Can. J. Math., Vol. XXVI, No. 1, 1974, pp. 145–176

DOUBLE COVERS AND METASTABLE IMMERSIONS OF SPHERES

ROBERT WELLS

Notation. The real line will be $\mathbf{R}$, Euclidean n-space will be $\mathbf{R}^n$, the unit ball in $\mathbf{R}^n$ will be E^n, the unit sphere in $\mathbf{R}^{n+1}$ will be S^n, and real projective n-space will be P_n. The canonical line bundle associated with the double cover $S^n \to P_n$ will be η_n. If γ is a vector bundle, $E(\gamma)$ will be its associated cell bundle, $S(\gamma)$ its associated sphere bundle, $P(\gamma)$ its associated projective space bundle $(P(\gamma) = S(\gamma)/(-1))$ and $T(\gamma) = E(\gamma)/S(\gamma)$ its Thom space. If $\varphi : \gamma \to \gamma'$ is a vector bundle map (i.e. linear isomorphism on each fiber) then we have $E(\varphi)$, $S(\varphi)$, $P(\varphi)$, $T(\varphi)$ by pulling a Riemann metric on γ' back to γ. If $f : M \to N$ is an immersion, we put a Riemann metric on N and write

$$v(f) = \{\xi \in \tau_{f(x)}N, \xi \perp df(x)\tau_x M\},$$

$f' : v(f) \to \tau(N)$ by $f'(x, \xi) = \xi$. If u is the outward normal of M along ∂M, then we say $x \to df(x)u(x)$ is the outward normal of f. If M is a submanifold of N, then $v(M : N)$ is the normal bundle of M in N. Finally, $KO^{\sim}(P_r) = Z_{2^{\varphi(r)}}$. We will abbreviate $c(r) = 2^{\varphi(r)}$.

Introduction. Let M be a compact smooth manifold of dimension m and $f : M \to N$ an immersion where N is a smooth manifold of dimension $n = m + k$; we say that f is metastable if $k \geqq (m + 3)/2$; we say that f is *generic* if f is metastable, f has only self-transverse double points, f embeds ∂M and $f^{-1}f(\partial M) = \partial M$. If $f : M \to N$ is a generic immersion, let $\Delta(f) \subset N$ be the set of double points of f and $D(f) = f^{-1}\Delta(f)$. Then $\Delta(f)$ is a closed smooth submanifold of dimension $d = m - k$ and

$$D(f) \xrightarrow{f} \Delta(f)$$

is a double cover. Clearly generic regular homotopies and cobordisms (suitably defined) of f lead to cobordisms of $D(f) \to \Delta(f)$, so that this double cover has a bearing on cobordism groups of immersions, especially since cobordism classes of metastable immersions and generic cobordism classes of immersions are in one-to-one correspondence by [**3**].

In [**10**], the author reduced to a homotopy problem the problem of computing the cobordism groups of immersions of closed manifolds in spheres' In [**8**] and [**9**], P. A. Schweitzer and F. Uchida (in the metastable case) succeed in

Received August 22, 1972 and in revised form, February 27, 1973.

https://doi.org/10.4153/CJM-1974-016-1 Published online by Cambridge University Press

computing the joint cobordism groups of immersions of closed manifolds in closed manifolds. That is, their objects are immersions

$$M \xrightarrow{f} N$$

where M and N are closed, and f is cobordant to f' if and only if there is an immersion $F : X \to Y$ with $\partial X = M + M'$, $\partial Y = N + N'$ and $F|M = f : M \to N$ and $F|M' = f' : M' \to N'$. In F. Uchida's treatment, and to a certain extent in P. A. Schweitzer's, the double points of metastable generic immersions play a crucial role. In particular, the following existence problem arises:

(1) Given a double cover of closed manifolds $D^d \to \Delta^d$ and a vector bundle $\xi \to D^d$ of dimension k, when does there exist a generic immersion of closed manifolds $f : M \to N$ such that the double point cover of f is $D \to \Delta$ and $v(D : M) = \xi$?

The answer that Uchida gives is that it is always possible to find such f; in actually constructing M, N and f, he makes crucial use of the fact that the only condition on N is that it be a closed manifold. In attempting to apply Uchida's techniques to the study of the cobordism groups of [**10**], we run into a more difficult problem:

(2) Given $D^d \to \Delta^d$ as above and $\xi \to D$ as above, when does there exist an immersion of closed manifolds $f : M \to S^n$ such that the double point cover of f is $D \to \Delta$ and $v(D : M) = \xi$?

It is the requirement that $N = S^n$ that makes this problem difficult, and, as we should expect, it does not always have a solution. A third version of the existence problem has been studied by J. G. Miller in [**6**]:

(3) Given a double cover $D \to \Delta$ as above, when does there exist a generic immersion of closed manifolds $f : M \to S^n$ such that the double point cover is $D \to A$?

Of course we may weaken problem (2) to problem (2″) in which we require M only to be compact, not necessarily closed. And between (2) and (2″) we have the problem (2′) in which we require M to be almost closed; that is, $\partial M = S^{m-1}$. Similarly, we obtain problems (3′) and (3″).

In the first part of this paper, we reduce the solution of (2″) to a KO-theoretic computation with the $KO^\sim$ transfer. Using this solution, we solve (2′) completely in the case that Δ is a homotopy projective space of dimension $\geqq$ 01 and D is connected:

THEOREM 1. *Let Δ be a homotopy projective space of dimension $d \geqq 10$, let $D \to \Delta$ be its universal cover and $\xi \to D$ a k-plane bundle. Then the virtual bundle $\xi - \dim \xi \in KO^\sim(D) = KO^\sim(S^d)$ is a certain multiple s of a generator of $KO^\sim(D)$; in the case that $KO^\sim(D) = 0$ we make the convention that $s = 0$. Then* *there is a generic immersion* $f : M^m \to S^n$ *with M almost closed,* $D(f) \to \Delta(f) = D \to \Delta$ *and* $v(D(f) : M) = \xi$ *if and only if:*

(1) $k = c(d) - d - 1 + rc(d)$ *where* $r = 0, 1, 2, \ldots$ *in the case that s is even, or*

(2) $k = c(d)/2 - d - 1 + rc(d)$ *where* $r = 0, 1, 2, \ldots$ *in the case that s is odd.*

In the case that Δ is a homotopy projective space of dimension $d \geqq 10$ and ξ is trivial, we recover Miller's solutions of problem 3 and extend them to a complete solution, with M in fact a sphere:

THEOREM 2. *Let* $D \to \Delta$ *be as in Theorem* 1. *Then there is a generic immersion* $f : S^m \to S^n$ *with* $D(f) \to \Delta(f) = D \to \Delta$ *if and only if* $k = c(d) - d - 1 + rc(d)$ *where* $r = 0, 1, 2, \ldots$.

THEOREM 3. *Let* $D \to \Delta$ *be as in Theorem* 1 *with the additional assumption that* $d \equiv 0 \bmod 4$. *Suppose that* $\xi \to D$ *is a k-plane bundle such that* $\xi - \dim \xi \neq 0$ *in* $KO^{\sim}(D)$. *Then there is no generic immersion* $f : M^m \to S^n$ *with M a closed orientable manifold,* $D(f) \to \Delta(f) = D \to \Delta$ *and* $v(D(f) : M) = \xi$.

We may combine the proofs of Theorem 1 and Theorem 3 with the Barratt-Mahowald Theorem to obtain a probably well-known corollary in homotopy theory. We proceed as follows: Let $d \equiv 0 \bmod 4$ and $d \geqq 10$, let $\xi \to S^d$ be a k dimensional representative of s times a generator of $KO^{\sim}(S^d)$ with k related to s and d as in Theorem 1. Then the proof of Theorem 1 provides us with a specific generic immersion $f(s) : M^m \to S^n$ which embeds $\partial M^m = S^{m-1}$. It will follow from the proof of Theorem 1 and the Barratt-Mahowald Theorem that if $[\xi - \dim \xi]$ is not in the kernel of the J-homomorphism, then $v(f(s))|S^{m-1}$ is non-trivial. But from the Barratt-Mahowald Theorem again, $v(f(s)|S^{m-1}) + 1$ is trivial. Thus we have explicitly constructed non-trivial elements of the kernel of $\pi_{m-1}BO(k) \to \pi_{m-1}BO(k+1)$ to obtain the following corollary:

COROLLARY. *For* $d \equiv 0 \bmod 4$ *and* $d \geqq 10$, *the maps*

$$\pi_{m-1}(BO(k)) \to \pi_{m-1}(BO(k+1))$$

all have non-trivial kernels, where $m = d + k$ *and* $k = c(d)/2 - d - 1 + rc(d)$ *for* $r = 0, 1, 2, \ldots$.

Though surely the corollary is known, presumably the explicit construction above is not. Of course, it would be nice to know the nature of the map $Z \to \pi_{m-1}(BO(k))$ given for instance by $s \to v(f(2s))|S^{m-1}$ in any of the cases $k = c(d) - d - 1 + rc(d)$ with $r = 0, 1, 2, \ldots$.

In the second section we begin with the observation that if M is c-parallelizable with $c \geqq d + 1$ and $f : M \to S^n$ is a generic immersion then $v(\Delta(f) : S^n) = k\eta + k$ where $\eta \to \Delta(f)$ is the canonical line bundle associated with the double cover $D(f) \to \Delta(f)$. In fact, if we fix a c-parallelization of M, each generic

https://doi.org/10.4153/CJM-1974-016-1 Published online by Cambridge University Press

immersion $f : M \to S^n$ gives rise to a bundle map $v(\Delta(f) : S^n) \to k\eta_\infty + k$ where $\eta_\infty \to P_\infty$ is the canonical line bundle over infinite dimensional projective space. Thus each generic immersion gives rise to a $k\eta_\infty$-manifold in the sense of [**10**]. In the case that M is actually c-connected, a choice of orientation for M determines uniquely a c-parallelization of M. And in that case we will obtain the following theorem:

THEOREM 4. *If M is c-connected with $c \geqq d + 1$ and $f : M \to S^n$ is a generic immersion, then any $k\eta_\infty$-surgery of the $k\eta_\infty$-manifold $v(\Delta(f) : S^n) + r \Rightarrow k\eta_\infty + k + r$ may be realized by a generic regular homotopy.*

For the sake of clarity, we should define generic regular homotopy. A *generic regular homotopy* is a regular homotopy $f_t : M \to N$ such that the immersion $F : I \times M \to I \times N$ satisfies the following four conditions:
(1) $f_t = f_0$ near ∂M for all t;
(2) $F^{-1}F(I \times \partial M) = I \times \partial M$;
(3) F has only self-transverse double points;
(4) f_0 is generic.
Then of course f_1 is generic and in our case the $k\eta_\infty$-manifold

$$v(\Delta(F) : I \times S^n) + r \to k\eta_\infty + k + r$$

is a $k\eta_\infty$-cobordism from $v(\Delta(f_0) : S^n) + r \to k\eta_\infty + k + r$ to $v(\Delta(f_1) : S^n) + r \to k\eta_\infty + k + r$. Ideally, this theorem should follow easily from Haefliger's Theorem in [**4**]. However, all that the author can extract from Haefliger's argument is the theorem that (subject to the hypotheses of Theorem 4) if there is a sequence of $k\eta_\infty$-surgeries of $v(\Delta(f) : S^n) + r \to k\eta_\infty + k + r$ leading to the empty $k\eta_\infty$-manifold, then there is a generic regular homotopy from f to an embedding. This consequence is also a corollary of Theorem 4, but apparently not enough to prove Theorem 4. A theorem similar to Theorem 4 but with weaker hypothesis and weaker conclusion has been proved by F. Connolly in [**2**]. Since neither Haefliger's nor Connolly's version implies Theorem 4, we will give a proof of Theorem 4 in the second section; essentially, it will parallel the proof of the author's theorem in [**11**]. Once Theorem 4 is available, we obtain the following realizability theorem as an easy corollary.

THEOREM 5. *Let Δ be a closed manifold of dimension d and suppose $k \geqq d + 3$. Let*

$$\Delta \xrightarrow{v(\Delta)} BO$$

be the classifying map for the stable normal bundle of Δ and let $k\eta' : P_\infty \to BO$ be the composition

$$P_\infty \xrightarrow{k\eta_\infty} BO(k) \to BO.$$

If there exists a homotopy commutative diagram

$$\begin{tikzcd} & P_\infty \arrow[d, "k\eta'"] \\ \Delta \arrow[ur] \arrow[r, "v(\Delta)"'] & BO \end{tikzcd}$$

then there is a generic immersion $S^m \to S^n$ *with* $D(f) \to \Delta(f) = D \to \Delta$ *where* $D \to \Delta$ *is the pullback under* $\Delta \to P_\infty$ *of the universal cover of* P_∞ (and, as always $m = d + k$ and $n = m + k$).

In [**6**], J. G. Miller refers to a realizability theorem in an unpublished paper of the author; Theorem 5 is that realizability theorem, and an easy proof will appear in section 2 below. With this theorem available, we may read in the results of [**13**] to obtain many more examples of double covers realized as the double point covers of generic immersions of spheres. The objects studied in [**13**] are free involutions—a *free involution* is a smooth map $\rho : M(\rho) \to M(\rho)$ such that $\rho^2 = \text{id}$ and $\rho(x) \neq x$ for all x; $Q(\rho)$ is the quotient of $M(\rho)$ by $x \sim x$ and $x \sim \rho x$. If $M(\rho)$ is oriented and ρ preserves the orientation, we say it is an *oriented free involution*; in this case, $Q(\rho)$ inherits an orientation. Then our largest class of examples is contained in the following theorem, which follows immediately from [**13**, Theorem 6] and Theorem 5 above:

THEOREM 6. *If* ρ *is any oriented free involution with* $M(\rho)$ *homotopy equivalent to* $S^l \times S^l$ *and* $l \equiv 4, 6 \bmod 8$, *then there is a generic immersion*

$$S^m \xrightarrow{f} S^n$$

with $D(f) \to \Delta(f) = M(\rho) \to Q(\rho)$.

