ON THE CONTINUITY AND SELF-INJECTIVITY OF A COMPLETE REGULAR RING

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0. Let S be a ring, and let (e_i) be an orthogonal system of a finite number of idempotents. Then $e = \sum e_i$ has the following properties:

(i) $Se = \sum Se_i$ and $eS = \sum e_i S$.

(ii) The mappings v: $Se \to \prod Se_i$ and w: $eS \to \prod e_i S$ defined by $v(x) = [xe_i]$ and $w(x) = [e_i x]$ respectively are isomorphisms.

Next assume that $(e_i)_{i \in I}$ is a set of idempotents indexed by a totally ordered set I such that $e_i e_j = 0$ for every $i < j$. If I is finite, it is evident that

$$e = \sum_k (-1)^{k+1} \sum_{i_1 > i_2 > \ldots > i_k} e_{i_1} e_{i_2} \ldots e_{i_k}$$

has the above two properties.

Suppose I is infinite; then the sum $\sum Se_i$ or $\sum e_i S$ is no longer generated by an idempotent in general. We assume that S is complete regular, and consider the following instead of (i):

(i)′ $Se = \cup Se_i$ and $eS = \cup e_i S$.

The aim of this paper is to show that (i)′, (ii), and some of their weakened forms may be used to characterize continuities and self-injectivities of S.

I am grateful to Professor I. Halperin, who kindly wrote to me about Theorem 2.4, which I had obtained only with some superfluous assumption.

1. A ring is called (von Neumann) *regular* if to any element x there exists y with $xyx = x$. A regular ring is said to be *complete* if the lattice of principal left ideals is complete. Thus, a regular ring is complete if and only if every annihilator left (right) ideal is principal. In this case the (lattice-theoretical) meet coincides with the (set-theoretical) intersection.

A module M is called an *essential extension* of a submodule N if $N \cap K \neq 0$ for every non-zero submodule K of M. We then say that M is *essential over* N, or N is essential in M.

In a regular ring, let A be a left ideal contained in a principal left ideal B. Then it is not hard to show that B contains every left ideal essential over A.

We denote the cardinal number of a set V by $\bar{V}$. The set of ordinals smaller than the initial ordinal for a cardinal number m is denoted by $I(m)$. Thus, $\overline{I(m)} = m$, and for any $i \in I(m)$ the cardinal number of the set of ordinals $\leqslant i$ is smaller than m.

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A regular ring S is said to be left *m-continuous* if it is complete, and if for any subset $(x_i)_{i\in I}$ of S with $\bar{I} = m$ the join $\bigcup_{i\in I} Sx_i$ is essential over the sum $\sum_{i\in I} Sx_i$. (We may of course define the left m-continuity of a left "m-complete" regular ring.) A right m-continuous regular ring is defined in the obvious way. We say that a regular ring is m-continuous if it is left and right m-continuous. Thus, a left continuous regular ring is a regular ring that is left m-continuous for every cardinal number m. A right continuous regular ring is defined similarly; see **(3)**.

It is obvious that a left (right) m-continuous regular ring is left (right) n-continuous for every $n < m$.

A set $(e_i)_{i\in I}$ of idempotents of a ring indexed by a chain I is called left (right) semi-orthogonal with respect to the order on I if $e_i e_j = 0$ ($e_j e_i = 0$) for every $i < j$. Thus, a set of idempotents is orthogonal if and only if it is both left and right semi-orthogonal with respect to a total order on the index set. If $(e_i)_{i\in I}$ is a left (right) semi-orthogonal system of idempotents of a ring S, then the sums $\sum_{i\in I} Sx_i$ and $\sum_{i\in I} x_i S$ are both direct.

1.1. LEMMA. *Let S be a regular ring, and suppose that it is left n-continuous for every $n < m$. Let $(x_i)_{i\in I(m)}$ be a subset of S. Then there exists a left semi-orthogonal system $(e_i)_{i\in I(m)}$ of idempotents such that*

(i) *$Se_i \subset Sx_i$ for every i,*

(ii) *$\sum_i Sx_i$ is essential over $\sum_i Se_i$, and*

(iii) *$\bigcup Sx_i = \bigcup Se_i$.*

Remark. Lemmas 1.1 and 1.2 are contained in their lattice-theoretical form in **(1**, Theorem 3.6**)**.

