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TORSION UNITS IN INTEGRAL GROUP RING OF THE MATHIEU SIMPLE GROUP M_{22}

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Abstract

We investigate the possible character values of torsion units of the normalized unit group of the integral group ring of the Mathieu sporadic group M_{22}. We confirm the Kimmerle conjecture on prime graphs for this group and specify the partial augmentations for possible counterexamples to the stronger Zassenhaus conjecture.

1. *Introduction, conjectures and main results*

Let $V(\mathbb{Z}G)$ be the normalized unit group of the integral group ring $\mathbb{Z}G$ of a finite group G. A long-standing conjecture of H. Zassenhaus **(ZC)** says that every torsion unit $u \in V(\mathbb{Z}G)$ is conjugate within the rational group algebra $\mathbb{Q}G$ to an element in G (see [**25**]).

For finite simple groups the main tool for the investigation of the Zassenhaus conjecture is the Luthar–Passi method, introduced in [**20**] for the case of A_5 and then applied in [**21**] for the case of S_5. Later M. Hertweck in [**15**] extended the method and applied it to $\mathrm{PSL}(2, p^n)$. The same method has also proved to be useful for some groups containing non-trivial normal subgroups. For some recent results we refer to [**5, 7, 14, 16, 15, 17**]. Some related properties and weakened variations of the Zassenhaus conjecture can be found in [**1, 3, 19**].

To define the conjectures we will investigate, and describe the methods we will use, we introduce some notation. By $\#(G)$ we denote the set of all primes dividing the order of G. The Gruenberg–Kegel graph (or the prime graph) of G is the graph $\pi(G)$ with vertices labeled by $\#(G)$ and an edge from p to q (for $p \neq q$) if there is an element of order pq in G. In [**19**], W. Kimmerle proposed the following weakened variation of the Zassenhaus conjecture:

(KC) If G is a finite group then $\pi(G) = \pi(V(\mathbb{Z}G))$.

It is easy to see that **(ZC)** implies **KC** since it implies that the set of orders of torsion units of $V(\mathbb{Z}G)$ is the same the set of orders of elements of G.

In ([**19**], §4) it is shown that **(KC)** holds for finite Frobenius and solvable groups. We remark that with respect to the so-called p-version of the Zassenhaus conjecture the investigation of Frobenius groups was completed by M. Hertweck and the first author in [**4**]. In [**6, 7, 8, 10**], **(KC)** was confirmed for the sporadic simple groups M_{11}, M_{12}, M_{23} and some Janko simple groups.

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Here we continue these investigations for the Mathieu simple group M_{22}. Although we cannot prove the rational conjugacy of torsion units of $V(\mathbb{Z}M_{22})$ with elements of M_{22}, our main result gives a lot of information on the orders and partial augmentations of these units. In particular, we confirm Kimmerle's conjecture for this group.

Let $G = M_{22}$. It is well known (see [**13, 24**]) that $|G| = 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$ and $exp(G) = 2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11$. The group G has 12 irreducible characters of the following degrees: 1, 21, 45, 45, 55, 99, 154, 210, 231, 280, 280 and 385. Let

$$\mathcal{C} = \{C_1, C_{2a}, C_{3a}, C_{4a}, C_{4b}, C_{5a}, C_{6a}, C_{7a}, C_{7b}, C_{8a}, C_{11a}, C_{11b}\}$$

be the collection of all conjugacy classes of M_{22}, where the first index denotes the order of the elements of this conjugacy class and $C_1 = \{1\}$. Suppose $u = \sum \alpha_g g \in V(\mathbb{Z}G)$ has finite order $k > 1$. Denote by $\nu_{nt} = \nu_{nt}(u) = \varepsilon_{C_{nt}}(u) = \sum_{g \in C_{nt}} \alpha_g$ the partial augmentation of u with respect to C_{nt}. From the Berman–Higman Theorem (see [**2**] and [**23**], Ch. 5, p. 102) one knows that $\nu_1 = \alpha_1 = 0$ and

$$\sum_{C_{nt} \in \mathcal{C}} \nu_{nt} = 1. \tag{1}$$

Hence, for any character χ of G, we get that $\chi(u) = \sum \nu_{nt} \chi(h_{nt})$, where h_{nt} is a representative of the conjugacy class C_{nt}.

Our main result is the following theorem.

Theorem 1. *Let G denote the Mathieu simple group M_{22}. Let u be a torsion unit of $V(\mathbb{Z}G)$ of order $|u|$. Denote by $\mathfrak{P}(u)$ the tuple*

$$(\nu_{2a}, \nu_{3a}, \nu_{4a}, \nu_{4b}, \nu_{5a}, \nu_{6a}, \nu_{7a}, \nu_{7b}, \nu_{8a}, \nu_{11a}, \nu_{11b}) \in \mathbb{Z}^{11}$$

of partial augmentations of u in $V(\mathbb{Z}G)$. The following properties hold.

- (i) *There are no elements of order* 10, 14, 15, 21, 22, 33, 35, 55 *or* 77 *in $V(\mathbb{Z}G)$. Equivalently, if $|u| \notin \{12, 24\}$, then $|u|$ is the order of some element $g \in G$.*
- (ii) *If $|u| \in \{2, 3, 5\}$, then u is rationally conjugate to some $g \in G$.*
- (iii) *If $|u| = 4$, then all components of $\mathfrak{P}(u)$ are zero except possibly ν_{2a}, ν_{4a} and ν_{4b}, and the triple $(\nu_{2a}, \nu_{4a}, \nu_{4b})$ is one of*

$$\{\ (-2,-1,4),\ (2,-1,0),\ (0,-1,2),\ (0,6,-5),\ (-2,6,-3),\ (0,5,-4),\ (-2,5,-2), (-2,4,-1), (2,4,-5), (0,4,-3), (-2,3,0), (2,3,-4), (0,3,-2),\ (2,0,-1),\ (-2,0,3),\ (0,0,1),\ (0,-6,7),\ (2,-6,5),\ (0,2,-1),\ (2,2,-3),\ (-2,2,1), (0,-5,6),\ (2,-5,4),\ (0,-4,5),\ (2,-4,3),\ (-2,-3,6),\ (0,-3,4),\ (2,-3,2),\ (-2,-2,5),\ (0,-2,3),\ (2,-2,1),\ (0,1,0),\ (-2,1,2),\ (2,1,-2)\ \}.$$