In Theorem 6 of course $d = 2l$. It is possible to say more about the codimension k; in [**13**] the concept of *type of* ρ is introduced for $M(\rho)$ homotopy equivalent to $S^l \times S^l$. It is defined as follows: By obstruction theory, there is a map $P_l \to Q(\rho)$ such that

$$\pi_1(P_l) \overset{\cong}{\to} \pi_1(Q(\rho)).$$

Under this map, the reduced normal bundle of $Q(\rho)$ pulls back to a certain multiple of the reduced canonical line bundle over P_l. As shown in [**13**], this multiple is well-defined modulo $c(l)$ and thus its class $K(\rho) \in Z_{c(l)}$ is well-defined. The type of ρ is $K(\rho)$. Then we may add to Theorem 6:

Addendum. For ρ as in Theorem 6 and $f : S^m \to S^n$ a generic immersion realizing $M(\rho) \to Q(\rho)$ we have that the codimension k may be any integer $\geqq 2l + 3$ such that $k \to K$ under $Z \to Z_{c(l)}$.

Actually, Theorem 6 is a consequence of a more delicate theorem. Let the set of oriented free involutions ρ with $M(\rho)$ homotopy equivalent to $S^l \times S^l$ and type K be factored by the relation $\rho \sim \rho'$ if and only if $Q(\rho)$ is orientation-preserving diffeomorphic to $Q(\rho')$. Denote the factor set by $I_{2l}(K)$. It follows from [**13**] that for l even there is an abelian group structure on $I_{2l}(K)$ and that there is a homomorphism $\alpha : I_{2l}(K) \to Z_2$ defined by $\alpha(\rho) = 0$ if the normal bundle of $Q(\rho)$ is stably a multiple of the canonical line bundle associated with $M(\rho) \to Q(\rho)$, and $\alpha(\rho) = 1$ otherwise. Thus we have the more delicate theorem,

THEOREM 6′. *Let l be even. Then for any representative ρ of the kernel of $\alpha : I_{2l}(K) \to Z_2$ there is a generic immersion*

$$S^m \xrightarrow{f} S^n$$

with $D(f) \to \Delta(f) = M(\rho) \to Q(\rho)$. Moreover, the codimension k may be any integer $\geqq 2l + 3$ such that $k \to K$ under $Z \to Z_{c(l)}$.

Then Theorem 6 is an immediate corollary of the fact that $\alpha = 0$ if $l \equiv 4$, 6 mod 8, which is proved in [**13**]. As it happens, there are more cases in which $\alpha = 0$: If $l \equiv 0$ mod 8, then $\alpha = 0$ if $K \equiv 2$, 6 mod 8, and if $l \equiv 2$ mod 8, then $\alpha = 0$ if $K \equiv 2$, 6 mod 8. Recently, in [**7**], H. Schneider has extended the results of [**13**] to all free involutions, orientation preserving or not, with $M(\rho)$ homotopy equivalent to $S^l \times S^l$ or $S^l \times S^{l+1}$ and l even or not; we may hope for more examples of double point covers from his results.

In the third section we examine what happens to elements not in the kernel of α. In [**12**], it is shown that $\alpha : I_{16r}(2^{4r} - 8r) \to Z_2$ is an epimorphism, so such examples occur when $l \equiv 0$ mod 8, and it is to this case that we confine our attention. By carrying out the KO-theoretic computation prescribed in the first section, we arrive at the following theorem.

THEOREM 7. *Suppose that ρ represents an element of $I_{2l}(K)$ with $l \equiv 0$ mod 8. Then there exists a generic immersion $f : M \to S^n$ with $D(f) \to \Delta(f) = M(\rho) \to Q(\rho)$.*

Unfortunately, we cannot determine whether M may be assumed to be closed or almost closed. Also, we are not able to determine the codimension k completely.

In the fourth section, we apply some of the theory above to the study of free involutions. That is, we seek functions defined on the ρ's which take on the same value on ρ and ρ' whenever $Q(\rho)$ has the homotopy type of $Q(\rho')$. The first such homotopy invariant is the type, and we have seen in [**12**] and [**13**] that it divides the equivalence classes of the ρ's, we are considering into the disjoint union of the $I_{2l}(K)$, where K runs over the even elements of $Z_{c(l)}$. The next possible homotopy invariant is α. Since we have not done so in an earlier paper, for the sake of completeness we prove the following theorem in the fourth section:

THEOREM 8. *Suppose that* $l \equiv 0 \bmod 8$ *and that* ρ *represents an element of* $I_{2l}(K)$ *not in the kernel of* α. *Then* $Q(\rho)$ *does not have the same homotopy type as* $Q(\rho')$ *for* ρ' *representing an element of the kernel of* $\alpha : I_{2l}(K') \rightarrow Z_2$.

The proof of this theorem actually depends on the work in [**13**] rather than the theory in the present paper. However, using the theory in :he present paper we are able to compute a lower bound for the Conner-Floyd coindex (see [**1**]) of $Q(\rho)$ for ρ representing a certain member of $I_{12}(4)$:

Example. There is a free orientation preserving involution $\rho : S^6 \times S^6 \rightarrow S^6 \times S^6$ such that the Conner-Floyd coindex of $Q(\rho)$ is $\geqq 9$.

As mentioned earlier, $I_{12}(4)$ is a group; the identity is represented by the covering transformation of the universal cover of $S(\gamma)$, where $\gamma \rightarrow P_6$ is a certain 7-plane bundle. Thus, if ρ_0 represents the identity, the coindex of $Q(\rho_0) = 6$ and $Q(\rho_0)$ does not have the homotopy type of $Q(\rho)$ since the coindex is obviously a homotopy invariant. In this case α would not have served to distinguish ρ_0 and ρ in any way since α is trivial on $I_{12}(4)$.

Finally, in the fifth section we note that the machinery of section 2 is precisely what is needed to generalize the Kervaire quadratic map, and therefore we do so in the following situation: N is a c-connected, stably parallelizable manifold. In the case $3m + 3 \leqq 2n$ and $c > 2m - n$, we have by Haefliger's Theorem that each element x of $\pi_m(N)$ is represented by a unique isotopy class of embeddings $\bar{x} : S^m \rightarrow N$. If $x, y \in \pi_m(N)$, we may define as in [**11**] the mutual intersection $\alpha(\bar{x}(S^m), \bar{y}(S^m)) \in \pi_{2m-n}{}^s = \pi_d{}^s$ using the conventions $d + k = m$ and $m + k = n$ as above. We may regard the intersection invariant as an element of $\pi_{d+k}{}^s(S^k)$. And letting $S^k \subset P_\infty/P_{k-1}$ be P_k/P_{k-1}, we obtain a map $\pi_{d+k}{}^s(S^k) \rightarrow \pi_{d+k}{}^s(P_\infty/P_{k-1})$. Denote the image of $\alpha(\bar{x}(S^m), \bar{y}(S^m))$ in $\pi_{d+k}{}^s(P_\infty/P_{k+1})$ by $x \cdot y$. Then $x \cdot y$ is clearly well-defined and bilinear in x and y. We define the generalized Kervaire quadratic map in two steps. First we define $q' : \pi_m(N) \rightarrow \pi_m(0/0(k))$ by letting $q'(x) = \bar{x}^* v(\bar{x}(S^m) : N)$. Then $q'(x) \in \pi_m BO(k)$ and $q'(x) \rightarrow 0$ under $\pi_m BO(k) \rightarrow \pi_m BO$. We assume $k \geqq 13$ (and so $m \geqq 23$) so that the Barrett-Mahowald Theorem applies and we may regard $\pi_m(0/0(k))$, as the kernel of $\pi_m BO(k) \rightarrow \pi_m BO$. Thus we obtain $q'(x) \in \pi_m 0/0(k))$, and it is clearly well defined. Second, we let Imm (S^m, S^n) be the space of immersion $S^m \rightarrow S^n$ modulo the relation of regular homotopy. Under connected sum, it becomes a group, and we have the Smale isomorphism Imm $(S^m, S^n) \rightarrow \pi_m(0/0(k))$. On the other hand, we may obtain a map

$$\text{Imm } (S^m, S^n) \xrightarrow{\varphi} \pi_m{}^s(P/P_{k-1})$$

by letting $P/P_{k-1} = T(k\eta_\infty)$ and for each $\beta \in \text{Imm}(S^m, S^n)$ choosing a generic representative $f : S^m \rightarrow S^n$. As in section two, we obtain a $k\eta_\infty$-manifold $\nu(\Delta(f) : S^n) + r \rightarrow k\eta_\infty + k + r$ and thus, by the Thom-Pontrjagin construction, we obtain an element $\varphi(\beta) \in \pi_m{}^s P/P_{k-1}$. Using the fact that metastable immersions and regular homotopies of closed manifolds may be approximated by generic ones [**3**], and Theorem 4, we see that φ is a well defined isomorphism $\varphi : \operatorname{Imm}(S^m, S^n) \to \pi_n^{\,s}(P/P_{k-1})$. Then composing the inverse Smale isomorphism with φ we obtain an isomorphism

$$\pi_m\left(\frac{0}{0(k)}\right) \xrightarrow{\varphi} \pi_m^{\,s}(P/P_{k-1}).$$

There is such an isomorphism, the James isomorphism, induced by a geometric map $P/P_{k-1} \to O/O(k)$. The author does not know whether the isomorphism above is the James isomorphism. At any rate, we define $q = \psi \circ q'$ and we obtain the following theorem.

Let N be an n-manifold; let m be such that $n - m \geqq 13$ and $2n \geqq 2m + 3$. Suppose N is $(2m - n + 1)$-connected. Set $k = n - m$ and let

$$\pi_m(N) \xrightarrow{q} \pi_m^{\,s}(P/P_{k-1}) \text{ and } \pi_m(N) \otimes \pi_m(N) \xrightarrow{\cdot} \pi_m^{\,s}(P/P_{k-1})$$

be the maps defined above.

THEOREM 9. $q(x + y) = q(x) + q(y) - x \cdot y$.

COROLLARY 1. *The composition*

$$\pi_m(N) \xrightarrow{q} \pi_m^{\,s}(P/P_k)$$

is a homomorphism.

This corollary follows from the fact that $x \cdot y$ comes from $\pi_m^{\,s}S^k$ in the cofibration $S^k \to P/P_{k-1} \to P/P_k$.

COROLLARY 2. *The composition*

$$\pi_m N \xrightarrow{q} \pi_m(0/0(k)) \to \pi_m(0/0(k + 1))$$

is a homomorphism.

This corollary follows from the commutative diagram

$$\begin{array}{ccc} \pi_m(0/0(k)) & \xrightarrow{\psi} & \pi_m^{\,s}(P/P_{k-1}) \\ \downarrow & & \downarrow \\ \pi_m(0/0(k+1)) & \xrightarrow{\psi} & \pi_m^{\,s}(P/P_k) \end{array}.$$

1. 1.1. We begin by defining the special case that we shall need of the transfer in KO theory, and developing some if its properties. Let

$$\bar{X} \xrightarrow{\pi} X$$

be a double cover with $\rho : \bar{X} \to \bar{X}$ the non-trivial covering transformation. Let $\xi \to \bar{X}$ be a vector bundle over $\bar{X}$. Then we may define a vector bundle map $\bar{\rho} : 1^*\xi \oplus \rho^*\xi \to 1^*\xi \oplus \rho^*\xi$ of order two, covering ρ by $\bar{\rho}((x, A_x), (x, B_{\rho x})) = ((\rho x, B_{\rho x}), (\rho x, A_x))$ where $x \in \bar{X}$ A_x is in the fiber of ξ over x and $B_{\rho x}$ is in the fiber of ξ over ρx. Then the quotient $1^*\xi \oplus \rho^*\xi 1\bar{\rho}$ of $1^*\xi \oplus \rho^*\xi$ by the equivalence relation $A \sim A$ and $A \sim \bar{\rho}A$ is a vector bundle $t(\xi) \to X$, which we shall call the *geometric transfer of* ξ. The most natural example of a geometric transfer arises in the case of a generic immersion $f : M \to N$. In this case it is easy to see that if ξ is either $\nu(f)|D(f)$ or $\nu(D(f) : M)$, then $t(\xi) = \nu(\Delta(f) : N)$. Conversely, let $\bar{X} \to X = D \to \Delta$ be a double cover of closed smooth manifolds. Then the linear monomorphism

$$\xi \xrightarrow{f} t(\xi)$$

given by $A_x \to [(x, A_x)(x, 0_{\rho x})]$ (where [ ] denotes equivalence class mod $\bar{\rho}$) is an immersion with only self transverse double points and double point cover $D \to \Delta$—this is Haefliger's construction in [**4**], and F. Uchida solves problem 1 simply by observing that the immersion $P(f + 1) : P(\xi + 1) \to P(t(\xi) + 1)$ has the desired properties. By taking associated cell bundles instead, we obtain the immersion

$$E(\xi) \xrightarrow{E(f)} E(t(\xi))$$

with double point cover $D \to \Delta$ and $\nu(D : E(\xi)) = \xi$. Thus, if dim $\xi \geqq$ dim $D + 3$, we see that problem 2″ has an affirmative solution if and only if $\nu(\Delta : S^n) = t(\xi)$. What we must find ways of determining then, is whether this equality holds or not. For this purpose then we develop some elementary properties of the geometric transfer; let $\approx$ denote bundle equivalence.

PROPOSITION 1. *Let $\bar{X} \to X$ be a double cover, let y be the associated canonical line bondle and t the associated geometric transfer map. Then*

(i) $t(\xi \oplus \zeta) \approx t(\xi) \oplus t(\zeta)$,
(ii) $t(k) \approx ky \oplus k$,
(iii) $t(\rho^*\xi) \approx t(\xi)$,
(iv) $t(\pi^*(\xi)) \approx \xi \otimes (y \oplus 1)$,
(v) $\pi^*t(\xi) \approx \xi \oplus \rho^*\xi$,
(vi) $t(\xi) \otimes t(\zeta) \approx t(\xi \otimes \zeta) \oplus t(\xi \otimes \rho^*\zeta)$,
(vii) $t(\xi) \otimes \eta \approx t(\xi)$,
(viii) *Suppose there is a vector bundle map $\rho' : \xi \to \xi$ of order two, covering ρ. Let ξ/ρ' be the obvious quotient vector bundle over X. Then $t(\xi) \approx \xi/\rho' \oplus \eta \otimes (\xi/\rho')$.*

Proof. All the proofs are straightforward exercises, which we leave to the reader, except for (vii). To prove (vii), set $t = t_+$ and $\bar{\rho} = \bar{\rho}_+$. Define $\bar{\rho} : 1^*\xi \oplus \rho^*\xi \to 1^*\xi \oplus \rho^*\xi$ by $\bar{\rho}_-(A) = -\bar{\rho}_+(A)$. Define $t_-(\xi)$ to be the quotient of $1^*\xi \oplus \rho^*\xi$ by the equivalence relation defined by $\bar{\rho}_-$. Clearly $t_-(\xi) \approx \eta \otimes t_+(\xi)$. However, the map $1^*\xi \oplus \rho^*\xi \to 1^*\xi \oplus \rho^*\xi$ defined by

$$((x, A_x), (x, B_{\rho x})) \to ((x, -A_x), (x, B_{\rho x}))$$

is an equivariant equivalence

$$(1^*\xi + \rho^*\xi, \bar{\rho}_+) \rightarrow (1^*\xi \oplus \rho^*\xi, \bar{\rho}_-).$$

Consequently $t_+(\xi) \approx t_-(\xi)$ and thus (vii) holds.

