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GENERAL ITERATIVE METHODS FOR NONLINEAR BOUNDARY VALUE PROBLEMS

RADHA SHRIDHARAN[1] and RAVI P. AGARWAL[2]

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Abstract

In this paper we shall develop existence-uniqueness as well as constructive theory for the solutions of systems of nonlinear boundary value problems when only approximations of the fundamental matrix of the associated homogeneous linear differential systems are known. To make the analysis widely applicable, all the results are proved component-wise. An illustration which dwells upon the sharpness as well as the importance of the obtained results is also presented.

1. Introduction

In this paper we shall consider the boundary value problem

$$x' = f(t, x), \qquad t \in J = [a, b], \tag{1.1}$$

$$g[x] = 0, \tag{1.2}$$

where x and f are n-dimensional vectors, g is an operator from $C(J)$ into R^n and $C(J)$ is the space of n vector functions which are continuous on J. The motivation to study (1.1), (1.2) comes from the fact that it includes various practical problems, including those arising in optimal control theory [21]. In existence-uniqueness as well as constructive theory of (1.1), (1.2) the explicit form of the fundamental matrix of the associated homogeneous linear differential systems plays an important role [1–3,5,7,9,10,13–17,19–25]. However, in [11] it has been noted that in practice the explicit form of this matrix is rarely known. Therefore, to have a wider applicability of the methods it is necessary to restudy (1.1), (1.2) when only approximations of the fundamental matrix are known.

[1]BLK 112, Tampines Street 11, # 08-195, Singapore 1852.
[2]Department of Mathematics, National University of Singapore, Singapore 0511 for correspondence.

The plan of this paper is as follows. In Section 2 we list some properties of square matrices which are used throughout this paper without further mention; state a contraction mapping theorem in generalized Banach spaces, and for the invertability of a linear operator which maps a generalized Banach space to another generalized Banach space provide necessary and sufficient conditions. In Section 3, we shall follow Hayashi [11] to obtain explicit representations of the solutions of the linear boundary value problems in terms of the exact as well as approximate fundamental matrices of the associated homogeneous differential systems. These explicit representations are used in Section 4 to prove the convergence of Picard's iterative methods for the boundary value problem (1.1), (1.2). The obtained results here are more general and precise than those available in [11]. Section 5 is devoted to the computational aspects of Picard's iterative schemes developed in Section 4. An example which dwells upon the importance as well as the sharpness of the obtained results is included in Section 6. All the results in this paper are proved in generalized (vector) normed spaces. The significance of such a study for systems is now well recognized from the fact that it enlarges the domain of existence and uniqueness of solutions, weakens the convergence conditions and provides sharper error estimates; for example, see [1–8,12,18,20,25].

2. Preliminaries

Throughout this paper we shall consider the inequalities between two vectors in R^n component-wise, but between two $n \times n$ matrices, elementwise. The following well-known properties of matrices will be used frequently.

(1) For any $n \times n$ matrix A, $\lim_{m\to\infty} A^m = 0$ if and only if $\rho(A) < 1$, where $\rho(A)$ denotes the spectral radius of A.

(2) For any $n \times n$ matrix A, $(I - A)^{-1}$ exists and $(I - A)^{-1} = \sum_{m=0}^{\infty} A^m$ if $\rho(A) < 1$, where I denotes the unit matrix. Also, if $A \geq 0$, then $(I - A)^{-1}$ exists and is nonnegative if and only if $\rho(A) < 1$.

(3) If $0 \leq B \leq A$ and $\rho(A) < 1$ then $\rho(B) < 1$.

(4) (Toeplitz Lemma). For a given $n \times n$ matrix $A \geq 0$ with $\rho(A) < 1$ and a sequence of vectors $\{d_m\}$, we define the sequence $\{s_m\}$, where $s_m = \sum_{i=0}^{m} A^{m-i} d_i$; $m = 0, 1, 2, \ldots$. Then $\lim_{m\to\infty} s_m = 0$ if and only if $d_m \to 0$.

THEOREM 2.1 (Contraction Mapping Theorem [1]). *Let B be a 'generalized (vector) Banach Space' and let $r \in R_+^n$, $r > 0 : \bar{S}(x_0, r) = \{x \in B : \|x - x_0\| \leq r\}$. Let T map $\bar{S}(x_0, r)$ into B and*

(i) *for all $x, y \in \bar{S}(x_0, r)$, $\|Tx - Ty\| \leq K\|x - y\|$, where $K \geq 0$ is an $n \times n$ matrix,*

(ii) $\rho(K) < 1$, *and* $r_0 = (I - K)^{-1}\|Tx_0 - x_0\| \leq r$.

Then the following hold:

(1) *T has a fixed point x^* in $\bar{S}(x_0, r_0)$,*
(2) *x^* is the unique fixed point of T in $\bar{S}(x_0, r)$,*
(3) *the sequence $\{x_m\}$ defined by $x_{m+1} = Tx_m$, $m = 0, 1, 2, \ldots$, converges to x^* with $\|x^* - x_m\| \le K^m r_0$,*
(4) *for any $x \in \bar{S}(x_0, r_0)$, $x^* = \lim_{m\to\infty} T^m x$,*
(5) *any sequence $\{\bar{x}_m\}$ such that $\bar{x}_m \in \bar{S}(x_m, K^m r_0)$, $m = 0, 1, 2, \ldots$, converges to x^*.*

For $x(t) = (x_1(t), \ldots, x_n(t)) \in C(J)$ we shall denote by $|x(t)| = (|x_1(t)|, \ldots, |x_n(t)|)$ and $\|x\| = (\sup_{t\in J}|x_1(t)|, \ldots, \sup_{t\in J}|x_n(t)|)$. The space $C(J)$ equipped with this norm is a generalized normed space. If $x \in R^n$, then obviously $x \in C(J)$, and hence $|x| = \|x\| = (|x_1|, \ldots, |x_n|)$. Let $M(J)$ denote the Banach space of all real $n \times n$ matrix valued functions $A(t)$ which are continuous on J with the norm analogous to the n vector functions.

For any fixed $t_0 \in J$, let $C_0(J) = \{x \in C(J) : x(t_0) = 0\}$. Then, $B_0 = C_0(J) \times R^n$ is a Banach space with the norm $\|y\| = \max(\|u\|, \|e\|)$ for $y = (u, e) \in B_0$. As usual $L(B_1, B_2)$ denotes the set of all bounded linear operators from the Banach space B_1 into the Banach space B_2.

Let $Q : C(J) \to C_0(J)$ and $F : C(J) \to B_0$ be the operators defined by

$$Qx = x(t) - x(t_0) - \int_{t_0}^{t} f\big(s, x(s)\big)\, ds, \tag{2.1}$$

$$Fx = (Qx, g[x]). \tag{2.2}$$

Clearly, $Qx = 0$ if and only if $x \in C(J)$ is a solution of (1.1). Thus, the boundary value problem (1.1), (1.2) is equivalent to finding a solution $x \in C(J)$ of the equation

$$Fx = 0. \tag{2.3}$$

In (1.1), (1.2) the function $f(t,x)$ is assumed to be continuous in $J \times R^n$ and continuously differentiable with respect to x, and $f_x(t,x)$ represents the Jacobian matrix of $f(t,x)$ with respect to x; $g[x]$ is continuously Frèchet differentiable in $C(J)$, and $g_x[x]$ denotes the Frèchet derivative of g at x.

For $h \in C(J)$, we define the linear operator $F_x(x) : C(J) \to B_0$ by

$$F_x(x)h = \big(Q_x(x)h, g_x[x]h\big), \tag{2.4}$$

where $F_x(x)$ denotes the Frèchet derivative of F at x and

$$Q_x(x)h = h(t) - h(t_0) - \int_{t_0}^{t} f_x\big(s, x(s)\big)h(s)\, ds. \tag{2.5}$$

Let $L \in L(C(J), B_0)$ be the operator independent of x which approximates $F_x(x)$ and is defined by

$$Lh = (Ph, \ell[h]), \qquad h \in C(J) \tag{2.6}$$

where the linear operator $P : C(J) \to C_0(J)$ in relation to $Q_x(x)$ is

$$Ph = h(t) - h(t_0) - \int_{t_0}^{t} A(s)h(s)ds, \;\; A \in M[J], \;\text{ and } \ell \in L\big(C(J), R^n\big). \tag{2.7}$$

Let B_1 and B_2 be two Banach spaces. A linear operator $T : B_2 \to B_1$ is said to be invertible if the equation $Tb_2 = b_1$ has a unique solution $b_2 \in B_2$ for each $b_1 \in B_1$.

LEMMA 2.2. *Let $L : B_1 \to B_2$ be a linear operator and $T : B_2 \to B_1$ be an invertible linear operator. Then, L is invertible if and only if there exists a nonnegative $n \times n$ matrix M, with $\rho(M) < 1$, such that*

$$\|I - TL\| \le M. \tag{2.8}$$

If L^{-1} exists, then

$$L^{-1} = \sum_{n=0}^{\infty}(I - TL)^n T \tag{2.9}$$

and

$$\|L^{-1}\| \le (I - M)^{-1}\|T\|. \tag{2.10}$$

PROOF. Assume that T, T^{-1} exist and (2.8) is satisfied. Since

$$\begin{aligned}\left\|\sum_{n=0}^{\infty}(I - TL)^n T\right\| &\le \|T\| + \|I - TL\|\|T\| + \|I - TL\|^2\|T\| + \cdots \\ &\le (I + M + M^2 + \cdots)\|T\| = (I - M)^{-1}\|T\|, \end{aligned} \tag{2.11}$$

the infinite series in (2.9) defines a bounded linear operator in B_2. Hence, for each $b_2 \in B_2$,

$$b^* = \sum_{n=0}^{\infty}(I - TL)^n Tb_2 \tag{2.12}$$

is a uniquely defined element of B_1.

