Appendix A
Useful formulae

A.1 The delta function

The Dirac delta function is a bi-local distribution defined by the relations

$$\delta(t,t') = \begin{cases} 0 & t-t' \neq 0 \\ \infty & t-t' = 0 \end{cases} \quad \begin{matrix} \text{(A.1)} \\ \text{(A.2)} \end{matrix}$$

$$\delta(x,x') = \delta(x-x') \tag{A.3}$$

$$\int_{-a}^{+a} \mathrm{d}x'\ \delta(x,x') f(x') = f(x) \tag{A.4}$$

$$\int_{-a}^{+a} \mathrm{d}x'\ \delta(x-x') = 1. \tag{A.5}$$

If $f(x)$ is a function which is symmetrical about x_0, then

$$\int_{-\infty}^{x_0} \delta(x_0-x') f(x')\mathrm{d}x' + \int_{x_0}^{\infty} \delta(x_0-x') f(x')\mathrm{d}x' = f(x_0); \tag{A.6}$$

thus, by symmetry,

$$\int_{-\infty}^{x_0} \delta(x_0-x') f(x')\mathrm{d}x' = \frac{1}{2} f(x_0). \tag{A.7}$$

A useful, integral representation of the delta function is given by the Fourier integral

$$\delta(x_1-x_1') = \int \frac{\mathrm{d}k}{2\pi} \mathrm{e}^{\mathrm{i}k(x_1-x_1')}. \tag{A.8}$$

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Various integral representations of the delta function are useful. For instance

$$\delta(x) = \lim_{\alpha \to 0} \frac{1}{2\pi\alpha} \mathrm{e}^{-\frac{1}{\alpha}x^2}. \tag{A.9}$$

The Fourier representation on the $(n+1)$ dimensional delta function

$$\begin{aligned}\delta(x, x') \equiv \delta(x - x') &= \int \frac{\mathrm{d}^{n+1}k}{(2\pi)^{n+1}}\, \mathrm{e}^{\mathrm{i}k(x-x')} \\ &= \delta(x^0 - x^{0'})\delta(x^1 - x^{1'}) \ldots \delta(x^n - x^{n'}) \end{aligned} \tag{A.10}$$

in particular is used in solving for Green functions. Here the shorthand notation $k(x - x')$ in the exponential stands for $k_\mu (x - x')^\mu$.

Derivatives of the delta function normally refer to derivatives of the test functions which they multiply. Meaning may be assigned to these as follows. Consider the boundary value of a function $f(x)$. From the property of the delta function,

$$\int \delta(x - a) f(x - a) \mathrm{d}x = f(0). \tag{A.11}$$

Now, differentiating with respect to a,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}a} f(0) &= \int \left[\frac{\mathrm{d}}{\mathrm{d}a} \delta(x - a) f(x - a) + \delta(x - a) \frac{\mathrm{d}}{\mathrm{d}a} f(x - a) \right] \mathrm{d}x \\ &= 0. \end{aligned} \tag{A.12}$$

From this, we discover that

$$f(x - a) \frac{\mathrm{d}}{\mathrm{d}a} \delta(x - a) = -\delta(x - a) \frac{\mathrm{d}}{\mathrm{d}a} f(x - a), \tag{A.13}$$

or

$$f(t)\; \partial_t \delta(t) = -\delta(t)\; \partial_t f(t), \tag{A.14}$$

which effectively defines the derivative of the delta function.