- (iv) *If $|u| = 6$, then all components of $\mathfrak{P}(u)$ are zero except possibly ν_{2a}, ν_{3a} and ν_{6a}, and the triple $(\nu_{2a}, \nu_{3a}, \nu_{6a})$ is one of*

$$\{\ (-4,6,-1),\ (-2,6,-3),\ (4,-9,6),\ (-4,9,-4),\ (-2,3,0),\ (-4,3,2),\ (0,3,-2),\ (2,0,-1),\ (-2,0,3),\ (0,0,1),\ (2,-6,5),\ (4,-6,3),\ (0,-3,4),\ (4,-3,0),\ (2,-3,2)\ \}.$$

- (v) *If $|u| = 7$, then all components of $\mathfrak{P}(u)$ are zero except possibly ν_{7a} and ν_{7b}*

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and the pair (ν_{7a}, ν_{7b}) *is one of*

$$\{\ (0,1),\ (2,-1),\ (1,0), (-1,2)\ \}.$$

(vi) *If* $|u| = 11$*, then all components of* $\mathfrak{P}(u)$ *are zero except possibly* ν_{11a} *and* ν_{11b} *and the pair* (ν_{11a}, ν_{11b}) *is one of*

$$\{\ (5,-4),\ (0,1),\ (-2,3),\ (2,-1),\ (-3,4),\\ (-4,5),\ (1,0),\ (3,-2),\ (-1,2),\ (4,-3)\ \}.$$

Note that using our implementation of the Luthar–Passi method (including Hertweck's extension), which we intend to make available in the GAP package LAGUNA [9], we are able to compute the set of 76 tuples containing (likely as a proper subset) possible tuples of partial augmentations for units of order 8, listed in Appendix 1. For the case of order 12 in Appendix 2 we listed 1166 tuples which can not be eliminated using the Luthar–Passi method.

As an immediate consequence of part (i) of the Theorem we obtain the corollary.

Corollary 1. *If* $G \cong M_{22}$ *then* $\pi(G) = \pi(V(\mathbb{Z}G))$.

2. *Preliminaries*

The following result allows a reformulation of the Zassenhaus conjecture in terms of vanishing of partial augmentations of torsion units.

Proposition 1 (see [20] and Theorem 2.5 in [22]). *Let* $u \in V(\mathbb{Z}G)$ *be of order* k*. Then* u *is conjugate in* $\mathbb{Q}G$ *to an element* $g \in G$ *if and only if for each* d *dividing* k *there is precisely one conjugacy class* C *with partial augmentation* $\varepsilon_C(u^d) \neq 0$.

The next results now serve to restrict the possible values of the partial augmentations of torsion units.

Proposition 2 (see [14], Proposition 3.1; [15], Proposition 2.2). *Let* G *be a finite group and let* u *be a torsion unit in* $V(\mathbb{Z}G)$*. If* x *is an element of* G *whose* p*-part, for some prime* p*, has order strictly greater than the order of the* p*-part of* u*, then* $\varepsilon_x(u) = 0$.

The next result is explained in detail in [20] and [4, 15].

Proposition 3 (see [15, 20]). *Let either* $p = 0$ *or* p *a prime divisor of* $|G|$*. Suppose that* $u \in V(\mathbb{Z}G)$ *has finite order* k *and assume* k *and* p *are coprime in case* $p \neq 0$*. If* z *is a complex primitive* k*th root of unity and* χ *is either a classical character or a* p*-Brauer character of* G*, then for every integer* l *the number*

$$\mu_l(u, \chi, p) = \tfrac{1}{k} \textstyle\sum_{d|k} Tr_{\mathbb{Q}(z^d)/\mathbb{Q}}\{\chi(u^d) z^{-dl}\} \qquad (2)$$

is a non-negative integer.

Note that if $p = 0$, we will use the notation $\mu_l(u, \chi, *)$ for $\mu_l(u, \chi, 0)$.

Finally, we shall use the well-known bound for orders of torsion units.

Proposition 4 (see [11]). *The order of a torsion element* $u \in V(\mathbb{Z}G)$ *is a divisor of the exponent of* G.

In case of units of prime power order, the following Proposition may also be useful to eliminate some tuples of partial augmentations.

Proposition 5 (see [**11**]). *Let p be a prime, and let u be a torsion unit of $V(\mathbb{Z}G)$ of order p^n. Then for $m \neq n$ the sum of all partial augmentations of u with respect to conjugacy classes of elements of order p^m is divisible by p.*

3. *Proof of the Theorem*

Throughout this section we denote M_{22} by G. The ordinary and p-Brauer character tables of G, which will be denoted by $\mathfrak{BCT}(p)$ where $p \in \{2, 3, 5, 7, 11\}$, can be found using the computational algebra system GAP [**13**], which derives its data from [**12, 18**]. For the characters and conjugacy classes we will use throughout the paper the same notation, including indexation, as used in the GAP Character Table Library.

Since the group G possesses elements of orders 2, 3, 4, 5, 6, 7, 8 and 11, we first investigate units of all of these orders except 8. After this, since by Proposition 4, the order of each torsion unit divides the exponent of G, it remains to consider units of orders 10, 12, 14, 15, 21, 22, 33, 35, 55 and 77. We prove that $V(\mathbb{Z}G)$ contains no units of any of these orders, except possibly for order 12 and 24.

Now we consider each case separately.

• Let u be a unit of order 2, 3 or 5. Using Proposition 2 we immediately obtain that all partial augmentations except one are zero. Thus by Proposition 1 part (ii) of Theorem 1 is proved.