From (2.12), we have

$$(I - TL)b^* = \sum_{n=1}^{\infty}(I - TL)^n Tb_2 = b^* - Tb_2,$$

and thus

$$TLb^* = Tb_2.$$

Since T^{-1} exists, $Lb^* = b_2$, so that $Lb_1 = b_2$ has at least one solution $b_1 = b^*$ for each $b_2 \in B_2$. To show that this solution is unique, we assume that $Lb_1 = b_2$ and $b_1 \neq b^*$. Then, $L(b_1 - b^*) = 0$ and hence

$$(I - TL)(b_1 - b^*) = b_1 - b^* \neq 0,$$

which implies that $\rho(\|I - TL\|) \geq 1$. But since $\rho(M) < 1$, this contradicts the assumption (2.8). Hence, L^{-1} exists and is given by (2.9). Inequality (2.10) follows immediately from (2.11) and (2.9).

To prove the necessity part, we assume that L^{-1} exists. For $T = L^{-1}$ it is clear that $T^{-1} = L$ exists and

$$\|I - TL\| = \|I - I\| = 0 \leq M,$$

so that (2.8) is satisfied.

COROLLARY 2.3. *In Lemma* 2.2, (2.8)–(2.10) *can be replaced by*

$$\|I - TL\| \leq M, \tag{2.13}$$

$$L^{-1} = \sum_{n=0}^{\infty} T(I - LT)^n \tag{2.14}$$

and

$$\|L^{-1}\| \leq \|T\|(I - M)^{-1}, \tag{2.15}$$

respectively.

3. Linear problems

Let $A \in M[J]$ and $\Phi(t)$ be the fundamental matrix solution of the homogeneous system $y' = A(t)y$ such that $\Phi(t_0) = I$. For $\ell \in L(C(J), R^n)$ we define an $n \times n$ matrix G by

$$G = \ell[\Phi], \tag{3.1}$$

whose column vectors are $\ell\phi_i$; $i = 1, \ldots, n$ and ϕ_i is the i-th column vector of Φ. If G is nonsingular, then we shall denote by

$$S = \Phi G^{-1}. \tag{3.2}$$

For $h \in C(J)$, let E be the element of $L(C(J), C(J))$ defined by

$$Eh = \int_{t_0}^{t} \Phi(t)\Phi^{-1}(s)h(s)\, ds. \tag{3.3}$$

Let $K : C(J) \to C(J)$ be the operator defined by

$$Kx = f\big(t, x(t)\big) - A(t)x(t), \qquad x \in C(J). \tag{3.4}$$

Now for $h \in C(J)$, $y = (u, e) \in B_0$, we consider the system

$$Ph = u(t), \tag{3.5}$$

together with

$$\ell[h] = e. \tag{3.6}$$

LEMMA 3.1. *If the matrix G is nonsingular, then* (3.5), (3.6) *has a unique solution $h(t)$, that is, for the operator L defined in* (2.6), *L^{-1} exists, and can be represented as*

$$L^{-1}y = h(t) = S_1 E_1 u + Se, \tag{3.7}$$

where

$$E_1 = I + EA, \qquad S_1 = I - S\ell, \qquad H = S_1 E. \tag{3.8}$$

PROOF. Any solution of (3.5) can be expressed as

$$h(t) = \Phi(t)c + u(t) + \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)A(s)u(s)\,ds, \tag{3.9}$$

where c is an arbitrary constant vector. The solution (3.9) satisfies (3.6) if and only if

$$Gc + \ell(I + EA)u = e. \tag{3.10}$$

Since $\det G \neq 0$, from (3.10) we get

$$c = G^{-1}e - G^{-1}\ell(I + EA)u.$$

Substituting this in (3.9) and following the definitions of S, E_1 and S_1, the result (3.7) follows.

COROLLARY 3.2. *Assume that the matrix G is nonsingular, and for $\phi \in C(J)$, let*

$$T\phi = \phi - S_1 E_1[Q\phi] - S[g[\phi]]. \tag{3.11}$$

Then

$$T\phi = \big(HK + S(\ell - g)\big)\phi. \tag{3.12}$$

PROOF. Since an integration by parts of $S_1 E_1 = [I + HA - S\ell]$ gives that $S_1 E_1 Q\phi = [I - HK - S\ell]\phi$, it follows that

$$T\phi = \phi - [I - HK - S\ell]\phi - S[g[\phi]] = [HK + S(\ell - g)]\phi.$$

In the existence and uniqueness theory of solutions of (1.1), (1.2) matrices $\Phi(t)$ and $\Phi^{-1}(t)$ play a vital role. However, in practical evaluation it often becomes necessary to approximate these matrices by the computed fundamental matrices. Let $\tilde{\Phi}(t)$ and $\hat{\Phi}(t)$ be the matrices that approximate $\Phi(t)$ and $\Phi^{-1}(t)$, respectively. Hereafter, for an operator $Z = Z(\Phi, \Phi^{-1})$ depending on Φ and Φ^{-1}, we shall denote by $\tilde{Z}$ the operator $Z(\tilde{\Phi}, \hat{\Phi})$.

We shall consider the following two cases:

Case 1. $\tilde{\Phi}(t)$ and $\hat{\Phi}(t)$ are continuous on J.
Case 2. $\tilde{\Phi}(t)$ and $\hat{\Phi}(t)$ are continuously differentiable on J.

Case 1. Let

$$\Gamma(t) = \tilde{\Phi}(t)\hat{\Phi}(t), \tag{3.13}$$

$$\Gamma_1(t) = I - \Gamma(t), \tag{3.14}$$

$$\sigma = \max(b - t_0, t_0 - a), \tag{3.15}$$

$$\Psi(t) = \tilde{\Phi}(t) - I - \int_{t_0}^{t} A(s)\tilde{\Phi}(s)\, ds, \tag{3.16}$$

$$\Psi_1(t) = \hat{\Phi}(t) - I + \int_{t_0}^{t} \hat{\Phi}(s)A(s)\, ds, \tag{3.17}$$

$$\Psi_2(t) = \Gamma_1(t) + \tilde{\Phi}(t)\Psi_1(t). \tag{3.18}$$

Let $R, R_1 \in L(C(J), C(J))$ and $R_2 : C(J) \to C(J)$ be the operators defined by

$$Rh = \Psi_2(t) \int_{t_0}^{t} h(s)\, ds - \tilde{\Phi}(t) \int_{t_0}^{t} \Psi_1(s)h(s)\, ds, \qquad h \in C(J) \tag{3.19}$$

$$R_1 h = \Psi_2 h(t_0) + RAh, \qquad h \in C(J) \tag{3.20}$$

$$R_2 x = \Psi_2 x(t_0) + Rf\big(t, x(t)\big), \qquad x \in C(J). \tag{3.21}$$

LEMMA 3.3. *Assume that*

$$\tilde{G} \text{ is nonsingular}, \tag{3.22}$$

$$\|\tilde{G}^{-1}\|\,\|\ell\| \exp\left(\sigma \|A\|\right)\|\Psi\| \le M, \tag{3.23}$$

and also

$$\|\tilde{S}_1 R_1\| \le \bar{M}, \tag{3.24}$$

where M and $\bar{M}$ are nonnegative $n \times n$ matrices with $\rho(M) < 1$ and $\rho(\bar{M}) < 1$. Then $\tilde{L}^{-1}$ exists and is invertible.

PROOF. Clearly (3.22) implies that $\tilde{L}^{-1}$ is defined. Since

$$\Phi(t) - I - \int_{t_0}^{t} A(s)\Phi(s)\,ds = 0$$

from (3.16), we have

$$\tilde{\Phi}(t) - \Phi(t) = \Psi(t) + \int_{t_0}^{t} A(s)[\tilde{\Phi}(s) - \Phi(s)]\,ds$$

and hence

$$\|\tilde{\Phi}(t) - \Phi(t)\| \leq \|\Psi\| + \left|\int_{t_0}^{t} \|A\|\,\|\tilde{\Phi}(s) - \Phi(s)\|\,ds\right|.$$

Thus, by Gronwall's inequality we find that

$$\|\tilde{\Phi} - \Phi\| \leq \exp\left(\sigma\|A\|\right)\|\Psi\|, \tag{3.25}$$

and since

$$\|I - \tilde{G}^{-1}G\| \leq \|\tilde{G}^{-1}\|\,\|\tilde{G} - G\| \leq \|\tilde{G}^{-1}\|\,\|\ell\|\,\|\tilde{\Phi} - \Phi\|,$$

in view of (3.25), (3.23) and Lemma 2.2, it follows that G is nonsingular, and in conclusion L is invertible.

