next we discuss some basic reduction steps.

Let $f\colon X \to S$ be a morphism that we would like to prove to be flat. We can usually harmlessly assume that S is local.

If f is of finite type, then flatness is an open property. Let $U \subset X$ denote the largest open set over which f is flat and set $Z := X \setminus U$. The situation is technically simpler if Z is a single closed point. To achieve this, one can use (10.56) to pass to a general hyperplane section of X and repeat if necessary, until Z becomes zero-dimensional. A potential drawback is that, while we can choose general hyperplanes, some fibers are nongeneral complete intersections, so may be harder to control.

Alternatively, we can localize at a generic point of Z. Then f is no longer of finite type, which can cause problems.

Once S and X are both local, we can take their completions. Now we have a local morphism of complete, local, Noetherian schemes. Note, however, that some of our results hold only over base schemes that are normal, seminormal, or reduced. These properties are preserved by completion for excellent schemes, but not in general.

Proposition 10.56 (Bertini theorem for flatness) (Matsumura, 1986, p.177) *Let $(x, X) \to (s, S)$ be a local morphism of local schemes, $r \in m_{x,X}$ and F a coherent sheaf on X. The following are equivalent.*

(10.56.1) *r is a non-zerodivisor on F and F/rF is flat over S.*

(10.56.2) *r is a non-zerodivisor on F_s and F is flat over S.* □

10.57 (Flatness and residue field extension) The following simple trick reduces most flatness questions for local morphisms $f: (x, X) \to (s, S)$ with finitely generated residue field extension $k(x)/k(s)$ to the special case when $k(x) = k(s)$ and they are infinite. (See 10.52–10.53 for other versions.)

If $k(x)/k(s)$ is a generated by n elements then there is a point $s' \in \mathbb{A}^n_{k(s)}$ such that $k(x) \subset k(s')$ and $k(s')$ is infinite.

Consider next the trivial lifting $f': X' := \mathbb{A}^n_X \to S' := \mathbb{A}^n_S$. Set $s' \in \mathbb{A}^n_{k(s)} \subset S'$ and $x' := (s', x) \in X'$ projecting to x. Thus we have a commutative diagram of pointed schemes

$$\begin{array}{ccc} (x' \in X') & \xrightarrow{\pi_X} & (x \in X) \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ (s' \in S') & \xrightarrow{\pi_S} & (s \in S) \end{array} \qquad (10.57.1)$$

where π_X, π_S are smooth, $k(x') = k(s')$ and f is flat at x iff f' is flat at x'.

Many properties of schemes and morphisms are preserved by composing with smooth morphisms; see Matsumura (1986, sec.23) for a series of such results. Thus the properties of (s, S) are inherited by (s', S'). Once we prove a result about (x', X') it descends to (x, X).

Over reduced bases, flatness is usually easy to check if we know all the fibers. For projective morphisms there are criteria using the Hilbert function (3.20). In the local case, we have the following.

Lemma 10.58 *Let S be a reduced scheme and $f: X \to S$ a morphism that is of finite type, pure dimensional and with geometrically reduced fibers. Then f is flat.*

Proof By (4.38), it is enough to show this when (s, S) is the spectrum of a DVR. In this case f is flat iff none of the associated points of X is contained in X_s. By assumption X_s is reduced, so only generic points of X_s could occur. Then the corresponding irreducible component of X_s is also an irreducible component of X, but we also assumed that f has pure relative dimension. □

10.59 (Format of flatness criteria) In many cases we have some information about the fibers of a morphism, but we do not fully understand them. So we are looking for results of the following type.

Let (s, S) be a local scheme, $f: X \to S$ a morphism and F a vertically pure coherent sheaf on X. Let $Z \subset X$ be a closed subset such that $F_{X \setminus Z}$ is flat over S. We make various assumptions on pure(F_s) (involving Z_s) and on S. The conclusion should be that F is flat and F_s is pure.

The natural way to organize the results is by the *relative codimension;* in the local case this equals $\operatorname{codim}_{X_s}(Z_s)$. The starting case is when $Z = X$, so the codimension is 0.

The main theorems are (10.60), (10.63), (10.67), (10.71) and (10.73).

Flatness in Relative Codimension 0

The basic result is the following, proved in Grothendieck (1971, II.2.3).

Theorem 10.60 *Let $f: (x, X) \to (s, S)$ be a local morphism of local, Noetherian schemes of the same dimension such that $f^{-1}(s) = x$ as schemes, that is, $m_{x,X} = m_{s,S}\mathscr{O}_X$. Assume that $k(x) \supset k(s)$ is separable and $\hat{S}$, the completion of S, is normal. (Note that if S is normal and excellent, then $\hat{S}$ is normal.)*

Then f is flat at x.

Proof We may replace S and X by their completions. As in (10.52.4), we can factor f as

$$f: (x, X) \xrightarrow{p} (y, Y) \xrightarrow{q} (s, S)$$

where (y, Y) is also complete, local, Noetherian, $k(x) = k(y)$, $m_{x,X} = m_{y,Y}\mathscr{O}_X$ and q is flat.

Thus $p^*: m_{y,Y}/m_{y,Y}^2 \to m_{x,X}/m_{x,X}^2$ is surjective, hence $p^*: \mathscr{O}_Y \to \mathscr{O}_X$ is surjective by the Nakayama lemma. Equivalently, $p: X \to Y$ is a closed embedding. It is thus an isomorphism, provided Y is integral.

In order to ensure these properties of Y we need to know more about q. If $k(x)/k(s)$ is finitely generated then q is the localization of a smooth morphism (10.52.3). Thus Y is normal and $\dim Y = \dim S$, as required. The general case is technically harder. We use that q is formally smooth and geometrically regular (10.52.4) to reach the same conclusions as before.

Thus p is an isomorphism, so $f = q$ and f is flat. □

Examples 10.61 These examples show that the assumptions in (10.60) and (10.63) are necessary.

(10.61.1) Assume that $\operatorname{char} k \neq 2$ and set $C := (y^2 = ax^2 + x^3)$ where $a \in k$ is not a square. Let $f\colon \bar{C} \to C$ denote the normalization. Then the fiber over the origin is the spectrum of $k(\sqrt{a})$, which is a separable extension of k. Here C is not normal and f is not flat.

(10.61.2) The extension $\mathbb{C}[x, y] \subset \mathbb{C}[\frac{x}{y}, y]_{(y)}$ is not flat yet $(x, y) \cdot \mathbb{C}[\frac{x}{y}, y]_{(y)}$ is the maximal ideal and the residue field extension is purely transcendental. However, the dimension of the larger ring is 1.

A similar thing happens with $\mathbb{C}[x, y] \hookrightarrow \mathbb{C}[[t]]$ given by $(x, y) \mapsto (t, \sin t)$. The fiber over the origin is the origin with reduced scheme structure.

