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THE KRALL POLYNOMIALS AS SOLUTIONS TO A SECOND ORDER DIFFERENTIAL EQUATION

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ABSTRACT. A popular problem today in orthogonal polynomials is that of classifying all second order differential equations which have orthogonal polynomial solutions. We show that the Krall polynomials satisfy a second order equation of the form

$$a_2(x, n)y''(x)+a_1(x, n)y'(x)+a_0(x, n)y(x)=0$$

1. **Introduction.** A problem of popular interest today in the area of orthogonal polynomials is the classification of all second order differential equations of the form:

$$a_2(x, n)y''(x)+a_1(x, n)y'(x)+a_0(x, n)y(x)=0 \tag{1.1}$$

which have a sequence of orthogonal polynomial solutions.

Hahn has shown [2] that if a sequence $\{y_n(x)\}$ of orthogonal polynomials satisfies a differential equation, then the equation may be reduced to one of order two or four and where this reduced equation still has $\{y_n(x)\}$ as a solution set. In fact, he has produced examples [3] of orthogonal polynomials that satisfy fourth order differential equations that cannot be reduced to order two. Along another vein, Atkinson and Everitt [1] have produced results concerning orthogonal polynomial solutions to an equation like (1.1).

In 1961, Shore developed a technique [10] whereby he showed the Legendre type polynomials [5] satisfy an equation of the form (1.1). Recently, Littlejohn and Shore [9] have shown that the Laguerre type and Jacobi type polynomials also satisfy equations like (1.1). All three of the above mentioned polynomial sets satisfy fourth order equations of the form

$$\sum_{i=0}^{4} \sum_{j=0}^{i} l_{ij}x^i y^{(i)}(x)=\lambda_n y(x);$$

see H. L. Krall [6 and 7]. These polynomials sets have been extensively studied by A. M. Krall [5].

This author has found a new set of orthogonal polynomials, called the Krall polynomials, that satisfy a sixth order differential equation. The various properties of these polynomials are developed in [7]. In this paper, we extend

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Shore's method to show that these polynomials satisfy an equation like (1.1). This is the first example of an equation of order higher than four being reduced to order two by this technique. It is clear that this procedure can be further extended to reduce even higher order equations to second order ones, thereby providing us with a powerful tool in the search for the solution to the above mentioned classification problem.

Before proceeding into the main body of this article, we feel it is necessary to discuss the relevancy of knowing the solution to this problem. Admittedly, the second order differential equation has little practical use since, in general, the properties of the associated orthogonal polynomials cannot be developed from the equation. However, determining all possible weight distributions is important. It is significant to be able to distinguish between a weight whose associated set of orthogonal polynomials satisfy a second order differential equation and a weight which lacks this property. So far, the literature seems to indicate that there is no general pattern for which the known weights satisfy. Recently, this author has come up with some interesting results (yet unpublished) which seem to indicate that such a pattern does indeed exist.

2. **The Krall polynomials.** The nth Krall polynomial is given by:

$$K_n(x)=\sum_{j=0}^{n}\frac{(-1)^{[j/2]}(2n-j)!\,Q(n,j)x^{n-j}}{2^{n+1}\left(n-\left[\frac{j+1}{2}\right]\right)!\left[\frac{j}{2}\right]!\,(n-j)!\,(n^2+n+AC+BC)}$$

where

$$Q(n,j)=\frac{1+(-1)^j}{2}\{(n^4+(2AC+2BC-1)n^2+4ABC^2)+2j(n^2+n+AC+BC)\}$$
$$+\frac{1-(-1)^j}{2}(4BC-4AC)$$

and $[\cdot]$ is the greatest integer function. These polynomials are orthogonal on $[-1,1]$ with respect to the weight distribution

$$w(x)=\frac{1}{A}\,\delta(x+1)+\frac{1}{B}\,\delta(x-1)+C$$

and they satisfy the sixth order differential equation:

$$\begin{aligned}
L_6(y)=&(x^2-1)^3y^{(vi)}(x)+18x(x^2-1)^2y^{(v)}(x)\\
&+(x^2-1)\{(3AC+3BC+96)x^2+(-3AC-3BC-36)\}y^{(iv)}(x)\\
&+x(x^2-1)(24AC+24BC+168)y'''(x)\\
&+\{(12ABC^2+42AC+42BC+72)x^2+(12BC-12AC)x\\
&+(-12ABC^2-30AC-30BC-72)\}y''(x)\\
&+\{(24ABC^2+12AC+12BC)x+(12BC-12AC)\}y'(x)\\
=&\mu_ny(x)
\end{aligned}\tag{2.1}$$

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where

$$\mu_n=(24ABC^2+12AC+12BC)n+(12ABC^2+42AC+42BC+72)n(n-1)$$
$$+(24AC+24BC+168)n(n-1)(n-2)$$
$$+(3AC+3BC+96)n(n-1)(n-2)(n-3)$$
$$+18n(n-1)(n-2)(n-3)(n-4)+n(n-1)(n-2)(n-3)(n-4)(n-5).$$

When $A=B=2$, $C=\alpha/2$, the polynomials are called the Legendre type polynomials. H. L. Krall [5] showed that these Legendre type polynomials satisfy the fourth order differential equation

$$\begin{aligned}(2.2)\quad &(x^2-1)^2y^{(iv)}(x)+8x(x^2-1)y'''(x)+(4\alpha+12)(x^2-1)y''(x)+8\alpha xy'(x)\\ &=[8\alpha n+(4\alpha+12)n(n-1)+8n(n-1)(n-2)+n(n-1)(n-2)(n-3)]y(x).\end{aligned}$$

Using (2.2) Shore showed that the Legendre type polynomials satisfy the second order equation $L_2(y)=0$ where:

$$\begin{aligned}(2.3)\quad L_2(y)=&(x^2-1)[(4\alpha^2+4\alpha+\varepsilon_n)x^2-(4\alpha^2-4\alpha+\varepsilon_n)]y''(x)\\ &+2x[(4\alpha^2+4\alpha+\varepsilon_n)x^2-(4\alpha^2-12\alpha+\varepsilon_n)]y'(x)\\ &-[(\delta_n+4\alpha+96)x^2-(\delta_n+4\alpha+96-4\varepsilon_n)]y(x)\end{aligned}$$

where $\varepsilon_n=n(n+1)(n^2+n+4\alpha-2)$ and $\delta_n=n(n+1)(n^4+2n^3-97n^2-98n+192-372\alpha-12\alpha^2)$. However, when $A\neq B$, the Krall polynomials do not satisfy an equation like (2.2). Hence, in order to find an equation like (1.1) for which the Krall polynomials are solutions, we begin with (2.1).

3. **The method.** To find the second order differential equation, we follow a technique similar to the one used in [9]. More specifically, we first find an eighth order differential equation for which the Krall polynomials are solutions. We then differentiate (2.1) twice to obtain another eighth order equation. By carefully combining these two equations, we eliminate the seventh and eighth order derivatives to obtain another sixth order equation, different from (2.1) Repeating this process, we combine the two sixth order equations, eliminate the fifth and sixth order derivatives, and obtain a fourth order equation $L_4(y)=0$. We then differentiate $L_4(y)=0$ twice to obtain another sixth order equation, different from the two previously calculated sixth order equations. By combining this new sixth order equation with (2.1) we eliminate the fifth and sixth order derivatives to obtain another fourth order equation different from $L_4(y)=0$. Finally, these two fourth order equations are combined to yield the desired second order equation.