We will now show that

$$\|I - \tilde{L}^{-1}L\| \leq \bar{M}. \tag{3.26}$$

Let

$$\Lambda(t) = \hat{\Phi}(t) - \Phi^{-1}(t), \qquad \text{and} \qquad \Omega(t) = \int_{t_0}^{t} \Lambda(s)A(s)\,ds. \tag{3.27}$$

Since

$$\Phi^{-1}(t) - I + \int_{t_0}^{t} \Phi^{-1}(s)A(s)\,ds = 0,$$

by (3.17) we have

$$\Lambda(t) + \Omega(t) = \Psi_1(t). \tag{3.28}$$

Let $u(t) = \int_{t_0}^{t} p(s)\,ds$, $p \in C(J)$. In view of $(\Phi^{-1})' = -\Phi^{-1}A$, an integration by parts gives that

$$\int_{t_0}^{t} \Phi^{-1}(s)A(s)u(s)\,ds = -\Phi^{-1}(t)u(t) + \int_{t_0}^{t} \Phi^{-1}(s)p(s)\,ds \tag{3.29}$$

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and

$$\int_{t_0}^{t} \Lambda(s)A(s)u(s)\,ds = \Omega(t)u(t) - \int_{t_0}^{t} \Omega(s)p(s)\,ds. \tag{3.30}$$

Thus, by (3.27)–(3.30) we have

$$\begin{aligned}\tilde{E}Au &= \tilde{\Phi}(t)\left[\int_{t_0}^{t} \Phi^{-1}(s)A(s)u(s)\,ds + \int_{t_0}^{t} \Lambda(s)A(s)u(s)\,ds\right]\\ &= -u + \tilde{E}p + Rp.\end{aligned} \tag{3.31}$$

From this and (3.20), we have

$$\tilde{E}_1 Ph = h - \tilde{\Phi}h(t_0) - R_1 h, \qquad h \in C(J). \tag{3.32}$$

Finally, since by (3.1) and (3.2)

$$\tilde{S}_1\tilde{\Phi} = (I - \tilde{\Phi}\tilde{G}^{-1}\ell)\tilde{\Phi} = 0 \tag{3.33}$$

from (3.7), (3.32) and (3.33) for $h \in C(J)$, we find that

$$(I - \tilde{L}^{-1}L)h = h - \tilde{S}_1\tilde{E}_1 Ph - \tilde{S}\ell[h] = \tilde{S}_1 R_1 h.$$

Hence, in view of (3.24) we find that (3.26) holds. Corollary 2.3 now implies that $\tilde{L}^{-1}$ is invertible.

COROLLARY 3.4. *Assume that* (3.22) *holds and let*

$$\tilde{T}\phi = \phi - \tilde{S}_1\tilde{E}_1[Q\phi] - \tilde{S}[g[\phi]], \qquad \phi \in C(J). \tag{3.34}$$

Then

$$\tilde{T}\phi = [\tilde{H}K + \tilde{S}(\ell - g) + \tilde{S}_1 R_2]\phi. \tag{3.35}$$

PROOF. By (3.31), we have

$$\tilde{E}_1 Q\phi = \phi - \tilde{E}K\phi - \tilde{\Phi}\phi(t_0) - R_2\phi \tag{3.36}$$

and hence (3.33) gives that

$$\tilde{S}_1\tilde{E}_1 Q\phi = \phi - \tilde{S}\ell[\phi] - \tilde{H}K\phi - \tilde{S}_1 R_2\phi. \tag{3.37}$$

Substituting (3.37) into (3.34), we obtain (3.35).

Case 2. Let

$$A_1(t) = \tilde{\Phi}'(t)\tilde{\Phi}^{-1}(t), \tag{3.38}$$

$$A_2(t) = -\hat{\Phi}^{-1}(t)\hat{\Phi}'(t); \tag{3.39}$$

$P_2, R_3, R_4 \in L(C(J), C(J))$ and $R_5 : C(J) \to C(J)$ be the operators defined by

$$P_2 h = h(t) - h(t_0) - \int_{t_0}^{t} A_1(s)h(s)\,ds, \qquad h \in C(J), \tag{3.40}$$

$$R_3 h = \tilde{E}(A - A_2)h + \Gamma_1 h, \qquad h \in C(J), \tag{3.41}$$

$$R_4 h = R_3(h - P_2 h) - \tilde{E}(A - A_1)h, \qquad h \in C(J), \tag{3.42}$$

$$R_5 x = R_3(x - Qx), \qquad x \in C(J). \tag{3.43}$$

LEMMA 3.5. *Assume that* (3.22) *holds. Then* $\tilde{L}^{-1}$ *is invertible if*

$$\sigma\left\|A_1 - \Gamma A\right\| \le N \tag{3.44}$$

or

$$\left\|\tilde{S}_1 R_4\right\| \le \bar{N}, \tag{3.45}$$

where N *and* $\bar{N}$ *are* $n \times n$ *nonnegative matrices with* $\rho(N) < 1$ *and* $p(\bar{N}) < 1$.

PROOF. Let L_1 be the operator defined by

$$L_1 h = (P_2 h, \ell[h]), \qquad h \in C(J). \tag{3.46}$$

Clearly, in view of (3.22), this operator L_1 is invertible.

For any $y = (u, e) \in B_0$ by (3.7), we find that

$$\tilde{L}^{-1} y = \tilde{E}_1 u - \tilde{S}\left[\ell[\tilde{E}_1 u] - e\right]. \tag{3.47}$$

Since $P_2\tilde{\Phi} = 0$ and $\tilde{G} = \ell[\tilde{\Phi}]$, we have

$$P_2\tilde{S} = (P_2\tilde{\Phi})\tilde{G}^{-1} = 0 \tag{3.48}$$

and

$$\ell[\tilde{S}] = \ell[\tilde{\Phi}])\tilde{G}^{-1} = I. \tag{3.49}$$

Now suppose that (3.44) holds. By (3.47) and (3.49), we get

$$\ell[\tilde{L}^{-1} y] = e \tag{3.50}$$

and by (3.47) and (3.48), $P_2\tilde{L}^{-1}y = P_2\tilde{E}_1 u$. Since $\tilde{\Phi}' = A_1\tilde{\Phi}$ and $u \in C_0(J)$, integration by parts gives that

$$P_2\tilde{L}^{-1}y = P_2\tilde{E}_1 u = u(t) - \int_{t_0}^{t} [A_1(s) - \Gamma(s)A(s)]u(s)\,ds.$$

Using this relation and (3.50), we find that

$$(I - L_1\tilde{L}^{-1})y = \left(\int_{t_0}^{t} [A_1(s) - \Gamma(s)A(s)]u(s)\,ds, 0\right).$$

Thus, it follows that $\|(I - L_1\tilde{L}^{-1})y\| \le \sigma\|A_1 - \Gamma A\|\,\|y\|$ and hence (3.44) and Lemma 2.2 implies that $\tilde{L}^{-1}$ is invertible.

Next suppose that (3.45) holds. For $q \in C(J)$, let $u(t) = \int_{t_0}^{t} q(s)\,ds$. Since $\hat{\Phi}' = -\hat{\Phi}A_2$, integration by parts gives that

$$\tilde{E}Au = \tilde{E}(A - A_2)u - \Gamma u + \tilde{E}q = -u + \tilde{E}q + R_3 u. \tag{3.51}$$

By this, for $h \in C(J)$ we get

$$\tilde{E}_1 P_2 h = h - \tilde{\Phi}\hat{\Phi}(t_0)h(t_0) - R_4 h. \tag{3.52}$$

Now substituting $u = P_2 h$ and $e = \ell[h]$ into (3.47) and using the resulting relation together with (3.52) and (3.33), we obtain $(I - \tilde{L}^{-1}L_1)h = \tilde{S}_1 R_4 h$. Thus, (3.45) and Corollary 2.3 imply that $\tilde{L}^{-1}$ is invertible.

COROLLARY 3.6. *Assume that* (3.22) *holds and let* $\tilde{T}\phi$ *be as in* (3.34). *Then*

$$\tilde{T}\phi = \left[\tilde{H}K + \tilde{S}(\ell - g) + \tilde{S}_1 R_5\right]\phi. \tag{3.53}$$

PROOF. By (3.51), we have

$$\tilde{E}_1 Q\phi = \phi - \tilde{E}K\phi - \tilde{\Phi}\hat{\Phi}(t_0)\phi(t_0) - R_5\phi \tag{3.54}$$

and hence (3.33) gives that

$$\tilde{S}_1\tilde{E}_1 Q\phi = \phi - \tilde{S}\ell[\phi] - \tilde{H}K\phi - \tilde{S}_1 R_5\phi. \tag{3.55}$$

Substituting (3.55) into (3.34), we get (3.53).

4. Convergence of Picard's Method

Using no approximation of Green's function in our earlier paper [20] we proved the following result, its proof being based on Theorem 2.1.

THEOREM 4.1. *With respect to the boundary value problem* (2.3) *assume that there is an approximate solution* $\bar{x}(t) \in C(J)$, *and*

(i) *there exists an* $n \times n$ *continuous matrix* $A(t)$ *and a bounded continuous linear operator* ℓ *such that* $G = \ell[\Phi(t)]$ *is nonsingular, where* $\Phi(t)$ *is the fundamental matrix solution of the homogeneous differential system* $y' = A(t)y$,

(ii) *there exist nonnegative* $n \times n$ *matrices* M_1 *and* M_2 *such that* $\|H\| \le M_1$ *and* $\|S\| \le M_2$,

(iii) *there exist nonnegative* $n \times n$ *matrices* M_3 *and* M_4 *and a positive vector* r *such that for all* $x(t) \in \bar{S}(\bar{x}, r) = \{z \in C(J) : \|z - \bar{x}\| \le r\}$, $\|f_x(t, x(t)) - A(t)\| \le M_3$ *and* $\|g_x[x] - \ell\| \le M_4$,

(iv) *there exists a nonnegative vector* η_o *such that* $\left\|S_1E_1[Q\bar{x}] + S[g[\bar{x}]]\right\| \le \eta_0$,

(v) $K_0 = M_1M_3 + M_2M_4$, $\rho(K_0) < 1$ *and* $r_0 = (I - K_0)^{-1}\eta_0 \le r$.