Proof. Note that (iii) follows directly from (ii). Let e_i be any idempotent with $Se_1 = Sx_1$. Suppose that we have defined a left semi-orthogonal system $(e_j)_{j<i}$ of idempotents for an ordinal $i \in I(m)$ in such a way that $Se_j \subset Sx_j$ and that $\sum_{k\leqslant j} Se_k$ is essential in $\sum_{k\leqslant j} Sx_k$ for every $j < i$. It is easily seen that then $\sum_{j<i} Se_j$ is essential in $\sum_{j<i} Sx_j$, and hence $\bigcup_{j<i} Se_j = \bigcup_{j<i} Sx_j$. Take an element a_i such that

$$Sx_i = ((\textstyle\bigcup_{j<i} Sx_j) \cap Sx_i) \oplus Sa_i.$$

Then

$$(\textstyle\bigcup_{j<i} Se_j) \oplus Sa_i = (\bigcup_{j<i} Sx_j) \oplus Sa_i = \bigcup_{j\leqslant i} Sx_j.$$

Hence there is an idempotent e_i such that $Sa_i = Se_i$ and $(\bigcup_{j<i} Se_j)e_i = 0$. Thus, $(e_j)_{j\leqslant i}$ is left semi-orthogonal, $Se_i = Sa_i \subset Sx_i$, and

$$\textstyle\bigcup_{j\leqslant i} Se_j = \bigcup_{j\leqslant i} Sx_j.$$

By assumption, $\bigcup_{j\leqslant i} Se_j$ is essential over $\sum_{j\leqslant i} Se_j$. Therefore $\sum_{j\leqslant i} Sx_j$ is essential over $\sum_{j\leqslant i} Se_j$.

By transfinite induction we define a left semi-orthogonal system $(e_i)_{i\in I(m)}$ of idempotents with the following properties: (i) $Se_i \subset Sx_i$; (ii)′ $\sum_{j\leqslant i} Se_j$ is essential in $\sum_{j\leqslant i} Sx_j$ for every $i \in I(m)$. It follows by (ii)$'$ that $\sum_{i\in I(m)} Se_i$ is essential in $\sum_{i\in I(m)} Sx_i$, completing the proof.

1.2. LEMMA. *Suppose the regular ring S is left n-continuous for every $n < m$. Then, for any subset $(x_i)_{i\in I(m)}$ of S such that the sum $\sum Sx_i$ is direct, there exists a left semi-orthogonal system $(e_i)_{i\in I(m)}$ of idempotents with $Sx_i = Se_i$ for every i.*

Proof. In the proof of Lemma 1.1, $(\bigcup_{j<i} Sx_j) \cap Sx_i = 0$ since $\bigcup_{j<i} Sx_j$ is essential over $\sum_{j<i} Sx_j$. Hence $Sx_i = Sa_i = Se_i$, as desired.

1.3. LEMMA. *Let S be a complete regular ring. Then the following conditions are equivalent for any set (e_i) of idempotents of S:*

(i) *There exists an idempotent e such that $Se = \bigcup Se_i$ and $eS \subset \bigcup e_i S$.*
(ii) *There is an idempotent f such that $Sf \supset \bigcup Se_i$ and $fS = \bigcup e_i S$.*
(iii) $(\bigcup Se_i) \cap (\bigcap S(1 - e_i)) = 0$.
(iv) $(\bigcup e_i S) + (\bigcap (1 - e_i)S) = S$.

Proof. Suppose we have (i). Then $S(1 - e) \supset \bigcap S(1 - e_i)$. Hence

$$(\bigcup Se_i) \cap (\bigcap S(1 - e_i)) \subset Se \cap S(1 - e) = 0,$$

proving (iii).

(iv) can be proved from (iii) by taking annihilators.

If we have (iv), there is an idempotent f with

$$fS = \bigcup e_i S \quad \text{and} \quad (1 - f)S \subset \bigcap (1 - e_i)S.$$

Then $Sf \supset \bigcup Se_i$, proving (ii).

Finally assume (ii). Let $\bigcup Se_i = Sg$, $g = g^2$. Then $Sg \subset Sf$. Set $e = fg$. Evidently, $Sg = Se$, and $eS \subset fS = \bigcup e_i S$, which proves (i), as desired.

