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HOMOTOPY OF THE EXCEPTIONAL LIE GROUP G_2

Dedicated to Prof. N. Shimada on his sixtieth birthday

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Let G be one of the following compact simply connected Lie groups: $SU(3), Sp(2), G_2$. In the first two cases there is a well known stable decomposition of G as $Q \vee S^d$ where $d = \dim G$ and Q is a certain subspace of G. For SU(3), Q is the stunted complex quasi-projective space $\Sigma(\mathbb{C}P^2/\mathbb{C}P^1)$ which fits into a cofibration sequence $S^3 \to Q \to S^5$ with stable attaching map $\eta: S^5 \to S^4$. For Sp(2), Q is the quaternionic quasi-projective space $\mathbb{H}\mathbb{Q}^1$ and fits into a cofibration sequence $S^3 \to Q \to S^7$ with stable attaching map $2\nu: S^7 \to S^4$. (Here η and ν are generators of $\pi_1^s(S^0)$ and $\pi_3^s(S^0)$ respectively.)

In this paper we describe a corresponding result for G_2. This time we have a cofibration $X^3 \to Q \to Y^{11}$ where X^3, Y^{11} are K-theory spheres to be described in Section 2. We compute the stable class of the attaching map $\phi: Y^{11} \to \Sigma X^3$ by using the complex Adams e-invariant

$$e: \{Y^{11}, \Sigma X^3\} \to \mathbb{Q}/\mathbb{Z}.$$

Theorem A. $\{Y^{11}, \Sigma X^3\} = \mathbb{Z}/60$ *with generator of e-invariant* $1/60 \in \mathbb{Q}/\mathbb{Z}$. *Hence e is monic.*

This theorem, proved as (3.2), is central. It enables us to extend to G_2 much of the theory already developed for SU(3) and Sp(2). First, by computing the Chern character on $K^*(G)$, we obtain, (4.12),

Theorem B. *Stably the attaching map* ϕ *is twice a generator, so of order* 30.

We then turn to the study of self-maps of G. $H^*(G; \mathbb{Q})$ is an exterior algebra on integral generators h_q and h_r say, where $(q, r) = (3, 5)$, $(3, 7)$, $(3, 11)$ in the three cases SU(3), Sp(2), G_2 respectively. For a self-map f of G, we define $d_i(f)$, the degree of f in

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Mathematicians were shocked and saddened to hear of the sudden death of Professor Oka on 30 October 1984. This paper is a revised version of a manuscript submitted by Professor Oka and returned to him with some suggestions for revision from the referee. Professor Oka died before completing this work. The editors are most grateful to Dr M. Crabb who was acquainted with Oka's work and who has revised the paper in accordance with the referee's suggestions.

dimension i ($i=q,r$), to be the integer in the equality $f^*(h_i)=d_i(f)h_i$. We then define the degree map

$$d=d_q\times d_r\colon [G,G]\to\mathbb{Z}\oplus\mathbb{Z}$$

and similarly the stable degree map

$$d^s=d_q^s\times d_r^s\colon \{G,G\}\to\mathbb{Z}\oplus\mathbb{Z}.$$

K-theory shows that $d_q(f)-d_r(f)$ is a multiple of the integer $\pi=2,12,30$ for $G=\mathrm{SU}(3)$, $\mathrm{Sp}(2), G_2$. It is known in the first two cases that this is the only restriction on d. This is also true for G_2, (5.6).

Theorem C. $\operatorname{Im} d=\operatorname{Im} d^s=\{(m,n)\mid m,n\in\mathbb{Z}, m\equiv n \bmod \pi\}$.

Our final application concerns H-maps. Let μ be an H-structure on G, for example the Lie group multiplication, and suppose that a self-map f is an H-map with respect to μ, that is, $\mu(f\times f)$ is homotopic to $f\mu$. Then there are additional restrictions on $d(f)$. For $G=\mathrm{SU}(3)$ or $\mathrm{Sp}(2)$, $d_3(f)\equiv 0,1 \bmod 4$, [**10**], [**11**]. For $G=G_2$ we prove that $d_3(f)\equiv 0,1 \bmod 4$ if $d_{11}(f)\equiv d_3(f) \bmod 2\pi$. With a recent result of Sawashita [**20**], this gives, (7.8),

Theorem D. *Suppose that a self-map f of G_2 is both a homotopy equivalence and an H-map for some H-structure on G_2. Then f is homotopic to the identity.*

(The same result holds for Sp(2), [**11**]; the situation for SU(3) is a little more complicated, [**10**].)

Most of the results on G_2 generalize easily to the H-spaces $G_{2,b}(-2\leqq b\leqq 5)$ introduced by Mimura–Nishida–Toda in [**13**].

The paper is organized as follows. In Section 1 we review the definition of the e-invariant in the form required for our application. In Section 2 we define the K-theory spheres X^3, Y^{11}. $\{Y^{11},\Sigma X^3\}$ is computed in Section 3. In Section 4 we discuss the complex representation ring and the K-theory of G_2; we compute the Chern character and prove Theorem B. The image of the degree map for self-maps of G_2 is determined in Section 5, with one computation deferred to Section 6. H-maps are discussed in Section 7.

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The book [**23**] is an excellent guide to the representation theory of the classical groups and the exceptional groups G_2 and F_4. Unfortunately it is written in Japanese and the author cannot find a similar work in English.

Spaces in this paper are always assumed to be homotopy equivalent to *CW* complexes and to have base points. Maps and homotopies also preserve base points. $[X, Y]$ denotes the set of homotopy classes of maps from X to Y, Σ^n the n-fold suspension, $\Sigma^1=\Sigma$, and $\{X, Y\}$ the additive group of stable maps from X to Y. We sometimes denote a map and its homotopy class by the same symbol. $\mathbb{Z}$ and $\mathbb{Z}/n$ denote additive groups isomorphic to the group of integers and integers modulo n, respectively; the generator is enclosed in braces $\{\ \}$.

1. The *e*-invariant

We begin by recalling the definition, in suitable generality, of the (complex) Adams e-invariant. Let Z be a finite complex and n a natural number with $\tilde{H}^{n-1}(Z;\mathbb{Q})=0$. We shall need the commutative diagram of Bockstein exact sequences:

$$\begin{array}{ccccccc}
\pi_s^{n-1}(Z)\otimes\mathbb{Q} & \to & \pi_s^{n-1}(Z;\mathbb{Q}/\mathbb{Z}) & \to & \pi_s^n(Z) & \to & \pi_s^n(Z)\otimes\mathbb{Q} \\
\downarrow d & & \downarrow d & & \downarrow d & & \downarrow d \\
\tilde{K}^{n-1}(Z)\otimes\mathbb{Q} & \to & \tilde{K}^{n-1}(Z;\mathbb{Q}/\mathbb{Z}) & \to & \tilde{K}^n(Z) & \to & \tilde{K}^n(Z)\otimes\mathbb{Q}
\end{array}$$

in which the vertical maps are Hurewicz homomorphisms (d-invariants) from stable cohomotopy (with $\mathbb{Z}, \mathbb{Q}, \mathbb{Q}/\mathbb{Z}$-coefficients) to complex K-theory. $\pi_s^n(Z)=\{Z, S^n\}$ is the group of stable maps from Z to S^n. By assumption, $\pi_s^{n-1}(Z)\otimes\mathbb{Q}=0$.

Now consider a torsion element $x\in\pi_s^n(Z)$ with $d(x)=0$. It lifts uniquely to a class $\tilde{x}\in\pi_s^{n-1}(Z;\mathbb{Q}/\mathbb{Z})$. We define the e-invariant of x to be $e(x)=d(\tilde{x})$ in

$$\operatorname{Coker}\{\tilde{K}^{n-1}(Z)\to\tilde{K}^{n-1}(Z)\otimes\mathbb{Q}\}\subseteq\tilde{K}^{n-1}(Z;\mathbb{Q}/\mathbb{Z}).$$

It is useful in practice to have another description of $e(x)$ using the Chern character, ch. Let $f:\Sigma^m Z\to S^{m+n}$ $(m\geqq 0)$ represent x. Form the mapping cone sequence

$$\Sigma^m Z\xrightarrow{f} S^{m+n}\xrightarrow{g} C_f\xrightarrow{h}\Sigma^{m+1}Z$$

and look at the associated exact sequences in K-theory and rational cohomology. We obtain a commutative diagram

$$\begin{array}{ccccccccc}
0\longrightarrow & \tilde{K}^{n-1}(Z) & \xrightarrow{h^*} & \tilde{K}^{m+n}(C_f) & \xrightarrow{g^*} & \mathbb{Z} & \longrightarrow 0 \\
 & \downarrow \text{ch} & & \downarrow \text{ch} & & \downarrow & \\
0\longrightarrow & \sum_i \tilde{H}^{n-1+2i}(Z;\mathbb{Q}) & \xrightarrow[h^*]{} & \sum_i \tilde{H}^{m+n+2i}(C_f;\mathbb{Q}) & \xrightarrow[g^*]{} & \mathbb{Q} & \longrightarrow 0,
\end{array}$$

in which the inclusion $\mathbb{Z}\to\mathbb{Q}$ is ch: $\tilde{K}^0(S^0)\to\tilde{H}^0(S^0;\mathbb{Q})$ and the summation is over $i\in\mathbb{Z}$.

The map $g^*:\tilde{H}^{m+n}(C_f;\mathbb{Q})\to\tilde{H}^0(S^0;\mathbb{Q})$ is an isomorphism. Let $b\in\tilde{H}^{m+n}(C_f;\mathbb{Q})$ map to

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$1\in\mathbb{Q}$. Now choose a class $a\in\tilde{K}^{m+n}(C_f)$ with $g^*(a)=1$. Then $\mathrm{ch}(a)-b=h^*(v)$ for some class v in the group $\sum_i \tilde{H}^{n-1+2i}(Z;\mathbb{Q})$. Since the Chern character is a rational isomorphism, we can think of v as an element of $\tilde{K}^{n-1}(Z)\otimes\mathbb{Q}$. Then

$$e(x)=v \bmod \tilde{K}^{n-1}(Z).$$

This was the original definition of $e(x)$.

In fact we must interpret the e-invariant in a somewhat broader context. Let X and Y be finite complexes with $\{\Sigma Y, X\}\otimes\mathbb{Q}=0$, $x\in\{Y,X\}$ a stable map from Y to X represented by a map $f:\Sigma^m Y\to\Sigma^m X$ (for some $m\geqq 0$). Assume that x is of finite order and with vanishing d-invariant in the sense that

$$(1\wedge x)^*:\tilde{K}^*(W\wedge X)\to\tilde{K}^*(W\wedge Y) \text{ is zero for every finite complex } W.$$

Then we have an e-invariant

$$e(x)\in \mathrm{Hom}(\tilde{K}^*(X)\otimes\mathbb{Q},\tilde{K}^{*-1}(Y)\otimes\mathbb{Q}) \quad \mathrm{mod}\, \mathrm{Hom}(\tilde{K}^*(X),\tilde{K}^{*-1}(Y)).$$

Here and elsewhere we think of K-theory as $\mathbb{Z}/2$-graded. Homomorphisms are of degree 0.