A useful relation for the one-dimensional delta function of a function $g(x)$ with several roots satisfying $g(x_i) = 0$ is:

$$\delta(g(x)) = \sum_i \frac{1}{g'(x_i)} \delta(g(x_i)), \tag{A.15}$$

where x_i are the roots of the function $g(x)$ and the prime denotes the derivative with respect to x. This is easily proven by change of variable. As with all delta-function relations, this is only strictly valid under the integral sign. Given

$$I = \int \mathrm{d}x f(x) \delta(g(x)), \tag{A.16}$$

change variables to $x' = g(x)$. This incurs a Jacobian in the measure $|J| = \frac{\partial x}{\partial x'} = \frac{\partial x}{\partial g(x)} = \left(\frac{\partial g(x)}{\partial x}\right)^{-1}$. So,

$$I = \int \mathrm{d}x' \frac{1}{g'(x')} f(g^{-1}(x))\delta(x'). \tag{A.17}$$

In replacing x by g^{-1}, we satisfy the rules of the change of variable, but the inverse function $g^{-1}(x')$ is not usually known. Fortunately, the singular nature of the delta function simplifies the calculation, since it implies that contributions can only come from the roots of $g(x')$, thus, the expression becomes,

$$I = \int \mathrm{d}x' \frac{1}{g'(x_i)} f(x_i)\delta(x'). \tag{A.18}$$

In summary, one may use this eqn. (A.15) under the integral sign generally, thanks to the extremely singular nature of the delta function, provided all multiplying functions in the integrand are evaluated at the roots of the original function $g(x)$.

A.2 The step function

$$\theta(t, t') = \begin{cases} 1 & t - t' > 0 \\ \frac{1}{2} & t = t' \\ 0 & t - t' < 0. \end{cases} \tag{A.19}$$

An integral representation of these may be expressed in two equivalent forms:

$$\begin{aligned} \theta(t - t') &= \mathrm{i} \int_{\infty}^{\infty} \frac{\mathrm{d}\alpha}{2\pi} \frac{\mathrm{e}^{-\mathrm{i}\alpha(t-t')}}{\alpha + \mathrm{i}\epsilon} \\ \theta(t' - t) &= -\mathrm{i} \int_{\infty}^{\infty} \frac{\mathrm{d}\alpha}{2\pi} \frac{\mathrm{e}^{-\mathrm{i}\alpha(t-t')}}{\alpha - \mathrm{i}\epsilon}, \end{aligned} \tag{A.20}$$

where the limit $\epsilon \to 0$ is understood. The derivative of the step function is a delta function,

$$\partial_t \theta(t - t') = \delta(t - t'). \tag{A.21}$$

A.3 Anti-symmetry and the Jacobi identity

The commutator (or indeed any anti-symmetrical quantity) has the purely algebraic property that:

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0. \tag{A.22}$$

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A.4 Anti-symmetric tensors in Euclidean space

Anti-symmetric tensors arise in many situations in field theory. In most cases, we shall only be interested in the two-, three- and four-dimensional tensors, defined respectively by

$$\epsilon_{ij} = \begin{cases} +1 & ij = 12 \\ -1 & ij = 21 \\ 0 & \text{otherwise,} \end{cases} \tag{A.23}$$

$$\epsilon_{ijk} = \begin{cases} +1 & ijk = 123 \text{ and even permutations} \\ -1 & ijk = 321 \text{ and other odd permutations} \\ 0 & \text{otherwise,} \end{cases} \tag{A.24}$$

$$\epsilon_{ijkl} = \begin{cases} +1 & ijkl = 1234 \text{ and even permutations} \\ -1 & ijkl = 1243 \text{ and other odd permutations} \\ 0 & \text{otherwise.} \end{cases} \tag{A.25}$$

There are as many values for the indices as there are indices on the tensors in the above relations. Because of the anti-symmetric properties, the following relations are also true.

$$\begin{aligned} \epsilon_{ij} &= = -\epsilon_{ji} \\ \epsilon_{ijk} &= \epsilon_{kij} = \epsilon_{jki} = -\epsilon_{kji} = \epsilon_{ikj} = -\epsilon_{jik} \\ \epsilon_{ii} &= 0 \\ \epsilon_{iij} &= 0 \\ \epsilon_{iijk} &= 0. \end{aligned} \tag{A.26}$$

The number of different permutations increases as the factorial of the number of indices on the tensor. The different permutations can easily be generated by computing the determinant

$$\begin{vmatrix} i & j & k & l \\ i & j & k & l \\ i & j & k & l \\ i & j & k & l \end{vmatrix} \tag{A.27}$$

as a mnemonic, but the signs will not automatically distinguish even and odd permutations, so this is not a practical procedure.

