J. Fluid Mech. (2024), *vol.* 983, A27, doi:10.1017/jfm.2024.126

JFM PAPERS

On elementary four-wave interactions in dispersive media

Stéphane Leblanc†

Department of Mathematics and Mechanics, University of Toulon, 83130 La Garde, France

(Received 29 August 2023; revised 11 December 2023; accepted 2 February 2024)

The cubic interactions in a discrete system of four weakly nonlinear waves propagating in a conservative dispersive medium are studied. By reducing the problem to a single ordinary differential equation governing the motion of a classical particle in a quartic potential, the complete explicit branches of solutions are presented, either steady, periodic, breather or pump, thus recovering or generalizing some already published results in hydrodynamics, nonlinear optics and plasma physics, and presenting some new ones. Various stability criteria are also formulated for steady equilibria. Theory is applied to deep-water gravity waves for which models of isolated quartets are described, including bidirectional standing waves and quadri-directional travelling waves, steady or not, resonant or not.

Key words: surface gravity waves, nonlinear instability, bifurcation

1. Introduction

The objective of the present study is to provide an account of the solutions of the system:

$$\left.\begin{aligned}
\mathrm{i}\frac{\mathrm{d}b_1}{\mathrm{d}t} &= (\omega_1 + T_{11}|b_1|^2 + 2T_{12}|b_2|^2 + 2T_{13}|b_3|^2 + 2T_{14}|b_4|^2)b_1 + 2Tb_2^*b_3b_4,\\
\mathrm{i}\frac{\mathrm{d}b_2}{\mathrm{d}t} &= (\omega_2 + 2T_{21}|b_1|^2 + T_{22}|b_2|^2 + 2T_{23}|b_3|^2 + 2T_{24}|b_4|^2)b_2 + 2Tb_1^*b_3b_4,\\
\mathrm{i}\frac{\mathrm{d}b_3}{\mathrm{d}t} &= (\omega_3 + 2T_{31}|b_1|^2 + 2T_{32}|b_2|^2 + T_{33}|b_3|^2 + 2T_{34}|b_4|^2)b_3 + 2Tb_4^*b_1b_2,\\
\mathrm{i}\frac{\mathrm{d}b_4}{\mathrm{d}t} &= (\omega_4 + 2T_{41}|b_1|^2 + 2T_{42}|b_2|^2 + 2T_{43}|b_3|^2 + T_{44}|b_4|^2)b_4 + 2Tb_3^*b_1b_2,
\end{aligned}\right\} \tag{1.1}$$

† Email address for correspondence: sl@univ-tln.fr

where $b_i(t)$, $i = 1, \ldots, 4$, are complex valued functions of time t, where an asterisk stands for complex conjugate and where the real-valued coefficients ω_i, $T_{ij} = T_{ji}$ and T are independent of time. System (1.1) is completed by initial conditions:

$$b_i(0) = \sqrt{q_i}\mathrm{e}^{\mathrm{i}\varphi_i}, \quad q_i \geq 0, \quad \varphi_i \in [0, 2\pi]. \tag{1.2}$$

The positive q_i are the initial wave actions.

Equations similar to (1.1) were first derived by Armstrong *et al.* (1962) in nonlinear optics (time being replaced by a spatial direction) and by Benney (1962) in hydrodynamics. Physically, (1.1) governs the evolution of an isolated quartet of weakly nonlinear waves with wave vectors $(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3, \boldsymbol{k}_4)$ satisfying

$$\boldsymbol{k}_1 + \boldsymbol{k}_2 = \boldsymbol{k}_3 + \boldsymbol{k}_4, \quad \boldsymbol{k}_i \neq \boldsymbol{k}_j, \tag{1.3}$$

propagating with linear frequency $\omega_i = \omega(\boldsymbol{k}_i) > 0$ in a conservative dispersive medium with non-decay dispersion law, i.e. where three-wave quadratic interactions are excluded. The coupling coefficients T_{ij} and T depend on the wave vectors and on the physical properties of the medium under consideration. Defining the frequency mismatch

$$\Delta\omega = \omega_1 + \omega_2 - \omega_3 - \omega_4, \tag{1.4}$$

the interaction (1.3) is said to be resonant if $\Delta\omega = 0$.

