J. Austral. Math. Soc. (*Series A*) **34** (1983), 356–363

A NOTE ON THE INDEX LAWS OF FRACTIONAL CALCULUS

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(Received 16 January 1981)

Communicated by E. Strzelecki

Abstract

Two index laws for fractional integrals and derivatives, which have been extensively studied by E. R. Love, are shown to be special cases of an index law for general powers of certain differential operators, by means of the theory developed in a previous paper. Discussion of the two index laws, which are rather different in appearance, can thus be unified.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*): 26 A 33.

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In [2], Love discussed two index laws for fractional integrals and derivatives and gave detailed conditions for their validity. These laws were also discussed in a distributional setting by Erdélyi [1] and later by McBride; see, for instance, [3, Chapter 3]. In order to state the index laws, we shall work, for convenience, with functions in the class $C^\infty(0,\infty)$ of smooth, complex-valued functions defined on $(0,\infty)$. Throughout, m will denote a positive real number.

Let α be a complex number and let ϕ $(\in C^\infty(0,\infty))$ be a suitably restricted function.

(i) For $\operatorname{Re}\alpha > 0$, we define $I_m^\alpha\phi$ by

$$(1.1)\qquad (I_m^\alpha\phi)(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x^m - t^m)^{\alpha-1}\phi(t)\,d(t^m)$$

https://doi.org/10.1017/S1446788700023776 Published online by Cambridge University Press

where $d(t^m) = mt^{m-1}\,dt$. The definition is extended step-by-step to the region $\operatorname{Re}\alpha \leqslant 0$ by repeated application of the formula

$$I_m^\alpha \phi = D_m I_m^{\alpha+1}\phi \tag{1.2}$$

where

$$D_m \equiv \frac{d}{dx^m} \equiv m^{-1}x^{1-m}\frac{d}{dx} \equiv m^{-1}x^{1-m}D. \tag{1.3}$$

(ii) For $\operatorname{Re}\alpha > 0$, we define $K_m^\alpha\phi$ by

$$(K_m^\alpha \phi)(x) = \frac{1}{\Gamma(\alpha)}\int_x^\infty (t^m - x^m)^{\alpha-1}\phi(t)\,d(t^m). \tag{1.4}$$

The definition is extended to $\operatorname{Re}\alpha \leqslant 0$ by repeated application of the formula

$$K_m^\alpha \phi = (-D_m)K_m^{\alpha+1}\phi. \tag{1.5}$$

We can now state the two index laws referred to above.

(i) (First Index Law). For any complex numbers α and β,

$$I_m^\alpha I_m^\beta \phi = I_m^{\alpha+\beta}\phi = I_m^\beta I_m^\alpha \phi, \tag{1.6}$$

$$K_m^\alpha K_m^\beta \phi = K_m^{\alpha+\beta}\phi = K_m^\beta K_m^\alpha \phi. \tag{1.7}$$

(ii) (Second Index Law). For complex numbers α, β and γ such that $\alpha + \beta + \gamma = 0$,

$$x^{m\alpha} I_m^\beta x^{m\gamma}\phi = I_m^{-\gamma} x^{-m\beta} I_m^{-\alpha}\phi, \tag{1.8}$$

$$x^{m\alpha} K_m^\beta x^{m\gamma}\phi = K_m^{-\gamma} x^{-m\beta} K_m^{-\alpha}\phi. \tag{1.9}$$

(Throughout we shall use x^λ to denote the operation of multiplying a function of the variable x by x^λ.)