Contractions of indices on anti-symmetric objects are straightforward to work out. The simplest of these are trivial to verify:

$$\begin{aligned} \epsilon^{ij}\epsilon_{kl} &= \delta^i{}_k\delta^j{}_l - \delta^i{}_l\delta^j{}_k \\ \epsilon^{ij}\epsilon_{jk} &= -\delta^i{}_k \\ \epsilon^{ij}\epsilon_{kj} &= \delta^i{}_k \\ \epsilon^{ij}\epsilon_{ij} &= 2. \end{aligned} \tag{A.28}$$

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More general contractions can be calculated by expressing the anti-symmetric tensor products as combinations of delta functions with varying signs and permutations of indices. These are most easily expressed using a notational shorthand for anti-symmetrization. Embedded square brackets are used to denote the anti-symmetrization over a set of indices. For example,

$$X_{[a}Y_{b]} \equiv \frac{1}{2!}(X_aY_b - X_bY_a), \tag{A.29}$$

$$\begin{aligned} X_{[a}Y_bZ_{c]} \equiv \frac{1}{3!}(&X_aY_bZ_c + X_cY_aZ_b + X_bY_cZ_a \\ &-X_cY_bZ_a - X_aY_cZ_b - X_bY_aZ_c), \end{aligned} \tag{A.30}$$

and higher generalizations.

Consider then the product of two three-dimensional Levi-Cevita symbols. It may be proven on the grounds of symmetry alone that

$$\epsilon^{ijk}\epsilon_{lmn} = 3!\delta^i{}_{[l}\delta^j{}_m\delta^k{}_{n]}, \tag{A.31}$$

where

$$\begin{aligned} \delta^i{}_{[l}\delta^j{}_m\delta^k{}_{n]} = \frac{1}{3!}\Big(&\delta^i{}_l\delta^j{}_m\delta^k{}_n + \delta^i{}_n\delta^j{}_l\delta^k{}_m + \delta^i{}_m\delta^j{}_n\delta^k{}_l \\ &-\delta^i{}_n\delta^j{}_m\delta^k{}_l - \delta^i{}_l\delta^j{}_n\delta^k{}_m - \delta^i{}_m\delta^j{}_l\delta^k{}_n\Big). \end{aligned} \tag{A.32}$$

Contracting this on one index (setting $i = l$), and writing the outermost permutation explicitly, we have

$$\epsilon^{ijk}\epsilon_{imn} = 2!\left(\delta^i{}_i\delta^j{}_{[m}\delta^k{}_{n]} - \delta^i{}_m\delta^j{}_{[i}\delta^k{}_{n]} - \delta^i{}_n\delta^j{}_{[m}\delta^k{}_{i]}\right). \tag{A.33}$$

Summing over i gives

$$\begin{aligned} \epsilon^{ijk}\epsilon_{imn} &= 2!(3-1-1)\delta^j{}_{[m}\delta^k{}_{n]} \\ &= \delta^j{}_m\delta^k{}_n - \delta^j{}_n\delta^k{}_m. \end{aligned} \tag{A.34}$$

It is not difficult to see that this procedure may be repeated for n-dimensional products,

$$\epsilon^{ij\ldots k}\epsilon_{lm\ldots n} = n!\delta^i{}_{[l}\delta^j{}_m \ldots \delta^k{}_{n]}. \tag{A.35}$$

Again, setting $i = l$ and expanding the outermost permutation gives,

$$\begin{aligned} \epsilon^{ij\ldots k}\epsilon_{lm\ldots n} &= (n-1)!(\delta^i{}_i\delta^j{}_{[m}\ldots\delta^k{}_{n]} - \cdots) \\ &= (n-1)!(\delta^i{}_i - (n-1))\delta^j{}_{[m}\ldots\delta^k{}_{n]}. \end{aligned} \tag{A.36}$$