Next, we define the algebraic transfer operation $t : KO^\sim(\bar{X}) \rightarrow KO^\sim(X)$ by $t[\xi - \dim \xi] = [t(\xi) - (\dim \xi)(\eta + 1)]$. Here [ ] denotes the element of $KO^\sim$ determined by the enclosed virtual bundle. Then we obtain Proposition 2 quite easily from Proposition 1.

PROPOSITION 2. *Let $\bar{X} \rightarrow X$ be a double cover, let $\eta \in KO(X)$ the associated canonical line bundle and $t : KO^\sim(\bar{X}) \rightarrow KO^\sim(X)$ the associated algebraic transfer. Then* (i)-(vii) *of Proposition* 1 *hold with $\xi, \zeta \in KO^\sim(\bar{X})$. If ξ is a vector bundle over $\bar{X}$ and $\rho' : \xi \rightarrow \xi$ as in* (viii) *of Proposition* 1, *then* $t[\xi - \dim \xi] = [\xi/\rho' - \dim \xi/\rho'] + \eta \cdot [\xi/\rho' - \dim \xi/\rho']$.

Remark. We write $+$ for the operation in $KO^\sim$ induced by Whitney sum $\oplus$ and $\cdot$ for the operation induced by $\otimes$. Notice that

$$KO(X) \otimes KO^\sim(X) \rightarrow KO^\sim(X)$$

so that the formulas involving η make sense.

In view of the fact that the geometric transfer does not preserve stable fiber homotopy triviality, it is interesting but trivial that the algebraic transfer does.

PROPOSITION 3. *Suppose $\alpha \in KO^\sim(\bar{X})$ is stably fiber homotopically trivial. Then so is $t(\alpha) \in KO^\sim(X)$.*

Proof. For k large enough, there is a k-plane bundle ξ such that $[\xi - k] = \alpha$ and a fiber homotopy equivalence

$$\begin{array}{ccc} \xi & \xrightarrow{F} & \bar{X} \times R^k \\ \downarrow & & \downarrow \\ \bar{X} & \xrightarrow{1} & \bar{X} \end{array}$$

(that is, the diagram commutes, and F maps each fiber of ξ properly with degree one onto the corresponding fiber of $\bar{X} \times \mathbf{R}^k$). By means of F we obtain

$$1^*\xi \oplus \rho^*\xi \xrightarrow{F \oplus \rho^* F} 1^*k + \rho^*k$$

equivariant with respect to $\bar{\rho}(\xi)$ on the left and $\bar{\rho}(k)$ on the right. Thus we obtain $t(\xi) \rightarrow k\eta + k$, which is a fiber homotopy equivalence. But then

$$t[\xi - k] = [t(\xi) - (k\eta + k)]$$

is stably fiber homotopically trivial.

For computations, the behavior of t with respect to the Atiyah-Hirzebruch spectral sequence is useful to know. Let A be an abelian group, and let $\overline{A + A}$ be the bundle of coefficients over X which on ρ acts by interchanging factors. Let A stand also for the trivial bundle of coefficients over X with fiber A. Then the fiberwise map $\overline{A + A} \to A$ given by sum induces a map $t : H^k(X : \overline{A + A}) \to H^k(X : A)$, which, by means of the spectral sequence of $\bar{X} \to X$ may be regarded as a map $t : H^k(\bar{X} : A) \to H^k(X : A)$. Then in the KO spectral sequence (with $A = 0, Z, Z_2$) we obtain $t : E_2^{pq}(\bar{X}) \to E_2^{pq}(X)$. We have

PROPOSITION 4. *The algebraic transfer is represented on the main diagonal of* E_2 *by* t.

Proof. First we define the relative transfer

$$t : KO^{\sim}(\bar{X}, \bar{A}) \to KO^{\sim}(X, A)$$

where $\bar{A} = \pi^{-1}(A)$. For this definition we use the definition of $KO^{\sim}(B, C)$ as equivalence classes of triples (ξ_1, ξ_2, φ) where ξ_i are bundles over B and $\varphi : \xi_1|C \to \xi_2|C$ is a bundle equivalence. If (ξ_1, ξ_2, φ) is such a triple over $\bar{X}$, $\bar{A}$ then $(t(\xi_1), t(\xi_2), t(\varphi))$ will be such a triple over X, A, where $t(\varphi)$ is defined in the obvious way. Then

$$t[\xi_1, \xi_2, \varphi] = [t(\xi_1), t(\xi_2), t(\varphi)]$$

defines $t : KO^{\sim}(\bar{X}, \bar{A}) \to KO^{\sim}(X, A)$. Now let

$$X = X^n \supset X^{n-1} \supset \ldots \supset X^0 \supset X^{-1} = \emptyset$$

be a skeleton filtration of X and set $\bar{X}^i = \pi^{-1}(X^i)$. Then it is easy to see that the following diagram commutes:

$$\begin{tikzcd}
KO^{\sim}(\bar{X}^r, \bar{X}^{-1}) \arrow[r, "t"] \arrow[d, "\| s"] & KO^{\sim}(X^r, X^{r-1}) \arrow[d, "\| s"] \\
C\,(X^r, X^{r-1} : \overline{KO^{-r}(*) + KO^{-r}(*)}) \arrow[r, "t_{\#}"] \arrow[d, equal] & C^r(X^r, X^{r-1} : KO^{-r}(*)) \\
C^r(\bar{X}^r, \bar{X}^{r-1} : KO^{-r}(*)) &
\end{tikzcd}$$

where the vertical isomorphisms are the ones occurring in the construction of the Atiyah-Hirzebruch spectral sequence and $t_{\#}$ is the representative of t on the chain level (defined by $\overline{A + A} \to A$). Then the proposition follows.

Remark. We have defined the transfer as a map $t : KO^{\sim}(\bar{X}) \to KO^{\sim}(X)$, not as a map $t : KO^{\sim *}(\rho) \to KO^{\sim *}(X)$. It may or may not be possible to define $t : KO^{\sim *}(\bar{X}) \to KO^{\sim *}(X)$, so the proposition says nothing about $t : E_2^{pq}(\bar{X}) \to E_2^{pq}(X)$ off the main diagonal.

1.2. Propositions 2 and 4 give us a weak, but sometimes adequate, mechanism for computing the algebraic transfer. For the cases that will arise in this paper it is adequate. In general, the following proposition reduces the solution of problem 2″ to a computation of the algebraic transfer. Recall that in problem 2″ we are given a double cover of closed manifolds $D \to \Delta$ of dimension d and a vector bundle $\xi \to D$ of dimension k, with $k \geqq d + 3$. We set $m = d + k$ and $m = n + k$.

PROPOSITION 5. *There is a generic immersion* $f : M \to S^n$ *with double point cover* $D(f) \to \Delta(f) = D \to \Delta$ *and* $\nu(D(f) : M) = \xi$ *if and only if* $t[\xi - k] = \nu(\Delta) - k[\eta - 1]$.

Proof. If there is such an immersion, we know that the geometric transfer $t(\xi)$ is $\nu(\Delta : S^n)$. Thus $t[\xi - k] = [t(\xi) - k\eta - k] = [\nu(\Delta : S^n) - 2k] + [k - k\eta]$. Conversely, if $t[\xi - k] = \nu(\Delta) - k[\eta - 1]$, then $[t(\xi) - k\eta - k] = \nu(\Delta) - k[\eta - 1]$ so $t(\xi)$ is stably equivalent to $\nu(\Delta : S^n)$. But since both are stable bundles, they are equivalent and we may assume that $E(t(\xi))$ is a tubular neighborhood of Δ in S^n. Let $g : \xi \to t(\xi)$ be Haefliger's immersion. Let f be the composition

$$M = E(\xi) \xrightarrow{E(g)} E(t(\xi)) \subset S^n$$

Then f is a generic immersion with $D(f) \to \Delta(f) = D \to \Delta$ and

$$\nu(D(f) : M) = \xi.$$

1.3. Now we specialize to the case that Δ is homotopy equivalent to P_d and D is its universal cover. Then D is homotopy equivalent to S^d. Then $KO^\sim(D)$ is cyclic and we pick a generator α; $\alpha = 0$ in case $KO^\sim(S^d) = 0$. Then the vector bundle $\xi \to D$ is a stable bundle and $[\xi - k] = s\alpha$ for some s—we make the convention that $s = 0$ if $\alpha = 0$ and $s = 0$ or 1 if $2\alpha = 0$. Having made these remarks, we may prove Theorem 1.

1.4 *Proof of Theorem* 1.

LEMMA 1. *If* $\alpha \neq 0$ *then* $t(\alpha) = 2^{\varphi-1}(\eta - 1)$.

Proof. The maps

$$H^d(D) \xrightarrow{t} H^d(\Delta) \text{ and } H^d(D : Z_2) \xrightarrow{t} H^d(\Delta : Z_2)$$

are epimorphisms. Now the lemma follows immediately from Proposition 4.

LEMMA 2. *Let* $d \geqq 10$. *Then there is a generic immersion* $f : E(\xi) \to S^n$ *if and only if*:

(i) s *is even and* $k = c(d) - d - 1 + rc(d)$ *where* $r = 0, 1, 2, \ldots$ *or*

(ii) s *is odd and* $k = c(d)/2 - d - 1 + rd(d)$ *where* $r = 0, 1, 2, \ldots$.

Proof. Suppose such an immersion exists. Then by Proposition 5 and Lemma 1 we have

$$s(c(d)/2)(\eta - 1) = (c(d) - d - 1 - k)(\eta - 1)$$

so, if s is even $d + k + 1 \equiv 0 \bmod c(d)$ and if s is odd, $c(d)/2 + d + k + 1 \equiv 0 \bmod c(d)$. Since the left hand side in each case is positive, we must have it equal to $(r + 1)c(d)$ for some $r \geqq 0$. Then one of the lemma follows. For the converse implication, notice that $k \geqq d + 3$ in case (i) and (ii). Whether (i) of (ii) holds, it follows that $t[\xi - k] = \nu(\Delta) - k[\eta - 1]$. Then the proof of the second half of Proposition 5 gives us our immersion $f : E(\xi) \to S^n$. Now Lemma 2 is proved.

To prove Theorem 1, suppose we have a generic immersion $f : M \to S^n$ with $D(f) \to \Delta(f) = D \to \Delta$ and $\xi = \nu(D : M)$, and $\partial M = S^{m-1}$. By restricting to a tubular neighborhood of D in M, we obtain a generic immersion $E(\xi) \to S^n$ and from Lemma 2 conclude that case (i) or case (ii) holds. Conversely, suppose that case (i) or case (ii) holds. Then Lemma 2 gives us a generic immersion $f : E(\xi) \to S^n$. Since $k + d < n = d + 2k$, and since $2k + 1 < n = d + 2k$, we may extend the embedding given by a fiber of $S(\xi) \to D$, $S^{k-1} \subset S(\xi) \to S^n$, to an embedding $E^k \subset S^n$ which meets the image of f exactly on $f(S^{k-1})$, with outward normal there equal to the inward normal of f. Now, $\nu(i(E^k) : S^n) = i(E^k) \times \mathbf{R}^{k+d}$. Let $\mathscr{F}$ be the framing of $\nu(s^{k-1} : S(\xi))$ which extends to a framing of $\nu(E^k : E(\xi))$ where E^k is a fiber of $E(\xi) \to D$. Using $\nu(i(E^k) : S^n) = E^k \times \mathbf{R}^{d+k}$, the framing $\mathscr{F}$ determines an element of $\pi_{k-1}[(SO(d + k))/(SO(k))] = 0$. Thus the framing $\mathscr{F}$ extends to a framing of a d-subbundle of $\nu(i(E^k) : S^n)$. It follows that if M is the smooth manifold obtained from $E(\xi)$ by attaching a handle along $S^{k-1}S(\xi)$ by means of $\mathscr{F}$, then the immersion f extends to a generic immersion $g : M \to S^n$. To see that $\partial M = S^{m-1}$, notice that ∂M is the manifold obtained for $S(\xi)$ by surgering $(S^{k-1}, \mathscr{F})$. But that surgery may be obtained by drilling out a thickened fiber of $E(\xi) \to D$. Thus ∂M is the boundary of a contractible manifold; since $\dim \partial M \geqq 5$ and $\pi_1(\partial M) = 0$, it follows that $\partial M = S^{m-1}$ by Smale's Theorem. Now Theorem 1 is proved.

1.5. *Proof of Theorem* 2. Suppose a generic immersion $f : S^m \to S^n$ exists with $D(f) \to \Delta(f) = D \to \Delta$. Since $D \subset S^m$ is a stable embedding of a homotopy sphere in S^m, we have $\nu(D : S^m)$ is trivial, and we are in case i). Conversely suppose $k = c(d) - d - 1 + rc(d)$ with $r = 0, 1, 2, \ldots$. Then $k \geqq d + 3$ and Lemma 2 applies to produce a generic immersion $f : D \times E^k \to S^n$ with double point cover $D \to \Delta$. Now, $D \times E^k = S^d \times E^k \subset S^m$. The complement is $E^{d+1} \times S^{k-1}$. Since $m + d < n = m + k$, the immersion f may be extended to a map $g : S^m \to S^n$ such that $g : E^{d+1} \times S^{k-1} \to S^n - T$ where T is a suitable tubular neighborhood of Δ in S^n, and such that g is a collared embedding near $S^d \times S^{k-1}$. We may then homotope g modulo a neighborhood of $D \times E^k$ to preserve those two properties and so that g becomes a generic map in the sense of Haefliger [**3**]. Then the singularities of g are in $E^{d+1} \times S^{k-1}$. Since $\pi_i(E^{d+1} \times S^{k-1}) = 0$ for $i < k - 1$ and $\pi_i(S^n - \Delta) = 0$ for $i < n - d - 1$, Haefliger's construction in [**3**] gives a homotopy modulo a neighborhood of $D \times E^k$ from g to a map h which extends f to an embedding $E^{d+1} \times S^{k-1} \to S^n - T$. Then h is a generic immersion $S^m \to S^n$ with double point cover $D \to \Delta$, and Theorem 2 is proved.