• Let u be a unit of order 4. By (1) and Proposition 2 we get $\nu_{2a} + \nu_{4a} + \nu_{4b} = 1$. Now using Proposition 3 we obtain the following system of inequalities:

$$\begin{aligned}
\mu_0(u, \chi_2, *) &= \tfrac{1}{4}(10\nu_{2a} + 2\nu_{4a} + 2\nu_{4b} + 26) \geqslant 0;\\
\mu_2(u, \chi_2, *) &= \tfrac{1}{4}(-10\nu_{2a} - 2\nu_{4a} - 2\nu_{4b} + 26) \geqslant 0;\\
\mu_0(u, \chi_5, *) &= \tfrac{1}{4}(14\nu_{2a} + 6\nu_{4a} - 2\nu_{4b} + 62) \geqslant 0;\\
\mu_2(u, \chi_5, *) &= \tfrac{1}{4}(-14\nu_{2a} - 6\nu_{4a} + 2\nu_{4b} + 62) \geqslant 0.
\end{aligned}$$

Put $t_1 = 5\nu_{2a} + \nu_{4a} + \nu_{4b}$ and $t_2 = 7\nu_{2a} + 3\nu_{4a} - \nu_{4b}$, then $t_1 \in \{2r+1 \mid -7 \leqslant r \leqslant 6\}$ and $t_2 \in \{2s + 1 \mid -16 \leqslant s \leqslant 15\}$. Thus, we obtain the system of linear equations $\nu_{2a} + \nu_{4a} + \nu_{4b} = 1$, $5\nu_{2a} + \nu_{4a} + \nu_{4b} = t_1$, $7\nu_{2a} + 3\nu_{4a} - \nu_{4b} = t_2$. Solving such systems for all possible combinations of values of t_1 and t_2, and considering additional inequalities

$$\begin{aligned}
\mu_0(u, \chi_5, 3) &= \tfrac{1}{4}(2\nu_{2a} - 6\nu_{4a} + 2\nu_{4b} + 50) \geqslant 0;\\
\mu_2(u, \chi_5, 3) &= \tfrac{1}{4}(-2\nu_{2a} + 6\nu_{4a} - 2\nu_{4b} + 50) \geqslant 0,
\end{aligned}$$

and also restrictions given by Proposition 5, we get only the 34 integer solutions $(\nu_{2a}, \nu_{4a}, \nu_{4b})$ listed in part (iii) of Theorem 1.

• Let u be a unit of order 6. By (1) and Proposition 2 we get $\nu_{2a} + \nu_{3a} + \nu_{6a} = 1$.

By Proposition 3 we obtain the following system of inequalities:

$$\mu_1(u,\chi_2,*) = \tfrac{1}{6}(5\nu_{2a} + 3\nu_{3a} - \nu_{6a} + 13) \geqslant 0;$$
$$\mu_3(u,\chi_2,*) = \tfrac{1}{6}(-10\nu_{2a} - 6\nu_{3a} + 2\nu_{6a} + 22) \geqslant 0;$$
$$\mu_0(u,\chi_4,7) = \tfrac{1}{6}(12\nu_{2a} + 60) \geqslant 0; \quad \mu_3(u,\chi_4,7) = \tfrac{1}{6}(-12\nu_{2a} + 48) \geqslant 0;$$
$$\mu_1(u,\chi_3,*) = \tfrac{1}{6}(-3\nu_{2a} + 48) \geqslant 0.$$

Using calculations similar to the previous case, we get only the 15 integer solutions $(\nu_{2a}, \nu_{3a}, \nu_{6a})$ listed in part (iv) of Theorem 1.

• Let u be a unit of order 7. By (1) and Proposition 2 we get $\nu_{7a} + \nu_{7b} = 1$. Using Proposition 3 we obtain the following system of inequalities:

$$\mu_1(u,\chi_3,*) = \tfrac{1}{7}(4\nu_{7a} - 3\nu_{7b} + 45) \geqslant 0; \quad \mu_3(u,\chi_3,*) = \tfrac{1}{7}(-3\nu_{7a} + 4\nu_{7b} + 45) \geqslant 0;$$
$$\mu_1(u,\chi_2,2) = \tfrac{1}{7}(4\nu_{7a} - 3\nu_{7b} + 10) \geqslant 0; \quad \mu_3(u,\chi_2,2) = \tfrac{1}{7}(-3\nu_{7a} + 4\nu_{7b} + 10) \geqslant 0.$$

Using that $\nu_{7a} + \nu_{7b} = 1$, we get that $-1 \leqslant \nu_{7a} \leqslant 2$, and after this it is easy to check that we have only the four integer solutions (ν_{7a}, ν_{7b}) listed in part (v) of Theorem 1.

• Let u be a unit of order 11. By (1) and Proposition 2 we get $\nu_{11a} + \nu_{11b} = 1$. Using Proposition 3 we obtain the following system of inequalities:

$$\mu_1(u,\chi_{10},*) = \tfrac{1}{11}(6\nu_{11a} - 5\nu_{11b} + 280) \geqslant 0;$$
$$\mu_2(u,\chi_{10},*) = \tfrac{1}{11}(-5\nu_{11a} + 6\nu_{11b} + 280) \geqslant 0;$$
$$\mu_1(u,\chi_5,2) = \tfrac{1}{11}(7\nu_{11a} - 4\nu_{11b} + 70) \geqslant 0;$$
$$\mu_2(u,\chi_5,2) = \tfrac{1}{11}(-4\nu_{11a} + 7\nu_{11b} + 70) \geqslant 0;$$
$$\mu_1(u,\chi_5,3) = \tfrac{1}{11}(6\nu_{11a} - 5\nu_{11b} + 49) \geqslant 0;$$
$$\mu_2(u,\chi_5,3) = \tfrac{1}{11}(-5\nu_{11a} + 6\nu_{11b} + 49) \geqslant 0.$$

Using calculations similar to the previous case, we get only the ten integer solutions for (ν_{11a}, ν_{11b}) listed in part (vi) of Theorem 1.