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Since $\delta^i{}_i = n$, the first bracket in the result above always reduces to unity. We may also write this result in a more general way, for the contraction of two p-index anti-symmetric products in n dimensions:

$$p!\delta^i{}_{[i}\delta^j{}_m \dots \delta^k{}_{n]} = (p-1)!(n-p+1)\delta^j{}_{[m} \dots \delta^k{}_{n]}. \tag{A.37}$$

This formula leads to a number of frequently used results:

$$\begin{aligned}
\epsilon^{ijk}\epsilon_{imn} &= \delta^j{}_m\delta^k{}_n - \delta^j{}_n\delta^k{}_m \\
\epsilon^{ijkl}\epsilon_{imnp} &= 3!\delta^j{}_{[m}\delta^k{}_n\delta^l{}_{p]} \\
\epsilon^{ijkl}\epsilon_{ijnp} &= 2!(n-2)\delta^k{}_{[n}\delta^l{}_{p]} \\
&= 2(\delta^k{}_n\delta^l{}_p - \delta^k{}_p\delta^l{}_n) \\
\epsilon^{ijkl}\epsilon_{ijkl} &= 2(4^2-4) = 24.
\end{aligned} \tag{A.38}$$

A.5 Anti-symmetric tensors in Minkowski spacetime

In Minkowski spacetime, we have to distinguish between up and down indices. It is normal to define

$$\epsilon^{\mu\nu} = \begin{cases} +1 & \mu\nu = 01 \\ -1 & \mu\nu = 10 \\ 0 & \text{otherwise,} \end{cases} \tag{A.39}$$

$$\epsilon^{\mu\nu\lambda} = \begin{cases} +1 & \mu\nu\lambda = 012 \text{ and even permutations} \\ -1 & \mu\nu\lambda = 210 \text{ and other odd permutations} \\ 0 & \text{otherwise,} \end{cases} \tag{A.40}$$

$$\epsilon^{\mu\nu\lambda\rho} = \begin{cases} +1 & \mu\nu\lambda\rho = 0123 \text{ and even permutations} \\ -1 & \mu\nu\lambda\rho = 0132 \text{ and other odd permutations} \\ 0 & \text{otherwise.} \end{cases} \tag{A.41}$$

Indices are raised and lowered using the metric for the appropriate dimensional spacetime. Since the zeroth component always incurs a minus sign,

$$\begin{aligned}
\epsilon^{\mu\nu\lambda\rho} &= g^{\mu\sigma}g^{\nu\kappa}g^{\lambda\tau}g^{\rho\delta}\epsilon_{\sigma\kappa\tau\delta} \\
\epsilon^{0123} &= -1.1.1.1.\epsilon_{0123},
\end{aligned} \tag{A.42}$$

one has all of the above definitions with indices lowered on the left hand side and minus signs changed on the right hand side. This also means that all of the contraction formulae incur an additional minus sign. This formula leads to a number of frequently used results:

$$\begin{aligned}
\epsilon^{\mu\nu\lambda}\epsilon_{\mu\rho\sigma} &= -\delta^\nu{}_\rho\delta^\lambda{}_\sigma + \delta^\nu{}_\sigma\delta^\lambda{}_\rho \\
\epsilon^{\mu\nu\lambda\rho}\epsilon_{\mu\sigma\tau\zeta} &= -3!\delta^\nu{}_{[\sigma}\delta^\lambda{}_\tau\delta^\rho{}_{\zeta]} \\
\epsilon^{\mu\nu\lambda\rho}\epsilon_{\mu\nu\sigma\tau} &= -2(\delta^\lambda{}_\sigma\delta^\rho{}_\tau - \delta^\lambda{}_\tau\delta^\rho{}_\sigma) \\
\epsilon^{\mu\nu\lambda\rho}\epsilon_{\mu\nu\lambda\rho} &= -24.
\end{aligned} \tag{A.43}$$