In section two, we will obtain a refinement of Theorem 2 as a corollary to Theorem 4.

1.6. *Proof of Theorem* 3. Assume that we do have a generic immersion $f : M \to S^n$ with M closed and orientable, $D(f) \to \Delta(f)$ the universal cover of a homotopy P_d with $d \equiv 0 \bmod 4$ and $[\nu(D(f) : M) - k] \not\simeq 0$. From Lemma 2 we see that k must be odd.

Let $X : M \to E(\nu(f))$ be section transverse regular to the zero section. Let $Z = X^{-1}(0)$; then Z is a closed orientable d-manifold and we may assume that $Z \cap D(f) = \emptyset$.

LEMMA 3. *With suitable orientations Z and $D(f)$ represent the same homology class in $H_d(M)$.*

Proof. Let $F : E(\nu(f)) \to S^n$ be an immersion extending f. Then $F \circ X$ and f are two generic immersions of M, and they intersect in $f(Z)\, f(D'(f))$, where $D'(f)\ M$ is isotopic to $D(f)$. Then $F \circ X(M)$ may be ambient isotoped in S^n until it is disjoint from $f(M)$. We may assume that the isotopy Φ_t satisfies

(1) $\Phi_t(F \circ X(M)) \cap \Delta(f) = \emptyset$ for $t \in [0, 1]$,

(2) $\Phi_t(\Delta(F \circ X)) \cap f(M) = \emptyset$ for $t \in [0, 1]$,

(3) $\Phi : (F \circ X(M) - \Delta(F \circ X)) \times I \to S^n \times I$,

is transverse regular along $(f(M) - \Delta(f)) \times I$. Then Φ generates an oriented cobordism in $M \times I$ from $D'(f)\ Z$ to the empty set, proving the lemma.

To prove the theorem, recall that $d \equiv 0 \bmod 4$ so that k is odd. Let $\chi \in H^k(M)$ be the Euler class of $\nu(f)$. Then $\chi \cap [M] = [Z]$. And $[Z] = \pm[D(f)]$ by the lemma, with $D(f)$ a homotopy d-sphere. We are assuming that $\nu(D(f) : M) = \xi$ is stably non-trivial. Let p be the $(d/4)$th Pontrjagin class of ξ; then $p[D(f)]$ is a non-zero integer. But ξ is equivalent to $\rho^*(\nu(f)|D(f))$ so that $\rho'[D(f)]$ is a non-zero integer for $\rho' = (d/4)$th Pontrijagin class of $\nu(f)|D(f)$. Finally, let $p'' = (d/4)$th Pontrjagin class of $\nu(f)$. Then

$$p'[D(f)] = p'' \cap [D(f)] = \pm p'' \cap [Z] = \pm p'' \cap (\chi \cap [M]) = \pm(p'' \cup \chi) \cap [M].$$

Consequently $(p'' \cap \chi) \cap [M]$ is a non-zero integer. But k is odd so $2\chi = 0$, so $2(p'' \cup \chi) = 0$ and consequently $(p'' \cup \chi) \cap [M] = 0$. This contradiction completes the proof of the theorem.

1.7. To obtain the corollary of Theorem 3, we consider the special case where $\xi \to S^d$ is a k-dimensional representative of s times the generator of $KO^\sim(S^d)$ and $d \equiv 0 \bmod 4$, and ξ is such that $[\xi - \dim \xi]$ is not in the kernel of the J-homomorphism. Once we have chosen the generator, this condition determines ξ up to equivalence since $k \geqq d + 3$. We are assuming that $d \geqq 10$ and that k is one of the numbers $c(d) - d - 1 + rc(d)$, $r = 0, 1, 2, \ldots$, if s is even, or one of the numbers $c(d)/2 - d - 1 + rc(d)$, $r = 0, 1, 2, \ldots$, if s is odd. Then Theorem 1 supplies us with a generic immersion $f : M \to S^n$ such that $\partial M = S^{m-1}$, $D(f) \to \Delta(f) = S^d \to P_d$ and $\nu(S^d : M) = \xi$. Suppose that $\nu(f)|S^{m-1}$ is trivial. Then we have an immersion-with-transverse field defined by $S^{m-1} = \partial M \to S^n$ and the transverse field equal to the outward normal of f. By Hirsch's Theorem, this immersion with transverse field is regularly homotopic to an immersion

$$S^{m-1} \xrightarrow{g} S^{n-1}$$

with transverse field the restriction of the outward normal of $S_+{}^n$ along S^{n-1}. We will have $\nu(g) = \nu(f)|S^{m-1}$; since we are supposing $\nu(f)|S^{m-1}$ is trivial, it follows from the Barratt-Mahowald Theorem by an argument of R. Lashof that we may take g to be a representative of the unique class of embeddings $S^{m-1} \subset S^{n-1}$: $\pi_{m-1}(O/O(k))$ represents the regular homotopy classes of immersions of S^{m-1} in S^{n-1} by Smale's Theorem. It is straightforward to check that the map $\pi_{m-1}(O/O(k)) \to \pi_{m-1}(BO(k))$ induced by $O/O(k) \to BO(k)$ assigns to each such immersion its normal bundle. Consequently the statement that normal bundles distinguish immersions is equivalent to the statement that $\pi_{m-1}(O/O(k)) \to \pi_{m-1}(BO(k))$ is a monomorphism. But in the dimension range above the Barratt-Mahowald Theorem states that $\pi_{m-1}(O/O(k)) \to \pi_{m-1}(BO(k))$ is a monomorphism. In particular then, since Haefliger's Embedding Theorem implies that in this dimension range there is only one isotopy class of embeddings $s^{m-1} \subset s^{n-1}$, it follows that an immersion $s^{m-1} \to s^{n-1}$ is regularly homotopic to an embedding if its normal bundle is trivial.

The diagram,

$$\begin{tikzcd} \pi_{m-1}\left(\dfrac{0}{0(k)}\right) \arrow[r] & \pi_{m-1}BO(k) \\ \mathrm{Imm}(S^{m-1}, S^{n-1}) \arrow[u] \arrow[ur, "\nu"'] & \end{tikzcd}$$

commutes, where vertical isomorphism is the Smale isomorphism and ν the map $f \to \nu(f)$. Thus there is a unique regular homotopy class of immersions $S^{m-1} \to S^{n-1}$ with trivial normal bundle, and it must be the class containing the embeddings $S^{m-1} \subset S^{n-1}$ (that there is a unique isotopy class of embeddings follows immediately from Haefliger's Theorem [**3**]). But the embedding

$S^{m-1} \subset S^{n-1}$ with transverse field the outward normal of S_+^n restricted to S^{m-1} may be capped by $E^m \subset S_-^n$ with inward normal equal to the outward normal of S_+^n restricted to S^{m-1}. Consequently, since the codimension is greater than zero and we have a covering isotopy theorem for immersions, it follows that the embedding $S^{m-1} = \partial M \subset S^n$ may be capped with an immersion

$$E^m \xrightarrow{h} S^n$$

with inward normal equal to the outward normal of f. We may assume that h is generic, that h misses $\Delta(f) = P_d \subset S^n$ and that h meets $f(M) - \Delta(f)$ transversally. Then we define a generic immersion $F : M \cup E^m \to S^n$ of the closed manifold $M \cap E^m$ by $F = f \cap h$. We will have that $D(F)$ is the disjoint union $D(f) \cup D(h) \cup h^{-1}f(M) \cup f^{-1}h(E^m)$ with $\rho : D(f) \to D(f)$ and $\rho : D(h) \to D(h)$ and ρ interchanging $h^{-1}f(M)$ and $f^{-1}h(E^m)$. Moreover $D(h) \cup h^{-1}f(M) \subset E^m$ so $\nu(F)|D(h) \cup h^{-1}f(M)$ is trivial. Thus $\nu(f^{-1}h(E^m) : M)$ is trivial. For brevity we write $L = f^{-1}h(E^m)$. By the methods of [**11**], we may assume that L is connected. By general position, we may assume $L \subset E(\xi)$ so that the composition $L \subset E(\xi) \to S^d$ gives us a map $j : L \to S^d$ such that $J^*(\nu(f)|S^d)$ is stably $\nu(L)$. On the other hand, the argument proving Theorem 3 shows that if p is the $(d/4)$th Pontrjagin class of $\nu(F)$, then $p[D(F)] = 0$. Since $\nu(F)|D(h) \cup h^{-1}f(M)$ is trivial, we must have $p[L] + p[S^d] = 0$. Let $\beta \in KO^\sim(S^d)$ be $[\nu(f) - k]$. Then we have $J^*\beta = [(\nu(F)|L) - k]$; and letting p' be the $(d/4)$th Pontrjagin class of β,

$$J^*p'[L] + p'[S^d] = 0$$

that is, $p'[J_*L] + p'(S^d] = 0$. But then J is a map of degree minus one and β must be in the kernel of the J-homomorphism. But then so must $-\beta = [\xi - k]$ be, contrary to hypothesis. This contradiction shows that our initial assumption that $\nu(f)|S^{m-1}$ is trivial must be false. Finally then, $\nu(f)|S^{m-1}$ is a nontrivial element of $\ker \pi_{m-1}(BO(k) \to \pi_{m-1}(BO(k+1))$, and the construction proves the corollary.

2. 2.1. As above, we continue to assume $k \geqq d + 3$ and we continue the convention $m = d + k$ and $n = m + k$. Now we add the assumption that M is c-parallelizable with $1 + d \leqq c$. We may assume $k \geqq c + 2$. Let M^c be a regular neighborhood of a c-skeleton of M. Then a c-parallelization of M amounts to a choice of a framing isotopy class of framings of the stable normal bundle of M restricted to M^c. Let $f : M \to S^n$ be a generic immersion. Since we are assuming that $k \geqq c + 2$, a c-parallelization of M amounts to a choice of a framing isotopy class of framings of $\nu(f)|M^c$. Since we are assuming that $c \geqq d + 1$ we may assume that $D(f)$ M^c; in fact this embedding is uniquely determined up to isotopy. Choose a c-parallelization $\mathscr{F}$ of M and let $\mathscr{F}' : \nu(f)|M^c \to R^k$ be a representative of $\mathscr{F}$. Then $\mathscr{F}'$ determines an

https://doi.org/10.4153/CJM-1974-016-1 Published online by Cambridge University Press

equivariant bundle map

$$1^*(\nu(f)|D(f)) \oplus \rho^*(\nu(f)|D(f)) \rightarrow S^\infty \times h^k \times R^k$$

by $((x, A_x), (x, B_{\rho x})) \rightarrow (\bar{\omega}(x), \mathscr{F}'(A_x), \mathscr{F}'(B_{\rho x}))$ where $\omega : \Delta(f) \rightarrow P_\infty$ is the classifying map of the double cover $D(f) \rightarrow \Delta(f)$ and

$$D(f) \xrightarrow{\bar{\omega}} S^\infty$$

a covering of ω. The Z_2 action on the left is defined by ρ and the Z_2 action on the right by $(y, u, v) \rightarrow (-y, v, u)$. Thus we obtain a bundle map $t(\nu(f)|D(f)) \rightarrow k\eta_\infty + k$. Another representative of $\mathscr{F}$ (involving another choice of M^c and $\mathscr{F}_0$) determines a bundle map $t(\nu(f)|D(f)) \rightarrow k\eta_\infty + k$ bundle isotopic to the first. Thus we have determined by $\mathscr{F}$ a bundle isotopy class of bundle maps $t(\nu(f)|D(f)) \rightarrow k\eta_\infty + k$. But then the canonical equivalence

$$t(\nu(f)|D(f)) \overset{\cong}{\rightarrow} \nu(\Delta(f): S^n)$$

transforms this class to a bundle isotopy class of bundle maps $\nu(\Delta(f) : S^n) \rightarrow k\eta_\infty + k$. But this class is precisely a $k\eta_\infty$-manifold in the sense of [**10**]; we will denote it by $\delta(f, \mathscr{F})$, and we will denote by $\alpha(f, \mathscr{F})$ the element of $\pi_m{}^s(T(k\eta_\infty))$ it determines via the Thom-Pontrjagin construction. Notice that $\alpha(f, \mathscr{F}) = \alpha(g, \mathscr{F})$ if f and g are generically regularly homotopic; in the case that $\partial M = \emptyset$, we have $\alpha(f, \mathscr{F}) = \alpha(g, \mathscr{F})$ if f and g are regularly homotopic.

2.2. Now we assume that M is c-connected and oriented. For M^c we may allow only copies of E^m and thus we see that there is only one c-parallelization compatible with the orientation. From now on we use only this c-parallelization. Consequently, we may omit further mention of $\mathscr{F}$ and write $\delta(f)$ and $\alpha(f)$.

2.3. *Proof of Theorem* 4. Before embarking on the proof, we assume that $f : M \rightarrow \mathbf{R}^n \subset S^n$. We give $\mathbf{R}^n$ the usual Riemann metric and pull it back to M so that all the manifolds that will appear in the proof have Riemann metrics. Our main use of this fact will be to define parallelization by means of orthonormal vector fields. For instance, we have $e_1, \ldots, e_k : M^c = E^m \rightarrow v(f)|M^c$ orthonormal sections and $\mathscr{F}' : \nu(f)|M^c \rightarrow \mathbf{R}^k$ may be defined by $\mathscr{F}'(\Sigma s_i e_i(x)) = (s_1, \ldots, s_k)$. Recall the map $f' : \nu(f) \rightarrow \tau(S^n)$. After some tedious isotoping and straightening we may assume that $df(x)(\nu(D(f) : M)_x) = f'(\nu(f)_{\rho x})$. We obtain an orthonormal framing of $\nu(D(f) : M)$ by defining $df(x)e_j'(x) = f'(e_j(\rho x))$. The fields $(e_1', \ldots, e_k')$ define an orthonormal framing of $\nu(D(f) : M)$.