(10.61.3) On $\mathbb{C}[x, y]$ consider the involution $\tau(x) = -x, \tau(y) = -y$. The invariant ring is $\mathbb{C}[x^2, xy, y^2] \subset \mathbb{C}[x, y]$. The fiber over the origin is the spectrum of $\mathbb{C}[x, y]/(x^2, xy, y^2)$; it has length 3 and embedding dimension 2. The fiber over any other point has length 2. Thus the extension is not flat.

(10.61.4) As in Kollár (1995a, 15.2), on $S := k[x_1, x_2, y_1, y_2]$ consider the involution $\tau(x_1, x_2, y_1, y_2) = (x_2, x_1, y_2, y_1)$. The ring of invariants is

$$R := k[x_1 + x_2, x_1x_2, y_1 + y_2, y_1y_2, x_1y_1 + x_2y_2].$$

The resulting extension is not flat along $(x_1 - x_2 = y_1 - y_2 = 0)$.

If $\operatorname{char} k = 2$, then $x_1 - x_2, y_1 - y_2$ are invariants. Set $P := (x_1 - x_2, y_1 - y_2)R$. Then $S/PS = S/(x_1 - x_2, y_1 - y_2)S \simeq k[x_1, y_1]$ and $R/P \simeq k[x_1^2, y_1^2]$.

Thus $S_P \supset R_P$ is a finite extension whose fiber over P is $k(x_1, y_1) \supset k(x_1^2, y_1^2)$. This is an inseparable field extension, generated by 2 elements.

(10.61.5) Set $X := (z = 0) \cup (z - x = z - y = 0) \subset \mathbb{A}^3$ with coordinate projection $\pi\colon X \to \mathbb{A}^2_{xy}$. Then π is finite, has curvilinear fibers, but not flat.

These examples leave open only one question: What happens with curvilinear fibers?

Definition 10.62 (Curvilinear schemes) Let k be a field and (A, m) a local, Artinian k-algebra. We say that $\operatorname{Spec}_k A$ is *curvilinear* if A is cyclic as a $k[t]$-module for some t. That is, if A can be written as a quotient of $k[t]$. It is easy to see that this holds if either A/m is a finite, separable extension of k and m is a principal ideal, or A is a field extension of k of degree = $\operatorname{char} k$.

Let B be an Artinian k-algebra. Then $\operatorname{Spec}_k B$ is called *curvilinear* if all of its irreducible components are curvilinear. If k is an infinite field, this holds iff B can be written as a quotient of $k[t]$. If K/k is a field extension and $\operatorname{Spec}_k B$ is curvilinear then so is $\operatorname{Spec}_K(B \otimes_k K)$.

Let $\pi\colon X \to S$ be a finite-type morphism. The embedding dimension of fibers is upper semicontinuous, thus the set $\{x \in X\colon X_{\pi(x)}$ is curvilinear at $x\}$ is open.

Theorem 10.63 *Let $f\colon X \to S$ be a finite type morphism with curvilinear fibers such that every associated point of X dominates S. Assume that either S is normal, or there is a closed $W \subset S$ such that* $\operatorname{depth}_W S \geq 2$ *and f is flat over $S \setminus W$. Then f is flat.*

Proof We start with the classical case when X, S are complex analytic, S is normal, f is finite, and $X \subset S \times \mathbb{C}$. Let $s \in S$ be a smooth point. Then $S \times \mathbb{C}$ is smooth along $\{s\} \times \mathbb{C}$ thus X is a Cartier divisor near X_s. In particular, f is flat over the smooth locus $S^{\mathrm{ns}} \subset S$. Set $d := \deg f$. For each $s \in S^{\mathrm{ns}}$ there is a unique monic polynomial $t^d + a_{d-1}(s)t^{d-1} + \cdots + a_0(s)$ of degree d whose zero set is precisely $X_s \subset \mathbb{C}$. As in the proof of the analytic form of the Weierstrass preparation theorem (see, for instance, Griffiths and Harris (1978, p.8) or Gunning and Rossi (1965, sec.II.B)) we see that the $a_i(s)$ are analytic functions on S^{ns}. By Hartogs's theorem, they extend to analytic functions on the whole of S; we denote these still by $a_i(s)$. Thus

$$X = (t^d + a_{d-1}(s)t^{d-1} + \cdots + a_0(s) = 0) \subset S \times \mathbb{C}$$

is a Cartier divisor and f is flat. This completes the complex analytic case.

In general, we argue similarly, but replace the polynomial $t^d + a_{d-1}(s)t^{d-1} + \cdots + a_0(s)$ by the point in the Hilbert scheme corresponding to X_s.

Assume first that f is finite. Again set $d := \deg f$ and let $S^\circ \subset S$ denote a dense open subset over which f is flat. Since f is finite, it is (locally) projective, thus we have

$$\begin{array}{ccc} \operatorname{Univ}_d(X/S) & \xrightarrow{\ p\ } & X \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle f} \\ \operatorname{Hilb}_d(X/S) & \xrightarrow{\ \pi\ } & S \end{array} \tag{10.63.1}$$

parametrizing length d quotients of the fibers of f. If $s \in S^\circ$ then $\mathscr{O}_{X_s}$ has length d, hence its sole length d quotient is itself. Thus π is an isomorphism over S°.

Let $s \to S$ be a geometric point. Then $X_s \simeq \operatorname{Spec} k(s)[t]/(\prod_i (t - a_i)^{m_i})$ for some $a_i \in k(s)$ and $m_i \in \mathbb{N}$. Thus the fiber of p over s is a finite set corresponding to length d quotients of $k(s)[t]/(\prod_i (t - a_i)^{m_i})$, equivalently, to solutions

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of the equation $\sum_i m'_i = d$ where $0 \le m'_i \le m_i$. We have not yet proved that $\mathrm{Hilb}_d(X/S)$ has no embedded points over $\mathrm{Sing}\, S$, but we obtain that $\mathrm{pure}(\mathrm{Hilb}_d(X/S)) \to S$ is finite and birational, hence an isomorphism if S is normal or if $S^\circ \supset S \setminus W$ in case (2) by (10.6.4). The natural map

$$\mathrm{pure}(p)\colon\ \mathrm{Univ}_d(X/S) \times_{\mathrm{Hilb}_d(X/S)} \mathrm{pure}(\mathrm{Hilb}_d(X/S)) \to X$$

is a closed immersion whose image is isomorphic to X over S°. Thus $\mathrm{pure}(p)$ is an isomorphism, so f is flat and $\mathrm{Hilb}_d(X/S) \simeq S$.