The first index law is very familiar but the second index law is much less familiar and seems, in the first instance, rather strange and unexpected. The object of this note is to point out that both laws can be brought under the same umbrella. In a recent paper [4], we have shown how it is possible to define general powers of an ordinary differential operator

$$L \equiv x^{a_1}Dx^{a_2}Dx^{a_3}\cdots x^{a_n}Dx^{a_{n+1}} \qquad \left(D \equiv \frac{d}{dx}\right) \tag{1.10}$$

of order n, as well as powers of the related operators

$$L' = (-1)^n x^{a_{n+1}}Dx^{a_n}\cdots x^{a_3}Dx^{a_2}Dx^{a_1}, \tag{1.11}$$

$$M = (-1)^n L \quad \text{and} \quad M' = (-1)^n L'. \tag{1.12}$$

This was done under the assumption that the complex numbers $a_1, \ldots, a_{n+1}$ were such that

$$a = \sum_{i=1}^{n+1} a_i \quad \text{is real} \tag{1.13}$$

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and

$$m = |a - n| > 0. \tag{1.14}$$

The powers satisfied a "first index law" so that, for instance,

$$L^{\alpha}L^{\beta}\phi = L^{\alpha+\beta}\phi = L^{\beta}L^{\alpha}\phi, \tag{1.15}$$

$$(L')^{\alpha}(L')^{\beta}\phi = (L')^{\alpha+\beta}\phi = (L')^{\beta}(L')^{\alpha}\phi, \tag{1.16}$$

under appropriate conditions. The two cases $a < n$ and $a > n$ produced different expressions for the general powers. We shall show that in the case $a < n$, (1.6) and (1.7) lead to (1.15) and (1.16) and, conversely, by choosing a suitable L, that (1.15) and (1.16) contain (1.6) and (1.7) so that, in a sense, (1.15) and (1.16) are equivalent to (1.6) and (1.7) in this case. More interestingly perhaps, in the case $a > n$, analogues of (1.15) and (1.16) for M and M' are equivalent to (1.8) and (1.9) so that the *first* index law for M and M' gives rise to the *second* index law for I_m^{α} and K_m^{α}.

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In what follows, ϕ will be a function in $C^{\infty}(0, \infty)$, such that all the subsequent formal analysis is valid. For instance, we may choose ϕ to be an element of the space $F_{p,\mu}$ defined in [3, Chapter 2]. Precise conditions under which the various steps can be justified within the framework of the $F_{p,\mu}$ spaces can be found in [3] and will not be detailed here.

For $m > 0$, $\mathrm{Re}\,\alpha > 0$ and suitable complex numbers η, we define the Erdélyi-Kober operators $I_m^{\eta,\alpha}$ and $K_m^{\eta,\alpha}$ by

$$I_m^{\eta,\alpha}\phi = x^{-m\eta-m\alpha}I_m^{\alpha}x^{m\eta}\phi, \tag{2.1}$$

$$K_m^{\eta,\alpha}\phi = x^{m\eta}K_m^{\alpha}x^{-m\eta-m\alpha}\phi, \tag{2.2}$$

where I_m^{α} and K_m^{α} are as in (1.1)–(1.5). Thus, for $\mathrm{Re}\,\alpha > 0$ and suitable η,

$$(I_m^{\eta,\alpha}\phi)(x) = \frac{1}{\Gamma(\alpha)}\int_0^1 (1 - u^m)^{\alpha-1}u^{m\eta}\phi(xu)\,d(u^m), \tag{2.3}$$

$$(K_m^{\eta,\alpha}\phi)(x) = \frac{1}{\Gamma(\alpha)}\int_1^{\infty} (u^m - 1)^{\alpha-1}u^{-m\eta-m\alpha}\phi(xu)\,d(u^m), \tag{2.4}$$

while, for any complex α,

$$I_m^{\eta,\alpha}\phi = (\eta + \alpha + 1)I_m^{\eta,\alpha+1}\phi + m^{-1}I_m^{\eta,\alpha+1}\delta\phi, \tag{2.5}$$

$$K_m^{\eta,\alpha}\phi = (\eta + \alpha)K_m^{\eta,\alpha+1}\phi - m^{-1}K_m^{\eta,\alpha+1}\delta\phi \tag{2.6}$$