We suppose that we are given a $k\eta_\infty$-surgery of $\delta(f)$. We may regard it as being given by a collared embedding $X \subset S^n \times I$ satisfying several conditions: We set $\partial_i X = \partial X \cap S^n \times i$; then we assume that $\partial_0 X = \Delta(f)$. We assume that there is an embedding $\iota : S^p \subset \Delta(f)$ that extends to an embedding

$\iota' : E^{p+1} \subset X$ such that X collapses to $\Delta(f) \cup E^{p+1}$. Moreover we assume that there is a bundle map $\nu(X : S^n \times I) \to k\eta_\infty + k$ extending a representative of $\delta(f)$; having made this assumption, we see that $\delta(f)$ has a representative with base map carrying $\iota(S^p)$ to a point in P_∞ and that we may assume that the base map of $\nu(X : S^n \times I) \to k\eta_\infty + k$ carries $\iota'(E^{p+1})$ to a point in P_∞. Then the map $\nu(X : S^n \times I)|\iota'(E^{p+1}) \to k\eta_\infty + k$ is a framing $\mathscr{S}$ of a subbundle $\nu(\iota'(E^{p+1}) : S^n \times I)$ which restricts to $\nu(\Delta(f) : S^n)$ over $\iota(S^p)$. Then ι and $\mathscr{S}$ determine the surgery $\nu(X : S^n \times I) \to k\eta_\infty + k$. In addition we have a framing $g_1, \ldots, g_{d-p}$ of $\nu(\iota(S^p) : \Delta(f))$ which extends to a framing of $\nu(\iota'(E^{p+1}) : X)$.

Since

$$S^p \xrightarrow{S} \Delta(f) \to P_\infty$$

is trivial, there are two lifts $\bar{\iota} : S^p \to D(f)$ and $\rho \circ \bar{\iota}$ of ι. The framing $\mathscr{S} \circ \iota$ may be taken to be

$$x \to f' \circ e_1 \circ \bar{\iota}(x), \ldots, f' \circ e_k \circ \bar{\iota}(x), f' \circ e_1 \circ \rho \circ \bar{\iota}(x), \ldots, f' \circ e_1 \circ \rho \circ \bar{\iota}(x).$$

The framing $g_1, \ldots, g_{d-p}$ defines framings $g_1^0, \ldots, g_{d-p}^0$ of $\nu(\bar{\iota}(S^p) : D(f))$ and $(g_1^1, \ldots, g_{d-p}^1)$ of $\nu(\rho \circ \bar{\iota}(S^p) : D(f))$ such that $d\rho(g_i^0) = g_i^1$.

From now on we will set $\bar{\iota} = \iota_0 = \iota_1$. Then ι_0 and ι_1 extend to disjoint embeddings $\iota_j' : E^{p+1} \to M$ with outward normals $e_1'|\iota_j(S^p)$. Since $p - 1 < d + k - p - 1 - (d - p)$ we have $\pi_{p-1}(V_{d-p,d+k-p-1}) = 0$ and framings $e_2', \ldots, e_k'|\iota_j(S^p)$ extends to framings $(e_2^0, \ldots, e_k^0)$ and $(e_2', \ldots, e_k')$ of subbundles of $\nu(\iota_0'(E^{p+1}) : M)$ and $\nu(\iota_1'(E^{p+1}) : M)$ respectively. The framings $(g_1^j, \ldots, g_{d-p}^j)$ extend to framings $(G_1^j, \ldots, G_{d-p}^j)$ of complementary subbundles of $\nu(\iota_j'(E^{p+1}) : M)$. Thus $(e_2^j, \ldots, e_k^j, G_1^j, \ldots, G_{d-p}^j)$ is an orthonormal framing of $\nu(\iota_j'(E^{p+1}) : M)$ for $j = 0, 1$.

Now we define three bundles over the topological $p + 1$ sphere $\Sigma = f(\iota_0'(E^{p+1})) \cup f(\iota_1'(E^{p+1}))$. We write $\Sigma_i = f(\iota_i'(E^{p+1}))$. Then the three bundles are:

$$\begin{aligned}
\xi\colon \xi|\Sigma_0 &= \text{span } (f' \circ e_2 \circ \iota_0', \ldots, f' \circ e_k \circ \iota_0')\\
\xi|\Sigma_1 &= \text{span } (df(e_2^0 \circ \iota_1'), \ldots, df(e_k^0 \circ \iota_1'))\\
\zeta\colon \zeta|\Sigma_0 &= df(\nu(\iota_0'(E^{p+1}) : M)\\
\zeta|\Sigma_1 &= df(\text{span } (G_1', \ldots, G_{d-p}')) \oplus \text{span } (f' \circ e_2 \circ \iota_1', \ldots, f' \circ e_k \circ \iota_1')\\
\sigma\colon \sigma|\Sigma_0 &= \tau(\Sigma_0) \oplus \text{span } (f' \circ e_1 \circ \iota_0')\\
\sigma|\Sigma_1 &= \tau(\Sigma_1) \oplus \text{span } (f' \circ e_1 \circ \iota_1').
\end{aligned}$$

We obtain a framing of ξ out of $(e_2, \ldots, e_k)$, so ξ is trivial. Moreover, $\xi \oplus \zeta \oplus \sigma = \tau(S^n)|\Sigma$ is trivial. Since $\zeta = k + d - p - 1 > p + 1$, we will have that ζ is trivial if and only if σ is trivial.

It is easy to see that there is a smooth $(p + 2)$-cell $\mathscr{D}^{p+2} \subset S^n$ with a corner along $\Sigma_0 \cap \Sigma_1$ and $\partial\mathscr{D}^{p+2} = \Sigma_0 \cup \Sigma_1$. We may assume that $\mathscr{D}^{p+2}$ has outward normal $f' \circ e_1 \circ \iota_0'$ along Σ_0 and $f' \circ e_1 \circ \iota_1'$ along Σ_1; later we will need that we may assume that $\mathscr{D}^{p+2} \cap f(M) = \Sigma$. Immediately we see that $\sigma = \tau(\mathscr{D}^{p+2})|\Sigma$ so that σ is trivial and consequently ζ is trivial.

Since ζ is trivial, it may be extended to a subbundle ζ' of $\nu(\mathscr{D}^{p+2} : S^n)$. The framing $(e_2{}^0, \ldots, e_k{}^0, G_1{}^0, \ldots, G_{d-p}{}^0)$ may be extended to a framing of all ζ'. Then that framing and the exponential map may be used to produce an embedding (of codimension zero)

$$(\mathscr{D}^{p+2} \times R^{d+k-p-1} \times R^{k-1}, \mathscr{D}^{p+2} \times R^{d+k-p-1} \times 0) \rightarrow (S^n, \exp \zeta').$$

Then the isotopy of [**11**, Section 5] carries over via this embedding to define a regular homotopy f_t of f. This regular homotopy realizes the given surgery.

2.4. *Proof of Theorem* 5. Now we apply Theorem 4 to the case of $M = S^m$. In the metastable range, Haefliger's general position theorems [**3**] show that the set Imm (S^m, S^n) of immersions modulo regular homotopy may be regarded as the set of generic immersions modulo generic regular homotopy. Regarding Imm (S^m, S^n) as defined in this latter way, α gives us a well defined map

$$\alpha : \text{Imm } (S^m, S^n) \rightarrow \pi_m{}^s T(k\eta_\infty).$$

Connected sum of immersions turns Imm (S^m, S^n) into a group (via the Smale isomorphism) and it is clear that α is a homomorphism with respect to this group structure. It follows immediately from Theorem 4 (or from Haefliger's Theorem in [**4**]) that the kernel of α is 0. The Smale isomorphism tells us that Imm $(S^m, S^n) \cong \pi_m(O/O(k))$ and the James isomorphism tells us that $\pi_m(O/O(k)) \cong \pi_m{}^s T(k\eta_\infty)$. But both are finite groups. Consequently, α is an epimorphism as well, and so an isomorphism.

Now let $\nu(\Delta : S^n) \rightarrow k\eta_\infty + k$ be an arbitrary $k\eta_\infty$ manifold of dimension d. By the Thom-Pontrjagin construction, it determines an element $\alpha_1 \in \pi_m{}^s T(k\eta_\infty)$. But we have just seen that there is a generic immersion $f_0 : S_m \rightarrow S^n$ such that $\alpha(f_0) = \alpha_1$. From the definition of α, it follows that the $k\eta_\infty$-manifold $\delta(f_0) : \nu(\Delta(f_0) : S^n) \rightarrow k\eta_\infty + k$ and $\nu(\Delta : S^n) \rightarrow k\eta_\infty + k$ are $k\eta_\infty$-cobordant. Consequently there is a finite sequence of $k\eta_\infty$-surgeries leading from $\delta(f_0)$ to $\nu(\Delta : S^n) \rightarrow k\eta_\infty + k$. Theorem 4 implies that there is a corresponding sequence of generic regular homotopies realizing these $k\eta_\infty$-surgeries. Then these generic regular homotopies may be put together to produce a generic regular homotopy f_t starting with f_0 and ending with f_1 such that $\delta(f_1) = \nu(\Delta : S^n) \rightarrow k\eta_\infty + k$.

Thus we have proved the following proposition, which is stronger than Theorem 5, and from which Theorem 5 follows as an immediate corollary:

PROPOSITION 6. *As always, let* $k \geqq d + 3$, $m = d + k$, $n = m + k$. *Suppose* $\nu(\Delta : S^n) \rightarrow k\eta_\infty + k$ *is a closed* $k\eta_\infty$*-manifold of dimension* d. *Then there is a generic immersion* $f : S^m \rightarrow S^n$ *such that* $\delta(f) = \nu(\Delta : S^n) \rightarrow k\eta_\infty + k$.

3. 3.1. As we have seen in the preceding section a closed d-manifold occurs as the double point set of some generic immersion if and only if the classifying map of the stable normal bundle

$$\Delta \xrightarrow{\nu(\Delta)} BO$$

factors through the map $P_\infty \to BO(k) \to BO$, where $P_\infty \to BO(k)$ classifies $k\eta_\infty$. In this paper we are concentrating our attention on the simplest examples of Δ available, namely homotopy P_d's and quotients of homotopy $S^l \times S^{l}$'s by free oriented involutions. In the first of these cases it is well-known that the factorization above exists for any $k = rc(d) - d - 1$, $r = 1, 2, \ldots$; hence the realization theorem for homotopy projective spaces. In the other of these cases, we have seen for example that if $l \equiv 4, 6 \bmod 8$ then such a factorization exists. However, in [**12**] the author constructs free oriented involutions ρ of homotopy $S^l \times S^{l}$'s with $l \equiv 0 \bmod 8$ such that $\nu(Q(\rho)) - k(\eta - 1)$ is not even stably fiber homotopically trivial for any k. Thus certainly $M(\rho) \to Q(\rho)$ does not appear as the double point cover of any generic immersion $S^m \to S^n$. Even more, no manifold Q' homotopy equivalent to $Q(\rho)$ can appear as the double point manifold of any generic immersion $S^m \to S^n$. At first glance, one may suspect that even problem 2″ cannot be solved affirmatively for these double covers, but as it happens, the KO-theoretic computation prescribed by Proposition 5 answers problem 2″ affirmatively for these involutions. In fact, that computation answers problem 2″ affirmatively for any free oriented involution ρ with $M(\rho)$ homotopy equivalent to $S^l \times S^l$ and $l \equiv 0 \bmod 8$, hence Theorem 7. Now we proceed to carry out that computation to prove Theorem 7—what we will not be able to answer are problems 2 and 2′ for these double covers.

3.2. *Proof of Theorem* 7. Let ρ be a free oriented involution with $M(\rho)$ homotopy equivalent to $S^l \times S^l$ with $l \equiv 0 \bmod 8$. Then $k(\rho) \equiv 0, 2, 4, 6 \bmod 8$. In the two cases $k \equiv 2, 6 \bmod 8$ we have seen by [**13**] that $\alpha : I_{2l}(k) \to Z_2$ is trivial so that if $k(\rho) \equiv 2, 6 \bmod 8$, then $\nu(Q(\rho))$ factors through the map

$$P_\infty \xrightarrow{k\eta_\infty} BO(k) \to BO$$

for any $k \to k(\rho)$ under $Z \to Z_{c(l)}$. If, in addition, $k \geqq 2l + 3$, then the realization theorems of Section 2 supply us with generic immersions

$$S^m \xrightarrow{f} S^n$$

such that $D(f) \to \Delta(f) = M(\rho) \to Q(\rho)$. Thus in this case the generic immersions of Theorem 7 exist, and are actually generic immersions of spheres.

Now we may assume that $k(\rho) \equiv 0 \bmod 4$. In this case, we will compute $t : KO^\sim(M(\rho)) \to KO^\sim(Q(\rho))$ in order to prove Theorem 7 via Proposition 5. To compute t, we will construct homotopy r-skeletons of $Q(\rho)$ for $r \leqq l + 2$; a *homotopy r-skeleton of a space X* is a *CW* complex K of dimension $\leqq r$ such that there exists a map $g : K \to X$ with the property that $\pi_i(M, K) = 0$ for $i \leqq r$, where M is the mapping cylinder of g.