Finally, (10.49) reduces the general case to the finite one. (Note that any finite type, quasi-finite morphism can be extended to a finite morphism, but there is no reason to believe that the extension still has curvilinear fibers. So we need to use the more difficult (10.49).) □

Over a nonnormal base there does not seem to be any simple analog of (10.63), but the following is quite useful.

Proposition 10.64 *Let $f\colon X \to (s, S)$ be a finite morphism with curvilinear fibers. Assume that*

(10.64.1) *the pair $(s \in S)$ is weakly normal (10.74),*

(10.64.2) *f is flat of constant degree d over $S \setminus \{s\}$,*

(10.64.3) *X has no associated points in* $\mathrm{Supp}\, f^{-1}(s)$*, and*

(10.64.4) *either $x := \mathrm{Supp}\, f^{-1}(s)$ is a single point and $k(x)/k(s)$ is purely inseparable, or f has well-defined specializations (4.2.9).*

Then f is flat.

Proof Again consider the diagram (10.63.1). By (2), p and π are isomorphisms over $S \setminus \{s\}$. We claim that π is an isomorphism. Since $(s \in S)$ is weakly normal, this holds if $\pi^{-1}(s)$ has a unique geometric point. If f has well-defined specializations, this holds by definition.

Otherwise, let $s' \to s$ be a geometric point. Since $k(x)/k(s)$ is purely inseparable, $X_{s'} \simeq \mathrm{Spec}\, k(s')[t]/(t^r)$ for some r, which has a unique subscheme of length d. Thus π is an isomorphism. As in the proof of (10.63), we conclude that p is also an isomorphism. □

The proof given in (4.21) applies with minor modifications to give the following result of Ramanujam (1963) and Samuel (1962); see also Grothendieck (1960, IV.21.14.1).

Theorem 10.65 (Principal ideals in power series rings) *Let (R, m) be a normal, complete, local ring and $P \subset R[[x_1, \dots, x_n]]$ a height 1 prime ideal that is not contained in $mR[[x_1, \dots, x_n]]$. Then P is principal.* □

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Corollary 10.66 (Unique factorization in power series rings) *Let* (R, m) *be a normal, complete, local ring and* $g \in R[[x_1, \ldots, x_n]]$ *a power series not contained in* $mR[[x_1, \ldots, x_n]]$. *Then* g *has a unique factorization as* $g = \prod_i p_i$ *where each* (p_i) *is a prime ideal.*

Proof Let P_i be a height 1 associated prime ideal of (g). Then P_i is not contained in $mR[[x_1, \ldots, x_n]]$ thus it is principal by (10.65). □

Example 10.66.1 A lemma of Gauss says that if R is a UFD then $R[t]$ is also a UFD. More generally, if Y is a normal scheme then $\mathrm{Cl}(Y \times \mathbb{A}^n) \simeq \mathrm{Cl}(Y)$. If $\mathbb{A}^n$ is replaced by a smooth variety X then there is an obvious inclusion

$$\mathrm{Cl}(Y) \times \mathrm{Cl}(X) \hookrightarrow \mathrm{Cl}(Y \times X),$$

but, as the next example shows, this map is not surjective, not even if $\mathrm{Cl}(Y) = \mathrm{Cl}(X) = 0$.

Let $E \subset \mathbb{P}^2$ be a cubic defined over $\mathbb{Q}$ such that $\mathrm{Pic}(E)$ is generated by a degree 3 point $P := E \cap L$ for some line $L \subset \mathbb{P}^2$. Let $S \subset \mathbb{A}^3$ be the affine cone over E and $E^\circ := E \setminus P$. Then $\mathrm{Cl}(S) = 0$ and $\mathrm{Cl}(E^\circ) = 0$. However, we claim that $\mathrm{Cl}(S \times E^\circ)$ is infinite.

To see this, pick any $\phi \in \mathrm{End}(E)$. (For example, for any m we have multiplication by $3m + 1$ which sends $p \in E(\bar{\mathbb{Q}})$ to the unique point $\phi(p) \sim (3m + 1)p - mP$.) The lines $\{\ell_p \times \{\phi(p)\} \colon p \in E\}$ sweep out a divisor in $S \times E$, where $\ell_p \subset S$ denotes the line over $p \in E$. It is not hard to see that this gives an isomorphism $\mathrm{End}(E) \simeq \mathrm{Cl}(S \times E^\circ)$.

As another application, let R denote the complete local ring of S at its vertex. The above considerations also show that R is a UFD, but $R[[t]]$ is not.

Flatness in Relative Codimension 1

The following result is stated in all dimensions, but we will have stronger theorems when the codimension is ≥ 2.

Theorem 10.67 *Let* $f \colon X \to S$ *be a finite type morphism of Noetherian schemes,* $s \in S$ *a closed point, and* $Z \subset X_s$ *a nowhere dense closed subset such that* f *is flat along* $X_s \setminus Z$. *Assume that*

(10.67.1) $\mathrm{pure}_Z(X_s)$ *is smooth,*

(10.67.2) $\dim S \geq 1$ *and* S *has no embedded points, and*

(10.67.3) X *has no embedded points.*

Then f *is smooth.*

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Proof Pick a closed point $x \in Z$. By (10.57) we may assume that $k(x) = k(s)$. Choose local coordinates $x_1, \dots, x_n \in m_{x,X_s}$ and apply (10.50). Then there is an elementary étale $\pi\colon (x', X') \to (x, X)$ such that $f \circ \pi$ factors as

$$(x', X') \xrightarrow{g} (y, Y) \xrightarrow{\tau} (s, S),$$

where g is finite, $g^{-1}(y') = \{x'\}$ (as sets), τ is smooth of relative dimension n, and $k(y) = k(s)$. We also know that g_s induces an embedding $\mathrm{pure}(X'_s) \to Y_s$.

We claim that g is an isomorphism. To see this, note first that, since $X' \to S$ is flat along $X'_s \setminus Z'$, (10.54) implies that there is a smallest closed subset $W \subset Y$ such that $g^{-1}(X'_s \cap W) \subset Z'$ and g is an isomorphism over $Y \setminus W$. Since $Y \to S$ is smooth, we are done if $W = \emptyset$.

To see this, pick a generic point $w \in W$ with projections $p_Y \in Y$ and $p \in S$. Since Y_p is smooth and $X'_p \to Y_p$ is an isomorphism outside W, we see that $\mathrm{pure}_W(X_p) \simeq Y_p$. Thus X'_p has an embedded point in $g^{-1}(W \cap Y_p)$. Therefore p is not a generic point of S by (3). Then

$$\mathrm{depth}_{p_Y} Y = \mathrm{depth}_{p_Y} Y_p + \mathrm{depth}_p S \geq 1 + 1 = 2,$$

and X' has no associated points contained in $g^{-1}(W)$ (3). Hence g is an isomorphism by (10.6). □

In codimension 1, an slc pair is either smooth or has nodes. Next we show that a close analog of (10.67) holds for nodal fibers if the base scheme is normal; the latter assumption is necessary by (10.70.1).