1.4. LEMMA. *Let S be a left m-continuous regular ring, and let $(e_i)_{i\in I}$ be a left (right) semi-orthogonal system of idempotents indexed by a chain I with $\bar{I} = (m)$. Then the system satisfies the conditions in Lemma 1.3.*

Proof. By the semi-orthogonality, we have

$$(\sum Se_i) \cap (\bigcap S(1 - e_i)) = 0.$$

$\bigcup Se_i$ is essential over $\sum Se_i$ by the left m-continuity. Hence

$$(\bigcup Se_i) \cap (\bigcap S(1 - e_i)) = 0,$$

as desired.

1.5. LEMMA. *Suppose that a regular ring S is left n-continuous for every $n < m$. Let $(f_i)_{i\in I(m)}$ be a right semi-orthogonal system of idempotents of S. Then there is an orthogonal system $(e_i)_{i\in I(m)}$ of idempotents such that $Se_i = Sf_i$ for every $i \in I(m)$, and $\bigcup_{i\in I(m)} e_i S = \bigcup_{i\in I(m)} f_i S$.*

Proof. Set $e_1 = f_1$. Let $i \in I(m)$. Suppose that we have defined an orthogonal system $(e_j)_{j<i}$ of idempotents in such a way that $Se_j = Sf_j$ and

$$\bigcup_{k\leqslant j} e_k S = \bigcup_{k\leqslant j} f_k S$$

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for every $j < i$. Then $\bigcup_{j<i} e_j S = \bigcup_{j<i} f_j S$. Since S is left n-continuous for every $n < m$,

$$(\bigcup_{j<i} e_j S) + (\bigcap_{j<i} (1 - e_j)S) = S$$

by Lemma 1.4. Now

$$\bigcup_{j\leqslant i} f_j S \supset \bigcup_{j<i} f_j S = \bigcup_{j<i} e_j S,$$

and hence

$$\bigcup_{j\leqslant i} f_j S = (\bigcup_{j<i} e_j S) \oplus a_i S$$

for some $a_i \in \bigcap_{j<i} (1 - e_j)S$. Then it is not difficult to see that

$$S = a_i S \oplus (1 - f_i)S.$$

There is an idempotent e_i such that $a_i S = e_i S$ and $(1 - f_i)S = (1 - e_i)S$. Thus $Sf_i = Se_i$, and $\bigcup_{j\leqslant i} f_j S = \bigcup_{j\leqslant i} e_j S$. Since

$$e_i \in a_i S \subset \bigcap_{j<i}(1 - e_j)S,$$

$e_j e_i = 0$ for every $j < i$. Moreover, since

$$\bigcup_{j<i} e_j S = \bigcup_{j<i} f_j S \subset (1 - f_i)S = (1 - e_i)S,$$

$e_i e_j = 0$ for every $j < i$. Therefore $(e_j)_{j\leqslant i}$ is orthogonal.

Thus, by transfinite induction, there exists an orthogonal system $(e_i)_{i\in I(m)}$ of idempotents such that $Se_i = Sf_i$ and $\bigcup_{j\leqslant i} e_j S = \bigcup_{j\leqslant i} f_j S$ for every $i \in I(m)$. Then, evidently,

$$\bigcup_{i\in I(m)} e_j S = \bigcup_{i\in I(m)} f_j S,$$

completing the proof.

2. The following is a direct consequence of Lemma 1.1.

2.1. THEOREM. *Let S be a complete regular ring. Then S is left m-continuous if and only if $\bigcup Se_i$ is essential over $\sum Se_i$ for any left semi-orthogonal system $(e_i)_{i\in I(m)}$ of idempotents of S.*

Proof. The "only if" part is evident by definition. We shall prove the "if" part by induction on m. Suppose that S is n-continuous for every $n < m$. Let $(x_i)_{i\in I} \subset S$, $\bar{I} = m$. Without loss of generality, we may assume that $I = I(m)$. By Lemma 1.1, there is a left semi-orthogonal system $(e_i)_{i\in I}$ of idempotents such that $Se_i \subset Sx_i$ for every $i \in I$, and $\bigcup_{i\in I} Se_i = \bigcup_{i\in I} Sx_i$. By assumption $\bigcup Se_i$ is essential over $\sum Se_i$, and hence also over $\sum Sx_i$, proving that S is left m-continuous, as desired.