The definition reduces by S-duality to the one we have already given. Let X' be an n-dual of X (for some $n\geqq 0$). Set Z equal to $X'\wedge Y$. Then $\{Y,X\}$ is identified with $\{Z,S^n\}$ and $\tilde{K}^{n-1}(Z)\otimes\mathbb{Q}$ with $\mathrm{Hom}(\tilde{K}^*(X)\otimes\mathbb{Q},\ \tilde{K}^{*-1}(Y)\otimes\mathbb{Q})$.

There is a direct interpretation of $e(x)$ by using the Chern character (and this may, for the purposes of this paper, be taken as the definition). Again form the mapping cone sequence

$$\Sigma^m Y\xrightarrow{f}\Sigma^m X\xrightarrow{g}C_f\xrightarrow{h}\Sigma^{m+1}Y.$$

We have an associated diagram

$$\begin{array}{ccccccccc}
0\longrightarrow & \tilde{K}^{-1}(Y) & \xrightarrow{h^*} & \tilde{K}^m(C_f) & \xrightarrow{g^*} & \tilde{K}^0(X) & \longrightarrow 0\\
& \downarrow{\scriptstyle \mathrm{ch}} & & \downarrow{\scriptstyle \mathrm{ch}} & & \downarrow{\scriptstyle \mathrm{ch}} & \\
0\longrightarrow & \sum_i \tilde{H}^{-1+2i}(Y;\mathbb{Q}) & \xrightarrow[h^*]{} & \sum_i \tilde{H}^{m+2i}(C_f;\mathbb{Q}) & \xrightarrow[g^*]{} & \sum_i \tilde{H}^{2i}(X;\mathbb{Q}) & \longrightarrow 0.
\end{array}$$

The cohomology sequence has a unique splitting b: $\sum_i \tilde{H}^{2i}(X;\mathbb{Q})\to\sum_i \tilde{H}^{m+2i}(C_f;\mathbb{Q})$ preserving the degree i, since $\mathrm{Hom}(\tilde{H}^{2i}(X;\mathbb{Q}),\ \tilde{H}^{-1+2i}(Y;\mathbb{Q}))=0$ by assumption. The K-theory sequence splits because $d(x)=0$ (in the strong sense); choose a splitting $a:\tilde{K}^0(X)\to\tilde{K}^m(C_f)$. If $u\in\tilde{K}^0(X)$, then $\mathrm{ch}(a(u))-b(\mathrm{ch}(u))=h^*(v)$ for some unique class v which we regard as an element of $\tilde{K}^{-1}(Y)\otimes\mathbb{Q}$. In this way we obtain a linear mapping $\tilde{K}^0(X)\to\tilde{K}^{-1}(Y)\otimes\mathbb{Q}$ which is well defined modulo homomorphisms $\tilde{K}^0(X)\to\tilde{K}^{-1}(Y)$ and represents the 0-component of $e(x)$. The other component is defined similarly.

In our applications X and Y will be K-theory spheres so that the groups concerned are very simple.

We remark that there is another definition of the e-invariant in terms of Adams operations in K-theory (in which one usually works locally at a fixed prime). The reader may prefer to rewrite proofs in the following sections from that point of view.

2. Some K-theory spheres

For our computations we quote some results on $\pi_*^s(S^0)$, including Toda brackets $\langle , , \rangle$, from [21].

(a) $\pi_0^s(S^0) = \mathbb{Z}\{\iota\}$,

(b) $\pi_1^s(S^0) = \mathbb{Z}/2\{\eta\}$,

(c) $\pi_2^s(S^0) = \mathbb{Z}/2\{\eta^2\}$, (c′) $\langle 2\iota, \eta, 2\iota \rangle = \eta^2$,

(d) $\pi_3^s(S^0) = \mathbb{Z}/24\{\nu\}$, (d′) $\eta^3 = 12\nu$, (d″) $\langle \eta, 2\iota, \eta \rangle \equiv 6\nu \bmod 12\nu$, (2.1)

(e) $\pi_4^s(S^0) = 0$, (e′) $\eta\nu = 0$,

(f) $\pi_5^s(S^0) = 0$,

(g) $\pi_6^s(S^0) = \mathbb{Z}/2\{\nu^2\}$, (g′) $\langle \eta, \nu, \eta \rangle = \nu^2$,

(h) $\pi_7^s(S^0) = \mathbb{Z}/240\{\sigma\}$.

(Note that ν and σ in [21] are the generators of the 2-primary parts.)

Denote the mod 2 Moore space by $M^n = S^n \bigcup_{2\iota} e^{n+1}$ and the usual cofibration by

$$S^n \xrightarrow{i} M^n \xrightarrow{p} S^{n+1}.$$

Since $2\eta = 0$, in the stable range there are elements

$$\bar{\eta}: M^{n+1} \to S^n, \tilde{\eta}: S^{n-1} \to M^{n-3}$$

such that

(a) $\bar{\eta} i = \eta, p\tilde{\eta} = \eta.$ (2.2)

By (2.1) (c′), (d′), (d″) and (g′),

(b) $2\bar{\eta} = \eta^2 p$, $2\tilde{\eta} = i\eta^2$,

(c) $\bar{\eta}\tilde{\eta} = \pm 6\nu$, (2.2)

(d) $\langle \bar{\eta}, i\nu p, \tilde{\eta} \rangle = \nu^2$.

Notice that $\bar{\eta}$ and $\tilde{\eta}$ are not uniquely determined by (2.2)(a): there are two choices differing in sign. Given $\bar{\eta}$, we may fix $\tilde{\eta}$ by requiring $\bar{\eta}\tilde{\eta} = 6\nu$.

We define complexes X^n $(n\geqq 3)$, Y^n $(n\geqq 6)$ as the mapping cones of $\bar{\eta},\tilde{\eta}$:

$$\begin{array}{c} M^{n+1}\xrightarrow{\bar{\eta}} S^n \xrightarrow{i} X^n \xrightarrow{j'} M^{n+2}, \\ \\ S^{n-1} \xrightarrow{\tilde{\eta}} M^{n-3} \xrightarrow{i'} Y^n \xrightarrow{j} S^n. \end{array} \tag{2.3}$$

The spaces X^n, Y^n are independent, up to homotopy equivalence, of the choice of attaching maps $\bar{\eta},\tilde{\eta}$. ($[M^4,S^3]$ and$[S^5,M^3]$ are both cyclic of order 4.)

We shall show that X^n and Y^n are K-theory spheres. Notice first that they are rationally equivalent to S^n with equivalences i and j respectively. Let $s_n\in\tilde{H}^n(S^n)=\mathbb{Z}$, $\sigma_n\in\tilde{K}^n(S^n)$ be generators with $\mathrm{ch}\,\sigma_n=s_n$. Let $x_n\in\tilde{H}^n(X^n)=\mathbb{Z}$, $y_n\in\tilde{H}^n(Y^n)=\mathbb{Z}$ be generators with $i^*x_n=s_n$, $y_n=j^*s_n$. We denote the rational classes of s_n, x_n, y_n by the same symbols.

Proposition 2.4. (a) $\tilde{K}^*(X^n)\cong\tilde{K}^*(S^n)$, $\tilde{K}^*(Y^n)\cong\tilde{K}^*(S^n)$ *as additive groups.*

(b) *There are generators* $\xi_n\in\tilde{K}^n(X^n)=\mathbb{Z}$, $\eta_n\in\tilde{K}^n(Y^n)=\mathbb{Z}$ *such that*

$$\mathrm{ch}\,\xi_n=2x_n,\qquad \mathrm{ch}\,\eta_n=\tfrac{1}{2}y_n,$$

$$i^*\xi_n=2\sigma_n,\qquad j^*\sigma_n=2\eta_n.$$

Proof. One readily checks that $\tilde{K}^n(M^{n+1})=\mathbb{Z}/2$, $\tilde{K}^{n-1}(M^{n+1})=0$. (2.3) then gives an exact sequence

$$0\to\tilde{K}^n(X^n)\xrightarrow{i^*}\tilde{K}^n(S^n)\to\mathbb{Z}/2\to\tilde{K}^{n+1}(X^n)\to 0.$$

Now the element η of $\pi_1^s(S^0)$ has non-trivial e-invariant, [2]: $e(\eta)=\frac{1}{2}\in\mathbb{Q}/\mathbb{Z}$. Form a homotopy-commutative diagram

$$\begin{array}{ccccc} M^{n+1} & \xrightarrow{\bar{\eta}} & S^n & \xrightarrow{i} & X^n \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle 1} & & \uparrow \\ S^{n+1} & \xrightarrow[\eta]{} & S^n & \longrightarrow & C_\eta \end{array}$$

and consider the induced maps in K-theory

$$\begin{array}{ccccccc} 0 & \longrightarrow & \tilde{K}^n(X^n) & \xrightarrow{i^*} & \tilde{K}^n(S^n)=\mathbb{Z} & & \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle 1} & & \\ 0\longrightarrow\tilde{K}^{n-1}(S^{n+1}) & \longrightarrow & \tilde{K}^n(C_\eta) & \longrightarrow & \tilde{K}^n(S^n) & \longrightarrow & 0. \end{array}$$

Since $i^*:\tilde{H}^n(X^n;\mathbb{Q})\to\tilde{H}^n(S^n;\mathbb{Q})$ is an isomorphism, it is clear from the Chern character interpretation of the e-invariant that $u.e(\eta)=0\in\mathbb{Q}/\mathbb{Z}$ for all $u\in\operatorname{Im} i^*\subseteq\mathbb{Z}$. It follows that $\operatorname{Im} i^*$ has index exactly 2; $\tilde{K}^n(X^n)\cong\mathbb{Z}$ and $\tilde{K}^{n+1}(X^n)=0$. The relations in (b) for ξ_n are easy. The case of Y^n is similar. □

X^n and Y^n are three-cell complexes: $X^n=S^n\bigcup_\eta e^{n+1}\bigcup_{2\iota}e^{n+2}$, $Y^n=S^{n-2}\bigcup_{2\iota}e^{n-1}\bigcup_\eta e^n$. Our construction may be generalized to produce other three-cell complexes which are K-theory spheres: $S^n\bigcup_\alpha e^{n+k}\bigcup_q e^{n+k+1}$ and $S^{n-k-1}\bigcup_q e^{n-k}\bigcup_\alpha e^n$, where $\alpha\in\pi^s_{k-1}(S^0)$ is any element satisfying the condition

$$k \text{ is even, } \alpha \text{ and } e(\alpha)\in\mathbb{Q}/\mathbb{Z} \text{ have the same order } q. \tag{2.5}$$

These are the simplest examples of K-theory spheres which are not homotopy equivalent to spheres. (There can be no two-cell complex W with $\tilde{K}^*(W)\cong\tilde{K}^*(S^n)$.)