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A.6 Doubly complex numbers

Complex numbers $z = x + \mathrm{i}y$ and the conjugates $z^* = x - \mathrm{i}y$ are vectors in the Argand plane. They form a complete covering of the two-dimensional space and, because of de Moivre's theorem,

$$\mathrm{e}^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta, \tag{A.44}$$

they are particularly suited to problems where rotation or circular symmetry is expected. But what of problems where rotation occurs in two separate, orthogonal planes? It seems logical to suppose that a complex representation of such rotation could be applied to each orthogonal plane individually. But such a description would require two separate kinds of vectors $x + \mathrm{i}y$ for one plane and $x + \mathrm{j}z$ for the orthogonal plane, where $\mathrm{i} = \sqrt{-1}$ and $\mathrm{j} = \sqrt{-1}$. We must treat these two imaginary numbers as independent vectors, such that

$$\begin{aligned} \mathrm{i}^2 &= -1 \\ \mathrm{j}^2 &= -1 \\ \mathrm{ij} &\neq -1. \end{aligned} \tag{A.45}$$

Using these quantities, we can formulate doubly complex numbers

$$\begin{aligned} w &= x + \mathrm{i}y - \mathrm{j}z \\ W &= X + \mathrm{i}Y - \mathrm{j}Z \end{aligned} \tag{A.46}$$

as an alternative representation to the three-dimensional vectors $\mathbf{w} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$. The final line in eqn. (A.45) above leads to an interesting question. What commutation properties should we assign to these objects? There are two possibilities:

$$\mathrm{ij} = \pm\mathrm{ji}. \tag{A.47}$$

Interestingly, these two signs correspond to the representations of two different groups. When i and j commute, the w form a representation of the group $U(1) \times U(1)$ which corresponds to independent rotations about two orthogonal axes z and y, but no rotation about the third axis x, in a three-dimensional space. This result is, in fact, trivial from de Moivre's theorem.

When i and j anti-commute, the w form a representation of $SU(2)$, the group of three-dimensional rotations. To show this we must introduce some notation for complex conjugation with respect to the i and j parts. Let us denote

$$\begin{aligned} \overset{\mathrm{i}}{w} &= x - \mathrm{i}y - \mathrm{j}z \\ \overset{\mathrm{j}}{w} &= x + \mathrm{i}y + \mathrm{j}z \\ \overset{\mathrm{ij}}{w} &= x - \mathrm{i}y + \mathrm{j}z, \end{aligned} \tag{A.48}$$

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which have the following algebraic products:

$$
\begin{aligned}
w\,\overset{\mathrm{ij}}{w} &= x^2 + y^2 z^2 - 2\mathrm{ij}yz \\
\overset{\mathrm{i}}{w}\overset{\mathrm{j}}{w} &= x^2 + y^2 + z^2 + 2\mathrm{ij}yz.
\end{aligned}
\tag{A.49}
$$

Thus, the length of a vector is

$$
\frac{1}{2}(w\,\overset{\mathrm{ij}}{w} + \overset{\mathrm{i}}{w}\overset{\mathrm{j}}{w}) = x^2 + y^2 + z^2 = \mathbf{w}\cdot\mathbf{w},
\tag{A.50}
$$

and the scalar product is

$$
\mathbf{w}\cdot\mathbf{W} = \frac{1}{4}\left(w\,\overset{\mathrm{ij}}{W} + \overset{\mathrm{i}}{w}\overset{\mathrm{j}}{W} + \overset{\mathrm{j}}{w}\overset{\mathrm{i}}{W} + \overset{\mathrm{ij}}{w}\,W\right).
\tag{A.51}
$$