It remains to prove part (i) of Theorem 1, considering units of $V(\mathbb{Z}G)$ of orders 10, 14, 15, 21, 22, 33, 35, 55 and 77.

• Let u be a unit of order 10. By (1) and Proposition 2 we get $\nu_{2a} + \nu_{5a} = 1$. Using Proposition 3 we obtain the following system of inequalities:

$$\mu_0(u,\chi_2,*) = \tfrac{1}{10}(20\nu_{2a} + 4\nu_{5a} + 30) \geqslant 0;$$
$$\mu_5(u,\chi_2,*) = \tfrac{1}{10}(-20\nu_{2a} - 4\nu_{5a} + 20) \geqslant 0;$$
$$\mu_1(u,\chi_3,*) = \tfrac{1}{10}(-3\nu_{2a} + 48) \geqslant 0,$$

that has no solutions such that all $\mu_i(u,\chi_j,*)$ are non-negative integers.

• Let u be a unit of order 14. Then by (1) and Proposition 2 we get $\nu_{2a}+\nu_{7a}+\nu_{7b} = 1$. We need to consider four cases defined by part (v) of Theorem 1, but in all of them using Proposition 3 we obtain the same system of inequalities:

$$\mu_0(u,\chi_2,*) = \tfrac{1}{14}(30\nu_{2a} + 26) \geqslant 0; \quad \mu_7(u,\chi_2,*) = \tfrac{1}{14}(-30\nu_{2a} + 16) \geqslant 0,$$

which has no solutions such that all $\mu_i(u,\chi_j,*)$ are non-negative integers.

• Let u be a unit of order 15. By (1) and Proposition 2 we get $\nu_{3a} + \nu_{5a} = 1$. Using

Proposition 3 we obtain the following system of inequalities:

$$\mu_0(u,\chi_2,*) = \tfrac{1}{15}(24\nu_{3a} + 8\nu_{5a} + 31) \geqslant 0;$$
$$\mu_5(u,\chi_2,*) = \tfrac{1}{15}(-12\nu_{3a} - 4\nu_{5a} + 22) \geqslant 0.$$

From this follows that $3\nu_{3a} + \nu_{5a} = -2$, and all conditions together leave us no integer solutions.

• Let u be a unit of order 21. Then by (1) and Proposition 2 we have

$$\nu_{3a} + \nu_{7a} + \nu_{7b} = 1.$$

We need to consider four cases determined by part (v) of Theorem 1. Using Proposition 3 we obtain the following systems of inequalities:

$$\mu_0(u,\chi_2,*) = \tfrac{1}{21}(36\nu_{3a} + 27) \geqslant 0; \qquad \mu_7(u,\chi_2,*) = \tfrac{1}{21}(-18\nu_{3a} + 18) \geqslant 0;$$
$$\mu_1(u,\chi_3,*) = \tfrac{1}{21}(3\nu_{7a} - 4\nu_{7b} + \alpha) \geqslant 0; \quad \mu_9(u,\chi_3,*) = \tfrac{1}{21}(-6\nu_{7a} + 8\nu_{7b} + \alpha) \geqslant 0;$$
$$\mu_3(u,\chi_3,*) = \tfrac{1}{21}(8\nu_{7a} - 6\nu_{7b} + \beta) \geqslant 0,$$

$$\text{where} \quad (\alpha,\beta) = \begin{cases} (49,42), & \text{when} \quad \chi(u^3) = \chi(7a); \\ (42,49), & \text{when} \quad \chi(u^3) = \chi(7b); \\ (56,35), & \text{when} \quad \chi(u^3) = 2\chi(7a) - \chi(7b); \\ (35,56), & \text{when} \quad \chi(u^3) = -\chi(7a) + 2\chi(7b), \end{cases}$$

which have no solutions such that all $\mu_i(u,\chi_j,*)$ are non-negative integers.

• Let u be a unit of order 22. Then by (1) and Proposition 2 we get

$$\nu_{2a} + \nu_{11a} + \nu_{11b} = 1.$$

We need to consider ten cases determined by part (vi) of Theorem 1. In each case using Proposition 3 we obtain the following system of inequalities:

$$\mu_0(u,\chi_2,*) = \tfrac{1}{22}(50\nu_{2a} - 10\nu_{11a} - 10\nu_{11b} + 16) \geqslant 0;$$
$$\mu_{11}(u,\chi_2,*) = \tfrac{1}{22}(-50\nu_{2a} + 10\nu_{11a} + 10\nu_{11b} + 6) \geqslant 0,$$

that has no solutions such that all $\mu_i(u,\chi_j,*)$ are non-negative integers.

• Let u be a unit of order 33. Then by (1) and Proposition 2 we get

$$\nu_{3a} + \nu_{11a} + \nu_{11b} = 1.$$

As in the previous case, we need to consider ten cases determined by part (vi) of Theorem 1. In each case using Proposition 3 we obtain the same system:

$$\mu_0(u,\chi_2,*) = \tfrac{1}{33}(60\nu_{3a} - 20\nu_{11a} - 20\nu_{11b} + 17) \geqslant 0;$$
$$\mu_{11}(u,\chi_2,*) = \tfrac{1}{33}(-30\nu_{3a} + 10\nu_{11a} + 10\nu_{11b} + 8) \geqslant 0,$$

that has no solutions such that all $\mu_i(u,\chi_j,*)$ are non-negative integers.

• Let u be a unit of order 35. Then by (1) and Proposition 2 we get $\nu_{5a}+\nu_{7a}+\nu_{7b} = 1$. We need to consider four cases defined by part (v) of Theorem 1, but in all of them using Proposition 3 we obtain the same system of inequalities:

$$\mu_0(u,\chi_2,*) = \tfrac{1}{35}(24\nu_{5a} + 25) \geqslant 0; \quad \mu_0(u,\chi_7,2) = \tfrac{1}{35}(-48\nu_{5a} + 90) \geqslant 0,$$

which has no solutions such that all $\mu_i(u,\chi_j,p)$ are non-negative integers.