Suppose the regular ring S is left n-continuous for every $n < m$. If S is not left m-continuous, then Theorem 2.1 assures that for some left semi-orthogonal system $(x_i)_{i\in I(m)}$ of idempotents the join $\bigcup Sx_i$ is not an essential extension over the sum $\sum Sx_i$. Thus there is a non-zero idempotent $x_0 \in \bigcup Sx_i$ such that $Sx_0 \cap (\sum Sx_i) = 0$. Set $Se = \bigcup Sx_i$, $e = e^2$. Let $J = \{0\} \cup I(m)$. Then, since the sum $\sum_{j \in J} Sx_j$ is direct, Lemma 1.2 implies that there is a left semi-orthogonal system $(e_j)_{j \in J}$ such that $Sx_j = Se_j$ for every j. $Se_0 = Sx_0 \subset Se$, and so $e_0 e = e_0$. Moreover, $e_0 e_i = 0$ for every $i \in I(m)$.

Case I. If we can assume that $(e_j)_{j \in J}$ is an orthogonal system of idempotents, then $e_i e_0 = 0$ for every $i \in I(m)$, and hence $ee_0 = 0$, since $Se = \bigcup_{i \in I(m)} Se_i$. Thus, $e_0 = e_0^2 = e_0 ee_0 = 0$, a contradiction, which shows that S is left m-continuous.

Case II. If there is an idempotent e such that $Se = \bigcup_{i \in I(m)} Se_i$ and $eS \subset \bigcup_{i \in I(m)} e_i S$, then $e_0 e \in e_0 \bigcup e_i S = 0$, and so $e_0 = 0$, a contradiction.

2.2. THEOREM. *Let S be a complete regular ring. Then the following conditions are equivalent:*

(i) *S is left m-continuous.*

(ii) *For any subset $(x_i)_{i \in I}$ with $\bar{I} = m$ such that the sum $\sum Sx_i$ is direct, there is an orthogonal system $(e_i)_{i \in I}$ of idempotents with the property that $Sx_i = Se_i$ for every $i \in I$.*

(iii) *For any left semi-orthogonal system $(e_i)_{i \in I(m)}$ of idempotents there is an orthogonal system $(g_i)_{i \in I(m)}$ of idempotents with $Se_i = Sg_i$ for every $i \in I(m)$.*

(iv) *For any left semi-orthogonal system $(e_i)_{i \in I(m)}$ of idempotents there exists an idempotent e such that $Se = \bigcup Se_i$ and $eS \subset \bigcup e_i S$.*

Proof. We prove (iii) $\Rightarrow$ (i) by induction. If S is left n-continuous for every $n < m$, then S is left m-continuous by Case I.

That (iv) $\Rightarrow$ (i) follows from Case II.

(ii) $\Rightarrow$ (iii) is obvious.

(i) $\Rightarrow$ (iv) is shown in Lemma 1.4.

(i) $\Rightarrow$ (ii): Since the sum $\sum Sx_i$ is direct, $Sx_i \cap (\sum_{j \neq i} Sx_j) = 0$. $\bigcup_{j \neq i} Sx_i$ is an essential extension of $\sum_{j \neq i} Sx_i$, by assumption. Hence $Sx_i \cap (\bigcup_{j \neq i} Sx_j) = 0$, and so there is an idempotent e_i such that $Sx_i = Se_i$ and $(\bigcup_{j \neq i} Sx_j)e_i = 0$. Therefore (e_i) is orthogonal, completing the proof.

The following is an immediate consequence of Lemma 1.5 and Theorem 2.2.

2.3. LEMMA. *Let S be a regular ring that is right n-continuous for every $n < m$. Suppose that for any orthogonal system $(e_i)_{i \in I}$ of idempotents with $\bar{I} = m$, there is an idempotent e such that $Se = \bigcup Se_i$ and $eS \subset \bigcup e_i S$. Then S is left m-continuous.*

Proof. Let $(x_i)_{i \in I(m)}$ be a left semi-orthogonal system of idempotents. Then by the right–left symmetry of Lemma 1.5, there is an orthogonal system $(e_i)_{i \in I(m)}$ of idempotents such that $\bigcup Se_i = \bigcup Sx_i$ and $\bigcup e_i S = \bigcup x_i S$.

Thus, by assumption, we can find an idempotent e with $Se = \bigcup Sx_i$ and $eS \subset \bigcup x_i S$. Therefore S is left m-continuous by Theorem 2.2.