It is interesting, and significant, that – concealed within these products are the vector and scalar products for Euclidean space. If we assume that i and j anti-commute, we have

$$
\begin{aligned}
(w, W) = \overset{\mathrm{i}}{w}\overset{\mathrm{j}}{W} &= (xX + yY + zZ) + \mathrm{i}(xY - Yx) \\
&\quad -\mathrm{j}(zX - xZ) - \mathrm{ij}(yZ - zY) \\
&= (\mathbf{w}\cdot\mathbf{W})1 + (\mathbf{w}\times\mathbf{W}),
\end{aligned}
\tag{A.52}
$$

where we have identified the complex numbers with Euclidean unit vectors as follows:

$$
\begin{aligned}
1 &\leftrightarrow \text{scalars} \\
\mathrm{i} &\leftrightarrow \hat{\mathbf{k}} \\
\mathrm{j} &\leftrightarrow -\hat{\mathbf{j}} \\
\mathrm{ij} &\leftrightarrow -\hat{\mathbf{i}}.
\end{aligned}
\tag{A.53}
$$

When the coupling between planes is unimportant, i and j commute and the power of this algebraic tool is maximal. An application of this method is given in section A.6.1.

A.6.1 Refraction in a magnetized medium

The addition of a magnetic field leads to the interesting phenomenon of plane wave rotation, studied in section 7.3.3. Neglecting attenuation, $\gamma = 0$, the forcing term can be written in the form of a general Lorentz force

$$
m\frac{\mathrm{d}^2\mathbf{s}}{\mathrm{d}t^2} + k\mathbf{s} = -e\left(\mathbf{E} + \frac{\mathrm{d}\mathbf{s}}{\mathrm{d}t}\times\mathbf{B}\right);
\tag{A.54}
$$

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or writing out the components and defining $\omega_0 = k/m$, $B = B_z$,

$$\frac{\mathrm{d}^2 s_x}{\mathrm{d}t^2} + \frac{e}{m} B \frac{\mathrm{d}s_y}{\mathrm{d}t} + \omega_0^2 s_x = -\frac{e}{m} E_x, \tag{A.55}$$

$$\frac{\mathrm{d}^2 s_y}{\mathrm{d}t^2} - \frac{e}{m} B \frac{\mathrm{d}s_x}{\mathrm{d}t} + \omega_0^2 s_y = -\frac{e}{m} E_y. \tag{A.56}$$

These two equations may be combined into a single equation by defining complex coordinates $s = s_x + \mathrm{i}s_y$ and $E = E_x + \mathrm{i}E_y$, provided ω_0 is an isotropic spring constant, i.e. $\omega_{0x} = \omega_{0y}$

$$\frac{\mathrm{d}^2 s}{\mathrm{d}t^2} - \mathrm{i}\frac{e}{m} B \frac{\mathrm{d}s}{\mathrm{d}t} + \omega_0^2 s = -\frac{e}{m} E. \tag{A.57}$$

Plane polarized waves enter the medium, initially with their E vector parallel to the x axis. These waves impinge upon the quasi-elastically bound electrons, forcing the motion

$$s = \mathrm{Re}\left[s_0 \mathrm{e}^{\mathrm{j}(kz - \omega t - \phi)}\right], \tag{A.58}$$

$$E = \mathrm{Re}\left[E_0 \mathrm{e}^{\mathrm{j}(kz - \omega t)}\right], \tag{A.59}$$

where $\mathrm{j}^2 = -1$, but $\mathrm{ij} \neq -1$. We use j as a vector, orthogonal to i and to the real line. Re is the real part with respect to the j complex part of a complex number. Substituting for E and s, we obtain

$$\left[-\omega^2 + \mathrm{ij}\omega \frac{eB}{m} + \omega_0^2\right] s_0 = -\frac{e}{m} E_0 \mathrm{e}^{\mathrm{j}\phi}. \tag{A.60}$$