Systems similar to (1.1) were also derived by Bretherton (1964) and Inoue (1975) from scalar model equations, by Boyd & Turner (1978) in plasma physics and by Chen & Snyder (1989) in nonlinear optics. Stiassnie & Shemer (2005) deduced (1.1) from the Zakharov equation (Zakharov 1966, 1968; Krasitskii 1990, 1994) that governs the evolution of discrete or continuous spectra of weakly nonlinear gravity waves, and which is generally used as the starting point for weak turbulence statistical theory (Yuen & Lake 1982; Zakharov, L'vov & Falkovich 1992; Zakharov 1999; Janssen 2004; Nazarenko & Lukaschuk 2016), even though Hasselmann (1962) in his pioneering work used primitive equations.

Complementary to (1.1), an isolated wave triad $(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)$ satisfying

$$\boldsymbol{k}_1 + \boldsymbol{k}_2 = 2\boldsymbol{k}_3, \quad \boldsymbol{k}_i \neq \boldsymbol{k}_j, \tag{1.5}$$

also interacts nonlinearly at third order and is governed by

$$\left.\begin{aligned} \mathrm{i}\frac{\mathrm{d}c_1}{\mathrm{d}t} &= (\omega_1 + T_{11}|c_1|^2 + 2T_{12}|c_2|^2 + 2T_{13}|c_3|^2)c_1 + Tc_2^*c_3^2, \\ \mathrm{i}\frac{\mathrm{d}c_2}{\mathrm{d}t} &= (\omega_2 + 2T_{21}|c_1|^2 + T_{22}|c_2|^2 + 2T_{23}|c_3|^2)c_2 + Tc_1^*c_3^2, \\ \mathrm{i}\frac{\mathrm{d}c_3}{\mathrm{d}t} &= (\omega_3 + 2T_{31}|c_1|^2 + 2T_{32}|c_2|^2 + T_{33}|c_3|^2)c_3 + 2Tc_3^*c_1c_2, \end{aligned}\right\} \tag{1.6}$$

as first derived by Benney (1962) for resonant gravity-wave interactions:

$$\omega_1 + \omega_2 = 2\omega_3, \tag{1.7}$$

and deduced by Shemer & Stiassnie (1985) from the Zakharov equation. In nonlinear optics, a spatial analogue of (1.6) has been derived by Cappellini & Trillo (1991), who also pointed out that the three-wave system (1.6) cannot be deduced from the four-wave system (1.1) by simple algebraic relations between the complex amplitudes $b_i(t)$ and $c_i(t)$.

System (1.1) may be solved following a procedure introduced by Armstrong *et al.* (1962) for wave triads in quadratic interaction and extended to quartets in cubic interaction by

Bretherton (1964): it consists of reducing (1.1) to a single scalar equation governing the evolution of an auxiliary variable, say $q(t)$. Bretherton proved that $q(t)$ is 'a periodic function of [t], with period depending on the initial conditions but which is only in exceptional cases infinite', but did not give explicit solutions. This was achieved by Inoue (1975), Boyd & Turner (1978), Turner (1980), Chen & Snyder (1989) and Stiassnie & Shemer (2005) using elliptic functions. Some of the 'exceptional cases' mentioned by Bretherton correspond to solutions now called 'pump' and 'breathers'; some of these were given by Inoue (1975) and Turner (1980), but some others were missing.