• Let u be a unit of order 55. Then by (1) and Proposition 2 we get

$$\nu_{5a} + \nu_{11a} + \nu_{11b} = 1.$$

We need to consider ten cases determined by part (vi) of Theorem 1. In each case using Proposition 3 we obtain the following system of inequalities:

$$\begin{aligned}
\mu_0(u,\chi_2,*) &= \tfrac{1}{55}(40\nu_{5a} - 40\nu_{11a} - 40\nu_{11b} + 15) \geqslant 0;\\
\mu_{11}(u,\chi_2,*) &= \tfrac{1}{55}(-10\nu_{5a} + 10\nu_{11a} + 10\nu_{11b} + 10) \geqslant 0;\\
\mu_1(u,\chi_{10},*) &= \tfrac{1}{55}(-6\nu_{11a} + 5\nu_{11b} + \alpha) \geqslant 0;\\
\mu_5(u,\chi_{10},*) &= \tfrac{1}{55}(24\nu_{11a} - 20\nu_{11b} + \alpha) \geqslant 0;\\
\mu_1(u,\chi_2,*) &= \tfrac{1}{55}(\nu_{5a} - \nu_{11a} - \nu_{11b} + 21) \geqslant 0,
\end{aligned}$$

$$\text{where} \quad \alpha = \begin{cases}
286, & \text{when} \quad \chi(u^5) = \chi(11a);\\
275, & \text{when} \quad \chi(u^5) = \chi(11b);\\
330, & \text{when} \quad \chi(u^5) = 5\chi(11a) - 4\chi(11b);\\
253, & \text{when} \quad \chi(u^5) = -2\chi(11a) + 3\chi(11b);\\
297, & \text{when} \quad \chi(u^5) = 2\chi(11a) - \chi(11b);\\
242, & \text{when} \quad \chi(u^5) = -3\chi(11a) + 4\chi(11b);\\
231, & \text{when} \quad \chi(u^5) = -4\chi(11a) + 5\chi(11b);\\
308, & \text{when} \quad \chi(u^5) = 3\chi(11a) - 2\chi(11b);\\
264, & \text{when} \quad \chi(u^5) = -\chi(11a) + 2\chi(11b);\\
319, & \text{when} \quad \chi(u^5) = 4\chi(11a) - 3\chi(11b),
\end{cases}$$

that has no solutions such that all $\mu_i(u,\chi_j,*)$ are non-negative integers.

- Let u be a unit of order 77. Then by (1) and Proposition 2 we have

$$\nu_{7a} + \nu_{7b} + \nu_{11a} + \nu_{11b} = 1.$$

We must consider 40 cases determined by parts (v) and (vi) of Theorem 1, but luckily in all of them using Proposition 3 we obtain the same system of inequalities:

$$\begin{aligned}
\mu_{11}(u,\chi_2,*) &= \tfrac{1}{77}(10\nu_{11a} + 10\nu_{11b} + 11) \geqslant 0;\\
\mu_0(u,\chi_2,*) &= \tfrac{1}{77}(-60\nu_{11a} - 60\nu_{11b} + 11) \geqslant 0,
\end{aligned}$$

which has no solutions such that all $\mu_i(u,\chi_j,*)$ are non-negative integers. This finishes the proof of Theorem 1.

Appendix 1.

Possible partial augmentations $(\nu_{2a},\nu_{4a},\nu_{4b},\nu_{8a})$ for units of order 8:

$(-2,-2,4,1)$,	$(-2,-1,3,1)$,	$(-2,0,2,1)$,	$(-2,0,4,-1)$,	$(-2,1,1,1)$,
$(-2,1,3,-1)$,	$(-2,2,0,1)$,	$(-2,2,2,-1)$,	$(-2,2,4,-3)$,	$(-2,3,-1,1)$,
$(-2,3,1,-1)$,	$(-2,3,3,-3)$,	$(-2,4,-2,1)$,	$(-2,4,0,-1)$,	$(-2,4,2,-3)$,
$(-2,5,-3,1)$,	$(-2,5,-1,-1)$,	$(-2,5,1,-3)$,	$(-2,6,-2,-1)$,	$(-2,6,0,-3)$,
$(0,-5,3,3)$,	$(0,-4,2,3)$,	$(0,-4,4,1)$,	$(0,-3,1,3)$,	$(0,-3,3,1)$,
$(0,-3,5,-1)$,	$(0,-2,0,3)$,	$(0,-2,2,1)$,	$(0,-2,4,-1)$,	$(0,-2,6,-3)$,
$(0,-1,-1,3)$,	$(0,-1,1,1)$,	$(0,-1,3,-1)$,	$(0,-1,5,-3)$,	$(0,0,-2,3)$,
$(0,0,0,1)$,	$(0,0,2,-1)$,	$(0,0,4,-3)$,	$(0,1,-3,3)$,	$(0,1,-1,1)$,
$(0,1,1,-1)$,	$(0,1,3,-3)$,	$(0,2,-4,3)$,	$(0,2,-2,1)$,	$(0,2,0,-1)$,
$(0,2,2,-3)$,	$(0,3,-5,3)$,	$(0,3,-3,1)$,	$(0,3,-1,-1)$,	$(0,3,1,-3)$,
$(0,4,-4,1)$,	$(0,4,-2,-1)$,	$(0,4,0,-3)$,	$(0,5,-3,-1)$,	$(0,5,-1,-3)$,
$(0,6,-2,-3)$,	$(2,-5,1,3)$,	$(2,-5,3,1)$,	$(2,-4,0,3)$,	$(2,-4,2,1)$,
$(2,-4,4,-1)$,	$(2,-3,-1,3)$,	$(2,-3,1,1)$,	$(2,-3,3,-1)$,	$(2,-2,-2,3)$,
$(2,-2,0,1)$,	$(2,-2,2,-1)$,	$(2,-1,-3,3)$,	$(2,-1,-1,1)$,	$(2,-1,1,-1)$,
$(2,0,-2,1)$,	$(2,0,0,-1)$,	$(2,1,-3,1)$,	$(2,1,-1,-1)$,	$(2,2,-2,-1)$,
$(2,3,-3,-1)$				

Appendix 2.