In [**13**] it is shown that

$$Q(\rho) \cong E(\gamma) \underset{\psi}{\cup} E(\gamma)$$

where $\gamma \to P_l$ is a certain l-plane bundle and $\psi : S(\gamma) \to S(\gamma)$ a diffeomorphism. The zero section of $E(\gamma)$ gives us an embedding $P_l \subset Q(\rho)$ which lifts to $S^l \subset M(\rho)$ representing an indivisible element of $H_l(M(\rho)) = Z + Z$. In [**12**] is constructed an embedding $S^l \subset Q(\rho)$ that meets P_l transversally at exactly one point. Thus we obtain a map $P_l \vee S^l \to Q(\rho)$ and using [**12**] it is easy to see that this map makes $P_l \vee S^l$ a homotopy l-skeleton of $Q(\rho)$. Now let $j \in \pi_l(P_l \vee S^l)$ be represented by the inclusion $S^l \subset P_l \vee S^l$ into the second term. Let $\rho \in \pi_1(P_l \vee S^l)$ be the non-zero element. Then it is straightforward to see that

$$Q_{l+1} = P_l \vee S^l \underset{j+\rho j}{\cup} D^{l+1}$$

is a homotopy $(l + 1)$-skeleton for $Q(\rho)$. Let ρ denote also a generator of $\pi_1(P_1 \vee S^l)$—this abuse of language, letting ρ stand for three different objects so far, is consistent. Then it is easy to see that we may write

$$Q_{l+1} = P_l \underset{P_1}{\cup} ((P_1 \vee S^l) \underset{j+\rho j}{\cup} D^{l+1}).$$

But $(P_1 \vee S^l) \cup_{j+\rho j} D^{l+1}$ is $S(\eta_1 + l) = S((l + 1)\eta_1) = \sigma$ in the notation of [**12**]. The embedding $P_1 \subset \sigma$ is given by any cross section of $S((l + 1)\eta_1) \to P_1$. Thus $Q_{l+1} = P_l \cup_{P_1} \sigma$. Let $\bar{Q}_{l+1} \to Q_{l+1}$ be the universal cover of Q_{l+1}; let $S^1 \times S^l \to \sigma$ be the double cover induced by $S^1 \to P_1$. Then we have that

$$\bar{Q}_{l+1} = S^l \underset{S^1}{\cup} (S^1 \times S^l),$$

which has the homotopy type of $S^l \vee S^l \vee S^{l+1}$.

Thus $KO^\sim(\bar{Q}_{l+1})$ is clear, though we will need to choose specific generators below. It is not hard to calculate $KO^\sim(Q_{l+1}) = KO^\sim\{P_l) \oplus Z_2$ where the additional Z_2 is generated by an element μ' of filtration exactly $l + 1$. At this point, it is convenient to recall from [**12**] that $KO^\sim(Q(\rho)) = KO^\sim(P_l) \oplus Z \oplus Z_2$ where the Z_2 is generated by an element μ of filtration exactly $l + 1$ and the Z by an element of filtration $2l$. Moreover, under $Q_{l+1} \to Q(\rho)$ we have $KO^\sim(Q(\rho)) \to KO^\sim(Q_{l+1})$ defined by

$$KO^\sim(P_l) \xrightarrow{\text{id}} KO^\sim(P_l), \mu \to \mu' \textit{ and } Z \to 0.$$

These properties are easy to check by means of the Atiyah-Hirzebruch spectral sequence, and the fact from [**12**] that $KO^\sim(\sigma) = Z_2 + Z_2$ with one generator the reduced canonical line bundle associated with the double cover $S^1 \times S_l \to \sigma$, and the other generator μ/σ. Finally, in the same way, the $Z_2 \subset KO^\sim(Q_{l+1})$ generated by μ', consisting of all elements of filtration exactly $l + 1$, maps isomorphically onto $Z_2 \subset KO^{\sim}(\sigma)$ generated by μ/σ, consisting of all elements of filtration exactly $l+1$.

Now we choose specific generators of $KO^{\sim}(\bar{Q}_{l+1})$ and we compute

$$t : KO^{\sim}(\bar{Q}_{l+1}) \to KO^{\sim}(Q_{l+1}).$$

First, since $\sigma = S((l+1)\eta_1)$, we have that σ is the quotient of $S^1 \times S^l$ by -1×-1. Let

$$S^1 \times S^l \xrightarrow{f_1} S^l$$

be the projection—it is equivariant with respect to the antipodal action on S^l. Thus we obtain the map of double covers

$$\begin{array}{ccc} S^1 \times S^l & \xrightarrow{f_1} & S^l \\ \downarrow & & \downarrow \\ \sigma & \longrightarrow & P^l. \end{array}$$

Since $t(\text{gen}_l) = (c(l)/2)(\eta_l - 1)$, where gen_l is a generator of $KO^{\sim}(S^l)$ and since $c(l)/2$ is even, and since the reduced canonical line bundle over σ has order two, we see that $t(f_1{}^*\text{gen}_l) = 0$.

The map f_1 extends to a map $g_1 : \bar{Q}_{l+1} \to S^l$, not necessarily equivariant, though we may assume that g_1 restricted to the S^l in $\bar{Q}_{l+1} = S^l \cup_{S^1} (S^1 \times S^l)$ is trivial. Let $\alpha \in KO^{\sim}(\bar{Q}_{l+1})$ be $g_1{}^*(\text{gen}_l)$. Then the map of covers

$$\begin{array}{ccc} S^1 \times S^l & \longrightarrow & \bar{Q}_{l+1} \\ \downarrow & & \downarrow \\ \sigma & \longrightarrow & Q_{l+1} \end{array}$$

$t(\alpha) = 0$ or $t(\alpha) = (c(l)/2)(\eta - 1)$. To see this fact suppose that $t(\alpha) = r(\eta - 1) + s\mu'$ with $s = 0$ or 1. Then since $t(\alpha) \to 0$ and μ' and $\eta - 1$ are carried to generators of distinct Z_2 summands in $KO^{\sim}(\sigma)$, we must have $s = 0$ and r is even. Thus $t(\alpha) = r(\eta - 1)$. But by Proposition 2 (vii), we have $\eta t(\alpha) = t(\alpha)$ so $\eta(r(\eta - 1)) = r(\eta - 1)$. But $y(r(\eta - 1)) = -r(\eta - 1)$. Thus $r = 0$ or $c(l)/2$.

For our second generator, observe that σ is also $S(\eta_1 + l)$. It is a section of this bundle that gives the P_1 in σ. On the double cover level, we see that the retraction $S^1 \times S^l \to S^1$ is equivariant, so we obtain a map $g_2 : \bar{Q}_{l+1} \to S^l$ equivariant with respect to the antipodal action on S^l, extending $S^l \xrightarrow{d} S^l$. Let $\beta = g_\lambda{}^*(\text{gen}_l)$. Then $t(\beta) = (c(l)/2)(\eta - 1)$.

Finally, we have the confibration

$$S^l \to \frac{S^1 \times S^l}{S^1 \times *} \to S^{l+1}.$$

There is a factorization

$$\begin{tikzcd} S^1 \times S^l \arrow[r, "f_1"] \arrow[d] & S^l \\ \dfrac{S^1 \times S^l}{S^1 \times *} \arrow[ur, "h"'] & \end{tikzcd}$$

which defines a splitting of the stable homotopy exact sequence long exact sequence into split short exact sequences:

$$0 \to \pi_{l+1}{}^s(S^l) \underset{h_*}{\rightleftarrows} \pi_{l+1}{}^s\left(\frac{S^1 \times S^l}{S^1 \times *}\right) \to \pi_{l+1}{}^s(S^{l+1}) \to 0.$$

We let γ be the pullback of a generator of $KO^{\sim}(S^{l+1})$ under a degree one map $\bar{Q}_{l+1} \to S^{l+1}$ (there is only one such map up to homotopy). Now, the kernel of h_* is infinite cyclic. Choose a generator $\iota : S^{l+1} \to S^1 \times S^l/S^1 \times *$. We call ι also the composition

$$S^{l+1} \xrightarrow{\iota} \frac{S^1 \times S^l}{S^1 \times *} \to \frac{S^1 \times S^l}{S^1 \times *} \vee S^l.$$

Then $\iota^*\gamma = \text{gen}_{l+1}$, the generator of $KO^{\sim}(S^{l+1})$. Now the Atiyah-Hirzebruch spectral sequence and Proposition 4 imply that $t(\gamma) = \mu'$. In fact, $t : H^{l+1}(\bar{Q}_{l+1} : Z_2) \to H^{l+1}(Q_{l+1} : Z_2)$ is an epimorphism and $t : KO^{\sim}(X) \to KO^{\sim}(X)$ does not lower filtration.

So far Q_{l+1} is a common homotopy $(l + 1)$-skeleton for all $Q(\rho)$, where ρ is a free oriented involution with $M(\rho) \sim S^l \times S^l$ and $l \equiv 0 \bmod 8$. However, there is no common $(l + 2)$-skeleton, and in constructing an $(l + 2)$-skeleton Q_{l+2} for a given ρ we will have to make use of our assumption that $k(\rho) \equiv 0 \bmod 4$. Recall that $H^*(Q(\rho) : Z_2) = Z_2[x]/(x^{l+1}) \otimes \Lambda(y)$, where $Z_2[x]/(x^{l+1})$ is the polynomial algebra on x over Z_2, truncated at x^{l+1} and $\Lambda(y)$ is the exterior algebra on y over Z_2; moreover, deg $x = 1$ and deg $y = l$. Since $k(\rho) \equiv 0 \bmod 4$, it follows that the first two Stiefel Whitney classes of $Q(\rho)$ are zero. Then by means of Wu's formula it follows that $Sq^2y = yx^2$. (In the case that $k(\rho) \equiv 2 \bmod 4$, it follows that $Sq^2u = 0$.) We will construct $Q_{l+2} = Q_{l+1} \cup D^{l+2}$, where the attaching map is in $\pi_{l+1}(Q_{l+1})$, so our next task is to examine $\pi_{l+1}(Q_{l+1})$.

We have obtained generators α, β, γ of $KO^{\sim}(\bar{Q}_{l+1})$ and we have obtained a homotopy class of maps $\iota : S^{l+1} \to \bar{Q}_{l+1}$ such that $\iota^*\gamma = \text{gen}_{l+1}$. Clearly $\pi_{l+1}(Q_{l+1}) = Z + Z_2 + Z_2$; and we have the following lemma.

LEMMA 4. *$\pi_1(Q_{l+1})$ acts on $\pi_{l+1}(Q_{l+1})$ by sign reversal.*

Proof. First observe that $\pi_{l+1}(P_l \vee S^l) = Z_2 + Z_2 + Z_2$. If $\delta : S^{l+1} \to S^l$ is the Hopf map, then one generator of $Z_2 + Z_2$ is $j \circ \delta$ and the other $\rho(j \circ \delta)$.

But since $j + \rho y \sim 0$ in $Q(\rho)$, we have that the exact sequence

$$\pi_{l+1}(P_l \vee S^l) \to \pi_{l+1}(Q(\rho)) \to 0$$

is the exact sequence $Z_2 + \overline{Z_2 + Z_2} \to Z_2 + Z_2 \to 0$ with kernel the diagonal elements in $\overline{Z_2 + Z_2}$. Thus $\pi_1(Q(\rho))$ acts on $\pi_{l+1}(Q(\rho)) = Z_2 + Z_2$ by the identity; since the elements have order two, we may just as well say "by sign reversal".

Now we find the action of ρ_* on $H_{l+1}(\bar{Q}_{l+1})$, where ρ now denotes the non-trivial covering transformation of $\bar{Q}_{l+1} \to Q_{l+1}$. To this end we inspect the exact sequence

$$\begin{array}{ccccccccc} 0 \to H_{l+1}(\bar{Q}_{l+1}) & \to & H_{l+1}(\bar{Q}_{l+1}, S^l \vee S^l \vee S^l) & \to & H_l(S^l \vee S^l \vee S^l) & \to & H_l(\bar{Q}_{l+1}) \to 0 \\ \| & & \| & & \| & & \| \\ 0 \to H_{l+1}(\bar{Q}_{l+1}) & \longrightarrow & \overline{Z + Z} & \longrightarrow & \bar{Z} + \overline{Z + Z} & \to & \bar{Z} + \bar{Z} \to 0 \end{array}$$

to conclude that $\rho_* : H_{l+1}(\bar{Q}_{l+1}) \to H_{l+1}(\bar{Q}_{l+1})$ is sign reversal.

Finally we look at the short exact sequence

$$0 \to \pi_{l+2}(Q(\rho), Q_{l+1}) \to \pi_{l+1}(Q_{l+1}) \to \pi_{l+1}(Q(\rho)) \to 0.$$

But

$$\pi_{l+2}(Q(\rho), Q_{l+1}) \cong \pi_{l+2}(M(\rho), \bar{Q}_{l+1}) \cong H_{l+1}(\bar{Q}_{l+1})$$

with each isomorphism equivariant, so we have a short exact sequence $0 \to \bar{Z} \to \pi_{l+1}(Q_{l+1}) \to Z_2 + Z_2 \to 0$ from which the lemma follows.

Now we choose specific generators of $\pi_{l+1}(Q_{l+1})$. Clearly ι generates an infinite cyclic summand

$$\pi_j(\bar{Q}_{l+1}) \overset{\cong}{\to} \pi_j(Q_{l+1})$$

for $j > 1$ so ι stands both for $S^{l+1} \to \bar{Q}_{l+1}$ and $S^{l+1} \to \bar{Q}_{l+1} \to Q_{l+1}$), and we take ι to be the first generator. For the second generator δ_1, we let $S_1{}^l \subset \sigma$ be a fiber of $\sigma \to P_1$ and we let δ_1 be the composition of the Hopf map $S^{l+1} \to S_1{}^l$ and the inclusion $S_1{}^l \subset Q_{l+1}$; then δ_1 generates a summand of order two. For the third generator δ_2, we take the composition of the Hopf map $S^{l+1} \to S^l$ followed by $S^l \to P_l$ and followed finally by $P_l \subset Q_{l+1}$; then δ_2 generates a summand of order two, and ι, δ_1, δ_2 generate $\pi_{l+1}(Q_{l+1})$. Finally, we write ι, δ_1, δ_2 for the corresponding generators of $\pi_{l+1}(\bar{Q}_{l+1})$. Then we have the following lemma.

LEMMA 5. $\iota^*\gamma = \text{gen}_{l+1}$, $\delta_1{}^*\alpha = \text{gen}_{l+1}$, $\delta_2{}^*\beta = \text{gen}_{l+1}$. *Also*, $\iota^*\alpha = \iota^*\beta = \delta_1{}^*\gamma = \delta_1{}^*\beta = \delta_2{}^*\gamma = \delta_2{}^*\alpha = 0$.

Proof. We have already see the first equation. The next two follow from the fact that if $\delta : S^{l+1} \to S^l$ is the Hopf map and $l \equiv 0 \bmod 8$, then $\delta^*\text{gen}_l = \text{gen}_{l+1}$.