Following a similar procedure, periodic solutions of (1.6) involving elliptic functions were found by Shemer & Stiassnie (1985) and Cappellini & Trillo (1991). These latter authors, who also found breather solutions, proved that the problem may be recast to an elegant one-degree integrable Hamiltonian system; see also Trillo & Wabnitz (1991). This approach allows one to plot phase portraits representing level-lines of the Hamiltonian from which interesting qualitative and quantitative results can be deduced: isolated points surrounded by closed orbits in phase space correspond respectively to stable equilibria and to periodic solutions, while saddle points connected by separatrices correspond respectively to unstable equilibria and to non-periodic solutions (Cappellini & Trillo 1991; Trillo & Wabnitz 1991; Andrade & Stuhlmeier 2023*b*). A similar Hamiltonian formulation is used in the present study, but phase portraits are not necessary for our purpose.

Steady equilibria of (1.1) or of (1.6) are of fundamental importance. By steady equilibria, we mean solutions with steady amplitudes $|b_i(t)|^2 = |b_i(0)|^2$. Benney (1962) noticed that the simplest of these is the finite-amplitude travelling wave discovered by Stokes in 1847. The existence of finite-amplitude bichromatic wavetrains $(\boldsymbol{k}_1, \boldsymbol{k}_2)$ was first proved by Phillips (1960), Longuet-Higgins & Phillips (1962) and Benney (1962). These steady bichromatic waves were one of the essential ingredients for the development of the theory of weakly nonlinear wave interactions (see also the historical survey by Phillips (1981)): in a few words, we recall that Phillips (1960) and Longuet-Higgins (1962) proved that the propagation of a finite-amplitude bichromatic gravity wavetrain, say $(\boldsymbol{k}_1, \boldsymbol{k}_3)$, leads initially by cubic (or 'tertiary') nonlinear resonant interaction to the spontaneous linear growth of a third wave $\boldsymbol{k}_2$ satisfying (1.5) and (1.7). As noticed by Phillips (1967), this behaviour may be deduced from (1.6) considering $|c_2| \ll |c_1|, |c_3|$.

The previous mechanism has to be distinguished from quadratic (or 'secondary') nonlinear interactions inside wave triads for which it has been proved by Galeev & Karpman (1963) and Hasselmann (1967) that a finite-amplitude wave $\boldsymbol{k}_3$ is exponentially unstable to a couple of infinitesimal disturbances $(\boldsymbol{k}_1, \boldsymbol{k}_2)$ if (see also Craik 1985, p. 131)

$$\boldsymbol{k}_1 + \boldsymbol{k}_2 = \boldsymbol{k}_3, \quad \omega_1 + \omega_2 = \omega_3. \tag{1.8}$$

Sometimes called 'decay instability', this mechanism cannot, however, operate in gravity waves, as proved by Phillips (1960) and Hasselmann (1962).

Back to cubic resonant interactions between three waves satisfying (1.5), (1.6) and (1.7), the continuous energy transfer from $(\boldsymbol{k}_1, \boldsymbol{k}_3)$ to $\boldsymbol{k}_2$ discovered by Phillips excludes therefore the possibility of steady states at resonance. However, the existence of non-resonant steady states in a system of three waves $(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)$ with finite constant amplitudes $|b_i|$ has been established near resonance by Cappellini & Trillo (1991), Liao, Xu & Stiassnie (2016) and Andrade & Stuhlmeier (2023*b*).

In the case of 'non-degenerate' quartets satisfying (1.1) and (1.3), the existence of steady equilibria has been established at resonance in numerical simulations by Liu & Liao (2014), observed experimentally by Liu *et al.* (2015) and identified off-resonance by Andrade & Stuhlmeier (2023*a*) using a Hamiltonian approach. The bidirectional standing wave of Okamura (1985) is also a particular case of such steady quartets. It was clear from these studies that the existence of steady equilibria is conditioned by certain constraints between the finite amplitudes $|b_i|$, but a general explicit formulation of these compatibility conditions was missing.