Possible partial augmentations $(\nu_{2a},\nu_{3a},\nu_{4a},\nu_{4b},\nu_{6a})$ for units of order 12:

$(-4,9,-4,-4,4)$,	$(-4,9,-3,-5,4)$,	$(-4,9,-2,-6,4)$,	$(-4,9,-1,-7,4)$,
$(-4,9,-1,-3,0)$,	$(-4,9,0,-8,4)$,	$(-4,9,0,-4,0)$,	$(-4,9,1,-9,4)$,
$(-4,9,2,-10,4)$,	$(-4,9,3,-11,4)$,	$(-4,12,-5,-5,3)$,	$(-4,12,-4,-6,3)$,
$(-4,12,-4,-4,1)$,	$(-4,12,-3,-7,3)$,	$(-4,12,-3,-5,1)$,	$(-4,12,-2,-8,3)$,
$(-4,12,-2,-6,1)$,	$(-4,12,-1,-9,3)$,	$(-4,12,-1,-7,1)$,	$(-4,12,0,-10,3)$,
$(-4,12,0,-8,1)$,	$(-3,3,0,1,0)$,	$(-3,3,1,0,0)$,	$(-3,3,2,-1,0)$,
$(-3,3,2,3,-4)$,	$(-3,3,3,-2,0)$,	$(-3,3,3,2,-4)$,	$(-3,3,4,-3,0)$,
$(-3,3,4,1,-4)$,	$(-3,3,5,-4,0)$,	$(-3,3,5,0,-4)$,	$(-3,3,6,-5,0)$,
$(-3,6,-4,-1,3)$,	$(-3,6,-3,-2,3)$,	$(-3,6,-2,-3,3)$,	$(-3,6,-2,1,-1)$,
$(-3,6,-1,-4,3)$,	$(-3,6,-1,0,-1)$,	$(-3,6,0,-5,3)$,	$(-3,6,0,-1,-1)$,