Then

(1) *there exists a solution* $x^*(t)$ *of* (2.3) *in* $\bar{S}(\bar{x}, r_0)$,

(2) $x^*(t)$ *is the unique solution of* (2.3) *in* $\bar{S}(\bar{x}, r)$,

(3) *the sequence* $\{x_m(t)\}$ *defined by*

$$x_{m+1}(t) = x_m(t) - S_1E_1[Qx_m] - S[g[x_m]],$$
$$x_0(t) = \bar{x}(t); \qquad m = 0, 1, \ldots \tag{4.1}$$

converges to $x^*(t)$ *with*

$$\left\|x^* - x_m\right\| \le K_0^m r_0,$$

(4) *for* $x_0(t) = x(t) \in \bar{S}(\bar{x}, r_0)$ *the iterative process* (4.1) *converges to* $x^*(t)$,

(5) *any sequence* $\{\bar{x}_m(t)\}$ *such that* $\bar{x}_m(t) \in \bar{S}(x_m, K_0^m r_0)$; $m = 0, 1, \ldots$ *converges to* $x^*(t)$.

Now we shall discuss the convergence of Picard's method when the fundamental matrices are replaced by the approximate fundamental matrices. First we shall consider Case 1.

THEOREM 4.2. *With respect to the boundary value problem* (2.3) *we assume that there is an approximate solution* $\bar{x}(t) \in C(J)$ *and*

(i) $\tilde{L}^{-1}$ *is invertible,*

(ii) *there exist nonnegative* $n \times n$ *matrices* M_1 *and* M_2 *such that* $\|\tilde{H}\| \le M_1$ *and* $\|\tilde{S}\| \le M_2$,

(iii) *there exist nonnegative* $n \times n$ *matrices* M_3, M_4 *and* M_5 *and a positive vector* r *such that for all* $x(t) \in \bar{S}(\bar{x}, r) = \{z \in C(J) : \|z - \bar{x}\| \le r\}$, $\|f_x(t, x(t)) - A(t)\| \le M_3$, $\|g_x[x] - \ell\| \le M_4$ *and* $\|f_x(t, x(t))\| \le M_5$,

(iv) *there exist nonnegative* $n \times n$ *matrices* M_6, M_7 *such that* $\|\tilde{S}_1 R\| \le M_6$ *and* $\|\tilde{S}_1 \Psi_2\| \le M_7$,

(v) *there exists a nonnegative vector* η_1 *such that*

$$\left\|\tilde{S}_1 \tilde{E}_1[Q\bar{x}] + \tilde{S}[g[\bar{x}]]\right\| \le \eta_1,$$

(vi) $K_1 = M_1 M_3 + M_2 M_4 + M_6 M_5 + M_7$, $\rho(K_1) < 1$ *and* $r_1 = (I - K_1)^{-1}\eta_1 \le r$.

Then

(1) *there exists a solution* $x^*(t)$ *of* (2.3) *in* $\bar{S}(\bar{x}, r_1)$,

(2) $x^*(t)$ *is the unique solution of* (2.3) *in* $\bar{S}(\bar{x}, r)$,

(3) *the sequence* $\{u_m(t)\}$ *defined by*

$$u_{m+1}(t) = \tilde{H}\big[K u_m\big] + \tilde{S}\big[\ell\big[u_m\big] - g\big[u_m\big]\big] + \tilde{S}_1\big[Rf\big(t, u_m(t)\big) + \Psi_2 u_m(t_0)\big],$$
$$u_0(t) = x_0(t) = \bar{x}(t), \qquad m = 0, 1, \ldots, \tag{4.2}$$

converges to $x^*(t)$ *with* $\left\|x^* - u_m\right\| \le K_1^m r_1$,

(4) *for* $u_0(t) = x(t) \in \bar{S}(\bar{x}, r_1)$ *the iterative process* (4.2) *converges to* $x^*(t)$,

(5) *any sequence* $\{\bar{u}_m(t)\}$ *such that* $\bar{u}_m(t) \in \bar{S}(u_m, K_1^m r_1)$, $m = 0, 1, \ldots$, *converges to* $x^*(t)$.

PROOF. Define an operator $\tilde{T} : \bar{S}(\bar{x}, r) \to C(J)$ by

$$\tilde{T}x(t) = \tilde{H}\big[Kx\big] + \tilde{S}\big[\ell\big[x\big] - g\big[x\big]\big] + \tilde{S}_1\big[Rf\big(t, x(t)\big) + \Psi_2 x(t_0)\big]. \tag{4.3}$$

If $x(t)$ is a solution of (4.3), that is, $\tilde{T}x(t) = x(t)$, then from Corollary 3.4 it is clear that $\tilde{S}_1 \tilde{E}_1[Qx] + \tilde{S}[g[x]] = 0$. But this is the same as $\tilde{L}^{-1}Fx = 0$. Thus, in view of condition (i) it follows that $Fx = 0$, that is, $x(t)$ is a solution of (2.3). Thus, it suffices to show that the operator $\tilde{T}$ satisfies the conditions of Theorem 2.1. For this, if $x(t), y(t) \in \bar{S}(\bar{x}, r)$ then we have

$$\begin{aligned}\tilde{T}x(t) - \tilde{T}y(t) = \tilde{H}&\left[\int_0^1 \left[f_x\Big(t, x(t) + \theta_1\big(y(t) - x(t)\big)\Big) - A(t)\right]\big(x(t) - y(t)\big)\, d\theta_1\right] \\ &+ \tilde{S}\left[\int_0^1 \big[\ell - g_x\big(x + \theta_2(y - x)\big)\big][x - y]\, d\theta_2\right] \\ &+ \tilde{S}_1\left[R\int_0^1 f_x\Big(t, x(t) + \theta_1\big(y(t) - x(t)\big)\Big)\big(x(t) - y(t)\big)\, d\theta_1 \right. \\ &\qquad \left. + \Psi_2\big(x(t_0) - y(t_0)\big)\right].\end{aligned}$$

Therefore, it follows that

$$\|\tilde{T}x - \tilde{T}y\| \le (M_1M_3 + M_2M_4 + M_6M_5 + M_7)\|x - y\| = K_1\|x - y\|.$$

Next from Corollary 3.4 and (4.3), we have

$$\tilde{T}x_0(t) - x_0(t) = \tilde{T}\bar{x}(t) - \bar{x}(t) = -\tilde{S}_1\tilde{E}_1[Q\bar{x}] - \tilde{S}[g[\bar{x}]]$$

and hence

$$\|\tilde{T}x_0 - x_0\| \le \| - \tilde{S}_1\tilde{E}_1[Q\bar{x}] - \tilde{S}[g[\bar{x}]]\| \le \eta_1.$$

Thus we find that $(I - K_1)^{-1}\|\tilde{T}x_0 - x_0\| \le (I - K_1)^{-1}\eta_1 = r_1 \le r$. Hence the conditions of Theorem 2.1 are satisfied and the conclusions (1)–(5) follow.

REMARK 4.1. For computational purposes let us assume the following: P_1 and P_2 are $n \times n$ nonnegative matrices such that

$$\sigma\left(\|\Psi_2\|\|A\| + \|\tilde{\Phi}\|\|\Psi_1 A\|\right) \le P_1,$$

$$\sigma\left(\|\Psi_2\| + \|\tilde{\Phi}\|\|\Psi_1\|\right) \le P_2.$$

Then, for any $h \in C(J)$ by (3.19), $\|RAh\| \le P_1\|h\|$, and $\|Rh\| \le P_2\|h\|$. Hence (3.24) can be replaced by

$$\|\tilde{S}_1\Psi_2\| + \|\tilde{S}_1\|P_1 \le \bar{M}. \tag{4.4}$$

Also, $\|\tilde{S}_1 R\| \le M_6$ in (iv) of Theorem 4.2 can be changed to

$$\|\tilde{S}_1\|P_2 \le M_6. \tag{4.5}$$

Next we shall consider Case 2.

THEOREM 4.3. *With respect to the boundary value problem* (2.3) *we assume that there is an approximate solution* $\bar{x}(t) \in C(J)$, *and*

(i) *the conditions* (i)–(iii) *and* (v) *of Theorem* 4.2 *are satisfied,*
(ii) *there exists an* $n \times n$ *nonnegative matrix* M_8 *such that* $\|\tilde{S}_1 R_3\| \le M_8$,
(iii) $K_2 = M_1M_3 + M_2M_4 + M_8(I + \sigma M_5)$; $\rho(K_2) < 1$ *and* $r_2 = (I - K_2)^{-1}\eta_1 \le r$.

Then

(1) *there exists a solution* $x^*(t)$ *of* (2.3) *in* $\bar{S}(\bar{x}, r_2)$,
(2) $x^*(t)$ *is the unique solution of* (2.3) *in* $\bar{S}(\bar{x}, r)$,

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(3) *the sequence* $\{u_m(t)\}$ *defined by*

$$u_{m+1}(t) = \tilde{H}[Ku_m] + \tilde{S}[\ell[u_m] - g[u_m]] + \tilde{S}_1[R_3(u_m - Qu_m)],$$
$$u_0(t) = x_0(t) = \bar{x}(t), \qquad m = 0, 1, \ldots, \tag{4.6}$$

converges to $x^*(t)$ *with* $\|x^* - u_m\| \le K_2^m r_2$,

(4) *for* $u_0(t) = x(t) \in \bar{S}(\bar{x}, r_2)$ *the iterative process* (4.6) *converges to* $x^*(t)$,

(5) *any sequence* $\{\bar{u}_m(t)\}$ *such that* $\bar{u}_m(t) \in \bar{S}(u_m, K_2^m r_2)$, $m = 0, 1, \ldots$, *converges to* $x^*(t)$.

PROOF. Define an operator $\tilde{T} : \bar{S}(\bar{x}, r) \to C(J)$ by

$$\tilde{T}x(t) = \tilde{H}[Kx] + \tilde{S}[\ell[x] - g[x]] + \tilde{S}_1[R_3(x - Qx)]. \tag{4.7}$$

Now as in Theorem 4.2, we can show that the operator $\tilde{T}$ satisfies the conditions of Theorem 2.1.