3. Computation of $\{Y^{n+7},X^n\}$

The maps $i\colon S^n\to X^n$ and $j\colon Y^{n+7}\to S^{n+7}$ of (2.3) induce a homomorphism

$$i_*j^*\colon\pi^s_7(S^0)\to\{Y^{n+7},X^n\}.$$

It is known [2] that the e-invariant on $\pi^s_7(S^0)$ is monic, more precisely

$$e(\sigma)=1/240\in\mathbb{Q}/\mathbb{Z}. \tag{3.1}$$

In the same way, the e-invariant as described in Section 1 is defined on the whole group $\{Y^{n+7},X^n\}$ and under the identification (2.4) of $\tilde{K}^n(X^n)$ and $\tilde{K}^{n-1}(Y^{n+7})$ with $\mathbb{Z}$ is a homomorphism

$$e\colon\{Y^{n+7},X^n\}\to\mathbb{Q}/\mathbb{Z}.$$

The purpose of this section is to show that this e-invariant is also monic, and simultaneously to determine the group structure of $\{Y^{n+7},X^n\}$.

Theorem 3.2. *The group $\{Y^{n+7},X^n\}$ is cyclic of order 60 with generator $i\sigma j$, which has e-invariant $e(i\sigma j)=1/60\in\mathbb{Q}/\mathbb{Z}$. In particular, the e-invariant is monic on $\{Y^{n+7},X^n\}$.*

The proof is divided into three steps:

$$j^*\colon\{S^{n+7},X^n\}\to\{Y^{n+7},X^n\} \text{ is epic.} \tag{3.2a}$$

$$i_*\colon\pi^s_7(S^0)\to\{S^{n+7},X^n\} \text{ is isomorphic.} \tag{3.2b}$$

$$\text{The kernel of } j^*\colon\{S^{n+7},X^n\}\to\{Y^{n+7},X^n\} \text{ is of order 4.} \tag{3.2c}$$

We shall need the following results of Mukai [16], which follow from (2.1), (2.2) and the universal coefficient exact sequences.

$$\text{(a)}\quad \{S^{n+4},M^n\}=\mathbb{Z}/2\{\tilde{\eta}\eta^2\},\qquad \text{(b)}\quad \{S^{n+5},M^n\}=0,$$
$$\text{(c)}\quad \{S^{n+6},M^n\}=\mathbb{Z}/2\{i\nu^2\}. \tag{3.3}$$

$$\text{(a)}\quad \{M^{n+3},S^n\}=\mathbb{Z}/2\{\eta^2\bar{\eta}\},\qquad \text{(b)}\quad \{M^{n+4},S^n\}=0,$$
$$\text{(c)}\quad \{M^{n+5},S^n\}=\mathbb{Z}/2\{\nu^2 p\}. \tag{3.4}$$

$$\text{(a)}\quad \{M^{n+2},M^n\}=\mathbb{Z}/2\{i\eta\bar{\eta}\}\oplus\mathbb{Z}/2\{\tilde{\eta}\eta p\}\oplus\mathbb{Z}/2\{i\nu p\},$$
$$\text{(b)}\quad \{M^{n+3},M^n\}=\mathbb{Z}/4\{\tilde{\eta}\bar{\eta}\}\oplus\mathbb{Z}/2\{\nu\wedge 1_M\},\quad \text{(b}'\text{)}\ 2\tilde{\eta}\bar{\eta}=i\eta^2\bar{\eta}=\tilde{\eta}\eta^2 p, \tag{3.5}$$
$$\text{(c)}\quad \{M^{n+4},M^n\}=\mathbb{Z}/2\{\tilde{\eta}\eta\bar{\eta}\}.$$

Proof of (3.2) **assuming** (3.2a), (3.2b), (3.2c). By (2.1)(h), the group is cyclic of order $240/4=60$ with generator $i\sigma j$, whose e-invariant is easily computed from (3.1) and naturality. □

Proof of (3.2a). Consider the commutative diagram with exact rows, which are induced by the first cofibration in (2.3):

$$\begin{array}{ccccccc}
\{M^{n+4},S^n\} & \xrightarrow{i_*} & \{M^{n+4},X^n\} & \xrightarrow{(j)_*} & \{M^{n+2},M^n\} & \xrightarrow{\bar{\eta}_*} & \{M^{n+3},S^n\} \\
\downarrow \tilde{\eta}^* & & \downarrow \tilde{\eta}^* & & \downarrow \tilde{\eta}^* & & \\
\pi_6^s(S^0) & \xrightarrow{i_*} & \{S^{n+6},X^n\} & \xrightarrow{(j')_*} & \{S^{n+4},M^n\}. & &
\end{array} \tag{3.6}$$

By (3.5)(a), (3.4)(a), (2.2)(a), (2.2)(c) and (2.1)(e′), the second and the third factors $\mathbb{Z}/2$ in (3.5)(a) generate the kernel of $\bar{\eta}_*$ in (3.6); hence, by (3.4)(b), (2.1)(g) and (3.3)(a), the diagram (3.6) becomes

$$\begin{array}{ccccccc}
0 & \longrightarrow & \{M^{n+4},X^n\} & \xrightarrow{(j)_*} & \mathbb{Z}/2\{\tilde{\eta}\eta p\}\oplus\mathbb{Z}/2\{i\nu p\} & \longrightarrow & 0 \\
\downarrow \tilde{\eta}^* & & \downarrow \tilde{\eta}^* & & \downarrow \tilde{\eta}^* & & \\
\mathbb{Z}/2\{\nu^2\} & \xrightarrow{i_*} & \{S^{n+6},X^n\} & \xrightarrow{(j)_*} & \mathbb{Z}/2\{\tilde{\eta}\eta^2\}. & &
\end{array}$$

By (2.2)(a) and (2.1)(e′), the right $\tilde{\eta}^*$ is epic and its kernel is generated by $i\nu p$. By definition of the Toda bracket, $\langle\bar{\eta},i\nu p,\tilde{\eta}\rangle=i_*^{-1}\tilde{\eta}^*(j')_*^{-1}(i\nu p)$, which is non-trivial by (2.2)(d). Therefore the middle $\tilde{\eta}^*$ is isomorphic. The second cofibration in (2.3) induces the exact sequence:

$$\{M^{n+5},X^n\}\xrightarrow{\tilde{\eta}^*}\{S^{n+7},X^n\}\xrightarrow{j^*}\{Y^{n+7},X^n\}\xrightarrow{(i')^*}\{M^{n+4},X^n\}\xrightarrow{\tilde{\eta}^*}\{S^{n+6},X^n\}, \tag{3.7}$$

where the last $\tilde{\eta}^*$ is isomorphic as before. Then j^* is epic. □

Proof of (3.2b). Consider the exact sequence:

$$\{S^{n+6}, M^n\} \xrightarrow{\bar{\eta}_*} \pi_7^s(S^0) \xrightarrow{i_*} \{S^{n+7}, X^n\} \xrightarrow{(j')_*} \{S^{n+5}, M^n\} = 0,$$

where the last term vanishes by (3.3)(b). By (3.3)(c), (2.2)(a), (2.1)(e′), $\bar{\eta}_* = 0$. Hence i_* is isomorphic. □

Proof of (3.2c). The element $\bar{\eta}\tilde{\eta}\bar{\eta}$ lies in $\{M^{n+4}, S^n\}$, which is trivial by (3.4)(b). Therefore $\bar{\eta}\tilde{\eta}\bar{\eta} = 0$. Similarly, $\tilde{\eta}\bar{\eta}\tilde{\eta} = 0$ by (3.3)(b). Thus the Toda bracket $\langle \bar{\eta}, \tilde{\eta}\bar{\eta}, \tilde{\eta} \rangle$ is defined. Then,

$$\langle \bar{\eta}, \tilde{\eta}\bar{\eta}, \tilde{\eta} \rangle = 60\sigma \text{ or } -60\sigma \bmod \text{zero}. \tag{3.8}$$

We will prove (3.8) after completing (3.2c). By (3.2b) and (2.1)(h),

$$\{S^{n+7}, X^n\} = \mathbb{Z}/240\{i\sigma\}.$$

This is cyclic, and hence, by the exact sequence (3.7), it is enough to show that the maximal order of elements in the image of the first $\tilde{\eta}^*$ in (3.7) is 4. By (3.8), there is an element γ in $\{M^{n+5}, X^n\}$ such that $(j')_*\gamma = \tilde{\eta}\bar{\eta}$ and $\tilde{\eta}^*(\gamma) = \pm 60 i\sigma$, an element of order 4. By (2.2)(b) and (2.1)(c), $4\tilde{\eta} = 0$, and hence the image is a $\mathbb{Z}/4$-module. □

Proof of (3.8). By (3.5)(b′) and (2.2)(a),

$$2\langle \bar{\eta}, \tilde{\eta}\bar{\eta}, \tilde{\eta} \rangle = \langle \bar{\eta}, \tilde{\eta}\eta, \eta^2 \rangle.$$

It is enough to show

$$\langle \bar{\eta}, \tilde{\eta}\eta, \eta^2 \rangle = 120\sigma \bmod 0.$$

By [**21**], the suspension $\Sigma^\infty : \pi_{12}(S^5) = \mathbb{Z}/30 \to \pi_7^s(S^0)$ is monic and the Hopf invariant $H : \pi_{12}(S^5) \to \pi_{12}(S^9) (\cong \pi_3(S^0) = \mathbb{Z}/24)$ is a monomorphism on 2-primary components. It is therefore sufficient to show that $\langle \bar{\eta}, \tilde{\eta}\eta, \eta^2 \rangle$ can be formed on S^5 with non-trivial Hopf invariant η^3. For $n \geqq 3$, $2\eta = 0$ holds on S^n. Therefore $\bar{\eta}$ on S^n and $\tilde{\eta}$ on M^n exist for $n \geqq 3$. Also, for $n \geqq 3$, η^3 on S^n is divisible by 4, $4\tilde{\eta} = 0$ on M^n and $\tilde{\eta}\eta^3 = 0$ on M^n. On S^3, $\bar{\eta}\tilde{\eta} = \nu'$ a generator of the 2-primary component of $\pi_6(S^3)$, and $\bar{\eta}(\tilde{\eta}\eta) = \nu'\eta$, which is non-zero on S^3, S^4 and becomes zero on S^5 [**21**]. Therefore the Toda bracket can be formed on S^5. We consider a part of the *EHP* exact sequence

$$\mathbb{Z}/2\{\eta\} = \pi_{10}(S^9) \xrightarrow{P} \pi_8(S^4) \xrightarrow{\Sigma} \pi_9(S^5).$$

($P = \Delta$, $\Sigma = E$ in [**21**]). The element $\bar{\eta}\tilde{\eta}\eta$ in the middle group vanishes at the right end. Hence $P(\eta) = \bar{\eta}\tilde{\eta}\eta$. By the formula [**21**, (2.6)], we conclude that

$$H(\langle \bar{\eta}, \tilde{\eta}\eta, \eta^2 \rangle \text{ on } S^5) = \eta\eta^2 = \eta^3 \in \pi_{12}(S^9). \qquad \square$$

4. The representation ring and the Chern character of G_2

We shall begin by quoting some results on G_2 and its complex representation ring from, for example, [23].