2.4. THEOREM (I. Halperin). *A complete regular ring S is m-continuous if and only if for any orthogonal system (e_i) of m idempotents there is an idempotent e such that $Se = \bigcup Se_i$ and $eS = \bigcup e_i S$.*

Proof. (i) Suppose S is m-continuous. By Theorem 2.2 there is an idempotent e such that $Se = \bigcup Se_i$ and $eS \subset \bigcup e_i S$. On the other hand, by the symmetry of Theorem 2.2, we have an idempotent f with $fS = \bigcup e_i S$ and $Sf \subset \bigcup Se_i$. Then it is easy to see that $e = f$, as desired.

(ii) We prove the sufficiency of our condition by induction on m. Suppose S is n-continuous for every $n < m$. Then by Lemma 2.3 and its symmetry, S is m-continuous, completing the proof.

3. Let S be a regular ring, and (e_i) a set of idempotents. We shall consider the homomorphism $v: S \to \prod Se_i$, which assigns to each $x \in S$ the element $[xe_i] \in \prod Se_i$. We call the above v and its restriction *canonical*. Let $fS = \bigcup e_i S$, $f = f^2$. Then $\ker v = S(1-f)$. Let $Se = \bigcup Se_i$, $e = e^2$, and denote the restriction of v on Se by w. $\ker w = Se \cap S(1-f)$. Hence w is a monomorphism if and only if there is an idempotent g with $Se = Sg$ and $fS \supset gS$.

On the other hand, note that $\prod Se_i$ is isomorphic to Hom $(\sum e_i S, S)$ if the sum $\sum e_i S$ is direct. Thus in this case v is an epimorphism if and only if every homomorphism $\sum e_i S \to S$ can be extended to an endomorphism of the right S-module S.

3.1. THEOREM. *Let S be a complete regular ring. Then the following conditions are equivalent:*

(i) *S is left m-continuous.*

(ii) *For any left semi-orthogonal system $(e_i)_{i \in I(m)}$ of idempotents, the canonical mapping $\bigcup Se_i \to \prod Se_i$ is a monomorphism.*

Proof. By virtue of the argument above, (i) $\Rightarrow$ (ii) follows from Lemma 1.4. (ii) $\Rightarrow$ (i) follows similarly from Theorem 2.2.

A unitary ring S is called *left self-m-injective* if any left S-homomorphism defined on a left ideal generated by m elements with the values in S can be extended to an endomorphism of the left S-module S. A *right self-m-injective* ring is defined in the obvious way. Thus a left (right) self-injective ring is a ring that is left (right) self-m-injective for every cardinal m.

3.2. THEOREM. *Let S be a complete regular ring. Then S is left self-m-injective if and only if for any left semi-orthogonal system $(e_i)_{i \in I(m)}$ of idempotents, any homomorphism $\sum Se_i \to S$ can be extended to an endomorphism of the left S-module S.*

In this case S is left m-continuous.

Proof. First we shall show that the condition of the theorem assures the left m-continuity of S. Let $(e_i)_{i \in I(m)}$ be a left semi-orthogonal system of idempotents. Set $Se = \cup Se_i$, and let $Sf \subset Se$, $Sf \cap \sum Se_i = 0$. Then the projection $Sf \oplus \sum Se_i \to Sf$ is given by the multiplication of a by Lemma 1.2. Let $aS = gS$, $g = g^2$. Then $\sum Se_i \subset S(1 - g)$, and so $Se \subset S(1 - g)$. Since $Sf \cap S(1 - g) = 0$, we have $Se \cap Sf = 0$, which implies that $Sf = 0$. Thus $\cup Se_i$ is essential over $\sum Se_i$. Therefore S is left m-continuous by Theorem 2.1.

The "only if" part of the theorem is obvious.

To prove the "if" part, let $(x_i)_{i \in I}$, $\bar{I} = m$, be a subset of S. Since we have already seen that S is left m-continuous, there is, by Lemma 1.1, a left semi-orthogonal system $(e_i)_{i \in I}$ of idempotents such that $\sum Se_i$ is essential in $\sum Sx_i$. Let p be a homomorphism $\sum Sx_i \to S$. Then the restriction of p to $\sum Se_i$ can be extended to an endomorphism of the left S-module S. The endomorphism is an extension of p, as desired.