REMARK 4.2. Again for computational purposes we assume that $\rho(\|\Gamma_1\|) < 1$ and let $Z = (I - \|\Gamma_1\|)^{-1}$. Then, since from Lemma 2.2, $\|\Gamma^{-1}\| \le Z$ we have the following:

$$\|A_1 - \Gamma A\| \le \|\tilde{\Phi}'\hat{\Phi} - \Gamma A \Gamma\| Z, \tag{4.8}$$

$$\|A - A_1\| \le \|A\Gamma - \tilde{\Phi}'\hat{\Phi}\| Z, \tag{4.9}$$

$$\|A - A_2\| \le Z\|\Gamma A + \tilde{\Phi}\hat{\Phi}'\|, \tag{4.10}$$

$$\|A_1\| \le \|\tilde{\Phi}'\hat{\Phi}\| Z. \tag{4.11}$$

Now let P_3 be an $n \times n$ nonnegative matrix such that

$$\|\tilde{H}\|\|A - A_2\| + \|\tilde{S}_1\|\|\Gamma_1\| \le P_3, \tag{4.12}$$

then

$$\|\tilde{S}_1 R_3 h\| \le P_3\|h\|, \qquad h \in C(J). \tag{4.13}$$

Hence it follows that

$$\|\tilde{S}_1 R_4\| \le P_3\big(I + \sigma\|A_1\|\big) + \|\tilde{H}\|\|A - A_1\| \tag{4.14}$$

and

$$\|\tilde{S}_1 R_3\|(I + \sigma M_5) \le P_3(I + \sigma M_5). \tag{4.15}$$

Thus, by (4.8)–(4.15) we can compute the left sides of (3.44), (3.45) and (ii) of Theorem 4.3 without computing $\tilde{\Phi}^{-1}$ and $\hat{\Phi}^{-1}$.

REMARK 4.3. If $\tilde{\Phi}(t) = \Phi(t)$ and $\hat{\Phi}(t) = \Phi^{-1}(t)$, then both the Theorems 4.2 and 4.3 reduce to Theorem 4.1.

5. Convergence of approximate Picard's Method

In our earlier paper [20], we have provided necessary and sufficient conditions for the convergence of the approximate Picard's sequence $\{y_m(t)\}$ generated by the scheme

$$y_{m+1}(t) = y_m(t) - S_1E_1\Big[y_m(t) - y_m(t_0) - \int_{t_0}^{t} f_m(s, y_m(s))\,ds\Big] - S[g_m[y_m]],$$
$$y_0(t) = x_0(t) = \bar{x}(t), \qquad m = 0, 1, \ldots, \tag{5.1}$$

to the solution $x^*(t)$ of the boundary value problem (2.3). In (5.1) for each m, the function f_m and the operator g_m, respectively, approximate f and g and are assumed to be continuous.

In relation to the function f_m we define $Q_m : C(J) \to C_0(J)$ by

$$Q_m x = x(t) - x(t_0) - \int_{t_0}^{t} f_m(s, x(s))\,ds, \qquad x \in C(J), \qquad m = 0, 1, \ldots.$$

LEMMA 5.1. *Assume that the matrix $\tilde{G}$ is nonsingular and let*

$$\tilde{T}\phi = \phi - \tilde{S}_1\tilde{E}_1[Q_m\phi] - \tilde{S}[g_m[\phi]], \qquad \phi \in C(J).$$

Then for Case 1,

$$\tilde{T}\phi = \tilde{H}\big[f_m\big(t, \phi(t)\big) - A(t)\phi(t)\big] + \tilde{S}\big[\ell[\phi] - g_m[\phi]\big]$$
$$+\tilde{S}_1\big[Rf_m\big(t, \phi(t)\big) + \Psi_2\phi(t_0)\big], \tag{5.2}$$

and for Case 2,

$$\tilde{T}\phi = \tilde{H}\big[f_m\big(t, \phi(t)\big) - A(t)\phi(t)\big] + \tilde{S}\big[\ell[\phi] - g_m[\phi]\big] + \tilde{S}_1\big[R_3(\phi - Q_m\phi)\big]. \tag{5.3}$$

PROOF. The proof of (5.2) is similar to that of Corollary 3.4, whereas (5.3) follows as in Corollary 3.6.

We shall now consider the following approximate Picard's scheme for Case 1.

$$v_{m+1}(t) = \tilde{H}\big[f_m\big(t, v_m(t)\big) - A(t)v_m(t)\big] + \tilde{S}\big[\ell[v_m] - g_m[v_m]\big]$$
$$+\tilde{S}_1\big[Rf_m\big(t, v_m(t)\big) + \Psi_2 v_m(t_0)\big],$$
$$v_0(t) = x_0(t) = \bar{x}(t), \qquad m = 0, 1, \ldots. \tag{5.4}$$

In (5.4) once again for each m, the function f_m and the operator g_m are assumed to be continuous. In addition, with respect to f_m and g_m we shall assume that the following conditions are satisfied.

CONDITION c_1: For all $t \in J$ and each $v_m(t)$ obtained from (5.4) the inequality

$$\left|f\big(t, v_m(t)\big) - f_m\big(t, v_m(t)\big)\right| \le \Delta_1 \left|f\big(t, v_m(t)\big)\right| \tag{5.5}$$

holds, where Δ_1 is an $n \times n$ nonnegative matrix with $\rho(\Delta_1) < 1$.
CONDITION c_2: For each $v_m(t)$ obtained from (5.4) the inequality

$$\left\|g[v_m] - g_m[v_m]\right\| \le \Delta_2 \left\|g[v_m]\right\| \tag{5.6}$$

holds, where Δ_2 is an $n \times n$ nonnegative matrix with $\rho(\Delta_2) < 1$.

Inequalities (5.5) and (5.6) correspond to the relative error in approximating f and g by f_m and g_m. Further, the above inequalities respectively imply that

$$\left|f\big(t, v_m(t)\big)\right| \le (I - \Delta_1)^{-1} \left|f_m\big(t, v_m(t)\big)\right| \tag{5.7}$$

and

$$\left\|g[v_m]\right\| \le (I - \Delta_2)^{-1} \left\|g_m[v_m]\right\|. \tag{5.8}$$

THEOREM 5.2. *With respect to the boundary value problem* (2.3) *we assume that there exists an approximate solution* $\bar{x}(t) \in C(J)$, *and*

(i) *the conditions* (i)–(iv) *of Theorem* (4.2) *are satisfied,*
(ii) *conditions* c_1 *and* c_2 *are satisfied,*
(iii) *there exists a nonnegative vector* $\bar{\eta}_1$ *such that* $\|\tilde{S}_1 \tilde{E}_1[Q_0 \bar{x}] + \tilde{S} g_0[\bar{x}]\| \le \bar{\eta}_1$ *and* $\eta = \max(\eta_1, \bar{\eta}_1)$,
(iv) $\bar{K}_1 = K_1 + (M_1 + M_6)\Delta_1(M_3 + \|A(t)\|) + M_2\Delta_2(M_4 + \|\ell\|)$; $\rho(\bar{K}_1) < 1$ *and* $\bar{r}_1 = (I - \bar{K}_1)^{-1}[\eta + 2(M_1 + M_6)\Delta_1(I - \Delta_1)^{-1}|f_0(t, v_0(t))| + 2M_2\Delta_2(I - \Delta_2)^{-1} \|g_0[v_0]\|] \le r$.

Then

(1) *all the conclusions* (1)–(5) *of Theorem* 4.2 *hold,*
(2) *the sequence* $\{v_m(t)\}$ *obtained from* (5.4) *remains in* $\bar{S}(\bar{x}, \bar{r}_1)$,
(3) *the sequence* $\{v_m(t)\}$ *converges to* $x^*(t)$, *the solution of* (2.3) *if and only if* $\lim_{m\to\infty} b_m = 0$, *where*

$$b_m = \Big\|v_{m+1}(t) - \tilde{H}\big[f\big(t, v_m(t)\big) - A(t)v_m(t)\big] - \tilde{S}\big[\ell[v_m] - g[v_m]\big] - \tilde{S}_1\big[Rf\big(t, v_m(t)\big) + \Psi_2 v_m(t_0)\big]\Big\|, \tag{5.9}$$

(4) *the following error estimate holds*

$$\left\|v_{m+1} - x^*\right\| \le (I - K_1)^{-1}\big[K_1\left\|v_{m+1} - v_m\right\| + (M_1 + M_6)\Delta_1(I - \Delta_1)^{-1}\left|f_m\big(t, v_m(t)\big)\right| + M_2\Delta_2(I - \Delta_2)^{-1}\left\|g_m[v_m]\right\|\big]. \tag{5.10}$$

PROOF. Since $K_1 \le \bar{K}_1$, $\rho(\bar{K}_1) < 1$ implies that $\rho(K_1) < 1$ and $\bar{r}_1 \ge r_1$. Hence the conditions of Theorem 4.2 are satisfied and (1) follows.

To show (2), it is obvious that $\bar{x}(t) \in \bar{S}(\bar{x}, \bar{r}_1)$. Further, from (5.4) and Lemma 5.1 we have

$$v_1(t) - v_0(t) = v_1(t) - \bar{x}(t) = -\tilde{S}_1 \tilde{E}_1[Q_0 \bar{x}] - \tilde{S}[g_0[\bar{x}]],$$

and hence in view of (iii) and (iv) it follows that $\|v_1 - v_0\| \le \bar{\eta}_1 \le \eta \le \bar{r}_1$, that is, $v_1(t) \in \bar{S}(\bar{x}, \bar{r}_1)$.