Corollary 10.68 *Let (s, S) be a normal, local scheme and $f\colon X \to S$ a finite type morphism of pure relative dimension 1. Assume that f is generically flat along X_s and* $\mathrm{pure}(X_s)$ *has a single singular point x, which is a node. Then, in a neighborhood of x, one of the following holds:*

(10.68.1) *f is flat and its fibers have only nodes.*

(10.68.2) *f is not flat, X is not S_2 and the normalization $\bar{f}\colon \bar{X} \to S$ is smooth.*

Proof By (10.51), after étale coordinate changes, we may assume that X is a partial normalization of a relative hypersurface $H = (h = 0) \subset \mathbb{A}^2_S$ such that h_s has a single node.

If the generic fiber H_g is smooth, then H is normal and so $X = H$. Otherwise, $\partial h/\partial x = \partial h/\partial y = 0$ is an étale section. After an étale base change, we may assume that the fibers are singular along the zero section $Z \subset \mathbb{A}^2_S \to S$. Blowing it up gives the normalization $\tau\colon \bar{H} \to H$, which is smooth over S. Furthermore, we have an exact sequence

$$0 \to \mathscr{O}_H \to \tau_* \mathscr{O}_{\bar{H}} \to \mathscr{O}_Z \to 0.$$

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Since X lies between $\bar{H}$ and H, there is an ideal sheaf $J \subset \mathscr{O}_Z$ such that $\mathscr{O}_X/\mathscr{O}_H \simeq J$.

If $J = 0$ then $X \simeq H$. If $J = \mathscr{O}_Z$, then $X \simeq \bar{H}$. The projection to S is flat in both cases. Otherwise $\mathrm{Supp}(\mathscr{O}_{\bar{H}}/\mathscr{O}_X) = \mathrm{Supp}(\mathscr{O}_Z/J)$ has codimension ≥ 2 in H, thus X is not S_2 by (10.6). □

With different methods, the following generalization of (10.68) is proved in Kollár (2011b). The projectivity assumption should not be necessary.

Theorem 10.69 *Let (s, S) be a normal, local scheme and $f: X \to S$ a projective morphism of pure relative dimension 1. Assume that X is S_2 and* $\mathrm{pure}(X_s)$ *is seminormal (resp. has only simple, planar singularities).*

Then f is flat with reduced fibers that are seminormal (resp. have only simple, planar singularities). □

See Arnol′d et al. (1985, I.p.245) for the conceptual definition of simple, planar singularities. For us, it is quickest to note that a plane curve singularity $(f(x, y) = 0)$ is simple iff $(z^2 + f(x, y) = 0)$ is a Du Val surface singularity.

Examples 10.70 The next examples show that (10.68–10.69) do not generalize to nonnormal bases or to other curve singularities.

10.70.1 (*Deformations of ordinary double points*) Let $C \subset \mathbb{P}^2$ be a nodal cubic with normalization $p: \mathbb{P}^1 \to C$. Over the coordinate axes $S := (xy = 0) \subset \mathbb{A}^2$ consider the family X that is obtained as follows.

Over the x-axis take a smoothing of C, over the y-axis take $\mathbb{P}^1 \times \mathbb{A}^1_y$ and glue them over the origin using $p: \mathbb{P}^1 \to C$ to get $f: X \to S$.

Then X is seminormal and S_2, the central fiber is C with an embedded point, yet f is not flat.

10.70.2 (*Deformations of ordinary triple points*) Consider the family of plane cubic curves

$$\mathbf{C} := ((x^2 - y^2)(x + t) + t(x^3 + y^3) = 0) \subset \mathbb{A}^2_{xy} \times \mathbb{A}^1_t.$$

For every t, the origin is a singular point, but it has multiplicity 3 for $t = 0$ and multiplicity 2 for $t \neq 0$. Thus blowing up the line $(x = y = 0)$ gives the normalization for $t \neq 0$, but it introduces an extra exceptional curve over $t = 0$. The normalization of $\mathbf{C}$ is obtained by contracting this extra curve. The fiber over $t = 0$ is then isomorphic to three lines though the origin in $\mathbb{A}^3$.

10.70.3 (*Deformations of ordinary quadruple points*) Let $\mathbf{C}_4 \to \mathbb{P}^{14}$ be the universal family of degree 4 plane curves and $\mathbf{C}_{4,1} \to S^{12}$ the 12-dimensional subfamily whose general members are elliptic curves with two nodes. We normalize both the base and the total space to get $\bar\pi\colon \bar{\mathbf{C}}_{4,1} \to \bar S^{12}$.

We claim that the fiber of $\bar\pi$ over the plane quartic with an ordinary quadruple point $C_0\colon\ = (x^3y - xy^3 = 0)$ is C_0 with at least two embedded points. Most likely, the family is not even flat.

We prove this by showing that in different families of curves through $[C_0] \in S^{12}$ we get different flat limits.

To see this, note that the seminormalization C_0^{sn} of C_0 can be thought of as four general lines through a point in $\mathbb{P}^4$. In suitable affine coordinates, we can write it as $k[x,y]/(x^3y - xy^3) \hookrightarrow k[u_1,\dots,u_4]/(u_iu_j\colon i \neq j)$ using the map $(x,y) \mapsto (u_1 + u_3 + u_4, u_2 + u_3 - u_4)$. Any three-dimensional linear subspace $\langle u_1,\dots,u_4\rangle \supset W_\lambda \supset \langle u_1 + u_3 + u_4, u_2 + u_3 - u_4\rangle$. corresponds to a projection of C_0^{sn} to $\mathbb{P}^3$; call the image $C_\lambda \subset \mathbb{P}^3$. Then C_λ is four general lines through a point in $\mathbb{P}^3$; thus it is a $(2,2)$-complete intersection curve of arithmetic genus 1. (Note that the C_λ are isomorphic to each other, but the isomorphism will not commute with the map to C_0 in general.) Every C_λ can be realized as the special fiber in a family $S_\lambda \to B_\lambda$ of $(2,2)$-complete intersection curves in $\mathbb{P}^3$ whose general fiber is a smooth elliptic curve.

By projecting these families to $\mathbb{P}^2$, we get a one-parameter family $S'_\lambda \to B_\lambda$ of curves in S^{12} whose special fiber is C_0 .

Let $\bar S'_\lambda \subset \bar{\mathbf{C}}_{4,1}$ be the preimage of this family in the normalization. Then $\bar S'_\lambda$ is dominated by the surface S_λ. In particular, the preimage of C_0 in $\bar{\mathbf{C}}_{4,1}$ is connected.