For $\iota^*\alpha = 0$ we observe that the composition

$$S^{l+1} \xrightarrow{\iota} \frac{S^1 \times S^l}{S^1 \times *} \xrightarrow{h} S^l$$

is trivial.

For $\iota^*\beta = 0$ we observe that the composition

$$S^{l+1} \xrightarrow{\iota} S^1 \times S^l \underset{S^1}{\cup} S^l \to S^l$$

carries S^{l+1} into a cone of S^1 in S^l. Consequently that composition is homotopically trivial.

For $\delta_1{}^*\gamma = 0$, recall that we have the cofibration

$$S^l \xrightarrow{\delta_1} \frac{S^1 \times S^l}{S^1 \times *} \to S^{l+1} \text{ split by } h: \frac{S^1 \times S^l}{S^1 \times *} \to S^l \text{ and } \iota: S^{l+1} \to \frac{S^1 \times S^l}{S^1 \times *}$$

the corresponding splitting map at the other end. Then $\gamma|(S^1 \times S^l/S^1 \times *)$ is just the pullback of gen_{l+1} by $(S^1 \times S^l/S^1 \times *) \to S^{l+1}$ so $\delta_1{}^*\gamma = \delta_1{}^*(\gamma|(S^1 \times S^l/S^1 \times *)) = 0$.

For $\delta_1{}^*\beta = 0$, observe that β is the pullback of gen_l under the equivariant retraction $S^1 \times S^l \cup_{S^1} S^l \to S^l$, and that under this retraction $\delta_1(S^l)$ is carried to a point.

For $\delta_2{}^*\gamma = 0$, we may regard $g_1 : (S^1 \times S^l/S^1 \times *) \to S^{l+1}$ as a map $g_1 : S^1 \times S^l \cup CS^1 \to S^{l+1}$, where CS^1 is a cone of S^1 in S^l. There is only one way (up to homotopy) to extend this map to a map $S^1 \times S^l \cup_{S^1} S^l \to S^{l+1}$, and this map carries δ_2 to the trivial map.

Finally, for $\delta_2{}^*\alpha = 0$ recall that the map $f_1 : S^1 \times S^l \to S^l$ was extended to $S^1 \times S^l \cup_{S^1} S^l \xrightarrow{g_1} S^l$ so that $g_1 \circ \delta_2$ is trivial, and the lemma is proved.

Next, we consider the exact sequence

$$\pi_{l+2}(Q(\rho), Q_{l+1}) \to \pi_{l+1}(Q_{l+1}) \to \pi_{l+1}(Q(\rho)) \to 0.$$

By passing to covering spaces and applying the Hurewicz isomorphism

$$\pi_{l+1}(M(\rho), \bar{Q}_{l+1}) \xrightarrow{\cong} H_{l+1}(M(\rho), \bar{\bar{Q}}_{l+1})$$

and the Hurewicz epimorphism $\pi_{l+1}(\bar{Q}_{l+1}) \to H_{l+1}(\bar{Q}_{l+1})$, we see that there is a commutative diagram

$$\begin{array}{ccccc}
\pi_{l+2}(Q(\rho), Q_{l+1}) & \to & \pi_{l+1}(Q_{l+1}) & \to & \pi_{l+1}(Q(\rho)) \to 0 \\
\downarrow \| s & & \downarrow & & \\
Z & \xrightarrow{\text{id}} & Z & &
\end{array}$$

with the top row exact and the second vertical an epimorphism. It follows that

https://doi.org/10.4153/CJM-1974-016-1 Published online by Cambridge University Press

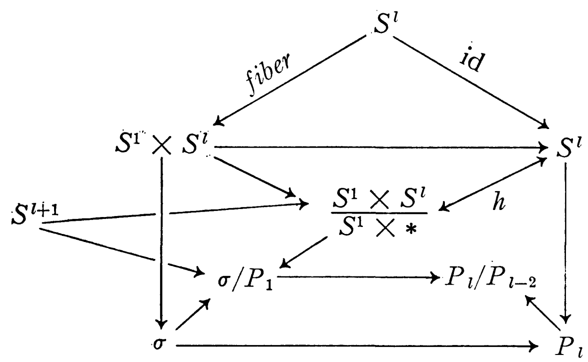
the kernel of $\pi_{l+1}(Q_{l+1}) \to \pi_{l+1}(Q(\rho))$ is generated by λ, where λ is one of the four maps ι, $\iota + \delta_1$, $\iota + \delta_2$ or $\iota + \delta_1 + \delta_2$. Then a homotopy $(l + 2)$-skeleton for $Q(\rho)$ is $Q_{l+1} \cup_\lambda D^{l+2} = K(\lambda)$. In order to decide which λ actually appears, we need the following lemma.

LEMMA 6. *The composition*

$$S^{l+1} \xrightarrow{\iota} Q_{l+1} \to Q_{l+1}/P_l$$

is homotopically trivial.

Proof. We observe that the composition of the lemma may be factored thus $S^{l+1} \to \sigma/P_1 \to Q_{l+1}/P_l$ so that it suffices to show that the map $S^{l+1} \to \sigma/P_1$ is trivial. We observe that that map appears in the following homotopy commutative diagram:



and that the composition

$$S^{l+1} \to \frac{S^1 \times S^l}{S^1 \times *} \xrightarrow{h} S^l$$

is homotopically trivial. Thus the composition $S^{l+1} \to \sigma/P_1 \to P_l/P_{l-2}$ is homotopically trivial. Consequently, it suffices to show that $\pi_{l+1}(\sigma/P_1) \to \pi_{l+1}(P_l/P_{l-2})$ is a monomorphism.

Notice that σ/P_1 is $S^l \cup_2 D^{l+1}$ with the generating sphere obtained from the composition $S^l \to S^1 \times S^l \to (S^1 \times S^l/S^1 \times *) \to \sigma/P_1$ in the diagram above. On the other hand, the composition $S^l \to P_l \to P_l/P_{l-2}$ must be nontrivial because it defines the cofibration $S^l \to P_l/P_{l-2} \to P_{l+1}/P_{l-2}$ in which

$$Sq^2 : H^{l-1}(P_{l+1}/P_{l-2} : Z_2) \to H^{l+1}(P_{l+1}/P_{l-2} : Z_2)$$

is nontrivial. From the Adams spectral sequence we see that $\pi_l(P_l/P_{l-2}) = Z_2$ so that the map $\sigma/P_1 \to P_l/P_{l-2}$ carries $\pi_l(\sigma/P_1) \xrightarrow{\cong} \pi_l(P_l/P_{l-2}) = Z_2$ isomorphically.

Let $S^l \to P_l/P_{l-2}$ be the above map, that is, the generator of $\pi_l(P_l/P_{l-2})$, and let $S^{l+1} \to S^l$ be the Hopf map. Then from the Adams spectral sequence we see that the composition $S^{l+1} \to S^l \to P_l/P_{l-2}$ is non-zero in $\pi_{l+1}(P_l/P_{l-2}) = Z_4$. On the other hand, the cofibration

$$S^l \xrightarrow{2} S^l \to \sigma/P_1$$

shows that the generator of $\pi_{l+1}(\sigma/P_1) = Z_2$ is the composition $S^{l+1} \to S^l \to \sigma/P_1$. Thus $\pi_{l+1}(\sigma/P_1) \to \pi_{l+1}(P_l/P_{l-2})$ is a monomorphism, and the lemma is proved.

Now we may prove the following proposition, from which Theorem 7 will follow immediately.

PROPOSITION 7. *Let ρ be a free involution with $M(\rho) \sim S^l \times S^l$ and $l \equiv 0$ mod 8 and $k(\rho) \equiv 0$ mod 4. Then either μ or $\mu + 2^{\varphi-1}(\eta - 1)$ is in the image of $t : KO^\sim(M(\rho)) \to KO^\sim(Q(\rho))$.*

Proof. Let $K(\lambda)$ be a homotopy $(l + 2)$ skeleton for $Q(\rho)$ and let $\overline{K(\lambda)} \to K(\lambda)$ be its double cover. We have the mapping of double covers

$$\begin{array}{ccc} \bar{Q}_{l+1} & \longrightarrow & \overline{K(\lambda)} \\ \downarrow & & \downarrow \\ Q_{l+1} & \longrightarrow & K(\lambda). \end{array}$$

Since Z_2 acts on $\pi_{l+1}(Q_{l+1})$ by sign reversal, it follows that $\overline{K(\lambda)}$ has the homotopy type of $(\bar{Q}_{l+1} \cup_\lambda D^{l+2}) \vee S^{l+2}$ where λ is now regarded as a map $S^{l+1} \to \bar{Q}_{l+1}$.

Since $k(\rho) \equiv 0$ mod 4, the first two Stiefel Whitney classes of $Q(\rho)$ must be zero, and by Wu's formula we must have $Sq^1 y = xy$ and $Sq^2 y = x^2 y$. But Sq^2 is the differential $d_2 : E_2^{l,-l-1} \to E_2^{l+2,-l-2}$ in the Atiyah-Hirzebruch spectral sequence for $Q(\rho)$. Since $H^j(Q(\rho) : Z_2) \to H^j(K(\lambda) : Z_2)$ is an isomorphism for $j \leqq l + 2$, the same thing happens for $K(\lambda)$ and it follows that we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & Z & \to & KO^\sim(M(\rho)) & \to & KO^\sim(K(\lambda)) & & \\ & & \downarrow \| s & & \downarrow t & & \downarrow t & & \\ 0 & \to & Z & \to & KO^\sim(Q(\rho)) & \to & KO^\sim(K(\lambda)) & \to & 0 \end{array}$$

with exact rows. The Z in each row consists of the elements of filtration $2l$. That $t : X \overset{\cong}{\to} Z$ follows from the following facts: (a) $\pi^* Z \to Z$ is multiplication

https://doi.org/10.4153/CJM-1974-016-1 Published online by Cambridge University Press

by 2; (b) t does not decrease filtration; (c) the diagram

$$\begin{tikzcd} M(\rho) \arrow[rr, "\rho"] \arrow[dr] & & M(\rho) \arrow[dl] \\ & S^{2l} & \end{tikzcd}$$

homotopy commutes for $M(\rho) \to S^{2l}$ a degree one map; and (d) $\pi^* t = id + \rho^*$.

Now we use again the fact that $Sq^2 : H^l(Q(\rho) : Z_2) \overset{\cong}{\to} H^{l+2}(Q(\rho) : Z_2)$. It will follow from the lemma below that $\lambda = \iota + \delta_1$ or $\lambda = \iota + \delta_1 + \delta_2$. But in either of these cases the element $\gamma + \alpha \in KO^\sim(\bar{Q}_{l+1})$ extends to an element $\alpha' \in KO^\sim(K(\lambda))$. Let $v' \in KO^\sim(K(\lambda))$ be the non-zero element of filtration $l + 2$. Since

$$d_2 : E^{l,-l-1}(K(\lambda)) \to E^{l+2,\, l-l-2}(K(\lambda))$$

is epimorphic in the Atiyah-Hirzebruch spectral sequence, and since t does not decrease filtration, we must have $t(\nu') = 0$. On the other hand, the sequence $KO^\sim(M(\rho)) \to KO^\sim(K(\lambda)) \to Z_2 \to 0$ is exact with ν' being carried to the generator of t_2. Thus, either α' or $\alpha' + \nu'$ extends to $\alpha'' \in KO^\sim(M(\rho))$. On the other hand, $t(\alpha') = t(\alpha' + \nu') \to \mu'$ or $\mu' + 2^{\varphi-1}(\eta - 1)$ under $KO^\sim(K(\lambda)) \to KO^\sim(Q_{l+1})$ so that $t(\alpha'') \to \mu'$ or $\mu' + 2^{\varphi-1}(\eta - 1)$ under $KO^\sim(Q(\rho)) \to KO^\sim(Q_{l+1})$. Since we may modify α'' by an element of $Z\, KO^\sim(M(\rho))$ and still obtain an extension of α' or $\alpha' + \nu'$ as the case may be, we see finally that there is $\alpha''' \in KO^\sim(M(\rho))$ such that $t(\alpha''') = \mu$ or $\mu + 2^{\varphi-1}(\eta - 1)$. Now the proposition is proved modulo the following lemma.

LEMMA 7. *$Sq^2 : H^l(K(\lambda) : Z_2) \to H^{l+2}(K(\lambda) : Z_2)$ is zero for $\lambda = \iota$ and $\lambda = \iota + \delta_2$; and it is a non-zero epimorphism for $\lambda = \iota + \delta_1$ and $\lambda = \iota + \delta_1 + \delta_2$.*

Proof. Since Q_{l+1} is a homotopy $(l + 1)$-skeleton of $Q(\rho)$ for any ρ as above, and since $H^{l+1}(Q_{l+1} : Z_2) = Z_2$, we have $H^{l+1}(Q(\rho) : Z_2) \overset{\cong}{\to} H^{l+1}(Q_{l+1} : Z_2)$. Thus the generator of $H^{l+1}(Q_{l+1} : Z_2)$ is the product of lower dimensional classes and the map $\lambda : S^{l+1} \to Q_{l+1}$ is zero in reduced cohomology. Then $Sq^2 : H^l(K(\lambda) : Z_2) \to H^{l+2}(K(\lambda) : Z_2)$ is zero or a non-zero epimorphism if and only if $Sq^2 : H^l(K(\lambda)/P_l : Z_2) \to H^{l+2}(K(\lambda)/P_l : Z_2)$ is zero or non-zero.

Let λ' be the composition

$$S^{l+1} \xrightarrow{\lambda} Q_{l+1} \to Q_{l+1}/P_l = \sigma/P_l.$$

Then we have the cofibration

$$S^{l+1} \xrightarrow{\lambda'} \sigma/P_1 \to K(\lambda)/P_l.$$

By Lemma 6, we have ι' trivial. Clearly δ_2' is trivial, so we have only two cases:

$$K(\iota)/P_l = K(\iota + \delta_2)/P_l = \sigma/P_1 \vee S^{l+2}$$

and

$$K(\iota + \delta_1)/P_l = K(\iota + \delta_1 + \delta_2)/P_l = \sigma/P_1 \bigcup_{\delta_1'} D^{l+2}$$

and the lemma follows immediately, and thus the proposition as well.