We denote by $\mathscr{C}$ the (non-associative) field of Cayley numbers. Let e_i $(0 \leqq i \leqq 7)$ be the usual $\mathbb{R}$-basis for $\mathscr{C}$ with multiplication rule:

$$e_0 = 1 \text{ (the unit)},$$

$$e_i^2 = -1 \ (i \geqq 1),\ e_i e_j = -e_j e_i \ (i \neq j, \quad i, j \geqq 1),$$

$$e_i e_j = e_k,\ e_j e_k = e_i,\ e_k e_i = e_j$$

$$\text{for} \quad (i, j, k) = (1,2,3), (1,4,5), (1,6,7), (2,5,7), (2,6,4), (3,4,7), (3,5,6).$$

An $\mathbb{R}$-linear isomorphism $g: \mathscr{C} \to \mathscr{C}$ is said to be an automorphism of $\mathscr{C}$ if it is multiplicative: $g(uv) = g(u)g(v)$, $u, v \in \mathscr{C}$. The compact, simply connected Lie group of type G_2 is realized as the automorphism group of $\mathscr{C}$:

$$G_2 = \operatorname{Aut} \mathscr{C}.$$

One introduces an $\mathbb{R}$-linear conjugation: $\bar{e}_0 = e_0, \bar{e}_i = -e_i$ $(i \geqq 1)$ and a norm $|u|$ by $u\bar{u} = |u|^2$. Let $\mathscr{C}_0 = \{u \in \mathscr{C} \mid \bar{u} = -u\} = \sum_{i=1}^{7} \mathbb{R}e_i$. Any element $g \in G_2$ satisfies $g(1) = 1$ and $|g(u)| = |u|, u \in \mathscr{C}$. Therefore G_2 is a closed subgroup of $O(\mathscr{C}_0) = O(7)$.

Let ρ be the 7-dimensional complex representation $\mathscr{C}_0 \otimes_{\mathbb{R}} \mathbb{C}$ of G_2. Put $\rho' = \wedge^2 \rho$, the exterior power. Then the complex representation ring of G_2 is the polynomial ring with generators ρ, ρ':

$$R(G_2) = \mathbb{Z}[\rho, \rho']. \tag{4.1}$$

Next, the subgroup $H_1 = \{g \in G_2 \mid g(e_1) = e_1\}$ can be identified with SU(3). (If we identify e_1 with the complex number $i \in \mathbb{C}$, then H_1 acts on $\mathbb{C}e_2 \oplus \mathbb{C}e_4 \oplus \mathbb{C}e_6$.) Let σ be the standard 3-dimensional representation of SU(3), $\hat{\sigma}$ its complex conjugate. Writing $j: \mathrm{SU}(3) \to G_2$ for the inclusion, we have

$$R(\mathrm{SU}(3)) = \mathbb{Z}[\sigma, \hat{\sigma}], \quad j^*\rho = \sigma + \hat{\sigma} + 1. \tag{4.2}$$

The subgroup $H_{1,2} = \{g \in G_2 \mid g(e_1) = e_1, g(e_2) = e_2\}$ can be identified in a similar way with $\mathrm{SU}(2) \subset \mathrm{SU}(3)$. σ and $\hat{\sigma}$ both restrict to $\tau + 1$, where $\tau = \hat{\tau}$ is the standard 2-dimensional representation of SU(2). Write $i: S^3 = \mathrm{SU}(2) \to G_2$ for the inclusion. Then

$$R(\mathrm{SU}(2)) = \mathbb{Z}[\tau],\ i^*\rho = 2\tau + 3,\ i^*\rho' = \tau^2 + 6\tau + 5. \tag{4.3}$$

Considering a representation simply as a continuous map to the infinite unitary group defines the β-construction $\beta: R(G) \to \tilde{K}^{-1}(G)$, and we have, by [9],

Theorem 4.4. (a) $K^*(G_2)=E(\beta(\rho),\beta(\rho'))$, $K^*(\mathrm{SU}(2))=E(\beta(\tau))$, *where* E *denotes the exterior algebra over* $\mathbb{Z}$.

(b) $i^*\beta(\rho)=2\beta(\tau)$, $i^*\beta(\rho')=10\beta(\tau)$.

Part (b) follows from (4.2) and the properties of β, [**9**].

The integral cohomology ring $H^*(G_2)$ and the mod 2 cohomology $H^*(G_2;\mathbb{Z}/2)$ were determined by Borel in [**5**].

Theorem 4.5. *There are integral classes* h_3 *and* h_{11} *in* $H^*(G_2)$, $\deg h_i=i$, *which generate the ring* $H^*(G_2)$ *with relations* $2h_3^2=0$, $h_3^4=0$, $h_3^2h_{11}=0$, $h_{11}^2=0$. *Hence the additive group structure of* $H^i=H^i(G_2)$ *is given as follows*:

$$H^0=\mathbb{Z},\ H^3=\mathbb{Z}\{h_3\},\ H^{3i}=\mathbb{Z}/2\{h_3^i\} \qquad (i=2,3),$$

$$H^{11}=\mathbb{Z}\{h_{11}\},\ H^{14}=\mathbb{Z}\{h_3h_{11}\},\ H^i=0 \text{ for other } i.$$

Let x_i *be the* mod 2 *reduction of* h_i *and* x_5 *be the* mod 2 *class whose Bockstein is* h_3^2. *Then the* mod 2 *cohomology* $H^*(G_2;\mathbb{Z}/2)$ *has the* $\mathbb{Z}/2$*-basis*

$$1, x_3, x_4, x_3^2, x_3x_5, x_3^3, x_{11}=x_3^2x_5, x_3x_{11}=x_3^3x_5$$

with

$$\mathrm{Sq}^2x_3=x_5,\ \mathrm{Sq}^1x_5=x_3^2,\ \mathrm{Sq}^1x_3x_5=x_3^3,\ \mathrm{Sq}^2x_3^3=x_{11}.$$

From (4.5), since G_2 is simply connected, we may construct a (minimal) CW complex

$$A=S^3\cup e^5\cup e^6\cup e^8\cup e^9\cup e^{11}\cup e^{14}$$

homotopy equivalent to G_2. Considering the squaring operations we see that the 6-skeleton $A^{(6)}$ of A is homotopy equivalent to X^3, (2.3), because η and 2ι are detected by Sq^2 and Sq^1 and the squaring operations determine the homotopy type of X^3. Similarly, $A^{(11)}/A^{(6)}$ is homotopy equivalent to Y^{11}. We have, therefore, a cofibration

$$X^3\xrightarrow{k}A^{(11)}\xrightarrow{l}Y^{11}. \tag{4.6}$$

Denote the next step of the sequence by $\phi\colon Y^{11}\to\Sigma X^3$.
By (2.4), $\phi^*=0$ in K-theory and

$$\tilde{K}^0(A^{(11)})=0,\ \tilde{K}^1(A^{(11)})=\mathbb{Z}\{\bar{\xi}_3\}\oplus\mathbb{Z}\{\bar{\eta}_{11}\}, \tag{4.7}$$

where $\bar{\eta}_{11}=l^*\eta_{11}$ and $\bar{\xi}_3$ is some element with $k^*\bar{\xi}_3=\xi_3$. Thus

$$\mathrm{ch}\,\bar{\eta}_{11}=\tfrac{1}{2}h_{11},\ \mathrm{ch}\bar{\xi}_3=2h_3+\lambda h_{11} \tag{4.8}$$

for some $\lambda \in \mathbb{Q}$, where, as before, we write h_i also for the rational class and we identify $H^*(A^{(11)};\mathbb{Q})$ with a subgroup of $H^*(A;\mathbb{Q}) = H^*(G_2;\mathbb{Q}) = E(h_3, h_{11})$, the exterior algebra over $\mathbb{Q}$.

Now, by the self-duality of $A = G_2$ [7], the attaching map $S^{13} \to A^{(11)}$ of the top cell in A is stably trivial. So G_2 splits stably as $A^{(11)} \vee S^{14}$. Hence we can write $\tilde{K}^*(A) = \tilde{K}^*(A^{(11)}) \oplus \tilde{K}^*(S^{14})$ and $\tilde{H}^*(A) = \tilde{H}^*(A^{(11)}) \oplus \tilde{H}^*(S^{14})$. (These decompositions are independent of the stable splitting, because the summands are in different dimensions.) In particular, $K^*(A)$ has no torsion and the Chern character for A is monic. Write x, y for the classes in $\tilde{K}^{-1}(A)$ corresponding to $\xi_3, \bar{\eta}_{11}$. By (4.8), $\mathrm{ch}(xy) = h_3 h_{11}$. This gives an alternative form of the theorem (4.4) of Hodgkin:

Theorem 4.9. (a) $K^*(A) = E(x, y)$, *the exterior algebra over* $\mathbb{Z}$.

(b) $\mathrm{ch}\, x = 2h_3 + \lambda h_{11}$, $\mathrm{ch}\, y = \frac{1}{2} h_{11}$ *for some* $\lambda \in \mathbb{Q}$.

The element x is determined modulo y, while y is unique. We wish to know the relation between the two generating systems $\{x, y\}$ here and $\{\beta(\rho), \beta(\rho')\}$ in (4.4).