Now we shall show that $v_{m+1}(t) \in \bar{S}(\bar{x}, \bar{r}_1)$ provided $v_m(t) \in \bar{S}(\bar{x}, \bar{r}_1)$. From (5.4) we have

$$\begin{aligned}
v_{m+1}(t) - v_0(t) &= \big(v_1(t) - v_0(t)\big) \\
&\quad + \tilde{H}\Big[\Big(f_m\big(t, v_m(t)\big) - f\big(t, v_m(t)\big)\Big) + \Big(f\big(t, v_0(t)\big) - f_0\big(t, v_0(t)\big)\Big) \\
&\qquad + \int_0^1 \Big(f_x\Big(t, v_0(t) + \theta_1\big(v_m(t) - v_0(t)\big)\Big) - A(t)\Big)\big(v_m(t) - v_0(t)\big)\, d\theta_1\Big] \\
&\quad + \tilde{S}\Big[\big(g[v_m] - g_m[v_m]\big) + \big(g_0[v_0] - g[v_0]\big) \\
&\qquad - \int_0^1 \big(g_x\,[v_0 + \theta_2\,(v_m - v_0)] - \ell\big)\,[v_m - v_0]\; d\theta_2\Big] \\
&\quad + \tilde{S}_1\Big[R\Big\{\Big(f_m\big(t, v_m(t)\big) - f\big(t, v_m(t)\big)\Big) + \Big(f\big(t, v_0(t)\big) - f_0\big(t, v_0(t)\big)\Big) \\
&\qquad + \int_0^1 f_x\Big(t, v_0(t) + \theta_1\big(v_m(t) - v_0(t)\big)\Big)\big(v_m(t) - v_0(t)\big)\, d\theta_1\Big\}\Big] \\
&\quad + \tilde{S}_1\big[\Psi_2\big(v_m(t_0) - v_0(t_0)\big)\big].
\end{aligned}$$

Thus in view of $v_0(t) + \theta_i(v_m(t) - v_0(t)) \in \bar{S}(\bar{x}, \bar{r}_1)$, $i = 1, 2$, we find that

$$\begin{aligned}
\big\|v_{m+1} - v_0\big\| \le \big\|v_1 - v_0\big\| \\
&+ (M_1 + M_6)\Big[\Delta_1\Big(\big|f\big(t, v_m(t)\big)\big| + \big|f\big(t, v_0(t)\big)\big|\Big)\Big] \\
&+ M_2\big[\Delta_2\big(\big\|g[v_m]\big\| + \big\|g[v_0]\big\|\big)\big] \\
&+ \big[M_1 M_3 + M_2 M_4 + M_6 M_5 + M_7\big]\big\|v_m - v_0\big\|.
\end{aligned}$$

However, since

$$\begin{aligned}
\big|f\big(t, v_m(t)\big)\big| &\le \big|f\big(t, v_m(t)\big) - f\big(t, v_0(t)\big) - A(t)\big(v_m(t) - v_0(t)\big)\big| \\
&\quad + \big|f\big(t, v_0(t)\big)\big| + \big\|A(t)\big\|\,\big\|v_m - v_0\big\| \\
&\le M_3\big\|v_m - v_0\big\| + \big\|A(t)\big\|\,\big\|v_m - v_0\big\| + \big|f\big(t, v_0(t)\big)\big|
\end{aligned}$$

and similarly

$$\big\|g[v_m]\big\| \le M_4\big\|v_m - v_0\big\| + \big\|\ell\big\|\,\big\|v_m - v_0\big\| + \big\|g[v_0]\big\|,$$

we obtain

$$\begin{aligned}\|v_{m+1}-v_0\| &\le \bar{\eta}_1+(M_1+M_6)\big[\Delta_1\big(M_3+\|A(t)\|\big)\|v_m-v_0\| \\ &\qquad\qquad +2\Delta_1(I-\Delta_1)^{-1}\,|f_0(t,v_0(t))|\,\big] \\ &\quad +M_2\big[\Delta_2\big(M_4+\|\ell\|\big)\|v_m-v_0\|+2\Delta_2(I-\Delta_2)^{-1}\|g_0[v_0]\|\big] \\ &\quad +K_1\|v_m-v_0\| \\ &\le \bar{K}_1\bar{r}_1+(I-\bar{K}_1)\bar{r}_1=\bar{r}_1.\end{aligned}$$

Thus, $v_{m+1}(t)\in\bar{S}(\bar{x},\bar{r}_1)$.

To prove (3), from (4.2) and (5.4) we have

$$\begin{aligned}u_{m+1}(t)-v_{m+1}(t) &= -v_{m+1}(t)+\tilde{H}\big[f\big(t,v_m(t)\big)-A(t)v_m(t)\big] \\ &\quad +\tilde{S}\big[\ell[v_m]-g[v_m]\big]+\tilde{S}_1\big[Rf\big(t,v_m(t)\big)+\Psi_2 v_m(t_0)\big] \\ &\quad +\tilde{H}\left[\int_0^1\Big(f_x\Big(t,v_m(t)+\theta_1\big(u_m(t)-v_m(t)\big)\Big)-A(t)\Big)\right. \\ &\qquad\qquad \left.\times\big(u_m(t)-v_m(t)\big)\,d\theta_1\right] \\ &\quad -\tilde{S}\left[\int_0^1\big(g_x[v_m+\theta_2(u_m-v_m)]-\ell\big)[u_m-v_m]\,d\theta_2\right] \\ &\quad +\tilde{S}_1\left[R\int_0^1 f_x\Big(t,v_m(t)+\theta_3\big(u_m(t)-v_m(t)\big)\Big)\big(u_m(t)-v_m(t)\big)d\theta_3\right. \\ &\qquad\qquad \left.+\Psi_2\big(u_m(t_0)-v_m(t_0)\big)\right]\end{aligned}$$

and hence

$$\|u_{m+1}-v_{m+1}\|\le b_m+K_1\|u_m-v_m\|.$$

The above inequality, on using the fact that $\|u_0-v_0\|=0$, gives that

$$\|u_{m+1}-v_{m+1}\|\le\sum_{i=0}^{m}K_1^{m-i}b_i.$$

Thus, in view of the triangle inequality, we get

$$\|x^*-v_{m+1}\|\le\sum_{i=0}^{m}K_1^{m-i}b_i+\|x^*-u_{m+1}\|. \tag{5.11}$$

In (5.11), Theorem 4.2 ensures that $\lim_{m\to\infty}\|x^*-u_{m+1}\|=0$. Thus the condition $\lim_{m\to\infty}b_m=0$ is necessary and sufficient for the convergence of the sequence $\{v_m(t)\}$ to $x^*(t)$ follows from the Toeplitz lemma.

Finally, we shall prove (4). For this, we have

$$
\begin{aligned}
v_{m+1}(t) - x^*(t)& \\
= \tilde{H}\Bigg[\Big(f_m\big(t, v_m(t)\big) - f\big(t, v_m(t)\big)\Big)& \\
+ \int_0^1 \Big(f_x\Big(t, x^*(t) + \theta_1\big(v_m(t) - x^*(t)\big)\Big) - A(t)\Big)\big(v_m(t) - x^*(t)\big)\, d\theta_1\Bigg]& \\
+ \tilde{S}\Bigg[\big(g[v_m] - g_m[v_m]\big) - \int_0^1 \big(g_x[x^* + \theta_2(v_m - x^*)] - \ell\big)[v_m - x^*]\, d\theta_2\Bigg]& \\
+ \tilde{S}_1\Bigg[R\Bigg[\int_0^1 f_x\Big(t, x^*(t) + \theta_3\big(v_m(t) - x^*(t)\big)\Big)\big(v_m(t) - x^*(t)\big)\, d\theta_3& \\
+ \Big(f_m\big(t, v_m(t)\big) - f\big(t, v_m(t)\big)\Big)\Bigg]\Bigg] + \tilde{S}_1\Big[\Psi_2\big(v_m(t_0) - x^*(t_0)\big)\Big].&
\end{aligned}
$$

Thus it follows that

$$
\begin{aligned}
\|v_{m+1} - x^*\| \le M_1\big[\Delta_1\big|f\big(t, v_m(t)\big)\big| + M_3\|v_m - x^*\|\big]& \\
+M_2\big[\Delta_2\|g[v_m]\| + M_4\|v_m - x^*\|\big]& \\
+M_6\big[\Delta_1\,\big|f\big(t, v_m(t)\big)\big| + M_5\|v_m - x^*\|\big] + M_7\|v_m - x^*\|.& \qquad (5.12)
\end{aligned}
$$

Using (5.7), (5.8) and the triangle inequality in (5.12), we obtain

$$
\begin{aligned}
\|v_{m+1} - x^*\| \le K_1\|v_{m+1} - x^*\| + K_1\|v_{m+1} - v_m\|& \\
+(M_1 + M_6)\Delta_1(I - \Delta_1)^{-1}\big|f_m\big(t, v_m(t)\big)\big|& \\
+M_2\Delta_2(I - \Delta_2)^{-1}\|g_m[v_m]\|,&
\end{aligned}
$$

which is the same as (5.10).

In our next result we shall need the following:

CONDITION c_3: Condition c_1 holds with (5.5) replaced by

$$\big|f\big(t, v_m(t)\big) - f_m\big(t, v_m(t)\big)\big| \le r_3, \qquad (5.13)$$

where r_3 is a nonnegative $n \times 1$ vector.

CONDITION c_4: Condition c_2 holds with (5.6) replaced by

$$\|g[v_m] - g_m[v_m]\| \le r_4, \qquad (5.14)$$

where r_4 is a nonnegative $n \times 1$ vector.

Inequalities (5.13) and (5.14) correspond to the relative error in approximating f and g by f_m and g_m.

THEOREM 5.3. *With respect to the boundary value problem* (2.3), *assume that there exists an approximate solution* $\bar{x}(t) \in C(J)$, *and*

(i) *conditions* (i), (iii) *of Theorem* 5.2 *are satisfied,*
(ii) *conditions* c_3 *and* c_4 *are satisfied,*
(iii) $\rho(K_1) < 1$ *and* $r_5 = (I - K_1)^{-1}[\eta + 2(M_1 + M_6)r_3 + 2M_2r_4] \le r$.