Concerning stability, it has been known since Zakharov (1966) and Phillips (1967) that the modulational instability discovered by Benjamin & Feir (1967) may be interpreted as the cubic interaction between a Stokes wave $\boldsymbol{k}_3$ with constant finite amplitude $|b_3|$ disturbed by a couple of 'satellites' with infinitesimal amplitudes $|b_1|, |b_2| \ll |b_3|$ and wave vectors $(\boldsymbol{k}_1, \boldsymbol{k}_2)$ satisfying (1.5), slightly off resonance (see reviews in Yuen & Lake 1982; Shemer & Stiassnie 1991; Janssen 2004; Zakharov & Ostrovsky 2009). It has also been known since Okamura (1984) and Ioualalen & Kharif (1994) that a standing wave or a bichromatic wavetrain $(\boldsymbol{k}_1, \boldsymbol{k}_2)$ with constant finite amplitudes $|b_1|, |b_2|$ may also be destabilized by a couple of infinitesimal satellites with wave vectors $(\boldsymbol{k}_3, \boldsymbol{k}_4)$ satisfying (1.3), at resonance or slightly off. A theory for this kind of modulational instability has been presented in Leblanc (2009).

The present study focuses only on four-wave interactions (1.3) inside a single quartet governed by (1.1). Motivated by the recent work of Andrade & Stuhlmeier (2023*a*) who established some interesting links between various aspects mentioned above, but who restricted their analysis to quartets with symmetric initial conditions $|b_1(0)| = |b_2(0)|$ and $|b_3(0)| = |b_4(0)|$, the present study aims at answering the questions that remain open, putting the various pieces of the puzzle together and finding the missing ones. In some sense, our work may be viewed as the extension to non-degenerate quartets (1.3) of the analyses of Shemer & Stiassnie (1985), Cappellini & Trillo (1991) and Andrade & Stuhlmeier (2023*b*) for the 'degenerate' case (1.5).

The paper is organized in two parts: the first one is generic to dispersive media (§§ 2–4), the second specific to deep-water gravity waves (§§ 5–7). More precisely: reduction to a single equation is carried out in § 2; steady equilibria and their stability are investigated in § 3; unsteady solutions are presented in § 4; models of steady and periodic solutions on deep water are described in § 5; examples of pump and breathers are elaborated in §§ 6 and 7. Results are summarized in § 8 and complements are given in the appendices.

2. Bretherton equation

Multiplying each equation of (1.1) respectively by $b_1^*, \dots, b_4^*$ and adding the complex conjugate equations yields (Bretherton 1964; Stiassnie & Shemer 2005)

$$\frac{\mathrm{d}}{\mathrm{d}t}|b_1|^2 = \frac{\mathrm{d}}{\mathrm{d}t}|b_2|^2 = -\frac{\mathrm{d}}{\mathrm{d}t}|b_3|^2 = -\frac{\mathrm{d}}{\mathrm{d}t}|b_4|^2 = -4T|b_1|\,|b_2|\,|b_3|\,|b_4| \sin p, \tag{2.1}$$

where the relative phase $p(t)$ is defined as

$$p(t) = p_1(t) + p_2(t) - p_3(t) - p_4(t), \quad p_i(t) = \arg b_i(t). \tag{2.2}$$

Individual phases $p_i(t)$ are governed by (Inoue 1975; Andrade & Stuhlmeier 2023*a*)

$$\frac{\mathrm{d}p_i}{\mathrm{d}t} = -\omega_i + T_{ii}|b_i|^2 - 2\sum_{j=1}^{4} T_{ij}|b_j|^2 - \frac{2T}{|b_i|^2}|b_1|\,|b_2||b_3|\,|b_4| \cos p. \tag{2.3}$$

The first three equalities in (2.1) yield integrals of motions known as the Manley–Rowe relations (Manley & Rowe 1956) that may be written, following Bretherton (1964) and