There are two possibilities. First, if $\bar S'_\lambda$ is isomorphic to S_λ, then the fiber of $\bar{\mathbf{C}}_{4,1} \to \bar S^{12}$ over $[C_0]$ is C_λ. This, however, depends on λ, a contradiction. Second, if $\bar S'_\lambda$ is not isomorphic to S_λ, then the fiber of $\bar S'_\lambda \to B_\lambda$ over the origin is C_0 with some embedded points. Since C_0 has arithmetic genus 3, we must have at least two embedded points.

Flatness in Relative Codimension ≥ 2

Once we know flatness at codimension 1 points of the fibers, the following general result, valid for coherent sheaves, can be used to prove flatness everywhere. We no longer need any restrictions on the base scheme S.

Theorem 10.71 *Let $f\colon X \to S$ be a finite type morphism of Noetherian schemes, (s,S) local. Let F be a vertically pure coherent sheaf on X and $Z \subset \operatorname{Supp} F_s$ a nowhere dense closed subset. Assume that*

(10.71.1) $\operatorname{depth}_Z \operatorname{pure}_Z(F_s) \geq 2$, *and*

(10.71.2) *F is flat over S along $X \setminus Z$.*

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Then F is flat over S and $\operatorname{tors}_Z(F_s) = 0$.

Proof Set $m := m_{s,S}$ and $X_n := \operatorname{Spec}_X(\mathscr{O}_X/m^n_{s,S}\mathscr{O}_X)$ and $F_n := F|_{X_n}$. We may assume that S is m-adically complete. There are natural complexes

$$0 \to (m^n/m^{n+1}) \cdot F_0 \to F_{n+1} \xrightarrow{r_n} F_n \to 0, \tag{10.71.3}$$

which are exact on $X \setminus Z$, but not (yet) known to be exact along Z, except that r_n is surjective. We also know that

$$(m^n/m^{n+1}) \cdot \operatorname{pure}_Z(F_0) \to \operatorname{pure}_Z(\ker r_n) \tag{10.71.4}$$

is an isomorphism on $X \setminus Z$. Since $\operatorname{depth}_Z \operatorname{pure}_Z(F_0) \geq 2$, this implies that (10.71.4) is an isomorphism on X by (10.6). Next we show that the induced

$$r_n\colon \operatorname{tors}_Z(F_{n+1}) \to \operatorname{tors}_Z(F_n) \quad \text{is surjective.} \tag{10.71.5}$$

Set $K_{n+1} := r_n^{-1}(\operatorname{tors}_Z(F_n))$. We have an exact sequence

$$0 \to \operatorname{pure}_Z(\ker r_n) \to K_{n+1}/\operatorname{tors}_Z(\ker r_n) \to \operatorname{tors}_Z(F_n) \to 0. \tag{10.71.6}$$

Using that (10.71.4) is an isomorphism, we have $\operatorname{depth}_Z \operatorname{pure}_Z(\ker r_n) \geq 2$, hence the sequence (10.71.6) splits by (10.6).

Thus $T := \varprojlim \operatorname{tors}_Z(F_n)$ is a subsheaf of F and $X_s \cap \operatorname{Supp} T \subset Z$. Thus $T = 0$ since F is vertically pure, and $\operatorname{tors}_Z(F_n) = 0$ for every n by (10.71.5).

Now (10.71.4) says that $(m^n/m^{n+1}) \cdot F_0 \simeq \ker r_n$. Therefore the sequences (10.71.3) are exact, F is flat and $\operatorname{tors}_Z(F_0) = 0$. □

Putting together the flatness criteria (10.60), (10.68), (10.69.1) and (10.71) gives the following strengthening of Hironaka (1958).

Theorem 10.72 *Let (s, S) be a normal, local, excellent scheme, X an S_2 scheme, and $f\colon X \to S$ a finite type morphism of pure relative dimension n. Assume that* $\operatorname{pure}(X_s)$ *is*

(10.72.1) *either geometrically normal*

(10.72.2) *or geometrically seminormal and S_2.*

Then f is flat with reduced fibers that are normal in case (1) and seminormal and S_2 in case (2). □

Flatness in Relative Codimension ≥ 3

The following gives an even stronger result in codimension ≥ 3; see Kollár (1995a, thm.12). Lee and Nakayama (2018) pointed out that the purity assumption in (2) is also necessary.

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Theorem 10.73 *Let $f: X \to S$ be a finite type morphism of Noetherian schemes, (s, S) local. Let F a coherent sheaf on X and $Z \subset \operatorname{Supp} F$ a closed subset such that $X_s \cap Z \subset \operatorname{Supp} F_s$ has codimension ≥ 3. Let $j: X_s \setminus Z \hookrightarrow X_s$ be the natural injection. Assume that*

(10.73.1) $\operatorname{depth}_{X_s \cap Z}\big(j_*(F_s|_{X_s \setminus Z})\big) \geq 3$,

(10.73.2) *$F|_{X \setminus Z}$ is flat over S with pure, S_2 fibers, and*

(10.73.3) $\operatorname{depth}_Z F \geq 2$.

Then F is flat over S and $F_s = j_*(F_s|_{X_s \setminus Z})$.

Proof Set $m := m_{s,S}$, $X_n := \operatorname{Spec}_X(\mathscr{O}_X/m^n\mathscr{O}_X)$ and $F_n := F|_{X_n}$. We may assume that $\mathscr{O}_S$ and $\mathscr{O}_X$ are m-adically complete. Set $G_n := F_n|_{X_n \setminus Z}$ and let j denote any of the injections $X_n \setminus Z \hookrightarrow X_n$. By assumption (2) we have exact sequences

$$0 \to (m^n/m^{n+1}) \cdot G_0 \to G_{n+1} \longrightarrow G_n \to 0. \tag{10.73.4}$$

Pushing it forward we get the exact sequences

$$\begin{aligned} 0 \to (m^n/m^{n+1}) \otimes j_*G_0 \to j_*G_{n+1} &\stackrel{r_n}{\to} j_*G_n \\ &\to (m^n/m^{n+1}) \otimes R^1 j_*G_0. \end{aligned} \tag{10.73.5}$$

Here j_*G_0 is coherent and assumption (1) implies (in fact is equivalent to) $R^1 j_*G_0 = 0$ by Grothendieck (1968, III.3.3, II.6 and I.2.9) or (10.29).

Thus the r_n are surjective. This shows that $G := \varprojlim j_*G_n$ is a coherent sheaf on X that is flat over S with S_2 fibers. Furthermore, the natural map $\varrho: F \to G$ is an isomorphism along $X_s \setminus Z$. Thus (10.6) implies that it is an isomorphism. So $F \simeq G$ is flat with central fiber $j_*G_0 = j_*(F_s|_{X_s \setminus Z})$. □

10.8 Seminormality and Weak Normality

Normalization is a very useful operation that can be used to "improve" a scheme X. However, the normalization $X^n \to X$ usually creates new points, and this makes it harder to relate X and X^n. The notions of semi and weak normalization intend to do as much of the normalization as possible, without creating new points.