4. **The second order differential equation.** Suppose the Krall polynomials $\{K_n(x)\}$ are solutions to the eighth order formally self adjoint differential

equation

$$\begin{aligned}
L_8(y) &= a_8(x)y^{(8)}(x)+4a_8'(x)y^{(vii)}(x)+a_6(x)y^{(vi)}(x)\\
&\quad+(3a_6'(x)-14a_8'''(x))y^{(v)}(x)+a_4(x)y^{(iv)}(x)\\
&\quad+(28a_8^{(v)}(x)-5a_6'''(x)+2a_4'(x))y''(x)+a_2(x)y''(x)\\
&\quad+(-17a_8^{(vii)}(x)+3a_6^{(v)}(x)-a_4'''(x)+a_2'(x))y'(x)\\
&=\lambda_n y(x).
\end{aligned}$$

By introducing the notation $w_{ij}=z^{(i)}y^{(j)}-z^{(j)}y^{(i)}$, where $z^{(i)}=d^iz/dx^i$, we can show [8], for $z=K_m$, $y=K_n$, $n\neq m$, that

$$\begin{aligned}
0&=\int_{-1}^{1}(zL_8(y)-yL_8(z))w\,dx\\
&=\sum_{0\le i<j\le 8} f_{ij}(a_2,a_4,a_6,a_8)w_{ij}(1)+\sum_{0\le i<j\le 8} g_{ij}(a_2,a_4,a_6,a_8)w_{ij}(-1)
\end{aligned}$$

In order to find a differential equation having $K_n(x)(n=0,1,\ldots)$ as a solution, it is sufficient that the coefficients of $w_{ij}(1)$ and $w_{ij}(-1)$ be zero. Assuming this and solving the resulting equations simultaneously allows us to determine a_2, a_4, a_6 and a_8. After some calculations, we find that $K_n(x)$ is a solution to the eighth order equation:

$$\begin{aligned}
L_8(y)&=(x^2-1)^4y^{(8)}(x)+32x(x^2-1)^3y^{(vii)}(x)+272(x^2-1)^2y^{(vi)}(x)\\
&\quad+96x(x^2-1)(55-49x^2)y^{(v)}(x)\\
&\quad+(x^2-1)\{(-30,648-1056AC-1056BC-8A^2C^2-8ABC^2\\
&\quad-8ABC^2-8B^2C^2)x^2+(16BC-16AC)x+(11,928+1072AC\\
&\quad+1072BC+8A^2C^2+8ABC^2+8B^2C^2)\}y^{(iv)}(x)+\{(-57,024\\
&\quad-8448AC-8448BC-64A^2C^2-64ABC^2-64B^2C^2)x^3\\
&\quad+(96BC-96AC)x^2+(57,024+8512AC+8512BC\\
&\quad+64A^2C^2+64ABC^2+64B^2C^2)x+(32AC-32BC)\}y'''(x)\\
&\quad+\{(-25,056-4352ABC^2-32A^2BC^3-32AB^2C^3-14,784AC\\
&\quad-14,784BC-112A^2C^2-112B^2C^2)x^2+(4128AC-4128BC\\
&\quad+32A^2C^2-32B^2C^2)x+(25,056+4352ABC^2+32A^2BC^3\\
&\quad+32AB^2C^3+10,656AC+10,656BC+80A^2C^2+80B^2C^2)\}y''(x)\\
&\quad+\{(-8512ABC^2-64A^2BC^3-64AB^2C^3\\
&\quad-4224AC-4224BC-32A^2C^2-32B^2C^2)x\\
&\quad+(4224AC-4224BC+32A^2C^2-32B^2C^2)\}y'(x)=\lambda_n y(x)
\end{aligned}$$

where

$$
\begin{aligned}
\lambda_n = {} & (-8512ABC^2 - 64A^2BC^3 - 64AB^2C^3 - 4224AC - 4224BC \\
& - 32A^2C^2 - 32B^2C^2)n + (-25,056 - 4352ABC^2 \\
& - 32A^2BC^3 - 32AB^2C^3 - 14,784AC \\
& - 14,784BC - 112A^2C^2 - 112B^2C^2)n(n-1) \\
& + (-57,024 - 8448AC - 8448BC \\
& - 64A^2C^2 - 64ABC^2 - 64B^2C^2)n(n-1)(n-2) \\
& + (-30,648 - 1056AC - 1056BC \\
& - 8A^2C^2 - 8ABC^2 - 8B^2C^2)n(n-1)(n-2)(n-3) \\
& - 4704n(n-1)(n-2)(n-3)(n-4) \\
& + 32n(n-1)(n-2)(n-3)(n-4)(n-5)(n-6) \\
& + n(n-1)(n-2)(n-3)(n-4)(n-5)(n-6)(n-7).
\end{aligned}
$$