The inclusion $i: S^3 = \mathrm{SU}(2) \to G_2$ is 4-connected. (We have a fibration $S^3 \to G_2 \to V_{7,2}$, where $V_{7,2}$ is the Stiefel manifold of 2-frames in $\mathbb{R}^7$, given by mapping $g \in G_2$ to $(g(e_1), g(e_2))$.) Hence we may identify i with the inclusion of the 3-skeleton $S^3 = A^{(3)} \to A$. Consider $i^*: K^{-1}(A) \to K^{-1}(S^3)$. By (2.4)(b) and the definition of x, y, we see that i^*x is twice a generator and y generates the kernel of i^*. From (4.4)(b) we obtain

Lemma 4.10. *There is a choice of* x *such that*

$$\beta(\rho) = x, \ \beta(\rho') = 5x \pm y.$$

The coefficient λ in (4.9)(b) determines the e-invariant of the stable class of ϕ in $\{Y^{11}, \Sigma X^3\}$. $e(\phi) = 2\lambda \pmod{\mathbb{Z}}$ in $\mathbb{Q}/\mathbb{Z}$.

Lemma 4.11. $e(\phi) = \pm 1/30 \in \mathbb{Q}/\mathbb{Z}$.

Proof. We compute the Adams operation ψ^2 on x, y. Apply [**1**, (5.1)(vi)] to (4.9)(b) to get

$$\mathrm{ch}\, \psi^2 x = 8h_3 + 64\lambda h_{11}, \ \mathrm{ch}\psi^2 y = 32 h_{11}.$$

Since ch is monic,

$$\psi^2 x = 4x + 120\lambda y. \tag{*}$$

On the other hand, $\psi^2 \rho = \rho^2 - 2 \wedge^2 \rho = \rho^2 - 2\rho'$. By (4.10),

$$\psi^2 x = \psi^2 \beta(\rho) = 14\beta(\rho) - 2\beta(\rho') = 4x \mp 2y.$$

Comparing this with (*) leads to $\lambda = \pm(1/60)$. □

By (3.2), the e-invariant faithfully determines the stable class of ϕ.

Theorem 4.12. *The stable class of the attaching map* $\phi: Y^{11} \to \Sigma X^3$ *is twice a generator, i.e.* $\pm 2i\sigma j$. *Hence it is of order* 30.

Remark 4.13. For a prime p, $G_2^{(6)} = X^3$ is a mod p stable retract of G_2 if and only if $p > 5$.

For $p=2$, (4.13) was recently obtained by Cohen and Peterson [**8**] by a different method. For $p=3$, (4.12) asserts that ϕ localized at 3 is detected by a secondary operation $\Phi: H^4(\Sigma G_2; \mathbb{Z}/3) \to H^{12}(\Sigma G_2; \mathbb{Z}/3)$; this is equivalent to the old result of Bott–Samelson [**6**] that $\pi_{10}(G_2)_{(3)} = 0$. Similarly, (4.12) localized at 5 is equivalent to the non-triviality of $\mathscr{P}^1: H^3(G_2; \mathbb{Z}/5) \to H^{11}(G_2; \mathbb{Z}/5)$, originally obtained by Bott [**6**].

Remark 4.14. By [**21**], $\pi_{14}(S^7) = \mathbb{Z}/120\{\sigma'\}$ and the generator σ' is not a suspension. However, the composite

$$Y^{14} \xrightarrow{j} S^{14} \xrightarrow{\sigma'} S^7 \xrightarrow{i} X^7$$

is a three-fold suspension. This is proved by showing that $i_* j^*: \pi_{14}(S^7) \to [Y^{14}, X^7]$ is epic. Then ϕ gives the required desuspension (up to multiplication by a unit in $\mathbb{Z}/120$).

Appendix. One would expect to be able to compute the Chern character on $K^*(G_2) = E(\beta(\rho), \beta(\rho'))$ by standard methods of representation theory. Professor H. Minami has kindly supplied the author with such a proof and we briefly describe his method here.

Recall that, for any compact Lie group G, the β-construction may be written, up to sign, as a composite [**9**]:

$$\beta: R(G) \xrightarrow{\alpha} K^0(BG) \longrightarrow \tilde{K}^0(BG) \xrightarrow{\sigma} \tilde{K}^{-1}(G).$$

σ is induced by the canonical map $\Sigma G \to BG$. To compute the Chern character on $\tilde{K}^{-1}(G_2)$ we work first on $K^0(BG_2)$. At this level we can restrict to a maximal torus T of G_2. For T we choose the standard maximal torus of $\mathrm{SU}(3) \subseteq G_2$ and then read off the information on $R(G_2) \to R(T)$ from (4.2)

To simplify the argument one can exploit the restriction to $\mathrm{SU}(2) \subseteq G_2$ and also use the fact [**4**] that $\mathrm{ch}(\beta(\rho)\beta(\rho')) = h_3 h_{11}$.

5. The degree of self-maps of G_2

Mimura, Nishida and Toda constructed in [**13**] simply connected H-spaces $G_{2,b}$ $(-2 \leqq b \leqq 5)$ of rank 2 with homology torsion, whose prototype is $G_2 = G_{2,0}$. $G_{2,b}$ is p-equivalent to G_2 for primes p other than 3 or 5, to $S^3 \times S^{11}$ or G_2 according as $b \equiv -2 \bmod p$ or not for $p = 3, 5$. On the homotopy type of $G_{2,b}$, they proved

Lemma 5.1. (a) [**13**, above (5.2), (5.2)] *$G_{2,b}$ is homotopy equivalent to a CW complex*

$$A_b = S^3 \cup e^5 \cup e^6 \cup e^8 \cup e^9 \cup e^{11} \cup e^{14},$$

which coincides up to the 9-skeleton with $A = A_0$ (in Section 4), homotopy equivalent to G_2: ${A_b}^{(9)} = A^{(9)}$.

(b) [**13**, (4.3)(ii), (5.3)] *The attaching map $\omega: S^{10} \to A^{(9)}$ of the top cell in $A^{(11)}$ is a generator of $\pi_{10}(A^{(9)}) = \mathbb{Z}/120$ and the attaching map $\omega_b: S^{10} \to A_b^{(9)} = A^{(9)}$ of the top cell in $A_b^{(11)}$ is $(8b+1)\omega$.*

(c) [**13**, (4.3)(iii)] *The image of ω in $\pi_{10}(A^{(9)}/A^{(6)}) = \pi_{10}(M^8) = \mathbb{Z}/4\{\tilde{\eta}\}$ is the generator $\pm\tilde{\eta}$; hence, so is the image of ω_b.*

(d) $A_b^{(6)} = X^3$, $A_b^{(11)}/A_b^{(6)} = Y^{11}$.

Since $(8b+1)\tilde{\eta} = \tilde{\eta}$, the second parts of (c) and (d) are clear. From (d) we get a map

$$\phi_b: Y^{11} \to \Sigma X^3 \quad (\phi_0 = \phi \text{ in Section 4})$$

extending the cofibration $A_b^{(6)} \to A_b^{(11)} \to A_b^{(11)}/A_b^{(6)}$.

Lemma 5.2 *Stably $\phi_b = (8b+1)\phi$.*

Proof. Consider the diagram

$$\begin{array}{ccccc} & & Y^{11} & \xrightarrow{j} & S^{11} \\ & & \Big\downarrow \phi_b & & \Big\downarrow \Sigma\omega_b \\ M^8 & \longrightarrow & \Sigma X^3 & \longrightarrow & \Sigma A^{(9)} \end{array}$$

where the lower sequence is a cofibration and the square is stably commutative. The difference $\phi_b - (8b+1)\phi$ in $\{Y^{11}, \Sigma X^3\}$ lifts to $\{Y^{11}, M^8\}$. Now the e-invariant is defined on the whole of $\{Y^{11}, M^8\}$ and is zero, because $\tilde{H}^*(M^8; \mathbb{Q}) = 0$ and $\tilde{K}_{11}(M^8) = 0$. So $\phi_b - (8b+1)\phi$ has trivial e-invariant and is zero by (3.2). □

The cohomology ring of $G_{2,b}$ is isomorphic to that of G_2, cf. [**13**, (2.2)], so we use h_3, h_{11} again for the multiplicative generators of $H^*(G_{2,b})$.

Theorem 5.3. *There are elements $x, y \in K^{-1}(G_{2,b})$ such that $K^*(G_{2,b}) = E(x, y)$, the exterior algebra over $\mathbb{Z}$, and*

$$\operatorname{ch} x = 2h_3 - ((8b+1)/60)h_{11}, \quad \operatorname{ch} y = (1/2)h_{11}.$$

The proof is like that of (4.9). The coefficient of h_{11} in $\operatorname{ch} x$ is $e(\phi_b)$. The sign of the e-invariant depends upon the orientation of the generators; we have fixed a choice here.

We shall study the image of the degree map d and stable degree map d^s for $G_{2,b}$:

$$d = d_3 \times d_{11} : [G_{2,b}, G_{2,b}] \to \mathbb{Z} \oplus \mathbb{Z},$$

$$d^s : \{G_{2,b}, G_{2,b}\} \to \mathbb{Z} \oplus \mathbb{Z},$$

defined as in the introduction for G_2. Both d and d^s preserve the addition, given by an H-structure on $G_{2,b}$ for $[G_{2,b}, G_{2,b}]$ and the usual track addition in the stable case, and the multiplication given by composition of maps.

Let $\pi(b)$ be the order of $8b+1$ mod 30, that is, by (3.2) and (5.2), the order of $e(\phi_b)$. For $-2 \leqq b \leqq 5$, $\pi(b)$ is given by:

b	-2	-1	0	1	2	3	4	5
$\pi(b)$	2	30	30	10	30	6	10	30

(5.4)

Proposition 5.5. $\operatorname{Im} d \subseteq \operatorname{Im} d^s \subseteq \{(m,n) \mid m, n \in \mathbb{Z},\ m \equiv n \bmod \pi(b)\}$.

Proof. The first inclusion is obvious. For a given stable map $f \in \{G_{2,b}, G_{2,b}\}$, let $m = d_3^s(f)$, $n = d_{11}^s(f)$. Thus $f^*(h_3) = mh_3, f^*(h_{11}) = nh_{11}$. We shall determine

$$f^* : \tilde{K}^{-1}(G_{2,b}) = \mathbb{Z}\{x\} \oplus \mathbb{Z}\{y\} \to \tilde{K}^{-1}(G_{2,b}).$$

From (5.3) it is immediate that $f^*(y) = ny, f^*(x) = mx + ky$ for some $k \in \mathbb{Z}$. One checks that $(8b+1)(m-n) = 30k$, which implies the congruence $m \equiv n \bmod \pi(b)$. □

We shall prove that the K-theoretic estimate for $\operatorname{Im} d$ given in (5.5) is the best result, namely,

Theorem 5.6. $\operatorname{Im} d = \operatorname{Im} d^s = \{(m,n) \mid m, n \in \mathbb{Z},\ m \equiv n \bmod \pi(b)\}$.