Definition 10.74 Let X be a Noetherian scheme and $Z \subset X$ a closed, nowhere dense subset. A *finite modification* of X *centered* at Z is a finite morphism $p: Y \to X$ such that the restriction $p: Y \setminus p^{-1}(Z) \to X \setminus Z$ is an isomorphism and none of the associated primes of Y is contained in $p^{-1}(Z)$.

A pair $(Z \subset X)$ is called *normal* if every finite modification $p\colon Y \to X$ centered at Z is an isomorphism. It is called *seminormal* (resp. *weakly normal*) if such a p is an isomorphism, provided $k(x) \hookrightarrow k(\operatorname{red} p^{-1}(x))$ is an isomorphism (resp. purely inseparable) for every $x \in X$.

A reduced scheme X is *normal* (resp. *seminormal* or *weakly normal*) if every pair $(Z \subset X)$ is normal (resp. seminormal or weakly normal).

Let X be a reduced scheme with normalization $X^{\rm n}$. There are unique

$$X^{\rm n} \longrightarrow X^{\rm sn} \xrightarrow{\pi_{\rm sn}} X \qquad \text{and} \qquad X^{\rm n} \longrightarrow X^{\rm wn} \xrightarrow{\pi_{\rm wn}} X,$$

where $X^{\rm sn}$ is seminormal, $X^{\rm wn}$ is weakly normal, and $k(x) \hookrightarrow k(\operatorname{red} \pi_{\rm sn}^{-1}(x))$ (resp. $k(x) \hookrightarrow k(\operatorname{red} \pi_{\rm wn}^{-1}(x))$) is an isomorphism (resp. purely inseparable) for every $x \in X$. Note that $X^{\rm wn} = X^{\rm sn}$ in characteristic 0.

For more details, see Kollár (1996, sec.I.7.2) and Kollár (2013b, sec.10.2).

Examples 10.75 The curve examples led to the general definition of seminormalization, but they do not adequately show how complicated seminormal schemes are in higher dimensions.

(10.75.1) The normalization of the higher cusps $C_{2m+1} := (x^2 = y^{2m+1})$ is

$$\pi_{2m+1}\colon \mathbb{A}^1_t \to C_{2m+1} \quad \text{given by} \quad t \mapsto (t^{2m+1}, t^2).$$

The map π_{2m+1} is a homeomorphism, so it is also the seminormalization. By contrast, the normalization of the higher tacnode $C_{2m} := (x^2 = y^{2m})$ is

$$\pi_{2m}\colon \mathbb{A}^1_t \times \{\pm 1\} \to C_{2m} \quad \text{given by} \quad (t, \pm 1) \mapsto (\pm t^m, t).$$

The map π_{2m} is not a homeomorphism since $(0,0) \in C_{2m}$ has two preimages, $(0,1)$ and $(0,-1)$. The seminormalization of C_{2m} is

$$\tau_{2m}\colon C_2 \simeq (s^2 = t^2) \to C_{2m} \quad \text{given by} \quad (s,t) \mapsto (s^m, t).$$

(10.75.2) Let $g(t) \in k[t]$ be a polynomial without multiple factors and set $C_g := \operatorname{Spec}_k(k + g \cdot k[t])$. We can think of C_g as obtained from $\mathbb{A}^1$ by identifying all roots of g. It is an integral curve whose normalization is $\mathbb{A}^1$. It has a unique singular point $c_g \in C_g$ and $k(c_g) = k$.

If g is separable then C_g is seminormal and weakly normal. If g is irreducible and purely inseparable then C_g is seminormal, but not weakly normal; the weak normalization is $\mathbb{A}^1$.

(10.75.3) If B is a seminormal curve, then every irreducible component of B is also seminormal, but an irreducible component of a seminormal scheme need not be seminormal. In fact, every reduced and irreducible affine variety that is

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smooth in codimension 1, occurs as an irreducible component of a seminormal complete intersection scheme, see Kollár (2013b, 10.12).

(10.75.4) If X is S_2 (but possibly nonreduced) and Z has codimension ≥ 2, then $(Z \subset X)$ is a normal pair by (10.6).

The following properties are proved in Kollár (2016c). The last equivalence is surprising since the completion of a normal local ring is not always normal.

Proposition 10.76 *For a Noetherian scheme X without isolated points, the following are equivalent.*

(10.76.1) *X is normal (resp. seminormal, weakly normal).*

(10.76.2) *$Z \subset X$ is a normal (resp. seminormal, weakly normal) pair for every closed, nowhere dense subset $Z \subset X$.*

(10.76.3) $\{x\} \subset \operatorname{Spec} \mathscr{O}_{x,X}$ *is a normal (resp. seminormal, weakly normal) pair for every nongeneric point $x \in X$.*

(10.76.4) $\{x\} \subset \operatorname{Spec} \widehat{\mathscr{O}}_{x,X}$ *is a normal (resp. seminormal, weakly normal) pair for every nongeneric point $x \in X$.* □

The next results show that many questions about schemes can be settled using points and specializations only, up to homeomorphisms.

Definition 10.77 Let $f: X \to Y$ be a morphism, R a DVR and q: $\operatorname{Spec} R \to Y$ a morphism. We say that q *lifts after a finite extension* if there is a DVR $R' \supset R$ that is the localization of a finite extension of R such that q': $\operatorname{Spec} R' \to \operatorname{Spec} R \to Y$ lifts to q'_X: $\operatorname{Spec} R' \to X$.

10.78 (Universal homeomorphism) A morphism $f: U \to V$ of S-schemes is a *universal homeomorphism* if $f \times_S 1_W: U \times_S W \to V \times_S W$ is a homeomorphism for every S-scheme W; see Stacks (2022, tag 04DC). Equivalently, if f is integral, surjective and geometrically injective, see Stacks (2022, tag 04DF).

The following characterization for local schemes is simple, but useful.

Claim 10.78.1 Let (s, S) be a local scheme and $f: U \to S$ a finite type morphism that is geometrically injective. Then f is a finite, universal homeomorphism iff every local, component-wise dominant (4.30) morphism from the spectrum of a DVR to S, lifts to U, after a finite extension.

Proof For any generic point $g_S \in S$ there is a q: $(t, T) \to (s, S)$ such that $q(t_g) = s_g$ and $q(t) = s$ where T is the spectrum of a DVR. Thus every irreducible component of S is dominated by a unique irreducible component of U. Let $V \subset U$ be their union. Extend $f|_V$ to a finite $h: \bar{V} \to S$.