Stiassnie & Shemer (2005), as

$$|b_1(t)|^2 - q_1 = |b_2(t)|^2 - q_2 = q_3 - |b_3(t)|^2 = q_4 - |b_4(t)|^2 \equiv q(t), \tag{2.4}$$

where, by construction, the (positive or negative) relative action $q(t)$ satisfies initially

$$q(0) = 0. \tag{2.5}$$

Furthermore, four-wave interactions governed by (1.1) are bounded since from (2.4) (conditions for 'explosive' four-wave interactions are detailed in Turner (1980), Verheest (1982) and Safdi & Segur (2007); none of these are fulfilled here)

$$-\min(q_1, q_2) \leq q(t) \leq \min(q_3, q_4). \tag{2.6}$$

From (2.1) and (2.4) we get also (here and below, $\dot{q}(t) = \mathrm{d}q/\mathrm{d}t$)

$$\dot{q} = -4T\sqrt{(q_1+q)(q_2+q)(q_3-q)(q_4-q)}\,\sin p, \tag{2.7}$$

while, from (2.2) and (2.3),

$$\dot{p} = -(2Aq + B) - 2TF(q)\cos p, \tag{2.8}$$

where

$$\left.\begin{aligned} A &= \tfrac{1}{2}(T_{11}+T_{22}+T_{33}+T_{44}) + 2(T_{12}-T_{13}-T_{14}-T_{23}-T_{24}+T_{34}),\\ B &= \Delta\omega + B_1q_1 + B_2q_2 - B_3q_3 - B_4q_4, \quad \Delta\omega = \omega_1+\omega_2-\omega_3-\omega_4,\\ B_1 &= T_{11} + 2(T_{12}-T_{13}-T_{14}), \quad B_2 = T_{22} + 2(T_{12}-T_{23}-T_{24}),\\ B_3 &= T_{33} + 2(T_{34}-T_{13}-T_{23}), \quad B_4 = T_{44} + 2(T_{34}-T_{14}-T_{24}) \end{aligned}\right\} \tag{2.9}$$

and

$$F(q) = \left(\frac{1}{q_1+q} + \frac{1}{q_2+q} - \frac{1}{q_3-q} - \frac{1}{q_4-q}\right)\sqrt{(q_1+q)(q_2+q)(q_3-q)(q_4-q)}. \tag{2.10}$$

If $T = 0$, (2.7) and (2.8) yield, together with (2.5)

$$q(t) = 0, \quad p(t) = (\Delta\omega + B_1q_1 + B_2q_2 - B_3q_3 - B_4q_4)t + p_0. \tag{2.11}$$

We shall therefore consider $T \neq 0$ from now on. The initial phase mismatch $p_0 = p(0)$ introduced above reads also, from (1.2)

$$p_0 = \arg b_1(0) + \arg b_2(0) - \arg b_3(0) - \arg b_4(0) = \varphi_1 + \varphi_2 - \varphi_3 - \varphi_4. \tag{2.12}$$

Equations (2.7) and (2.8) may be written in canonical form:

$$\dot{p} = -\partial H/\partial q, \quad \dot{q} = \partial H/\partial p, \tag{2.13}$$

where the Hamiltonian $H(p, q)$, defined up to an additive constant, may be written as

$$H(p,q) = G(q) - G(0) + 4T\sqrt{(q_1+q)(q_2+q)(q_3-q)(q_4-q)}\,\cos p, \tag{2.14}$$

with

$$\begin{aligned} G(q) = q\Delta\omega &+ \tfrac{1}{2}(T_{11}(q_1+q)^2 + T_{22}(q_2+q)^2 + T_{33}(q_3-q)^2 + T_{44}(q_4-q)^2)\\ &+ 2T_{12}(q_1+q)(q_2+q) + 2T_{13}(q_1+q)(q_3-q) + 2T_{14}(q_1+q)(q_4-q)\\ &+ 2T_{23}(q_2+q)(q_3-q) + 2T_{24}(q_2+q)(q_4-q) + 2T_{34}(q_3-q)(q_4-q). \end{aligned} \tag{2.15}$$

The degree of the Hamiltonian system (2.13) is one so that it is integrable. Since $H(p, q)$ is conserved along the flow of (p, q), we have, using (2.5) and (2.14)

$$H(p(t), q(t)) = H(p(0), 0) = 4T\sqrt{q_1q_2q_3q_4}\cos p_0. \tag{2.16}$$

Since p_0 intervenes through its cosine, we shall consider $p_0 \in [0, \pi]$ from now on. (The Hamiltonian derived by Andrade & Stuhlmeier (2023*a*) is a particular case of (2.14) in which, with the present notations, $q_1 = q_2$ and $q_3 = q_4$.)