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(−3, 6, 1, −6, 3), (−3, 6, 1, −2, −1), (−3, 6, 2, −7, 3), (−3, 6, 2, −3, −1),
(−3, 6, 3, −8, 3), (−3, 6, 3, −4, −1), (−3, 6, 4, −9, 3), (−3, 6, 4, −5, −1),
(−3, 6, 5, −10, 3), (−3, 6, 5, −6, −1), (−3, 6, 6, −11, 3), (−3, 9, −7, −4, 6),
(−3, 9, −6, −5, 6), (−3, 9, −6, −3, 4), (−3, 9, −5, −6, 6), (−3, 9, −5, −4, 4),
(−3, 9, −5, −2, 2), (−3, 9, −4, −7, 6), (−3, 9, −4, −5, 4), (−3, 9, −4, −3, 2),
(−3, 9, −4, −1, 0), (−3, 9, −3, −8, 6), (−3, 9, −3, −6, 4), (−3, 9, −3, −4, 2),
(−3, 9, −3, −2, 0), (−3, 9, −2, −9, 6), (−3, 9, −2, −7, 4), (−3, 9, −2, −5, 2),
(−3, 9, −2, −3, 0), (−3, 9, −1, −10, 6), (−3, 9, −1, −8, 4), (−3, 9, −1, −6, 2),
(−3, 9, −1, −4, 0), (−3, 9, 0, −11, 6), (−3, 9, 0, −9, 4), (−3, 9, 0, −7, 2),
(−3, 9, 0, −5, 0), (−3, 9, 1, −12, 6), (−3, 9, 1, −10, 4), (−3, 9, 1, −8, 2),
(−3, 9, 1, −6, 0), (−3, 9, 2, −13, 6), (−3, 9, 2, −11, 4), (−3, 9, 2, −9, 2),
(−3, 9, 2, −7, 0), (−3, 9, 3, −14, 6), (−3, 9, 3, −10, 2), (−3, 9, 3, −8, 0),
(−3, 9, 4, −15, 6), (−3, 12, −7, −6, 5), (−3, 12, −7, −4, 3), (−3, 12, −6, −7, 5),
(−3, 12, −6, −5, 3), (−3, 12, −5, −8, 5), (−3, 12, −5, −6, 3), (−3, 12, −4, −9, 5),
(−3, 12, −4, −7, 3), (−3, 12, −3, −10, 5), (−3, 12, −3, −8, 3), (−3, 12, −2, −11, 5),
(−3, 12, −2, −9, 3), (−3, 12, −1, −12, 5), (−3, 12, −1, −10, 3), (−3, 12, 0, −13, 5),
(−3, 12, 0, −11, 3), (−3, 12, 1, −12, 3), (−2, 0, 0, 4, −1), (−2, 0, 1, 3, −1),
(−2, 0, 2, 2, −1), (−2, 0, 2, 6, −5), (−2, 0, 3, 1, −1), (−2, 0, 3, 5, −5),
(−2, 0, 4, 0, −1), (−2, 0, 4, 4, −5), (−2, 0, 5, −1, −1), (−2, 0, 5, 3, −5),
(−2, 0, 6, −2, −1), (−2, 0, 6, 2, −5), (−2, 0, 7, −3, −1), (−2, 0, 7, 1, −5),
(−2, 0, 8, −4, −1), (−2, 0, 8, 0, −5), (−2, 3, −4, 2, 2), (−2, 3, −3, 1, 2),
(−2, 3, −2, 0, 2), (−2, 3, −2, 4, −2), (−2, 3, −1, −1, 2), (−2, 3, −1, 1, 0),
(−2, 3, −1, 3, −2), (−2, 3, 0, −2, 2), (−2, 3, 0, 0, 0), (−2, 3, 0, 2, −2),
(−2, 3, 0, 4, −4), (−2, 3, 1, −3, 2), (−2, 3, 1, −1, 0), (−2, 3, 1, 1, −2),
(−2, 3, 1, 3, −4), (−2, 3, 2, −4, 2), (−2, 3, 2, −2, 0), (−2, 3, 2, 0, −2),
(−2, 3, 2, 2, −4), (−2, 3, 3, −5, 2), (−2, 3, 3, −3, 0), (−2, 3, 3, −1, −2),
(−2, 3, 3, 1, −4), (−2, 3, 4, −6, 2), (−2, 3, 4, −4, 0), (−2, 3, 4, −2, −2),
(−2, 3, 4, 0, −4), (−2, 3, 5, −7, 2), (−2, 3, 5, −3, −2), (−2, 3, 5, −1, −4),
(−2, 3, 6, −8, 2), (−2, 3, 6, −4, −2), (−2, 3, 6, −2, −4), (−2, 3, 7, −9, 2),
(−2, 3, 7, −5, −2), (−2, 3, 8, −10, 2), (−2, 3, 8, −6, −2), (−2, 6, −7, −1, 5),
(−2, 6, −6, −2, 5), (−2, 6, −6, 0, 3), (−2, 6, −5, −3, 5), (−2, 6, −5, −1, 3),
(−2, 6, −5, 1, 1), (−2, 6, −4, −4, 5), (−2, 6, −4, −2, 3), (−2, 6, −4, 0, 1),
(−2, 6, −4, 2, −1), (−2, 6, −3, −5, 5), (−2, 6, −3, −3, 3), (−2, 6, −3, −1, 1),
(−2, 6, −3, 1, −1), (−2, 6, −2, −6, 5), (−2, 6, −2, −4, 3), (−2, 6, −2, −2, 1),
(−2, 6, −2, 0, −1), (−2, 6, −1, −7, 5), (−2, 6, −1, −5, 3), (−2, 6, −1, −3, 1),
(−2, 6, −1, −1, −1), (−2, 6, 0, −8, 5), (−2, 6, 0, −6, 3), (−2, 6, 0, −4, 1),
(−2, 6, 0, −2, −1), (−2, 6, 1, −9, 5), (−2, 6, 1, −7, 3), (−2, 6, 1, −5, 1),
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(−2, 6, 2, −4, −1), (−2, 6, 3, −11, 5), (−2, 6, 3, −9, 3), (−2, 6, 3, −7, 1),
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(−2, 6, 4, −6, −1), (−2, 6, 5, −13, 5), (−2, 6, 5, −9, 1), (−2, 6, 5, −7, −1),
(−2, 6, 6, −10, 1), (−2, 6, 6, −8, −1), (−2, 9, −9, −5, 8), (−2, 9, −9, −3, 6),
(−2, 9, −8, −6, 8), (−2, 9, −8, −4, 6), (−2, 9, −7, −7, 8), (−2, 9, −7, −5, 6),
(−2, 9, −7, −3, 4), (−2, 9, −7, −1, 2), (−2, 9, −6, −8, 8), (−2, 9, −6, −6, 6),
(−2, 9, −6, −4, 4), (−2, 9, −6, −2, 2), (−2, 9, −5, −9, 8), (−2, 9, −5, −7, 6),
(−2, 9, −5, −5, 4), (−2, 9, −5, −3, 2), (−2, 9, −4, −10, 8), (−2, 9, −4, −8, 6),
(−2, 9, −4, −6, 4), (−2, 9, −4, −4, 2), (−2, 9, −3, −11, 8), (−2, 9, −3, −9, 6),
(−2, 9, −3, −7, 4), (−2, 9, −3, −5, 2), (−2, 9, −2, −12, 8), (−2, 9, −2, −10, 6),
(−2, 9, −2, −8, 4), (−2, 9, −2, −6, 2), (−2, 9, −1, −13, 8), (−2, 9, −1, −11, 6),
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(−1, 6, −7, −4, 7), (−1, 6, −7, −2, 5), (−1, 6, −7, 0, 3), (−1, 6, −7, 2, 1),
(−1, 6, −6, −5, 7), (−1, 6, −6, −3, 5), (−1, 6, −6, −1, 3), (−1, 6, −6, 1, 1),
(−1, 6, −5, −6, 7), (−1, 6, −5, −4, 5), (−1, 6, −5, −2, 3), (−1, 6, −5, 0, 1),
(−1, 6, −4, −7, 7), (−1, 6, −4, −5, 5), (−1, 6, −4, −3, 3), (−1, 6, −4, −1, 1),
(−1, 6, −3, −8, 7), (−1, 6, −3, −6, 5), (−1, 6, −3, −4, 3), (−1, 6, −3, −2, 1),
(−1, 6, −2, −9, 7), (−1, 6, −2, −7, 5), (−1, 6, −2, −5, 3), (−1, 6, −2, −3, 1),
(−1, 6, −1, −10, 7), (−1, 6, −1, −8, 5), (−1, 6, −1, −6, 3), (−1, 6, −1, −4, 1),
(−1, 6, 0, −11, 7), (−1, 6, 0, −9, 5), (−1, 6, 0, −7, 3), (−1, 6, 0, −5, 1),
(−1, 6, 1, −12, 7), (−1, 6, 1, −10, 5), (−1, 6, 1, −8, 3), (−1, 6, 1, −6, 1),
(−1, 6, 2, −11, 5), (−1, 6, 2, −9, 3), (−1, 6, 2, −7, 1), (−1, 6, 3, −12, 5),
(−1, 6, 3, −10, 3), (−1, 6, 3, −8, 1), (−1, 6, 4, −9, 1), (−1, 6, 5, −10, 1),
(−1, 9, −10, −7, 10), (−1, 9, −10, −5, 8), (−1, 9, −9, −8, 10), (−1, 9, −9, −6, 8),
(−1, 9, −9, −4, 6), (−1, 9, −9, −2, 4), (−1, 9, −8, −9, 10), (−1, 9, −8, −7, 8),
(−1, 9, −8, −5, 6), (−1, 9, −8, −3, 4), (−1, 9, −7, −10, 10), (−1, 9, −7, −8, 8),