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Pick a point $\bar{v} \in g^{-1}(s)$. There is a $q\colon T \to \bar{V}$ such that $q(t) = \bar{v}$ and $q(t_g)$ is a generic point of $\bar{V}$. Then q is the only possible lifting of $h \circ q$, hence $\bar{v} \in V$. Thus $V = \bar{V}$ and h is a universal homeomorphism. Since f is geometrically injective we must have $V = U$. □

The following is a special case of Stacks (2022, tag 0CNF).

Claim 10.78.2 A finite morphism $Y \to X$ of $\mathbb{F}_p$-schemes is a universal homeomorphism iff it factors a power of the Frobenius $F_q\colon X_q \to Y \to X$. □

Definition 10.79 For a scheme X let $|X|$ denote its underlying point set. Let X, Y be reduced schemes and $\phi\colon |X| \to |Y|$ a set-map of the underlying sets. We say that ϕ is a *morphism* if there is a morphism $\Phi\colon X \to Y$ inducing ϕ. Note that such a Φ is unique since its graph is determined by its points.

Our aim is to find simple conditions that guarantee that a subset is Zariski closed or that a set-map is a morphism.

We say that ϕ is a *morphism on points* if the natural inclusion $k(x) \hookrightarrow k(x, \phi(x))$ is an isomorphism for every $x \in X$, where we view $(x, \phi(x))$ as a point in $X \times Y$. (This in effect says that there is a natural injection $k(\phi(x)) \hookrightarrow k(x)$.)

We say that ϕ is a *morphism on DVRs* (resp. component-wise dominant DVRs) if the composite $\phi \circ h$ is a morphism whenever $h\colon T \to X$ is a morphism (resp. a component-wise dominant morphism (4.30)) from the spectrum of a DVR to X.

Lemma 10.80 (Valuative criterion of being a section) *Let $h\colon X \to S$ be a separated morphism of finite type and $B \subset |X|$ a subset. Then there is a Zariski closed $Z \subset X$ such that $B = |Z|$ and $h|_Z\colon Z \to S$ is a finite, universal homeomorphism (10.78) iff every point $s \in S$ has a unique preimage $b_s \in B$, $k(b_s)/k(s)$ is purely inseparable, and the following holds.*

Let R be an excellent DVR and $q\colon \operatorname{Spec} R \to S$ a component-wise dominant morphism. Then q lifts after a finite extension (10.77) to $q'\colon \operatorname{Spec} R' \to X$ whose image is in B.

Proof By assumption, $h|_B\colon B \to S$ is a universal bijection. Let $s_g \in S$ be a generic point and $b_g \in B$ its preimage. We claim that $\bar{b}_g \subset B$. For any $b_0 \in \bar{b}_g$, there is a component-wise dominant morphism $\tau\colon (t, T) \to S$ that maps the generic point to $h(b_g)$ and the closed point to $h(b_0)$, where T is the spectrum of a DVR. Lifting it shows that $b_0 \in B$.

Thus Z is the union of all $\bar{b}_g$, hence Zariski closed and $h|_Z\colon Z \to S$ is a finite, universal bijection, hence a homeomorphism. □

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Lemma 10.81 (Valuative criterion of being a morphism) *Let X, Y be schemes of finite type, X seminormal, and Y separated. Then a set-map $\phi\colon |X| \to |Y|$ is a morphism iff it is a morphism on points and on component-wise dominant DVRs.*

Proof Let $Z \subset X \times Y$ be the graph of ϕ and $h\colon X \times Y \to X$ the projection. By (10.80) $h|_Z\colon Z \to X$ is a finite, universal homeomorphism that is residue field preserving since ϕ is a morphism on points. Thus $h|_Z\colon Z \to X$ is an isomorphism since X is seminormal. □

Definition 10.82 A morphism $p\colon X \to Y$ is *geometrically injective* if for every geometric point $\bar{y} \to Y$ the fiber $X \times_Y \bar{y}$ consists of at most one point.

Equivalently, for every point $y \in Y$, its preimage $p^{-1}(y)$ is either empty or a single point and $k(p^{-1}(y))$ is a purely inseparable extension of $k(y)$.

If, furthermore, $k(p^{-1}(y))$ equals $k(y)$ then we say that p *preserves residue fields.* The two notions are equivalent in characteristic 0.

A morphism of schemes $f\colon X \to Y$ is a *monomorphism* if for every scheme Z, the induced map of sets $\mathrm{Mor}(Z, X) \to \mathrm{Mor}(Z, Y)$ is an injection.

A monomorphism is geometrically injective. The normalization of the cusp $\pi\colon \operatorname{Spec} k[t] \to \operatorname{Spec} k[t^2, t^3]$ is geometrically injective, but not a monomorphism. The problem is with the fiber over the origin, which is $\operatorname{Spec} k[t]/(t^2) \simeq \operatorname{Spec} k[\varepsilon]$ (where $\varepsilon^2 = 0$). The two maps $g_i\colon \operatorname{Spec} k[\varepsilon] \to \operatorname{Spec} k[t]$ given by $g_0^*(t) = 0$ and $g_1^*(t) = \varepsilon$ are different, but $\pi \circ g_0 = \pi \circ g_1$. A similar argument shows that a morphism is a monomorphism iff it is geometrically injective and unramified; see Grothendieck (1960, IV.17.2.6).

As this example shows, in order to understand when a map between moduli spaces is a monomorphism, the key is to study the corresponding functors over $\operatorname{Spec} k[\varepsilon]$ for all fields k.

See (1.64) for an example that is geometrically bijective but, unexpectedly, not a monomorphism.

A closed, open or locally closed embedding is a monomorphism. A typical example of a monomorphism that is not a locally closed embedding is the normalization of the node with a point missing, that is $\mathbb{A}^1 \setminus \{-1\} \to (y^2 = x^3 + x^2)$ given by $(t \mapsto (t^2 - 1, t^3 - t)$.

Claim 10.82.1 (Stacks, 2022, tag 04XV) A proper monomorphism $f\colon X \to Y$ is a closed embedding. □

Definition 10.83 A morphism $g\colon X \to Y$ is a *locally closed embedding* if it can be factored as $g\colon X \to Y^\circ \hookrightarrow Y$ where $X \to Y^\circ$ is a closed embedding and $Y^\circ \hookrightarrow Y$ is an open embedding.

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A monomorphism $g\colon X \to Y$ is called a *locally closed partial decomposition* of Y if the restriction of g to every connected component $X_i \subset X$ is a locally closed embedding.

If g is also surjective, it is called a *locally closed decomposition* of Y. For reduced schemes, the key example is the following.

Claim 10.83.1 Let $h\colon Y \to \mathbb{Z}$ be a constructible, upper semi-continuous function and set $Y_i := \{y \in Y : h(y) = i\}$. Then $\amalg_i Y_i \to Y$ is a locally closed decomposition. □

The following direct consequence of (10.82.1) is quite useful.