Since d is a homomorphism and $d(\mathrm{id}) = (1,1)$, the theorem is equivalent to the existence of a self-map f of $G_{2,b}$ with $d_3(f) = 0$, $d_{11}(f) = \pi(b)$. To construct such a map, we need the following lemmas.

Lemma 5.7. $\pi_{11}(G_{2,b}) = \mathbb{Z}\{\gamma\} \oplus \mathbb{Z}/2\{\tau_1\}$ *and the image of the Hurewicz homomorphism* $\pi_{11}(G_{2,b}) \to \tilde{H}_{11}(G_{2,b})$ *has index* $4\pi(b)$.

Lemma 5.8. $[Y^{11}, G_{2,b}] = \mathbb{Z}\{\gamma'\} \oplus \mathbb{Z}/2\{j^*\tau_1\}$ *with* $j^*\gamma = 4\gamma'$. (j: $Y^{11} \to S^{11}$ *as in* (2.3).)

As we need a number of computations to prove (5.8), we shall delay its proof until the next section.

Proof of (5.7). The homotopy group is computed in [14, (3.3)]. Since $\pi_{11}(G_{2,b}) = \pi_{11}(A_b^{(11)})$, the index equals the order of ω_b, the attaching map of the top cell in $A_b^{(11)}$, which is $4\pi(b)$ by (5.1)(b). □

Proof of (5.6) (assuming (5.8)). The attaching map of the top cell in $G_{2,b}$, as in G_2, is stably trivial. Hence, since $\pi_{13}(Y^{11})$ is already in the stable range, $G_{2,b}/G_{2,b}^{(6)}=Y^{11}\vee S^{14}$. Let $g: G_{2,b}\to Y^{11}$ be the projection. Then the composite $f=\gamma' g: G_{2,b}\to G_{2,b}$ clearly has $d_3(f)=0$. By (5.8), $4d_{11}(f)=d_{11}(\gamma jg)$, which is equal to $4\pi(b)$, by (5.7), since $j^*: \tilde{H}^{11}(S^{11})\to\tilde{H}^{11}(Y^{11})$ is isomorphic. Hence $d_{11}(f)=\pi(b)$ and f has the required degree. □

Remark. It is rather easier to compute $\operatorname{Im} d^s$. From the cofibration

$$A_b^{(11)}\longrightarrow Y^{11}\xrightarrow{\phi_b}\Sigma X^3$$

we obtain an exact sequence

$$\{Y^{11},A_b^{(11)}\}\to\{Y^{11},Y^{11}\}\to\{Y^{11},\Sigma X^3\}.$$

$\pi(b)\,\mathrm{id}\in\{Y^{11},Y^{11}\}$ lifts to $\{Y^{11},A_b^{(11)}\}$. By composing with g and the inclusion $A_b^{(11)}\to A_b=G_{2,b}$, we obtain a stable class f with $d_3^s(f)=0$, $d_{11}^s(f)=\pi(b)$.

6. Proof of Lemma 5.8

We recall the cofibrations

$$S^n\xrightarrow{2\iota}S^n\xrightarrow{i}M^n\xrightarrow{p}S^{n+1}\qquad(n\geqq 1),$$

$$S^{n-1}\xrightarrow{\tilde{\eta}}M^{n-3}\xrightarrow{i'}Y^n\xrightarrow{j}S^n\qquad(n\geqq 6).$$

The second induces the exact sequence

$$[M^9,G_{2,b}]\xrightarrow{\tilde{\eta}^*}\pi_{11}(G_{2,b})\xrightarrow{j^*}[Y^{11},G_{2,b}]\xrightarrow{(i')^*}[M^8,G_{2,b}]\xrightarrow{\tilde{\eta}^*}\pi_{10}(G_{2,b}).$$

The above groups except the middle one were computed by Mimura and Sawashita [**14**,(3.3),(3.5)]. But note that the M^n in [**13**], [**14**] is different from ours: we write $M^n=S^n\bigcup_{2\iota}e^{n+1}$, while $M^n=S^{n-1}\bigcup_{2\iota}e^n$ in [**13**], [**14**]. From their results (with our notation for M^n),

$$\pi_{11}(G_{2,b})=\mathbb{Z}\{\gamma\}\oplus\mathbb{Z}/2\{\tau_1\},$$

$$[M^9,G_{2,b}]=\mathbb{Z}/2\{\tau_2'\tilde{\eta}\},$$

$$[M^8,G_{2,b}]=\mathbb{Z}/4\{\tau_2\},$$

$\pi_{10}(G_{2,b})$ is an odd torsion group, where γ and τ_1 are denoted by $\langle 2\Delta\iota_{13}\rangle$ and $i_*[\nu_5^2]$, $\tilde{\eta}$ is the element in (2.2) for suitable n, τ_2', denoted by $\langle\eta_6^2\rangle$ in **[14]**, is a generator of $\pi_8(G_{2,b})=\mathbb{Z}/2$ and τ_2 is an extension of τ_2', i.e. $\tau_2 i=\tau_2'$.

By (2.2)(c), $\tilde{\eta}^*[M^9, G_{2,b}]=0$. Hence we get a short exact sequence

$$0\to\mathbb{Z}\{\gamma\}\oplus\mathbb{Z}/2\{\tau_1\}\to[Y^{11}, G_{2,b}]\to\mathbb{Z}/4\{\tau_2\}\to 0.$$

The group extension at $[Y^{11}, G_{2,b}]$ is a 2-local problem because no odd torsion group is involved in the short exact sequence. Since $G_{2,b}$ is 2-equivalent to G_2, the above sequence is equivalent to

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_{11}(A) & \xrightarrow{j^*} & [Y^{11}, A] & \xrightarrow{(i')^*} & [M^8, A] & \longrightarrow & 0 \\ & & \| & & & & \| & & \\ & & \mathbb{Z}\{\gamma\}\oplus\mathbb{Z}/2\{\tau_1\} & & & & \mathbb{Z}/4\{\tau_2\}. & & \end{array} \qquad (6.1)$$

For a CW complex W, we denote the exact sequence

$$\longrightarrow\pi_n(W)\xrightarrow{j^*}[Y^n, W]\xrightarrow{(i')^*}[M^{n-3}, W]\longrightarrow$$

by $[n, W]$; thus $(6.1)=[11, A]$.

Since A is homotopy equivalent to G_2, there is a well known fibration

$$SU(3)\xrightarrow{i''}A\xrightarrow{p''}S^6$$

classified by the generator $[2\iota]\in\pi_5(SU(3))=\mathbb{Z}$ with $d_5([2\iota])=2$. To determine the group extension of (6.1), we examine the exact sequences $[11, SU(3)]$, $[11, S^6]$, $[10, SU(3)]$.

Lemma 6.2. (a) *$\pi_{11}(SU(3))=\mathbb{Z}/4\{\tau_1'\}$, and $j^*:\pi_{11}(SU(3))\to[Y^{11}, SU(3)]$ is epic, where $\tau_1=(i'')_*\tau_1'$.*

(b) *$\pi_{10}(SU(3))=\mathbb{Z}/30\{\tau_3\}$, and the element $j^*\tau_3$ in $[Y^{10}, SU(3)]$ is of order 15.*

(c) *The image of $(i'')_*:[Y^{11}, SU(3)]\to[Y^{11}, A]$ is $\mathbb{Z}/2$, generated by $j^*\tau_1=(i'')_*j^*\tau_1'$.*

Proof. The results on $\pi_i(SU(3))$ are in **[15]**, where τ_1' is denoted by $[\nu_5^2]$ and the 2-primary part of τ_3 by $[\nu_5\eta_8^2]$. By **[12**, (6.1)**]**, the generator $\tau_1\in\pi_{11}(G_2)=\pi_{11}(A)$ in (6.1) is in the image of $(i'')_*$, that is $\tau_1=(i'')_*\tau_1'$. Then part (c) is immediate from (a), (b). The sequences $[11, SU(3)]$ and $[10, SU(3)]$ are connected by $\tilde{\eta}^*:[M^8, SU(3)]\to\pi_{10}(SU(3))$. To show (a), (b), it is enough to prove that the $\tilde{\eta}^*$ is an isomorphism at 2, because $[M^8, SU(3)]$ is a 2-group. Let

$$S^3\xrightarrow{i_1}SU(3)\xrightarrow{p_1}S^5$$

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be the usual S^3-bundle with characteristic element $\eta \in \pi_4(S^3)$. Consider the exact sequence

$$\xrightarrow{\partial}[M^8,S^3]\xrightarrow{(i_1)_*}[M^8,\mathrm{SU}(3)]\xrightarrow{(p_1)_*}[M^8,S^5]\xrightarrow{\partial'},$$

where the boundary homomorphisms ∂, ∂' satisfy $\partial(\Sigma\alpha)=\eta\alpha, \alpha\in[M^8,S^4]$, $\partial'(\Sigma\alpha')=\eta\alpha'$, $\alpha'\in[M^7,S^4]$. We have, from the results on $\pi_i(S^3)$, $\pi_i(S^5)$, $i=8,9$ in **[21]**, that

$$[M^8,S^3]=\mathbb{Z}/2\{\nu'\eta\bar{\eta}\}$$

$$[M^8,S^5]=\mathbb{Z}/2\{\nu\eta p\}\oplus\mathbb{Z}/2\{\eta^2\bar{\eta}\},$$

where ν' is a generator of the 2-primary part $\mathbb{Z}/4$ of $\pi_6(S^3)$, and satisfies

$$\text{(a) } 2\nu'=\eta^3 \text{ in } \pi_6(S^3), \qquad \text{(b) } \eta\nu=\nu'\eta \text{ in } \pi_7(S^3). \tag{6.3}$$

Then we have

$$\nu'\eta\bar{\eta}=\eta\nu\bar{\eta}=\partial(\nu\bar{\eta})\in\partial[M^9,S^5].$$

Hence $(i_1)_*=0$, and

$$\partial'(\nu\eta p)=\eta\nu\eta p=\nu'\eta^2 p, \quad \partial'(\eta^2\bar{\eta})=\eta^3\bar{\eta}=\nu'(2\bar{\eta})=\nu'\eta^2 p$$

by an unstable version of (2.2)(b). Since $\nu'\eta^2\in\pi_8(S^3)$ cannot be halved **[21]**, $\nu'\eta^2 p\neq 0$; hence

$$\operatorname{Ker}\partial'=\mathbb{Z}/2\{\nu\eta p+\eta^2\bar{\eta}\},$$

$$(p_1)_*:[M^8,\mathrm{SU}(3)]\to\operatorname{Ker}\partial' \text{ is isomorphic.}$$