One can check that $K_n(x)$ satisfies the fourth order equation:

$$
\begin{aligned}
& L_8(y) - (x^2-1)L_6''(y) - 2xL_6'(y) + \left\{(3AC+3BC+354) + \frac{4}{x^2-1}\right\}L_6(y) \\
& \qquad = \lambda_n y - (x^2-1)\mu_n y'' - 2x\mu_n y' + \left\{(3AC+3BC+354) + \frac{4}{x^2-1}\right\}\mu_n y.
\end{aligned}
$$

Simplified, this equation is:

$$
\begin{aligned}
L_4(y) = {} & (x^2-1)^2\{(AC-BC)^2x^2 + (4BC-4AC)x \\
& + [4AC+4BC-(AC-BC)^2]\}y^{(iv)}(x) \\
& + (x^2-1)\{8(AC-BC)^2x^3 + (36BC-36AC)x^2 \\
& + [40AC+40BC-8(AC-BC)^2]x + (4BC-4AC)\}y'''(x) \\
& + \{(14A^2C^2 - 20ABC^2 + 14B^2C^2 + 4A^2BC^3 + 4AB^2C^3 + \mu_n)x^4 \\
& + (-72AC + 72BC - 4A^2C^2 + 4B^2C^2)x^3 \\
& + (-24A^2C^2 + 64ABC^2 - 24B^2C^2 - 8A^2BC^3 \\
& - 8AB^2C^3 + 96AC + 96BC - 2\mu_n)x^2 \\
& + (24AC - 24BC + 4A^2C^2 - 4B^2C^2)x \\
& + (10A^2C^2 - 44ABC^2 + 10B^2C^2 + 4A^2BC^3 + 4AB^2C^3 \\
& - 48AC - 48BC + \mu)\}y''(x) \\
& + \{(4A^2C^2 + 8ABC^2 + 4B^2C^2 + 8A^2BC^3 + 8AB^2C^3 + 2\mu_n)x^3 \\
& + (-24AC + 24BC - 4A^2C^2 + 4B^2C^2)x^2 \\
& + (-4A^2C^2 + 88ABC^2 - 4B^2C^2 - 8A^2BC^3 - 8AB^2C^3
\end{aligned}
$$

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$$
\begin{aligned}
&+48AC+48BC-2\mu_n)x\\
&+(-24AC+24BC+4A^2C^2-4B^2C^2)\}y'(x)\\
&+\{(-\lambda_n-3AC\mu_n-3BC\mu_n-354\mu_n)x^2\\
&+(\lambda_n+3AC\mu_n+3BC\mu_n+350\mu_n)\}y(x)\\
=\,&0.
\end{aligned}
$$

The required second order differential equation is given by:

$$
\begin{aligned}
\frac{1}{x^2-1}&[\{(AC-BC)^2x^2+(4BC-4AC)x+(4AC+4BC-(AC-BC)^2)\}^2L_6(y)\\
&-\{(AC-BC)^2x^2+(4BC-4AC)x+(4AC+4BC\\
&-(AC-BC)^2)\}(x^2-1)L_4''(y)\\
&+\{2(AC-BC)^2x+(4BC-4AC)\}(x^2-1)L_4'(y)\\
&-\{(3A^3C^3+3B^3C^3-7A^2BC^3-7AB^2C^3\\
&-8ABC^2-\mu_n)x^2+(8B^2C^2-8A^2C^2)x\\
&+(-3A^3C^3-3B^3C^3+7A^2BC^3+7AB^2C^3+8A^2C^2\\
&+8B^2C^2+8ABC^2+\mu_n)\}L_4(y)]=0.
\end{aligned}
$$