[3, formulae (3.14) and (3.18)], where

$$\delta \equiv x\frac{d}{dx}. \tag{2.7}$$

We can prove that

$$I_m^{\eta,\alpha}I_m^{\xi,\beta}\phi = I_m^{\xi,\beta}I_m^{\eta,\alpha}\phi \tag{2.8}$$

under appropriate conditions. For $\operatorname{Re}\alpha > 0$, $\operatorname{Re}\beta > 0$, (2.8) follows on using (2.3) and interchanging the order of integration; the restriction on α and β can then be removed by repeated application of (2.5) together with the fact that δ commutes with each I operator. Also, by (1.6) and (2.1),

$$I_m^{\eta-\gamma,-\alpha}I_m^{\eta,-\gamma}\phi = I_m^{\eta,-\alpha-\gamma}\phi. \tag{2.9}$$

Hence by (2.8) and (2.9), for suitable functions ψ,

$$I_m^{\eta,-\gamma}I_m^{\eta-\gamma,-\alpha}\psi = I_m^{\eta,-\alpha-\gamma}\psi.$$

(2.1) then gives

$$x^{-m\eta+m\gamma}I_m^{-\gamma}x^{m\eta}x^{-m\eta+m\gamma+m\alpha}I_m^{-\alpha}x^{m\eta-m\gamma}\psi = x^{-m\eta+m\alpha+m\gamma}I_m^{-\alpha-\gamma}x^{m\eta}\psi. \tag{2.10}$$

If we write $\phi(x) = x^{m\eta-m\gamma}\psi(x)$ and $\beta = -\alpha-\gamma$ so that $\alpha+\beta+\gamma = 0$, (2.10) becomes (1.8). Thus we may say that the first index law for I_m^{α} together with the commutativity of the Erdélyi-Kober operators leads to the second index law for I_m^{α}. Similarly, we can show that (1.7) and the result

$$K_m^{\eta,\alpha}K_m^{\xi,\beta}\phi = K_m^{\xi,\beta}K_m^{\eta,\alpha}\phi \tag{2.11}$$

lead to (1.9). This gives us one way of viewing the second index laws for I_m^{α} and K_m^{α}.

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Now we show how the two index laws for I_m^{α} and K_m^{α} are related to the first index law for general powers of the operators L, L', M and M' defined by (1.10)–(1.12). As indicated above, the two cases $a < n$ and $a > n$ need separate treatment. We shall consider L and L' for $a < n$ and M and M' for $a > n$.

The method used in [4] relied on rewriting the operator L, defined by (1.10), in the equivalent form

$$L = m^n x^{a-n}\prod_{k=1}^{n} x^{m-mb_k}D_m x^{mb_k} \tag{3.1}$$

where

$$b_k = \frac{1}{m}\left(\sum_{i=k+1}^{n+1} a_i + k - n\right) \qquad (k = 1,\dots,n). \tag{3.2}$$

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In the case $a < n$, induction shows that, for $r = 1, 2, \ldots,$

$$L^r = m^{nr} x^{-mr} \prod_{k=1}^{n} x^{mr-mb_k} (D_m)^r x^{mb_k}$$

and, since $(D_m)^r = I_m^{-r}$ under appropriate conditions, we can use (2.1) to write

$$L^r = m^{nr} x^{-mr} \prod_{k=1}^{n} I_m^{b_k, -r}$$

which in turn leads to the definition of L^α, for any complex number α, as the operator

$$(3.3) \qquad L^\alpha = m^{n\alpha} x^{-m\alpha} \prod_{k=1}^{n} I_m^{b_k, -\alpha}.$$