We then get the commutative diagram

$$\begin{array}{ccc}
[M^8,\mathrm{SU}(3)] & \xrightarrow{\tilde{\eta}^*} & \pi_{10}(\mathrm{SU}(3))=\mathbb{Z}/30 \\
\cong\downarrow (p_1)_* & & \downarrow (p_1)_* \\
\operatorname{Ker}\partial' & \xrightarrow{\tilde{\eta}^*} & \pi_{10}(S^5)=\mathbb{Z}/2\{\nu\eta^2\},
\end{array}$$

where the upper $\tilde{\eta}^*$ is the one we are investigating. The lower $\tilde{\eta}^*$ is isomorphic, because

$$\begin{aligned}\tilde{\eta}^*(\nu\eta p+\eta^2\bar{\eta})&=\nu\eta p\tilde{\eta}+\eta^2\bar{\eta}\tilde{\eta}\\ &=\nu\eta^2+\eta^2(6\nu)=\nu\eta^2\end{aligned}$$

by (2.2)(a), (c). As in [**15**, (4.1)], the right $(p_1)_*$ is an isomorphism at 2; hence, so is the upper $\tilde{\eta}^*$. □

Lemma 6.4. (a) $(i')^*:[Y^{10},SU(3)]\to[M^7,SU(3)]$ *is an isomorphism on 2-primary components.*

(b) $[Y^{10},SU(3)]=\mathbb{Z}/30\{\tau_4\}$, $[M^7,SU(3)]=\mathbb{Z}/2\{\tau_5\}$, *where*

$$j^*\tau_3=2\tau_4,\quad (i')^*\tau_4=\tau_5=[2\iota]\nu p=i_1\nu'\tilde{\eta}.$$

Proof. (a) By [**15**], $\pi_9(SU(3))$ is an odd torsion group, while $[M^7,SU(3)]$ is a 2-group. Hence $\tilde{\eta}^*=0:[M^7,SU(3)]\to\pi_9(SU(3))$ and $(i')^*$ is epic. The result is then immediate from (6.2)(b).

(b) We compute $[M^7,SU(3)]$ in two different ways, one using the fibration of SU(3) and the other the cofibration of M^7. By [**15**], $\pi_7(SU(3))=0$ and $\pi_8(SU(3))=\mathbb{Z}/12\{[2\iota]\nu\}$. The exact sequence induced from the cofibration of M^7 then leads to $[M^7,SU(3)]=\mathbb{Z}/2\{[2\iota]\nu p\}$. We next consider the exact sequence induced from the fibration:

$$[M^8,S^5]\xrightarrow{\partial'}[M^7,S^3]\xrightarrow{(i_1)_*}[M^7,SU(3)]\xrightarrow{(p_1)_*}[M^7,S^5]\xrightarrow{\partial''}[M^6,S^3],$$

where ∂', ∂'' satisfy $\partial'(\Sigma\alpha')=\eta\alpha'$, $\partial''(\Sigma\alpha')=\eta\alpha''$ for $\alpha'\in[M^7,S^4]$, $\alpha''\in[M^6,S^4]$ with $\eta\in\pi_4(S^3)$. From the results on $\pi_i(S^3)$, $\pi_i(S^5)$ $i=6$, 7, 8, 9, in [**21**], we obtain

$$[M^8,S^5]=\mathbb{Z}/2\{\nu\eta p\}\oplus\mathbb{Z}/2\{\eta^2\tilde{\eta}\},\qquad [M^7,S^3]=\mathbb{Z}/4\{\nu'\tilde{\eta}\},$$

$$[M^7,S^5]=\mathbb{Z}/2\{\nu p\}\oplus\mathbb{Z}/2\{\eta\tilde{\eta}\},\qquad [M^6,S^3]=\mathbb{Z}/2\{\nu'\eta p\}\oplus\mathbb{Z}/2\{\eta^2\tilde{\eta}\}.$$

In particular, by (2.2)(b), (6.3)(a), the following relations hold.

$$2\nu'\tilde{\eta}=\eta^3\tilde{\eta}=\nu'\eta^2p \text{ in } [M^7,S^3]. \tag{6.5}$$

We then have

$$\partial'(\nu\eta p)=\eta\nu\eta p=\nu'\eta^2p=2\nu'\tilde{\eta},\qquad \partial'(\eta^2\tilde{\eta})=\eta^3\tilde{\eta}=2\nu'\tilde{\eta},$$

$$\partial''(\nu p)=\eta\nu p=\nu'\eta p,\qquad \partial''(\eta\tilde{\eta})=\eta^2\tilde{\eta},$$

by (6.3)(b) and (6.5). Therefore $\operatorname{Coker}\partial'=\mathbb{Z}/2\{\nu'\tilde{\eta}\}$ and $\operatorname{Ker}\partial''=0$, proving $[M^7,SU(3)]=\mathbb{Z}/2\{i_1\nu'\tilde{\eta}\}$.

The odd primary part of $[Y^{10},SU(3)]$ is isomorphic to that of $\pi_{10}(SU(3))$ via j^*, while the 2-primary part is $\mathbb{Z}/2$. Hence $[Y^{10},SU(3)]=\mathbb{Z}/30\{\tau_4\}$ with $j^*\tau_3=2\tau_4$. The last relation $(i')^*\tau_4=\tau_5$ is immediate from (a). □

We next compute the exact sequence $[11,S^6]$.

Lemma 6.6. (a) *The sequence* $[11, S^6]$ *is short exact, where the marginal terms are*

$$\pi_{11}(S^6) = \mathbb{Z}\{[\iota_6, \iota_6]\}, \quad [M^8, S^6] = \mathbb{Z}/2\{\eta\bar{\eta}\} \oplus \mathbb{Z}/2\{\nu p\}.$$

($[\iota_6, \iota_6]$ *is the Whitehead square.*)

(b) $[Y^{11}, S^6] = \mathbb{Z}\{\gamma''\} \oplus \mathbb{Z}/2\{\Sigma\tau_6\}, \quad j^*[\iota_6, \iota_6] = 2\gamma'', \quad (i')^*\gamma'' = \nu p, \quad (i')^*(\Sigma\tau_3) = \eta\bar{\eta}.$

Proof. (a) This is clear from the table of $\pi_i(S^6)$ in [**21**], since $\pi_{10}(S^6) = 0$ and $[M^9, S^6]$ is finite.

(b) We first compute $[10, S^5], [9, S^5]$. Extending the lower $\tilde{\eta}^*$ to the right in the commutative diagram in the proof of (6.2), we see that

$$(i')^* : [Y^{10}, S^5] \to [M^7, S^5]$$

is monic. As

$$[M^7, S^5] = \mathbb{Z}/2\{\eta\bar{\eta}\} \oplus \mathbb{Z}/2\{\nu p\}, \quad \pi_9(S^5) = \mathbb{Z}/2\{\nu\eta\},$$

$$\eta\bar{\eta}\tilde{\eta} = 0 \quad \text{and} \quad \nu p\tilde{\eta} = \nu\eta \quad \text{in} \quad \pi_9(S^5),$$

the image of $(i')^*$ is $\mathbb{Z}/2\{\eta\bar{\eta}\}$. Hence

$$[Y^{10}, S^5] = \mathbb{Z}/2\{\tau_6\}, \quad (i')^*\tau_6 = \eta\bar{\eta}.$$

By a similar computation, we obtain

$$[Y^9, S^5] = 0.$$

We next study the EHP exact sequence

$$[Y^{10}, S^5] \xrightarrow{\Sigma} [Y^{11}, S^6] \xrightarrow{H} [Y^{11}, S^{11}] \xrightarrow{P} [Y^9, S^5] = 0$$

in order to determine $[Y^{11}, S^6]$. Since $\eta\bar{\eta}$ is still non-trivial in $[M^8, S^5]$, $\Sigma\tau_6$ is non-trivial and generates $\operatorname{Im}\Sigma = \mathbb{Z}/2$. Clearly, $[Y^{11}, S^{11}] = \mathbb{Z}\{j\}$. Therefore there is an element γ'' with $H(\gamma'') = j$ for which

$$[Y^{11}, S^6] = \mathbb{Z}\{\gamma''\} \oplus \mathbb{Z}/2\{\Sigma\tau_6\}.$$

The Hopf invariant of $[\iota_6, \iota_6]$ is known to be $2\iota_{11}$; hence

$$H(j^*[\iota_6, \iota_6]) = 2j,$$

$$j^*[\iota_6, \iota_6] \equiv 2\gamma'' \bmod \Sigma\tau_6.$$

Clearly $(i')^*j^*[\iota_6,\iota_6]=(ji')^*[\iota_6,\iota_6]=0$. Since $[M^8,S^6]$ is a $\mathbb{Z}/2$-module, $(i)'^*(2\gamma'')=0$. Therefore $j^*[\iota_6,\iota_6]=2\gamma''$ because $(i')^*(\Sigma\tau_6)=\eta\bar{\eta}\neq 0$. As $(i')^*:[Y^{11},S^6]\to[M^8,S^6]$ is epic, there is a choice of γ'' which satisfies $(i')^*\gamma''=\nu p$ as well as the other relations. □

Now we are ready to prove Lemma 5.8.