Claim 10.83.2 A proper, locally closed partial decomposition $g\colon X \to Y$ is a closed embedding. If Y is reduced, then a proper, locally closed decomposition $g\colon X \to Y$ is an isomorphism. □

Proposition 10.84 (Valuative criterion of locally closed embedding) *For a geometrically injective morphism of finite type $f\colon X \to Y$, the following are equivalent.*

(10.84.1) *$f(X) \subset Y$ is locally closed and $X \to f(X)$ is finite.*

(10.84.2) *Every component-wise dominant morphism, from the spectrum of an excellent DVR to $f(X)$, lifts to X, after a finite extension (10.77).*

If f is a monomorphism, then these are further equivalent to

(10.84.3) *f is a locally closed embedding.*

Proof It is clear that (1) ⇒ (2). Next assume (2). A geometrically injective morphism of finite type is quasi-finite, hence, by Zariski's main theorem, there is a finite morphism $\bar f\colon \bar X \to Y$ extending f. Set $Z := \bar X \setminus X$.

If $Z \neq \bar f^{-1}\bar f(Z)$ then there are points $z \in Z$ and $x \in X$ such that $\bar f(z) = \bar f(x)$. Let T be the spectrum of a DVR and $h\colon T \to \bar X$ a component-wise dominant morphism. Set $g := \bar f \circ h$. Then $g(T) \subset f(X)$ and the only lifting of g to $T \to \bar X$ is h, but $h(T) \not\subset X$.

Thus $Z = \bar f^{-1}\bar f(Z)$ hence $X \to Y \setminus \bar f(Z)$ is proper, proving (1). A proper monomorphism is a closed embedding by (10.82.1), showing the equivalence with (3). □

A major advantage of seminormality over normality is that seminormalization $X \mapsto X^{\mathrm{sn}}$ is a functor from the category of excellent schemes to the category of excellent seminormal schemes. (The injection $\operatorname{Sing} X \hookrightarrow X$ rarely lifts to the normalizations.) It is thus reasonable to expect that taking the coarse moduli space commutes with seminormalization. This is indeed the case for coarse moduli spaces satisfying the following mild condition.

Definition 10.85 A functor $\mathcal{M}$: (schemes) → (sets) with coarse moduli space M has *enough one-parameter families* if the following holds.

(10.85.1) Let R be a DVR and $\operatorname{Spec} R \to M$ a morphism. Then there is a DVR $R' \supset R$ that is the localization of a finite extension of R and $F' \in \mathcal{M}(\operatorname{Spec} R')$ such that $\operatorname{Spec} R' \to \operatorname{Spec} R \to M$ is the moduli map of F'.

This condition holds if M is obtained as a quotient $M = E/G$, where G is an algebraic group acting properly on E and there is a universal family over E. Thus it is satisfied by all moduli spaces considered in this book.

Proposition 10.86 *Let $\mathcal{M}$: (schemes) → (sets) be a functor defined on finite type schemes over a field of characteristic 0. Assume that $\mathcal{M}$ has a finite type coarse moduli space M and enough one-parameter families.*

Then M^{sn} is the coarse moduli space for $\mathcal{M}^{sn}$, the restriction of $\mathcal{M}$ to the category Sch^{sn} *of finite type, seminormal schemes.*

Proof Since seminormalization is a functor, every morphism $W \to M$ lifts to $W^{\rm sn} \to M^{\rm sn}$. Thus we have a natural transformation $\Phi\colon \mathcal{M}^{\rm sn} \to \operatorname{Mor}(*, M^{\rm sn})$.

Assume that M' is a finite type, seminormal scheme and we have another natural transformation $\Psi\colon \mathcal{M}^{\rm sn} \to \operatorname{Mor}(*, M')$. Every geometric point $s \mapsto M^{\rm sn}$ comes from a scheme X_s. Let $Z \subset M^{\rm sn} \times M'$ denote the union of the points $(s, \Phi[X_s])$. Since M is a coarse moduli space and $M^{\rm sn} \to M$ is geometrically bijective, the coordinate projection $Z \to M^{\rm sn}$ is also geometrically bijective. Since $\mathcal{M}$ has enough one-parameter families, $Z \to M^{\rm sn}$ is a universal homeomorphism by (10.80). Thus $Z \to M^{\rm sn}$ is an isomorphism since $M^{\rm sn}$ is seminormal and the characteristic is 0.

Thus we get a morphism $M^{\rm sn} \to M'$ and Ψ factors through Φ. □

The next examples show that the characteristic 0 assumption is likely necessary in (10.86) and that the analogous claim for the underlying reduced subscheme is likely to be false.

Examples 10.87 Let $\mathcal{D}$ be any diagram of schemes with direct limit $\lim \mathcal{D}$. Since seminormalization is a functor, we get a diagram $\mathcal{D}^{\rm sn}$ and a natural morphism $\lim(\mathcal{D}^{\rm sn}) \to (\lim \mathcal{D})^{\rm sn}$. However, this need not be an isomorphism.

(10.87.1) Consider the diagram of all maps $\phi_a\colon \operatorname{Spec} k[x] \to \operatorname{Spec} k[(x-a)^2, (x-a)^3]$ for $a \in k$ where k is an infinite field.

If $\operatorname{char} k = 0$ then the direct limit is $\operatorname{Spec} k$. After seminormalization, the maps ϕ_a become isomorphisms $\phi_a^{\rm sn}\colon \operatorname{Spec} k[x] \simeq \operatorname{Spec} k[x]$. Now the direct limit is $\operatorname{Spec} k[x]$.

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(10.87.2) If $\operatorname{char} k = p > 0$ then $x^p - a^p = (x-a)^p \in k[(x-a)^2, (x-a)^3]$ shows that the direct limit is $\operatorname{Spec} k[x^p]$. After seminormalization, the direct limit is again $\operatorname{Spec} k[x]$. Here $\operatorname{Spec} k[x^p]$ behaves like a coarse moduli space.

(10.87.3) Consider the maps $\sigma_i \colon k[x] \to k[x, \varepsilon]$ given by $\sigma_0(g(x)) = g(x)$ and $\sigma_1(g(x)) = g(x) + g'(0)\varepsilon$. We get a universal push-out diagram

$$\begin{array}{ccc} \operatorname{Spec} k[x,\varepsilon] & \xrightarrow{\sigma_0} & \operatorname{Spec} k[x] \\ {\scriptstyle \sigma_1}\downarrow & & \downarrow \\ \operatorname{Spec} k[x] & \longrightarrow & \operatorname{Spec} k[x^2, x^3]. \end{array}$$

If we pass to the underlying reduced subspaces, the push-out is $\operatorname{Spec} k[x]$.

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