(−1, 9, −7, −6, 6),	(−1, 9, −7, −4, 4),	(−1, 9, −6, −11, 10),	(−1, 9, −6, −9, 8),
(−1, 9, −6, −7, 6),	(−1, 9, −6, −5, 4),	(−1, 9, −5, −12, 10),	(−1, 9, −5, −10, 8),
(−1, 9, −5, −8, 6),	(−1, 9, −5, −6, 4),	(−1, 9, −4, −13, 10),	(−1, 9, −4, −11, 8),
(−1, 9, −4, −9, 6),	(−1, 9, −4, −7, 4),	(−1, 9, −3, −14, 10),	(−1, 9, −3, −12, 8),
(−1, 9, −3, −10, 6),	(−1, 9, −3, −8, 4),	(−1, 9, −2, −13, 8),	(−1, 9, −2, −9, 4),
(−1, 9, −1, −14, 8),	(−1, 9, −1, −10, 4),	(−1, 9, 0, −11, 4),	(−1, 9, 1, −12, 4),
(−1, 12, −10, −7, 7),	(−1, 12, −9, −8, 7),	(−1, 12, −8, −9, 7),	(−1, 12, −7, −10, 7),
(−1, 12, −6, −11, 7),	(−1, 12, −5, −12, 7),	(−1, 12, −4, −13, 7),	(−1, 12, −3, −14, 7),
(0, −9, 5, 11, −6),	(0, −9, 6, 10, −6),	(0, −9, 7, 9, −6),	(0, −9, 8, 8, −6),
(0, −9, 9, 7, −6),	(0, −9, 10, 6, −6),	(0, −9, 11, 5, −6),	(0, −9, 12, 4, −6),
(0, −6, 0, 10, −3),	(0, −6, 1, 9, −3),	(0, −6, 2, 8, −3),	(0, −6, 2, 12, −7),
(0, −6, 3, 7, −3),	(0, −6, 3, 11, −7),	(0, −6, 4, 6, −3),	(0, −6, 4, 10, −7),
(0, −6, 5, 5, −3),	(0, −6, 5, 9, −7),	(0, −6, 6, 4, −3),	(0, −6, 6, 8, −7),
(0, −6, 7, 3, −3),	(0, −6, 7, 7, −7),	(0, −6, 8, 2, −3),	(0, −6, 8, 6, −7),
(0, −6, 9, 1, −3),	(0, −6, 9, 5, −7),	(0, −6, 10, 0, −3),	(0, −6, 10, 4, −7),
(0, −6, 11, 3, −7),	(0, −6, 12, 2, −7),	(0, −3, −4, 8, 0),	(0, −3, −3, 7, 0),
(0, −3, −2, 6, 0),	(0, −3, −2, 10, −4),	(0, −3, −1, 5, 0),	(0, −3, −1, 7, −2),
(0, −3, −1, 9, −4),	(0, −3, 0, 4, 0),	(0, −3, 0, 6, −2),	(0, −3, 0, 8, −4),
(0, −3, 0, 10, −6),	(0, −3, 1, 3, 0),	(0, −3, 1, 5, −2),	(0, −3, 1, 7, −4),
(0, −3, 1, 9, −6),	(0, −3, 2, 2, 0),	(0, −3, 2, 4, −2),	(0, −3, 2, 6, −4),
(0, −3, 2, 8, −6),	(0, −3, 3, 1, 0),	(0, −3, 3, 3, −2),	(0, −3, 3, 5, −4),
(0, −3, 3, 7, −6),	(0, −3, 4, 0, 0),	(0, −3, 4, 2, −2),	(0, −3, 4, 4, −4),
(0, −3, 4, 6, −6),	(0, −3, 5, −1, 0),	(0, −3, 5, 1, −2),	(0, −3, 5, 3, −4),
(0, −3, 5, 5, −6),	(0, −3, 6, −2, 0),	(0, −3, 6, 0, −2),	(0, −3, 6, 2, −4),
(0, −3, 6, 4, −6),	(0, −3, 7, −3, 0),	(0, −3, 7, −1, −2),	(0, −3, 7, 1, −4),
(0, −3, 7, 3, −6),	(0, −3, 8, −4, 0),	(0, −3, 8, −2, −2),	(0, −3, 8, 0, −4),
(0, −3, 8, 2, −6),	(0, −3, 9, −1, −4),	(0, −3, 9, 1, −6),	(0, −3, 10, −2, −4),
(0, −3, 10, 0, −6),	(0, 0, −7, 5, 3),	(0, 0, −6, 4, 3),	(0, 0, −6, 6, 1),
(0, 0, −5, 3, 3),	(0, 0, −5, 5, 1),	(0, 0, −5, 7, −1),	(0, 0, −4, 2, 3),
(0, 0, −4, 4, 1),	(0, 0, −4, 6, −1),	(0, 0, −4, 8, −3),	(0, 0, −3, 1, 3),
(0, 0, −3, 3, 1),	(0, 0, −3, 5, −1),	(0, 0, −3, 7, −3),	(0, 0, −2, 0, 3),
(0, 0, −2, 2, 1),	(0, 0, −2, 4, −1),	(0, 0, −2, 6, −3),	(0, 0, −1, −1, 3),
(0, 0, −1, 1, 1),	(0, 0, −1, 3, −1),	(0, 0, −1, 5, −3),	(0, 0, 0, −2, 3),
(0, 0, 0, 0, 1),	(0, 0, 0, 2, −1),	(0, 0, 0, 4, −3),	(0, 0, 1, −3, 3),
(0, 0, 1, −1, 1),	(0, 0, 1, 1, −1),	(0, 0, 1, 3, −3),	(0, 0, 2, −4, 3),
(0, 0, 2, −2, 1),	(0, 0, 2, 0, −1),	(0, 0, 2, 2, −3),	(0, 0, 3, −5, 3),
(0, 0, 3, −3, 1),	(0, 0, 3, −1, −1),	(0, 0, 3, 1, −3),	(0, 0, 4, −6, 3),
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