Proof of (5.8). The commutative diagram of exact sequences

$$[11,SU(3)]\xrightarrow{(i'')_*}[11,A]\xrightarrow{(p'')_*}[11,S^6]\xrightarrow{\Delta}[10,SU(3)],$$

where the boundary homomorphism Δ satisfies $\Delta\Sigma=[2\iota]_*$, becomes, by previous computations:

$$\begin{array}{ccccc}
 & & 0 & & \\
 & & \downarrow & & \\
\mathbb{Z}/4\{\tau_1'\} & \xrightarrow{j^*} & \mathbb{Z}/2\{j^*\tau_1'\} & \longrightarrow & 0 \\
\downarrow (i'')_* & & \downarrow (i'')_* & & \\
0\longrightarrow\mathbb{Z}\{\gamma\}\oplus\mathbb{Z}/2\{\tau_1\} & \xrightarrow{j^*} & [Y^{11},A] & \xrightarrow{(i')^*} & \mathbb{Z}/4\{\tau_2\}\longrightarrow 0 \\
\downarrow (p'')_* & & \downarrow (p'')_* & & \downarrow (p'')_* \\
0\longrightarrow\mathbb{Z}\{[\iota_6,\iota_6]\} & \xrightarrow{j^*} & \mathbb{Z}\{\gamma''\}\oplus\mathbb{Z}/2\{\Sigma\tau_6\} & \xrightarrow{(i')^*} & \mathbb{Z}/2\{\eta\bar{\eta}\}\oplus\mathbb{Z}/2\{\nu p\}\longrightarrow 0 \\
\downarrow \Delta & & \downarrow \Delta & & \downarrow \Delta \\
\mathbb{Z}/30\{\tau_3\} & \xrightarrow{j^*} & \mathbb{Z}/30\{\tau_4\} & \xrightarrow{(i')^*} & \mathbb{Z}/2\{\tau_5\}\longrightarrow 0
\end{array}$$

Here we have also proved

$$(i'')_*\tau_1'=\tau_1,$$

$$j^*[\iota_6,\iota_6]=2\gamma'',\quad (i')^*\gamma''=\nu p,\quad (i')^*(\Sigma\tau_6)=\eta\bar{\eta},$$

$$j^*\tau_3=2\tau_4,\quad (i')^*\tau_4=\tau_5=[2\iota]\nu p=i_1\nu'\bar{\eta}.$$

Since $\pi_{10}(A)=\pi_{10}(G_2)=0$, the left Δ is epic, hence

$$(p'')_*\gamma=30[\iota_6,\iota_6].$$

The right Δ is computed as follows.

$$\Delta(\eta\bar{\eta})=[2\iota]\eta\bar{\eta}=i_1\nu'\bar{\eta}=\tau_5, \quad \Delta(\nu p)=[2\iota]\nu p=\tau_5,$$

because of the formula $\Delta(\Sigma\alpha)=[2\iota]\alpha$ and the relation $\Delta\eta=i_1\nu'$ in [**12**, (6.3) with $\alpha=\eta_6$]. Therefore

$$(p'')_*(\tau_2)=\eta\bar{\eta}+\nu p, \quad \Delta(\gamma'')=\tau_4, \quad \Delta(\Sigma\tau_6)=15\tau_4.$$

Then we can find an element $\gamma'\in[Y^{11},A]$ with

$$(p'')_*\gamma'=15\gamma''+\Sigma\tau_6, \quad (i')^*\gamma'=\tau_2,$$

$$[Y^{11},A]=\mathbb{Z}\{\gamma'\}\oplus\mathbb{Z}/2\{j^*\tau_1\},$$

where we may replace τ_2 by $-\tau_2$ if necessary. Then

$$(p'')_*j^*\gamma=60\gamma''=4(p'')_*\gamma' \quad \text{and} \quad j^*\gamma\equiv 4\gamma' \bmod \operatorname{Im}(i'')_*.$$

We may replace γ by $\gamma+\tau_1$ to get the exact relation $j^*\gamma=4\gamma'$. □

7. Self-H-maps of $G_{2,b}$

In this section we fix b, $-2\leqq b\leqq 5$, and write $G=G_{2,b}$. Let $\mu:G\times G\to G$ be an H-space multiplication. (We assume only the existence of a unit for μ.) Let $[G,G]_\mu$ (respectively $\mathscr{E}(G)$) denote the set of homotopy classes of H-maps (respectively homotopy equivalences) $G\to G$. We put $\mathscr{E}_H(G;\mu)=[G,G]_\mu\cap\mathscr{E}(G)$. All three sets are closed under composition; $\mathscr{E}(G)$ and $\mathscr{E}_H(G;\mu)$ become groups.

The group $\mathscr{E}(G)$ was determined, up to extension, by Mimura and Sawashita [**14**], and, for $b\neq -2$, we settled in [**17**] the group extension. Recent work of Sawashita [**20**] states that $\mathscr{E}_H(G;\mu)$ (for any b, any multiplication μ) is either trivial or of order 2, and that in the second case the non-trivial element, f say, has $d_3(f)=-1$. The purpose of this section is to eliminate the order 2 case if $b\neq -2$ by estimating the image of the degree map on $[G,G]_\mu$ using the same method as in [**10**], [**11**]. Our result is:

Theorem 7.1. *For $-2\leqq b\leqq 5$ and for arbitrary multiplication μ,*

$$d[G_{2,b},G_{2,b}]_\mu\subseteq\{(m,m+l\pi(b))\,|\,l,m\in\mathbb{Z} \quad \textit{and} \quad m\equiv 0,1 \bmod 4 \quad \textit{if } l \textit{ is even}\}.$$

Proof. Let P be the projective plane of the H-space G with multiplication μ. P is the cofibre of the Hopf construction on μ, $H:\Sigma G\wedge G\simeq G*G\to\Sigma G$, and we have the cofibration

$$\Sigma G\wedge G\xrightarrow{H}\Sigma G\xrightarrow{i}P\xrightarrow{j}\Sigma^2 G\wedge G.$$

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The Künneth formula holds for $K^*(G\times G)$, [3], and $K^*(G)$ becomes a primitively generated Hopf algebra. H is the reduced co-multiplication map (via the suspension isomorphism σ).

We conclude that $\tilde{K}^0(P)$ is a free $\mathbb{Z}$-module with basis $\{\alpha,\beta,\alpha^2,\alpha\beta,\beta^2,\gamma\}$, where

$$i^*\alpha=x,\quad i^*\beta=y,\quad \gamma=j^*(\sigma^2(xy\otimes xy)),$$

$$\alpha^3=\alpha^2\beta=\alpha\beta^2=\beta^3=0,$$

$$\alpha\gamma=\beta\gamma=\gamma^2=0.$$

From the Chern character formula in (5.3),

$$\psi^2x=4x-2(8b+1)y,\quad \psi^2y=64y.$$

Therefore we may put

$$\begin{aligned}\psi^2\alpha&\equiv 2\beta+t\alpha^2+u\alpha\beta+v\beta^2+w\gamma \bmod 4,\\ \psi^2\beta&\equiv a\alpha^2+b\alpha\beta+c\beta^2+d\gamma \bmod 4,\end{aligned}\tag{7.2}$$

for some integers t,u,v,w,a,b,c,d. We also have

$$\psi^2\alpha^2\equiv 0,\quad \psi^2\alpha\beta\equiv 0,\quad \psi^2\beta^2\equiv 0,\quad \psi^2\gamma\equiv 0 \bmod 4.\tag{7.3}$$

Since $\psi^2\beta\equiv\beta^2 \bmod 2$,

$$c \text{ is odd; } a,\ b \text{ and } d \text{ are even.}\tag{7.4}$$

Now let $f:G\to G$ be an H-map with $d_3(f)=m$, $d_{11}(f)=n$. As in the proof of (5.5),

$$f^*(x)=mx+ky,\quad f^*(y)=ny,$$

where $30k=(8b+1)(m-n)$ and in particular $2k\equiv m-n \bmod 4$.

Since f is an H-map, there is a map $g:P\to P$ fitting into a commutative diagram

$$\begin{array}{ccccc}\Sigma G & \xrightarrow{i} & P & \xrightarrow{j} & \Sigma^2 G\wedge G\\ \downarrow{\scriptstyle \Sigma G} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \Sigma^2 f\wedge f}\\ \Sigma G & \xrightarrow{i} & P & \xrightarrow{j} & \Sigma^2 G\wedge G.\end{array}$$

Then

$$g^*\alpha\equiv m\alpha+k\beta,\quad g^*\beta\equiv n\beta \bmod \alpha^2,\alpha\beta,\beta^2,\gamma.$$

$$\begin{aligned}g^*\alpha^2&=m^2\alpha^2+2mk\alpha\beta+k^2\beta^2,\qquad g^*\beta^2=n^2\beta^2,\\ g^*\alpha\beta&=mn\alpha\beta+kn\beta^2,\qquad g^*\gamma=(mn)^2\gamma.\end{aligned}\tag{7.5}$$

We compare the coefficients modulo 4 of β^2 in $\psi^2 g^*\beta$ and in $g^*\psi^2\beta$. Let D be the subgroup generated by $\alpha, \beta, \alpha^2, \alpha\beta, 4\beta^2, \gamma$. Then

$$\psi^2 g^*\beta \equiv n\psi^2\beta \equiv nc\beta^2 \bmod D,$$

$$g^*\psi^2\beta \equiv ag^*\alpha^2 + bg^*\alpha\beta^2 + cg^*\beta + dg^*\gamma$$

$$\equiv (ak^2 + bkn + cn^2)\beta^2 \bmod D,$$

by (7.2), (7.3) and (7.5). Hence

$$c(n^2 - n) + (ak + bn)k \equiv 0 \bmod 4.$$

By (7.4), $n^2 - n \equiv 2sk \bmod 4$ for some $s \in \mathbb{Z}$. So $n^2 - n \equiv s(m-n) \bmod 4$. If $m-n$ is divisible by 4, then $m \equiv n \equiv 0, 1 \bmod 4$. Since $\pi(b) \equiv 2 \bmod 4$, (5.4), the theorem follows. □

Remark 7.6. When $G = G_2$ (so $b = 0$) and μ is the Lie group multiplication, we can use the standard map $P \to BG_2$ and the method of the appendix to Section 4 to determine ψ^2 for P. The result is

$$\psi^2\alpha = 4\alpha - 2\beta + \alpha^2, \quad \psi^2\beta = 64\beta - 12\alpha^2 + 12\alpha\beta + \beta^2,$$

from which we can obtain further (complicated) restrictions on the degree of H-maps.

Corollary 7.7. *For $-1 \leqq b \leqq 5$ and arbitrary multiplication μ, $d(\mathscr{E}_H(G_{2,b}; \mu)) = \{(1,1)\}$.*

Proof. If $f \in \mathscr{E}_H(G_{2,b}; \mu)$, then $m = d_3(f) = \pm 1$, $n = d_{11}(f) = \pm 1$. $m \equiv n \bmod \pi(b)$. But $\pi(b) > 2$, by (5.4). Hence $m = n$. By (7.1) $m = 1$. □

The result of Sawashita [**20**, (5.6)] states that, for $-2 \leqq b \leqq 5$ and any μ, the map $G_{2,b} \to K(\mathbb{Z}, 3)$ which kills all the homotopy groups except π_3 induced a monomorphism

$$\mathscr{E}_H(G_{2,b}; \mu) \to \mathscr{E}_H(K(\mathbb{Z}, 3); \mu_K).$$

The multiplication μ_K on the Eilenberg–MacLane space $K(\mathbb{Z}, 3)$ is unique and $\mathscr{E}_H(K(\mathbb{Z}, 3); \mu_K)$ is of order 2 with generator g acting non-trivially on π_3, H_3 and H^3. If there were a lift h of g to $\mathscr{E}_H(G_{2,b}; \mu)$, the action of h on H^3 would have to be non-trivial, which is impossible if $b \neq -2$ by (7.7). In consequence, we have

Theorem 7.8. *For $-1 \leqq b \leqq 5$ and arbitrary multiplication μ, $\mathscr{E}_H(G_{2,b}; \mu) = \{\text{id}\}$.*

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