The condition that the family of surfaces $\phi(x, y, z, \lambda)=0$ may be one of an orthogonal triad.

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Cayley has shown that if the family of surfaces $\phi(x, y, z) = \lambda$ is one of an orthogonal triad, ϕ satisfies a differential equation of the third order. If, however, the parameter λ is involved implicitly in the equation of the family, the condition requires modification: for example, although

$$\phi \equiv \frac{x^2}{a+\lambda} + \frac{y^2}{b+\lambda} + \frac{z^2}{c+\lambda} - 1 = 0$$

is one of an orthogonal triad, ϕ does not satisfy Cayley's equation.

In a previous paper* I have discussed a method of obtaining Cayley's equation. The first modification that is necessary occurs on p. 49. The condition that C may lie on the consecutive surface is now

$$\phi(x + \mathrm{L}\rho,\ y + \mathrm{M}\rho,\ z + \mathrm{N}\rho,\ \lambda + d\lambda) = 0$$

or

$$\mathrm{R}^2\rho + \phi' d\lambda = 0,$$

ϕ' denoting $\dfrac{\partial\phi}{\partial\lambda}$. Thus $\mathrm{R}^2\rho/\phi'$ is constant as we proceed along the surface λ, and so

$$\frac{d\rho}{\rho} + \frac{d(\mathrm{R}^2)}{\mathrm{R}^2} - \frac{d\phi'}{\phi'} = 0,$$

where

$$d\phi' = \left(dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y} + dz\frac{\partial}{\partial z}\right)\frac{\partial\phi}{\partial\lambda} = dx\frac{\partial \mathrm{L}}{\partial\lambda} + dy\frac{\partial \mathrm{M}}{\partial\lambda} + dz\frac{\partial \mathrm{N}}{\partial\lambda}$$
$$= dx\mathrm{L}' + dy\mathrm{M}' + dz\mathrm{N}'.$$

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Also the value of a at the point C is

$$a+\rho\delta a+\frac{\partial a}{\partial\lambda}d\lambda$$

or

$$a+\rho\left(\delta a-\frac{R^2a'}{\phi'}\right).$$

In the conjugate relation on the consecutive surface (p. 50) we must therefore replace δa by $\delta a-\frac{R^2a'}{\phi'}$ and so on, and add new terms arising from the parts of $d_1\rho$ and $d_2\rho$ which involve ϕ'. The additional terms are

$$\frac{d_1\phi'}{\phi'}(d_2x\delta L+d_2y\delta M+d_2z\delta N)+\frac{d_2\phi'}{\phi'}(d_1x\delta L+d_1y\delta M+d_1z\delta N)$$

i.e. $$\frac{1}{\phi'}\{(d_1xL'+d_1yM'+d_1zN')(d_2x\delta L+d_2y\delta M+d_2z\delta N)$$
$$+(d_2xL'+d_2yM'+d_2zN')(d_1x\delta L+d_1y\delta M+d_1z\delta N)\}$$

i.e. $$d_1xd_2x\frac{2L'\delta L}{\phi'}+\ldots+\ldots+(d_1yd_2z+d_2yd_1z)\frac{M'\delta N+N'\delta M}{\phi'}+\ldots+\ldots.$$

The other equations involving d_1xd_2x, etc., are unchanged, so that the first row of the modified determinant is

$$\delta a-2A-\frac{R^2a'}{\phi'}+\frac{2L'\delta L}{\phi'},\ \ldots,\ \ldots,\ 2\left(\delta f-2F-\frac{R^2f'}{\phi'}+\frac{M'\delta N+N'\delta M}{\phi'}\right),\ldots,\ldots,$$

and the other rows are the same as before.

Hence the condition that $\phi(x,y,z,\lambda)=0$ may be a family belonging to an orthogonal triad is

$$\mathfrak{A}\left(\delta a-2A-\frac{R^2a'}{\phi'}+\frac{2L'\delta L}{\phi'}\right)+\ldots+\ldots$$
$$+2\mathfrak{F}\left(\delta f-2F-\frac{R^2f'}{\phi'}+\frac{M'\delta N+N'\delta M}{\phi'}\right)+\ldots+\ldots=0.$$