Then

(1) *all the conclusions* (1)–(5) *of Theorem* 4.2 *hold,*
(2) *the sequence* $\{v_m(t)\}$ *obtained from* (5.4) *remains in* $\bar{S}(\bar{x}, r_5)$,
(3) *conclusion (3) of Theorem* 5.2 *holds,*
(4) *the following error estimate holds:*

$$\left\| v_{m+1} - x^* \right\| \le (I - K_1)^{-1}\left[K_1 \left\| v_{m+1} - v_m \right\| + (M_1 + M_6)r_3 + M_2r_4\right].$$

PROOF. The proof is contained in Theorem 5.2.

Next we shall consider the following approximate Picard's scheme for Case 2.

$$\begin{aligned} v_{m+1}(t) = {} & \tilde{H}[f_m(t, v_m(t)) - A(t)v_m(t)] + \tilde{S}\left[l[v_m] - g_m[v_m]\right] \\ & + \tilde{S}_1\left[R_3(v_m - Q_m v_m)\right], \quad v_0(t) = x_0(t) = \bar{x}(t), \quad m = 0, 1, \ldots. \end{aligned} \tag{5.15}$$

In (5.15), for each m the function f_m and the operator g_m are assumed to be continuous. In addition, with respect to f_m and g_m we shall assume that the following conditions are satisfied.

CONDITION c_5: For all $t \in C(J)$ and each $v_m(t)$ obtained from (5.15), the inequality (5.5) holds, where Δ_1 is an $n \times n$ nonnegative matrix with $\rho(\Delta_1) < 1$.

CONDITION c_6: For each $v_m(t)$ obtained from (5.15), the inequality (5.6) holds, where Δ_2 is an $n \times n$ nonnegative matrix with $\rho(\Delta_2) < 1$.

THEOREM 5.4. *With respect to the boundary value problem* (2.3), *assume that there exists an approximate solution* $\bar{x}(t) \in C(J)$, *and*

(i) *the conditions* (i) *and* (ii) *of Theorem* 4.3 *are satisfied,*
(ii) *the condition* (iii) *of Theorem* 5.2 *and conditions* c_5 *and* c_6 *are satisfied,*
(iii) $\bar{K}_2 = K_2 + (M_1 + \sigma M_8)\Delta_1(M_3 + \|A(t)\|) + M_2\Delta_2(M_4 + \|\ell\|)$; $\rho(\bar{K}_2) < 1$ *and* $\bar{r}_2 = (I - K_2)^{-1}[\eta + 2(M_1 + \sigma M_8)\Delta_1(I - \Delta_1)^{-1}|f_0(t, v_0(t))| + 2M_2\Delta_2(I - \Delta_2)^{-1}\|g_0[v_0]\|] \le r$.

Then

(1) *all the conclusions* (1)–(5) *of Theorem* 4.3 *hold,*
(2) *the sequence* $\{v_m(t)\}$ *obtained from* (5.15) *remains in* $\bar{S}(\bar{x}, \bar{r}_2)$,

(3) *the sequence* $\{v_m(t)\}$ *converges to* $x^*(t)$, *the solution of (2.3) if and only if* $\lim_{m\to\infty} c_m = 0$, *where*

$$c_m = \left\| v_{m+1}(t) - \tilde{H}\left[f\left(t, v_m(t)\right) - A(t)v_m(t)\right] - \tilde{S}\left[\ell[v_m] - g[v_m]\right] - \tilde{S}_1\left[R_3\left(v_m - Qv_m\right)\right]\right\|, \tag{5.16}$$

(4) *the following error estimate holds:*

$$\left\| v_{m+1} - x^* \right\| \le (I - K_2)^{-1}\Big[K_2 \left\| v_{m+1} - v_m \right\| + (M_1 + \sigma M_8)\Delta_1(I - \Delta_1)^{-1} \left| f_m\left(t, v_m(t)\right)\right| + M_2\Delta_2(I - \Delta_2)^{-1}\left\| g_m[v_m]\right\|\Big]. \tag{5.17}$$

PROOF. The proof is similar to that of Theorem 5.2.

In our next result we shall need the following:
CONDITION c_7: Condition c_5 holds with (5.5) replaced by (5.13).
CONDITION c_8: Condition c_6 holds with (5.6) replaced by (5.14).

THEOREM 5.5. *With respect to the boundary value problem* (2.3), *assume that there exists an approximate solution* $\bar{x}(t) \in C(J)$, *and*

(i) *condition* (i) *of Theorem* 5.4 *and* (iii) *of Theorem* 5.2 *are satisfied,*
(ii) *conditions* c_7 *and* c_8 *are satisfied,*
(iii) $\rho(K_2) < 1$ *and* $r_5 = (I - K_2)^{-1}[\eta + 2(M_1 + \sigma M_8)r_3 + 2M_2 r_4] \le r$.

Then

(1) *all the conclusions* (1)–(5) *of Theorem* 4.3 *hold,*
(2) *the sequence* $\{v_m(t)\}$ *obtained from* (5.15) *remains in* $\bar{S}(\bar{x}, r_5)$,
(3) *conclusion* (3) *of Theorem* 5.4 *holds,*
(4) *the following error estimate holds:*

$$\left\| v_{m+1} - x^* \right\| \le (I - K_2)^{-1}[K_2 \left\| v_{m+1} - v_m \right\| + (M_1 + \sigma M_8)r_3 + M_2 r_4].$$

PROOF. The proof is similar to that of Theorem 5.2.

6. An example

The following example illustrates the sharpness as well as the importance of our results.

EXAMPLE 6.1. The boundary value problem

$$u'' + u + (u-t)^3 = t + 0.1 \qquad (6.1)$$
$$u(-1) = -0.9, \qquad u(1) = 1.1$$

is due to Urabe [22], and has also appeared in [1, 5, 11, 25].

In system form (6.1) is the same as

$$x_1' = x_2$$
$$x_2' = -x_1 - (x_1 - t)^3 + t + 0.1 \qquad (6.2)$$
$$x_1(-1) = -0.9, \qquad x_1(1) = 1.1.$$

For (6.2) choose $\bar{x}(t) = (t + 0.1, 1)^T$

$$A(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and

$$l[x(t)] = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1(-1) \\ x_2(-1) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1(1) \\ x_2(1) \end{pmatrix}.$$

As in [11], let $\epsilon = 10^{-3}$, $\mu = 1 + \epsilon$, $\nu = 1 - \epsilon$,

$$\tilde{\Phi}(t) = \begin{pmatrix} \mu \cos(1+t) & \mu \sin(1+t) \\ -\mu \sin(1+t) & \nu \cos(1+t) \end{pmatrix}$$

and

$$\hat{\Phi}(t) = \begin{pmatrix} \cos(1+t) & -\nu \sin(1+t) \\ \sin(1+t) & \mu \cos(1+t) \end{pmatrix}$$

so that

$$\tilde{G} = \begin{pmatrix} \mu & 0 \\ \mu \cos 2 & \mu \sin 2 \end{pmatrix},$$

$$\tilde{S}(t) = \frac{1}{\mu \sin 2} \begin{pmatrix} \mu \sin(1-t) & \mu \sin(1+t) \\ -C(t) & \nu \cos(1+t) \end{pmatrix}$$

and

$$\tilde{H}[\phi(t)] = \int_{-1}^{1} \tilde{H}(t,s)\phi(s)\, ds,$$

where

$$\sin 2\, \tilde{H}(t,s) = \begin{pmatrix} \mu \sin(1-t)\cos(1+s) & -\mu\nu \sin(1-t)\sin(1+s) \\ -C(t)\cos(1+s) & \nu C(t) \sin(1+s) \end{pmatrix}, \quad -1 \le s \le t \le 1$$

$$= -\begin{pmatrix} \mu \sin(1+t)\cos(1-s) & \mu \sin(1+t)\, D(s) \\ \nu \cos(1+t)\cos(1-s) & \nu \cos(1+t)\, D(s) \end{pmatrix}, \quad -1 \le t \le s \le 1$$

and $C(t) = \cos(1-t) - \epsilon\cos(3+t)$, $\quad D(t) = \sin(1-t) + \epsilon\sin(3+t)$.