Now we may prove Theorem 7. We have already dealt with the case $k(\rho) \equiv 2 \bmod 4$, so we assume $k(\rho) \equiv 0 \bmod 4$. Thus Proposition 7 applies and there is an element $\xi_0 \in KO^\sim(M(\rho))$ such that $t(\xi_0) = \mu$ or $\mu + 2^{\varphi-1}(\eta - 1)$.

If $\nu(Q(\rho)) = k(\rho)(\eta - 1)$, we are done, as in the case $k(\rho) \equiv 2 \bmod 4$. The subgroup $Z\ KO^\sim(Q(\rho))$ of filtration $2l$ elements cannot make any contribution to $\nu(Q(\rho))$ since the index of $Q(\rho)$ is zero. Thus, if $\nu(Q(\rho)) \neq k(\rho)(\eta - 1)$, then $\nu(Q(\rho)) = k(\rho)(\eta - 1) + \mu$. In the case that $t(\xi_0) = \mu$ we have that

$$t(\xi_0) = \nu(Q(\rho)) - k(\rho)(\eta - 1)$$

and in the case that $t(\xi_0) = \mu + (c(l)/2)(\eta - 1)$, we have that

$$t(\xi_0) = \nu(Q(\rho)) - (k(\rho) + c(l)/2)(\eta - 1).$$

It follows then from Proposition 5 that there exist generic immersions $M^{2l+k} \to S^{2l+2k}$ for $k \geqq 2l + 3$ if and only if $k \to k(\rho)$ in the first case and $k \to k(\rho) + 2^{\varphi-1}$ in the second case. In either case, the theorem is proved.

4. 4.1 *Proof of Theorem* 8. We have that $l \equiv 0 \bmod 8$ and that g represents an element of $I_{2l}(K)$ not in the kernel of α. Then $K = k(\rho) \equiv 0 \bmod 4$. It follows from [**12**] that $KO^\sim(Q(\rho)) = KO^\sim(P_l) \oplus (Z + Z_2)$, a direct sum with respect to Adams operations. Since μ the generator of Z_2 is not stably fiber homotopically trivial, it follows from Quillen's Theorem that $J|KO^\sim(P_l) \oplus Z_2$ is a monomorphism. Moreover, as pointed out in the proof of Theorem 7, we have $\nu(Q(\rho)) = k(\rho)(\eta - 1) + \mu$. Suppose that $Q(\rho) \to Q(\rho')$ is a homotopy equivalence. Then it follows that $J(k(\rho) - k(\rho'))(\eta - 1) + \mu) = 0$, which is a contradiction, and the theorem is proved.

4.2. We will obtain the example by means of the following proposition:

PROPOSITION 8. *Let*

$$M \xrightarrow{f} S^n$$

be a generic immersion. Let $c : \Delta(f) \to P_r$ *be transverse to* P_{r-1} *and such that the composition* $\Delta(f) \xrightarrow{c} P_r \to P_\infty$ *classifies* $D(f) \to \Delta(f)$. *Then there is a generic immersion* $g : M \to S^{n+1}$, *regularly homotopic modulo a neighborhood of* ∂M *to* $M \xrightarrow{f} S^n \subset S^{n+1}$, *with* $\Delta(g) = C^{-1}(P_{r-1})$ *and with*

$$\Delta(g) \xrightarrow{c|\Delta(g)} P_{r-1} \to P_\infty$$

classifying $D(g) \to \Delta(g)$.

https://doi.org/10.4153/CJM-1974-016-1 Published online by Cambridge University Press

Proof. We may replace S^n with $\mathbf{R}^n$ and $S^n \to S^{n+1}$ with $\mathbf{R}^n \to \mathbf{R}^{n+1}$ so that $\mathbf{R}^n = \{(x_1, \dots, x_{1+1}) | x_{1+1} = 0\}$. We have the maps of double covers

$$\begin{tikzcd} D(f) \arrow[r] \arrow[d] & S^r \arrow[d] & S^{r-1} \arrow[r, phantom, "\subset"] \arrow[d] & S^r \arrow[d] \\ \Delta(f) \arrow[r] & P_r & P_{r-1} \arrow[r, phantom, "\subset"] & P_r. \end{tikzcd}$$

Let $h : S^r \to \mathbf{R}$ be projection on the $(r + 1)$st coordinate. This map is equivariant with respect to the antipodal action on S^r and sign reversal on S^r; we may assume $S^{r-1} = h^{-1}(0)$. The equivariant smooth function

$$D(f) \to S^r \xrightarrow{h} R$$

may be extended to a smooth function $H : M \to \mathbf{R}$, zero near ∂M. Let the immersion g be given by $g(x) = (f(x), H(x))$. Then clearly g satisfies the conclusion of the proposition, which is now proved.

COROLLARY. *Let $f : S^m \to S^n$ be a generic immersion and suppose that there is a map $c : \Delta(f) \to P_r$ such that*

$$\Delta(f) \xrightarrow{c} P_r \to P_\infty$$

classifies $D(f) \to \Delta(f)$. Then the immersion

$$S^m \xrightarrow{f} S^n \subset S^{n+r+1}$$

is regularly homotopic to the embedding.

Now we recall the definition of the Conner-Floyd coindex: If

$$\bar{X} \xrightarrow{\pi} X$$

is a double cover, the *coindex* of π is the smallest r for which there is $X \to P_r$ such that $X \to P_r \to P_\infty$ classifies π.

To obtain the example, we recall that by composing the James and Smale isomorphisms we obtain a commutative diagram:

$$\begin{tikzcd} \mathrm{Imm}(S^m, S^n) \arrow[r] \arrow[d, "\cong"] & \mathrm{Imm}(S^m, S^{n+1}) \arrow[d, "\cong"] \\ \pi_m{}^s(P/P_{k-1}) \arrow[r] & \pi_m{}^s(P/P_k) \end{tikzcd}$$

Inspection of Mahowald's Tables reveals that there is an element $x \in \pi_{16}{}^sP/P_3$ (represented by $h_1 \cdot {}_{11}l$ in his notation) which is carried by $\pi_{16}{}^s(P/P_3) \to \pi_{16}{}^s(P/P_{12})$ to a non-zero element. But there is a commutative diagram by James periodicity:

$$\begin{tikzcd} \pi_{16}{}^s(P/P_s) \arrow[r] \arrow[d, "\cong"] & \pi_{16}{}^s(P/P_{12}) \arrow[d, "\cong"] \\ \pi_{144}{}^s(P/P_{131}) \arrow[r] & \pi_{144}{}^s(P/P_{140}). \end{tikzcd}$$

https://doi.org/10.4153/CJM-1974-016-1 Published online by Cambridge University Press

Thus we obtain a generic immersion $f : S^{144} \to S^{276}$ such that the immersion

$$S^{144} \xrightarrow{f} S^{276} \subset S^{285}$$

is not regularly homotopic to an embedding. Consequently, the corollary to Proposition 8 implies that the Conner-Floyd coindex of $D(f) \to \Delta(f)$ must be $\geqq 9$.

Finally Theorem 4 and the corollary to Theorem 2 of [**13**] imply that we may assume that $D(f)$ is 5 connected and $H_6(D(f)) = Z + Z$. Since $\nu(\Delta(f)) = 4(\eta - 1)$, it follows that $\nu(D(f)) = 0$ so that

$$D(f) = (S^6 \times E^6 \underset{E^6 \times E^6}{\cup} E^6 \times S^6) \underset{\psi}{\cup} E^{12}$$

where $\psi : S^{11} \to S^{11}$ is a diffeomorphism. Since $\pi_{12}{}^s = 0$ and $bP_{13} = 0$, it follows that ψ is concordant to the identity and then that $D(f)$ is diffeomorphic to $S^6 \times S^6$. Then the non trivial covering transformation of $D(f) \to \Delta(f)$ defines a free oriented involution ρ of $S^6 \times S^6$ such that the coindex of $S^6 \times S^6 \to Q(\rho)$ is $\geqq 9$. That is the example we sought.

4.3. We conclude this section with a remark on the periodicity

$$\pi_m{}^s(P/P_{k-1}) \xrightarrow{\cong} \pi_{m+r2^\varphi}{}^s(P/P_{k-1+r2^\varphi})$$

where $\varphi = \varphi(m + 1 - k)$. We let $m = 2l + k$ with both l and k even, and we take $k \geqq 2l + 3$. Let $\gamma \to P_l$ be the $(l + 1)$-plane bundle stably equivalent to $(2^{\varphi(l)} - l - 1 - k)\eta_l$. Then there is a bundle map $\nu(E(\gamma) : S_+{}^{n+1}) \to k\eta_\infty + k$. Twisting the restriction $\nu(S(\gamma) : S^n) \to k\eta_\infty + k$ by means of $KO^{\sim -1}(S(\gamma))$ in the usual way, we obtain a j-homomorphism $j : KO^{\sim -1}(S(\gamma)) \to \pi_{2l+k}{}^2 T(k\eta_\infty) = \pi_m{}^s(P/P_{k-1})$. Let $\Lambda_{2l}(k)$ be the cokernel of this map. According to [**13**], the group $\Lambda_{2l}(k)$ is isomorphic to the kernel of

$$I_{2l}(\eta) \xrightarrow{\alpha} Z_2$$

where κ is the image of k under $Z \to Z_{2^{\varphi(l)}}$. Consequently we obtain a faster periodicity of $\Lambda_{2l}(k)$ than implied by James periodicity:

$$\Lambda_{2l}(k) \cong \Lambda_{2l}(k + c(l))$$

instead of

$$\Lambda_{2l}(k) \cong \Lambda_{2l}(k + c(2l + 1)).$$

5. 5.1. Let $\mathrm{Imm}(S^m, N)$ be the immersions $S^m \to N$ modulo regular homotopy. Under connected sum this set becomes a group and the definition of q' may be extended to $q' : \mathrm{Imm}(S^m, N) \to \pi_m(0/0(k))$. Notice that q' is a homomorphism. If $f : S^m \to N$ is a generic immersion, then we obtain as in Section 2 a uniquely defined class of bundle maps $\nu(\Delta(f) : N) \to k\eta_\infty + k$. However, we may take a c-skeleton of N to be $E^n \subset N$, and then $\Delta(f) \subset N$ defines a unique isotopy class of embeddings and consequently we have a unique class of bundle maps $\nu(\Delta(f) : E^n) \to k\eta_\infty + k$. That is, we have a $k\eta_\infty$ manifold $\delta(f)$ as in Section 2, and consequently an element $\alpha(f) \in \pi_m{}^s(P/P_{k-1})$.

Using again the fact that N is c-connected, so that E^n N is a c-skeleton for N, we see that the proof of Theorem 4 carries over, replacing M by S^m and S^n by N to give us the following proposition.

PROPOSITION 9. *Any $k\eta_\infty$-surgery of $\delta(f)$ may be realized by a generic regular homotopy.*

5.2. *Proof of Theorem* 9. Let $\bar{x} : S^m \to N$ and $\bar{y} : S^m \to N$ be embeddings representing $x, y \in \pi_m(N)$ respectively. We may assume that they are mutually transverse. Let $+$ denote connected sum. Then $\alpha(\bar{x} + \bar{y}) = x \circ y$ and $q'(\bar{x} + \bar{y}) = q'(\bar{x}) + q'(\bar{y})$.

By Theorem 4, we may choose a generic immersion $f : S^m \to E^n \subset N$ with $\alpha(f) = -x \cdot y$. We may assume that E^n is a small regular neighborhood of a point not in $\bar{x} + \bar{y}(S^m)$ so $\bar{x} + \bar{y}$ and f are disjoint immersion into $E^n \subset S^n$, notice that it follows from the definition of ψ that $\psi q'(f) = \alpha(f) = -x \cdot y$. Now, since $\bar{x} + \bar{y}$ and f are disjoint we have $\alpha(x + \bar{y} + f) = 0$. Thus there is, by Proposition 9, an embedding $\bar{z} : S^m \to N$ regularly homotopic to $\bar{x} + \bar{y} + f$. Since $f : S^m \to E^n$, the embedding $\bar{z}$ represents $x + y$. Thus $q(z + y = \psi(q'(\bar{z})) = \psi(q'(\bar{x} + \bar{y} + f)) = \psi(q'(\bar{x}) + q'(\bar{y}) + q'(f)) = q(x) + q(y) - x \cdot y$, and the theorem is proved.

REFERENCES

1. P. E. Conner and E. E. Floyd, *Fixed point free involutions and equivariant maps*, Bull. Amer. Math. Soc. *66* (1960), 416–441; and *105* (1962), 222–228.
2. F. Connolly, *From immersions to embeddings of smooth manifolds*, Trans. Amer. Math. Soc. *152* (1970), 253–271.
3. A. Haefliger, *Plongements différentiables de variétés dans variétés*, Comment. Math. Helv. *36* (1961), 47–82.
4. ——— *Plongements différentiables dans le domaine stable*, Comment. Math. Helv. *37* (1962/63), 155–176.
5. M. A. Kervaire, *A manifold which does not admit any differentiable structure*, Comment. Math. Helv. *34* (1960), 257–270.
6. J. G. Miller, *Self-intersections of some immersed manifolds*, Trans. Amer. Math. Soc. *136* (1969), 329–338.
7. H. Schneider, *Free involutions on homotopy* $S^{[h/2]} \times S^{h-[n/2]}$'s, Ph.D. Thesis, University of Chicago, 1972.
8. P. A. Schweitzer, *Joint cobordism of immersions, The Steenrod algebra and its applications*: A conference to celebrate N. E. Steenrod's sixtieth birthday. Proceedings of the conference held at the Batelle Memorial Institute (Springer-Verlag, 1972).
9. F. Uchida, *Exact sequences involving cobordism groups of immersions*, Osaka J. Math. *6* (1969), 397–408.
10. R. Wells, *Cobordism groups of immersions*, Topology *5* (1966), 281–294.
11. ——— *Modification of intersections*, Illinois Math. *11* (1967), 389–399.
12. ——— *Some examples of free involutions on homotopy* $S^l \times S^l$'s, Illinois J. Math. *15* (1971), 542–550.
13. ——— *Free involutions of Homotopy* $S^l \times S^l$'s, Illinois Math. *15* (1971), 160–184.

Pennsylvania State University,
University Park, Pennsylvania