The amplitudes s_0 and E_0 are purely real both in i and j. Comparing real and imaginary parts in j, we obtain two equations:

$$(\omega_0^2 - \omega^2) s_0 = -\frac{e}{m} E_0 \cos\phi, \tag{A.61}$$

$$\mathrm{i}\frac{eB\omega}{m} s_0 = -\frac{e}{m} E_0 \sin\phi. \tag{A.62}$$

The phase ϕ is i complex, and this leads to rotation of the polarization plane – but this is not the best way to proceed. We shall show below that it is enough that the wavevector k be an i complex number to have rotation of the polarization plane vector E. To find k, we must find the dispersion relation for waves in a magnetized dielectric. It is assumed that the resonant frequency of the system is greater than the frequency of the electromagnetic waves ($\omega_0 > \omega$). For low-energy radiation, this is reasonable. Defining the usual relations

$$\begin{aligned}\mathbf{P} &= -\rho_N e\mathbf{s}\\ \mathbf{D} &= \mathbf{P} + \epsilon_0\mathbf{E}\\ \mathbf{B} &= \mu\mathbf{H}\\ c^2 &= \frac{1}{\epsilon_0\mu_0},\end{aligned} \tag{A.63}$$

from Maxwell‘s equations, one has that

$$\nabla^2\mathbf{E} = \mu_0\frac{\partial^2\mathbf{D}}{\partial t^2} = \mu_0\frac{\partial^2\mathbf{P}}{\partial t^2} + \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}. \tag{A.64}$$

Differentiating twice with respect to t allows one to substitute for s in terms of P and therefore E, so that eliminate s altogether to obtain a dispersion relation for the waves.

$$\left[\frac{\partial^2}{\partial t^2} - \mathrm{i}\frac{eB}{m}\frac{\partial}{\partial t} + \omega_0^2\right]\left(\nabla^2\mathbf{E} - \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}\right) = \frac{\mu_0\rho_N e^2}{m}\frac{\partial^2\mathbf{E}}{\partial t^2}. \tag{A.65}$$

For linearly polarized plane waves, it then follows that

$$\left(-\omega^2 + \mathrm{ij}\frac{eB}{m}\omega + \omega_0^2\right)\left(-k^2 + \frac{\omega^2}{c^2}\right)E = -\frac{\mu_0\rho_N e^2 E\omega^2}{m}. \tag{A.66}$$

This is the dispersion relation. The ij complex nature is a direct result of the coupling to the magnetic field. Re-arranging:

$$k^2 = \frac{\omega^2}{c^2}\left[1 + \frac{\mu_0\rho_N e^2c^2/m}{(-\omega^2 + \mathrm{ij}\frac{eB}{m}\omega + \omega_0^2)}\right] \quad (\omega_0 > \omega). \tag{A.67}$$

The wavevector is therefore a complex number. Writing the wavenumber with real and imaginary parts separated:

$$k = k_\mathrm{r} - \mathrm{ij}k_\mathrm{i} \quad (k_\mathrm{i} > 0), \tag{A.68}$$

one can substitute back into the plane wave:

$$\begin{aligned}E = E_0\cos(kz - \omega t) &= \mathrm{Re}E_0\,\mathrm{e}^{\mathrm{j}(kz-\omega t)}\\ &= \mathrm{Re}E_0\exp\mathrm{j}(k_\mathrm{r}z - \omega t + \mathrm{ij}k_\mathrm{i}z)\\ &= E_0\exp(\mathrm{i}k_\mathrm{i}z)\mathrm{Re}\exp(\mathrm{j}(k_\mathrm{r}z - \omega t))\\ E = E_0\cdot{}& \underbrace{[\cos(k_\mathrm{i}z) + \mathrm{i}\sin(k_\mathrm{i}z)]}_{\text{rotation} \propto \text{z}}\cdot\underbrace{\cos(k_\mathrm{r}z - \omega t)}_{\text{travelling wave}}.\end{aligned} \tag{A.69}$$

Thus, we have a clockwise rotation of the polarization plane.