3.3. THEOREM. *Let S be a complete regular ring. Then the following conditions are equivalent:*

(i) *S is right self-m-injective.*

(ii) *For any system (e_i) of m idempotents such that the sum $\sum e_i S$ is direct, the canonical mapping $\cup Se_i \to \prod Se_i$ is an epimorphism.*

(iii) *For any right semi-orthogonal system $(f_i)_{i \in I(m)}$ of idempotents, the canonical homomorphism $\cup Sf_i \to \prod Sf_i$ is an epimorphism.*

Proof. (i) $\Rightarrow$ (ii). By the remark at the beginning of this section the canonical mapping v: $S \to \prod Se_i$ is an epimorphism. By Theorem 3.2, S is right m-continuous. Hence by the symmetry of Lemma 1.4, there is an idempotent e such that $Se \subset \cup Se_i$ and $eS = \cup e_i S$. The kernel of the above canonical mapping v is $S(1 - e)$. Thus

$$\prod Se_i = v(Se) = v(\cup Se_i).$$

This proves (ii).

(ii) $\Rightarrow$ (iii) is obvious. Suppose (iii). Then any homomorphism defined on $\sum f_i S$ with values in S can be extended to an endomorphism of the right S-module S again by the argument at the beginning of this section. By the symmetry of Theorem 3.2, this implies that S is right self-m-injective, completing the proof.

3.4. THEOREM. *Let S be a complete regular ring. Then it is left m-continuous and right self-m-injective if and only if the canonical mapping w: $\cup Se_i \to \prod Se_i$ is an isomorphism for any orthogonal system (e_i) of m idempotents.*

Proof. The necessity follows from Theorems 3.1 and 3.3. It remains to prove the sufficiency. By assumption, we may conclude that (i) there exists an idempotent e such that $Se = \cup Se_i$ and $eS \subset \cup e_i S$, and (ii) any homomor-

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phism $\sum e_i S \to S$ can be extended to an endomorphism of the right S-module S. By Lemma 1.3, there is an idempotent f such that $fS = \bigcup e_i S$ and $Sf \supset \bigcup Se_i$. Since the kernel of the canonical mapping $S \to \prod Se_i$ is $S(1-f)$, the canonical mapping $Sf \to \prod Se_i$ is a monomorphism. However, the restriction of this mapping to $\bigcup Se_i$ is an isomorphism by assumption. Thus, $\bigcup Se_i = Sf$. Suppose that S is n-continuous for every $n < m$. Then Theorem 2.4 implies that S is m-continuous. Let $(g_i)_{i \in I}$ be a set of idempotents with $\bar{I} = m$ such that the sum $\sum g_i S$ is direct. Then, since S is right m-continuous, it follows by the symmetry of Theorem 2.2 that there is an orthogonal system $(e_i)_{i \in I}$ of idempotents such that $g_i S = e_i S$ for every $i \in I$. Thus every homomorphism defined on $\sum g_i S (= \sum e_i S)$ with values in S can be extended to an endomorphism of the right S-module S. This implies by Theorem 3.2 that S is right self-m-injective, completing the proof.

3.5. COROLLARY. *Let S be a left m-continuous regular ring. Then*

$$\text{(i)} \Rightarrow \text{(ii)} \Rightarrow \text{(iii)}$$

where

(i) *S is right self-m-injective.*

(ii) *For any orthogonal system (e_i) of m idempotents, $\prod Se_i$ is essential over $\sum Se_i$.*

(iii) *S is right m-continuous.*

Proof. Suppose (i). Then by Theorem 3.4 the canonical mapping

$$\bigcup Se_i \to \prod Se_i$$

is an isomorphism for any orthogonal system (e_i) of m idempotents. In this mapping, each element of $\sum Se_i$ corresponds to itself. Since S is left m-continuous, $\bigcup Se_i$ is essential over $\sum Se_i$, and hence $\prod Se_i$ also is essential over $\sum Se_i$, which proves (ii).

Suppose (ii). By assumption, there is an idempotent f such that $fS = \bigcup e_i S$ and $Sf \supset \bigcup Se_i$. Then the canonical mapping $Sf \to \prod Se_i$ is a monomorphism. Hence Sf is essential over $\sum Se_i$. This means that $Sf = \bigcup Se_i$, and that S is m-continuous by Theorem 2.4, as desired.

By **(4**, Theorem 4.2**)** if a regular ring is left continuous, and right self-injective, then it is left self-injective. Thus, the following is an immediate consequence of the above corollary.

3.6. THEOREM. *Let S be a left self-injective regular ring. Then it is right self-injective if and only if, for any orthogonal system (e_i) of idempotents, the direct product $\prod Se_i$ is an essential extension of the sum $\sum Se_i$.*

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