Written out, this equation is:

$$
\begin{aligned}
(x^2-1)\{Ex^2+Fx+G\}y''(x)+\{2Ex^3+3Fx^2+Hx+F\}y'(x)&\\
+\{Jx^2+Kx+L\}y(x)=0&
\end{aligned}
\tag{4.1}
$$

where

$$
\begin{aligned}
E=&-32A^4B^2C^6+128A^3B^3C^6-32A^2B^4C^6-16A^4BC^5+80A^3B^2C^5\\
&+80A^2B^3C^5-16AB^4C^5+16A^3BC^4+32A^2B^2C^4+16AB^3C^4\\
&+8A^2BC^3\mu_n+8AB^2C^3\mu_n\\
&+356A^2C^2\mu_n-696ABC^2\mu_n+356B^2C^2\mu_n\\
&+A^2C^2\lambda_n-2ABC^2\lambda_n+B^2C^2\lambda_n+\mu_n^2\\
F=&-32A^4BC^5+160A^3B^2C^5-160A^2B^3C^5\\
&+32AB^4C^5+32A^3BC^4-32AB^3C^4\\
&-8A^2C^2\mu_n+8B^2C^2\mu_n\\
&-1416AC\mu_n+1416BC\mu_n-4AC\lambda_n+4BC\lambda_n\\
G=&\,32A^4B^2C^6-128A^3B^3C^6+32A^2B^4C^6\\
&-16A^4BC^5+16A^3B^2C^5+16A^2B^3C^5\\
&-16AB^4C^5+16A^3BC^4+32A^2B^2C^4\\
&+16AB^3C^4-8A^2BC^3\mu_n-8AB^2C^3\mu_n\\
&-348A^2C^2\mu_n+744ABC^2\mu_n-348B^2C^2\mu_n
\end{aligned}
$$

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$$+1416AC\mu_n+1416BC\mu_n-A^2C^2\lambda_n$$
$$+2ABC^2\lambda_n-B^2C^2\lambda_n+4AC\lambda_n+4BC\lambda_n-\mu_n^2$$

$$\begin{aligned}H={}&64A^4B^2C^6-256A^3B^3C^6+64A^2B^4C^6\\&-96A^4BC^5+224A^3B^2C^5+224A^2B^3C^5\\&-96AB^4C^5+96A^3BC^4+192A^2B^2C^4\\&+96AB^3C^4-16A^2BC^3\mu_n-16AB^2C^3\mu_n\\&-680A^2C^2\mu_n+1584ABC^2\mu_n-680B^2C^2\mu_n\\&+5664AC\mu_n+5664BC\mu_n-2A^2C^2\lambda_n\\&+4ABC^2\lambda_n-2B^2C^2\lambda_n+16AC\lambda_n+16BC\lambda_n-2\mu_n^2\end{aligned}$$

$$\begin{aligned}J={}&8A^4C^4\mu_n-8A^3BC^4\mu_n-48A^2B^2C^4\mu_n\\&-8AB^3C^4\mu_n+8B^4C^4\mu_n+1056A^3C^3\mu_n\\&-2496A^2BC^3\mu_n-2496AB^2C^3\mu_n+1056B^3C^3\mu_n\\&-708A^2C^2\mu_n-1416ABC^2\mu_n\\&-708B^2C^2\mu_n-3AC\mu_n^2-3BC\mu_n^2-354\mu_n^2+3A^3C^3\lambda_n-7A^2BC^3\lambda_n\\&-7AB^2C^3\lambda_n\\&+3B^3C^3\lambda_n-2A^2C^2\lambda_n-4ABC^2\lambda_n-2B^2C^2\lambda_n-\mu_n\lambda_n\end{aligned}$$