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A.7 Vector identities in $n = 3$ dimensions

For general vectors **A** and **B**, and scalar ϕ,

$$\nabla \cdot (\phi \mathbf{A}) = \phi(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla \phi) \tag{A.70}$$
$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}) \tag{A.71}$$
$$\nabla \times \phi \mathbf{A} = \phi(\nabla \times \mathbf{A}) + (\nabla \phi) \times \mathbf{A} \tag{A.72}$$
$$\nabla \times \nabla \phi = 0 \tag{A.73}$$
$$\nabla \cdot (\nabla \times \mathbf{A}) = 0 \tag{A.74}$$
$$(\nabla \times (\nabla \times \mathbf{A})) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}. \tag{A.75}$$

A.8 The Stokes and Gauss theorems

Stokes' theorem in three spatial dimensions states that

$$\int_R (\nabla \times \mathbf{A}) \cdot \mathrm{d}\mathbf{S} = \oint_C \mathbf{A} \cdot \mathrm{d}\mathbf{l}, \tag{A.76}$$

i.e. the integral over a surface region R of the curl of a vector, also called the flux of the curl of that vector, is equal to the value of the vector integrated along a loop which encloses the region.

The Gauss divergence theorem in three-dimensional vector language states that

$$\int_\sigma (\nabla \cdot \mathbf{A}) \mathrm{d}\sigma_x = \int_S \mathbf{A} \cdot \mathrm{d}\mathbf{S}; \tag{A.77}$$

i.e. the integral over a spatial volume, σ, of the divergence of a vector is equal to the integral over the surface enclosing the volume of the vector itself. In index notation this takes on the trivial form:

$$\int \mathrm{d}\sigma \, \partial^i A_i = \int \mathrm{d}S^i A_i, \tag{A.78}$$

and the spacetime generalization to $n + 1$ dimensions (which we use frequently) is

$$\int \mathrm{d}V_x \; \partial^\mu A_\mu = \int \mathrm{d}\sigma^\mu A_\mu. \tag{A.79}$$

Notice that Gauss' law is really just the generalization of integration by parts in a multi-dimensional context. In action expressions we frequently use the quantity $(\mathrm{d}x) = \mathrm{d}V_t = \frac{1}{c}\mathrm{d}V_x$, whence

$$\int (\mathrm{d}x) \; \partial^\mu A_\mu = \frac{1}{c} \int \mathrm{d}\sigma^\mu A_\mu. \tag{A.80}$$

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A.9 Integrating factors

Differential equations of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + f(x)\, y = g(x) \tag{A.81}$$

can often be solved by multiplying through by a factor $I(x)$

$$I(x)\,\frac{\mathrm{d}y}{\mathrm{d}x} + f(x)\, I(x)\; y = g(x)\, I(x), \tag{A.82}$$

which makes the left hand side a perfect differential:

$$\frac{\mathrm{d}(uv)}{\mathrm{d}x} = u\,\frac{\mathrm{d}V}{\mathrm{d}x} + v\,\frac{\mathrm{d}u}{\mathrm{d}x}. \tag{A.83}$$

Comparing these equations and identifying $u = I$ and $v = y$, one finds

$$\frac{\mathrm{d}I}{\mathrm{d}x} = I(x) f(x), \tag{A.84}$$

which solves to give

$$I(x) = \exp\left(\int_0^x f(x')\,\mathrm{d}x'\right). \tag{A.85}$$

Thus the differential equation (A.81) may be written

$$\frac{\mathrm{d}}{\mathrm{d}x}\,(I(x)y) = g(x)\, I(x). \tag{A.86}$$

A.10 Matrix formulae

The so-called Baker–Campbell–Hausdorf identity for non-singular matrices A and B states that

$$\mathrm{e}^{-A} B \mathrm{e}^{A} = B + \frac{1}{1!}[B, A] + \frac{1}{2!}[[B, A], A] + \cdots. \tag{A.87}$$

A.11 Matrix factorization

A formula which is useful in diagonalizing systems is:

$$\begin{pmatrix} \Delta_1 & A \\ B & \Delta_2 \end{pmatrix} = \begin{pmatrix} \Delta_1 - A\Delta_2^{-1}B & A\Delta_2^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ B & \Delta_2 \end{pmatrix}. \tag{A.88}$$

$$\det \begin{pmatrix} \Delta_1 & A \\ B & \Delta_2 \end{pmatrix} = \det(\Delta_1 - A\Delta_2^{-1}B)\det\Delta_2. \tag{A.89}$$

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