Thus it follows that

$$\|\ell\| = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}; \quad \|\tilde{\Phi}\| \le \begin{pmatrix} 1.0010 & 0.9103 \\ 0.9103 & 0.9990 \end{pmatrix}; \quad \|\tilde{G}^{-1}\| \le \begin{pmatrix} 0.9991 & 0.0000 \\ 0.4573 & 1.0987 \end{pmatrix};$$

$$\begin{aligned} \|\tilde{H}\| \le \left\| \int_{-1}^{1} |\tilde{H}(t,s)| ds \right\| &\le \frac{1}{\sin 2} \begin{pmatrix} 2\mu & \mu(2\sin 1 - \sin 2 + 2\epsilon\sin 2) \\ 2\mu & 2\mu\sin^2 1 + 2\nu\epsilon \end{pmatrix} \\ &\le \begin{pmatrix} 2.2017 & 0.8537 \\ 2.2017 & 1.5612 \end{pmatrix} = M_1; \end{aligned}$$

$$\|\tilde{S}\| \le \frac{1}{\mu\sin 2} \begin{pmatrix} \mu & \mu \\ \mu & \nu \end{pmatrix} \le \begin{pmatrix} 1.0998 & 1.0998 \\ 1.0998 & 1.0976 \end{pmatrix} = M_2;$$

$$\|\tilde{S}_1\| \le \begin{pmatrix} 3.1996 & 0 \\ 2.1974 & 1 \end{pmatrix};$$

for (t, x) such that $t \in [-1, 1]$, $x \in \bar{S}(\bar{x}, r)$, $r = (r_{01}, r_{02})$

$$\|f_x(t,x) - A(t)\| = \left\| \begin{pmatrix} 0 & 0 \\ 3\big(x_1(t) - t\big)^2 & 0 \end{pmatrix} \right\| \le \begin{pmatrix} 0 & 0 \\ 3(0.1 + r_{01})^2 & 0 \end{pmatrix} = M_3;$$

$$M_4 = 0;$$

$$M_5 = \begin{pmatrix} 0 & 1 \\ 1 + 3(0.1 + r_{01})^2 & 0 \end{pmatrix};$$

$$\|\Gamma_1\| \le 10^{-3} \begin{pmatrix} 1 & 1.001 \\ 1 & 0.001 \end{pmatrix}; \quad \|\Psi\| \le 10^{-3} \begin{pmatrix} 1 & 1.8186 \\ 0 & 1.8323 \end{pmatrix};$$

$$\|\Psi_1\| \le 10^{-3} \begin{pmatrix} 1.4162 & 0.9093 \\ 0.9093 & 1.0000 \end{pmatrix}; \quad \|\Psi_2\| \le 10^{-3} \begin{pmatrix} 2.6557 & 0 \\ 0.9103 & 1 \end{pmatrix};$$

$$\sigma\big(\|\Psi_2\|\|A\| + \|\tilde{\Phi}\|\|\Psi_1 A\|\big) \le \begin{pmatrix} 0.0037 & 0.0099 \\ 0.0057 & 0.0063 \end{pmatrix} = P_1;$$

$$\sigma\big(\|\Psi_2\| + \|\tilde{\Phi}\|\|\Psi_1\|\big) \le \begin{pmatrix} 0.0099 & 0.0037 \\ 0.0063 & 0.0057 \end{pmatrix} \le P_2;$$

$$\|\tilde{S}_1\| P_2 \le \begin{pmatrix} 0.0317 & 0.0119 \\ 0.0281 & 0.0139 \end{pmatrix} = M_6; \qquad \text{(cf. Remark 4.1)};$$

$$\|\tilde{S}_1 \Psi_2\| \le \begin{pmatrix} 0.0085 & 0.0000 \\ 0.0068 & 0.0010 \end{pmatrix} = M_7;$$

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$$e^{\sigma\|A\|} \le \begin{pmatrix} \cosh 2 & \sinh 2 \\ \sinh 2 & \cosh 2 \end{pmatrix} \le \begin{pmatrix} 3.7622 & 3.6269 \\ 3.6269 & 3.7622 \end{pmatrix};$$

$$\|\tilde{G}_1\|\|\ell\| \exp(\sigma\|A\|)\|\Psi\| \le \begin{pmatrix} 0.0038 & 0.0135 \\ 0.0059 & 0.0210 \end{pmatrix} = M;$$

$$\|\tilde{S}_1\Psi_2\| + \|\tilde{S}_1\|P_1 \le \begin{pmatrix} 0.0204 & 0.0317 \\ 0.0207 & 0.0291 \end{pmatrix} = \bar{M};$$

$$\|\tilde{S}_1\tilde{E}_1[Q\bar{x}] + \tilde{S}[g[\bar{x}]]\| \le 10^{-3}\begin{pmatrix} 5.8246 \\ 6.0002 \end{pmatrix} = \eta_1;$$

$$K_1 = \begin{pmatrix} 0.0204 + 2.5968(0.1 + r_{01})^2 & 0.0317 \\ 0.0207 + 4.7253(0.1 + r_{01})^2 & 0.0291 \end{pmatrix}; \tag{6.3}$$

$$\sigma\|A_1 - \Gamma A\| \le 0.002\begin{pmatrix} 0.0021 & 1.0021 \\ 2.0010 & 2.0041 \end{pmatrix} = N;$$

$$(I - \|\Gamma_1\|)^{-1} \le \begin{pmatrix} 1.0011 & 0.0011 \\ 0.0011 & 1.0001 \end{pmatrix} = Z;$$

$$\|\Gamma A + \tilde{\Phi}\hat{\Phi}'\| \le 10^{-3}\begin{pmatrix} 1.001 & 1.001 \\ 1.001 & 2.000 \end{pmatrix}$$

$$\|\tilde{H}\|Z\|\Gamma A + \tilde{\Phi}\hat{\Phi}'\| + \|\tilde{S}_1\|\|\Gamma_1\| \le 10^{-3}\begin{pmatrix} 6.2640 & 7.1225 \\ 6.9708 & 7.5362 \end{pmatrix} = P_3 = M_8$$

(cf. (4.10), (4.12), (4.13));

$$K_2 = \begin{pmatrix} 0.0206 + 2.6039(0.1 + r_{01})^2 & 0.0197 \\ 0.0221 + 4.7289(0.1 + r_{01})^2 & 0.0215 \end{pmatrix}. \tag{6.4}$$

To apply Theorem 4.2 we note that from the above computation, $\rho(M) < 1$ and $\rho(\bar{M}) < 1$, and therefore in view of Remark 4.1, conditions of Lemma 3.3 are satisfied, and in conclusion $\tilde{L}^{-1}$ is invertible. Next, $\rho(K_1) < 1$ if and only if

$$0 \le r_{01} < 0.4965. \tag{6.5}$$

Further, the condition $(I - K_1)^{-1}\eta_1 \le r$ implies that

$$6.35204 \times 10^{-3} \le r_{01} \le 0.492783 \tag{6.6}$$

and

$$r_{02} \ge \frac{10^{-3}(5.9984 + 11.9417(0.1 + r_{01})^2)}{0.9504 - 2.671025(0.1 + r_{01})^2}. \tag{6.7}$$

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For $r_{01} = 6.4 \times 10^{-3}$, both (6.5) and (6.6) are satisfied. Also, for this value of r_{01} from (6.3) and (6.7), we have

$$K_1 = \begin{pmatrix} 0.0498 & 0.0317 \\ 0.0742 & 0.0291 \end{pmatrix} \qquad \text{and} \qquad r_{02} \leq 6.66578 \times 10^{-3}.$$

In conclusion, the following hold:

(1) there exists a solution $x^*(t)$ of (6.2) in $\bar{S}(\bar{x}, r) = \{(x_1, x_2) : |x_1 - (t + 0.1)| \leq 6.4 \times 10^{-3}, |x_2 - 1| \leq 6.66578 \times 10^{-3}\}$,

(2) $x^*(t)$ is the unique solution of (6.2) in $\bar{S}(\bar{x}, r) = \{(x_1, x_2) : |x_1 - (t + 0.1)| \leq 0.492783\}$,

(3) the sequence $\{u_m(t)\}$ generated by (4.2) for the problem (6.2) remains in $\bar{S}(\bar{x}, r) = \{(x_1, x_2) : |x_1 - (t + 0.1)| \leq 6.4 \times 10^{-3}, |x_2 - 1| \leq 6.66578 \times 10^{-3}\}$ and converges to $x^*(t)$,

(4) the following error estimate is valid:

$$\left|x^*(t) - u_m(t)\right| \leq 10^{-3} \begin{pmatrix} 0.0498 & 0.0317 \\ 0.0742 & 0.0291 \end{pmatrix}^m \begin{pmatrix} 6.40000 \\ 6.66578 \end{pmatrix}.$$

To apply Theorem 4.3 from the above computation, it is clear that $\rho(N) < 1$, and therefore conditions of Lemma 3.5 are satisfied, and in conclusion $\tilde{L}^{-1}$ is invertible. Next, $\rho(K_2) < 1$ if and only if

$$0 \leq r_{01} < 0.502243. \tag{6.8}$$

Further, the condition $(I - K_2)^{-1}\eta_1 \leq r$ implies that

$$6.26839 \times 10^{-3} \leq r_{01} \leq 0.498563 \tag{6.9}$$

and

$$r_{02} \geq \frac{10^{-3}\left(6.005 + 11.9200(0.1 + r_{01})^2\right)}{0.9579 - 2.6411(0.1 + r_{01})^2}. \tag{6.10}$$

Once again for $r_{01} = 6.4 \times 10^{-3}$ both (6.8) and (6.9) are satisfied. Also, for this value of r_{01} from (6.4) and (6.10), we have

$$K_2 = \begin{pmatrix} 0.0501 & 0.0197 \\ 0.0757 & 0.0215 \end{pmatrix} \qquad \text{and} \qquad r_{02} \geq 6.61632 \times 10^{-3}.$$

In conclusion, the following hold

(1) there exists a solution $x^*(t)$ of (6.2) in $\bar{S}(\bar{x}, r) = \{(x_1, x_2) : |x_1 - (t + 0.1)| \leq 6.4 \times 10^{-3}, |x_2 - 1| \leq 6.61632 \times 10^{-3}\}$,

(2) $x^*(t)$ is the unique solution of (6.2) in $\bar{S}(\bar{x}, r) = \{(x_1, x_2) : |x_1 - (t + 0.1)| \leq 0.498563$,

(3) the sequence $\{u_m(t)\}$ generated by (4.2) for the problem (6.2) remains in $\bar{S}(\bar{x}, r) = \{(x_1, x_2) : |x_1 - (t + 0.1)| \le 6.4 \times 10^{-3}, |x_2 - 1| \le 6.61632 \times 10^{-3}\}$ and converges to $x^*(t)$,

(4) the following error estimate is valid:

$$|x^*(t) - u_m(t)| \le 10^{-3} \begin{pmatrix} 0.0501 & 0.0197 \\ 0.0757 & 0.0215 \end{pmatrix}^m \begin{pmatrix} 6.40000 \\ 6.61632 \end{pmatrix}.$$

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