$$\begin{aligned}K={}&-16A^3C^3\mu_n-48A^2BC^3\mu_n+48AB^2C^3\mu_n\\&+16B^3C^3\mu_n-2832A^2C^2\mu_n\\&+2832B^2C^2\mu_n-8A^2C^2\lambda_n+8B^2C^2\lambda_n\end{aligned}$$

and

$$\begin{aligned}L={}&-8A^4C^4\mu_n+8A^3BC^4\mu_n+48A^2B^2C^4\mu_n\\&+8AB^3C^4\mu_n-8B^4C^4\mu_n-1040A^3C^3\mu_n\\&+2512A^2BC^3\mu_n+2512AB^2C^3\mu_n-1040B^3C^3\mu_n\\&+2132A^2C^2\mu_n+4296ABC^2\mu_n\\&+2132B^2C^2\mu_n+2832AC\mu_n+2832BC\mu_n\\&+3AC\mu_n^2+3BC\mu_n^2+350\mu_n^2-3A^3C^3\lambda_n\\&+7A^2BC^3\lambda_n+7AB^2C^3\lambda_n-3B^3C^3\lambda_n\\&+6A^2C^2\lambda_n+12ABC^2\lambda_n+6B^2C^2\lambda_n\\&+8AC\lambda_n+8BC\lambda_n+\mu_n\lambda_n.\end{aligned}$$

Lastly, we note that when $A=B=2$, $C=\alpha/2$, (4.1) can be written as:

$$\begin{aligned}&[n^8+4n^7+(8\alpha+4)n^6+(24\alpha-2)n^5+(24\alpha^2+20\alpha-5)n^4+(48\alpha^2-2)n^3\\&\quad+(32\alpha^3+32\alpha^2-4\alpha)n^2+(32\alpha^3+8\alpha^2)n+16\alpha^4+16\alpha^3]L_2(y)=0\end{aligned}$$

where $L_2(y)$ is given by (2.3).

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5. **Conclusion.** Recently, Koornwinder has generalized the Krall polynomials [4]. He has generated a sequence of polynomials orthogonal on $[-1, 1]$ with respect to the weight distribution $(1/A)\,\delta(x+1)+(1/B)\,\delta(x-1)+C(1-x)^{\alpha}(1+x)^{\beta}$. Using properties of these polynomials, he has shown that they satisfy a second order differential equation. Our method of deriving the second order equation does not rely on properties of the polynomials.

References

1. F. V. Atkinson and W. N. Everitt, *Orthogonal Polynomials Which Satisfy Second Order Differential Equations*, Christoffel Festschrift, 1981, Birhaüser, Basel.

2. W. Hahn, *On Differential Equations for Orthogonal Polynomials*, Funkcialaj Ekvacioj, **21** (1978) 1–9.

3. ——, *Über Orthogonalpolynome mit drei Parametern*, Deutsche Math., **5** (1940) 273–278.

4. T. H. Koornwinder, *A Further Generalization of Krall's Jacobi type Polynomials*, Mathematisch Centrum, Amsterdam, 1982.

5. A. M. Krall, *Orthogonal Polynomials Satisfying Fourth Order Differential Equations*, Proc. Roy. Soc. Edin., **87(A)** (1981) 271–288.

6. H. L. Krall, *Certain Differential Equations for Tchebycheff Polynomials*, Duke Math. Jour., **4** (1938) 705–718.

7. ——, *On Orthogonal Polynomials Satisfying a Certain Fourth Order Differential Equation*, The Pennsylvania State College Studies, No. 6, The Pennsylvania State College, State College, Pa., 1940.

8. L. L. Littlejohn, *The Krall Polynomials: A new Class of Orthogonal Polynomials*, Quaestione Mathematicae, **5** (1982), 255–265.

9. —— and S. D. Shore, *Nonclassical Orthogonal Polynomials as Solutions to Second Order Differential Equations*, Can. Math. Bull., **25** (3) 1982.

10. S. D. Shore, *On the Second Order Differential Equation which has Orthogonal Polynomial Solutions*, Bull. Calcutta Math. Soc., **56** (1964) 195–198.

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