(The product on the right-hand side is unambiguous in view of (2.8).) (2.1) and (2.8) give

$$(3.4) \qquad \begin{aligned} L^\alpha L^\beta \phi &= m^{n\alpha} x^{-m\alpha} \prod_{k=1}^{n} I_m^{b_k, -\alpha} m^{n\beta} x^{-m\beta} \prod_{k=1}^{n} I_m^{b_k, -\beta} \phi \\ &= m^{n(\alpha+\beta)} x^{-m(\alpha+\beta)} \prod_{k=1}^{n} I_m^{b_k - \beta, -\alpha} \prod_{k=1}^{n} I_m^{b_k, -\beta} \phi \\ &= m^{n(\alpha+\beta)} x^{-m(\alpha+\beta)} \prod_{k=1}^{n} I_m^{b_k - \beta, -\alpha} I_m^{b_k, -\beta} \phi \end{aligned}$$

while

$$(3.5) \qquad L^{\alpha+\beta} \phi = m^{n(\alpha+\beta)} x^{-m(\alpha+\beta)} \prod_{k=1}^{n} I_m^{b_k, -(\alpha+\beta)} \phi.$$

That the right-hand sides of (3.4) and (3.5) are equal is a consequence of (2.9), which in turn is a consequence of (1.6). Thus we may say that

$$(3.6) \qquad L^\alpha L^\beta = L^{\alpha+\beta}$$

is a consequence of the first index law for I_m^α in this case. Conversely, we may regard (1.6) as a special case of (3.6) corresponding to

$$(3.7) \qquad L = mD_m \equiv x^{1-m} D.$$

In the notation of (1.10), $n = 1$, $a_1 = 1 - m$, $a_2 = 0$, $b_1 = 0$ so that

$$L^{-\alpha} = m^{-\alpha} x^{m\alpha} I_m^{0,\alpha} = m^{-\alpha} I_m^\alpha.$$

Thus $L^{-\alpha} L^{-\beta} = L^{-(\alpha+\beta)} \Rightarrow I_m^\alpha I_m^\beta = I_m^{\alpha+\beta}$ as required.

In a similar fashion, for $a < n$, we define $(L')^\alpha$ by

$$(3.8) \qquad (L')^\alpha \phi = m^{n\alpha} \prod_{k=1}^{n} K_m^{b_k + 1 - 1/m, -\alpha} x^{-m\alpha} \phi.$$

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The index law

$$(3.9) \qquad (L')^{\alpha}(L')^{\beta} = (L')^{\alpha+\beta}$$

is a consequence of (1.7). Conversely, with L as in (3.7), we find that

$$(L')^{-\alpha} = m^{-\alpha}x^{m-1}K_m^{\alpha}x^{1-m}$$

so that (1.7) is a special case of (3.9).

We now consider the case $a > n$. (1.12) and (3.1) give

$$M = m^n x^m \prod_{k=1}^{n} x^{m-mb_k}(-D_m)x^{mb_k}.$$

By induction, we obtain, for $r = 1, 2, \ldots,$

$$M^r = m^{nr}x^{mr} \prod_{k=1}^{n} x^{m-mb_k}(-D_m)^r x^{mb_k+m(r-1)}$$

and, since $(-D_m)^r = K_m^{-r}$ under appropriate conditions, (2.1) gives

$$M^r = m^{nr}x^{mr} \prod_{k=1}^{n} K_m^{1-b_k,-r}.$$

(The inductive step requires the result $(-D_m)^r x^{m(r+1)}(-D_m) = x^m(-D_m)^{r+1}x^{mr}$, a special case of (1.9) which can be established by Leibnitz' formula.) This suggests that we define M^{α}, for any complex number α, to be the operator

$$(3.10) \qquad M^{\alpha} = m^{n\alpha}x^{m\alpha} \prod_{k=1}^{n} K_m^{1-b_k,-\alpha}.$$

From (2.2) and (2.11), we obtain

$$(3.11) \qquad \begin{aligned} M^{\alpha}M^{\beta}\phi &= m^{n\alpha}x^{m\alpha} \prod_{k=1}^{n} K_m^{1-b_k,-\alpha} m^{n\beta}x^{m\beta} \prod_{k=1}^{n} K_m^{1-b_k,-\beta}\phi \\ &= m^{n(\alpha+\beta)}x^{m(\alpha+\beta)} \prod_{k=1}^{n} K_m^{1-b_k-\beta,-\alpha} \prod_{k=1}^{n} K_m^{1-b_k,-\beta}\phi \\ &= m^{n(\alpha+\beta)}x^{m(\alpha+\beta)} \prod_{k=1}^{n} K_m^{1-b_k-\beta,-\alpha}K_m^{1-b_k,-\beta}\phi \end{aligned}$$

while

$$(3.12) \qquad M^{\alpha+\beta}\phi = m^{n(\alpha+\beta)}x^{m(\alpha+\beta)} \prod_{k=1}^{n} K_m^{1-b_k,-(\alpha+\beta)}\phi.$$

The right-hand sides of (3.11) and (3.12) are equal provided that

$$K_m^{1-b_k-\beta,-\alpha}K_m^{1-b_k,-\beta}\phi = K_m^{1-b_k,-(\alpha+\beta)}\phi,$$

which, by (2.2), is equivalent to

$$(3.13)\qquad x^{m-mb_k-m\beta}K_m^{-\alpha}x^{-m+mb_k+m\beta+m\alpha}x^{m-mb_k}K_m^{-\beta}x^{-m+mb_k+m\beta}\phi$$
$$= x^{m-mb_k}K_m^{-(\alpha+\beta)}x^{-m+mb_k+m\alpha+m\beta}\phi, \quad \text{or}$$
$$K_m^{-\alpha}x^{m(\alpha+\beta)}K_m^{-\beta}\psi = x^{m\beta}K_m^{-(\alpha+\beta)}x^{m\alpha}\psi$$

where $\psi(x) = x^{-m+mb_k+m\beta}\phi(x)$. (3.13) is simply (1.9) with α, β, γ and ϕ replaced by β, $-(\alpha+\beta)$, α and ψ respectively. Thus the equation

$$(3.14)\qquad M^\alpha M^\beta = M^{\alpha+\beta}$$

is a consequence of the second index law for K_m^α. Conversely, (1.9) is a special case of (3.14), corresponding to

$$(3.15)\qquad L = xDx^m.$$

In the notation of (1.10), $n = 1$, $a_1 = 1$, $a_2 = m$, $b_1 = 1$ so that

$$M^\alpha = m^\alpha x^{m\alpha}K_m^{0,-\alpha} = m^\alpha x^{m\alpha}K_m^{-\alpha}x^{m\alpha}.$$

If $\alpha+\beta+\gamma = 0$, then

$$M^{\gamma+\alpha}\psi = M^\gamma M^\alpha\psi$$
$$\Rightarrow m^{\gamma+\alpha}x^{m(\gamma+\alpha)}K_m^{-(\gamma+\alpha)}x^{m(\gamma+\alpha)}\psi = m^\gamma x^{m\gamma}K_m^{-\gamma}x^{m\gamma}m^\alpha x^{m\alpha}K_m^{-\alpha}x^{m\alpha}\psi$$
$$\Rightarrow x^{m\alpha}K_m^\beta x^{m\gamma}\phi = K_m^{-\gamma}x^{-m\beta}K_m^{-\alpha}\phi$$

where $\phi(x) = x^{m\alpha}\psi(x)$. This gives (1.9).

Similarly, for $a > n$, we define $(M')^\alpha$ by

$$(3.16)\qquad (M')^\alpha\phi = m^{n\alpha}\prod_{k=1}^{n} I_m^{-b_k+1/m,-\alpha}x^{m\alpha}\phi.$$

The index law

$$(3.17)\qquad (M')^\alpha(M')^\beta = (M')^{\alpha+\beta}$$

is a consequence of (1.8). Conversely, with L as in (3.15),

$$(M')^\alpha = m^\alpha x^{m\alpha+m-1}I_m^{-\alpha}x^{m\alpha-m+1}$$

and, as above, we can show that (1.8) is a special case of (3.17).

Finally, we mention that the results are valid in the setting of distribution theory, for instance in the spaces $F'_{p,\mu}$ introduced in [3, Chapter 2].

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