Squaring (2.7) and taking (2.14) and (2.16) into account leads to

$$\dot{q}^2 = 16T^2(q_1 + q)(q_2 + q)(q_3 - q)(q_4 - q) - (G(q) - G(0) - H(p_0, 0))^2, \tag{2.17}$$

or equivalently to the Bretherton equation:

$$\dot{q}^2 = f(q), \quad f(q) = 16T^2(q_1 + q)(q_2 + q)(q_3 - q)(q_4 - q) - (Aq^2 + Bq + C)^2, \tag{2.18}$$

where A and B are defined in (2.9) and where

$$C = -4T\sqrt{q_1q_2q_3q_4}\cos p_0. \tag{2.19}$$

Equations similar to (2.18) have been derived by Inoue (1975), Boyd & Turner (1978), Chen (1989) and Stiassnie & Shemer (2005). (Substituting $q(t) = 4TZ(t)$ in (2.18) yields equation (3.9) in Stiassnie & Shemer (2005).) Expanding $f(q)$ gives

$$f(q) = aq^4 + bq^3 + cq^2 + dq + e, \tag{2.20}$$

with

$$\left.\begin{aligned} a &= 16T^2 - A^2, \\ b &= 16T^2(q_1 + q_2 - q_3 - q_4) - 2AB, \\ c &= 16T^2(q_1q_2 + q_3q_4 - (q_1 + q_2)(q_3 + q_4)) - B^2 - 2AC, \\ d &= 16T^2((q_1 + q_2)q_3q_4 - q_1q_2(q_3 + q_4)) - 2BC, \\ e &= 16T^2q_1q_2q_3q_4\sin^2 p_0. \end{aligned}\right\} \tag{2.21}$$

According to Jeffreys & Jeffreys (1956, p. 667), Pars (1965, p. 4) or Craik (1985, p. 138), (2.18) may therefore be interpreted as the energy conservation equation of a unit-mass Newtonian particle $q(t)$ initially at $q(0) = 0$ moving rectilinearly in a quartic potential $U(q)$ defined by

$$U(q) = -\tfrac{1}{2}f(q). \tag{2.22}$$

3. Steady solutions and their stability

3.1. *Steady equilibria*

From the definition adopted in § 1, a steady equilibrium is a solution of (1.1) with constant amplitudes $|b_i|$. From the Manley–Rowe relations (2.1), it corresponds to a solution of (2.18) such that $\dot{q}(t) = 0$ for all time. Since $q(0) = 0$, a steady equilibrium therefore corresponds in our problem to the null solution $q(t) = 0$, $\forall t \geq 0$, for which (2.18) implies $f(0) = 0$. But since (2.18) is equivalent to 'Newton's second law':

$$\ddot{q} = \tfrac{1}{2}f'(q) = -U'(q), \tag{3.1}$$

steady equilibria also satisfy $U'(0) = 0$ (here and below, $f'(q) = \mathrm{d}f/\mathrm{d}q$), i.e. they are critical points of the potential energy (Arnold 1989, p. 99). For (2.20), conditions $f(0) = 0$

and $f'(0) = 0$ are respectively equivalent to $e = 0$ and $d = 0$. Since we now consider $T \neq 0$, condition $e = 0$ yields either $q_1q_2q_3q_4 = 0$, or $p_0 = 0$ or π, while condition $d = 0$ becomes

$$2T((q_1 + q_2)q_3q_4 - q_1q_2(q_3 + q_4)) + B\sqrt{q_1q_2q_3q_4}\cos p_0 = 0. \tag{3.2}$$

Suppose first $q_1q_2q_3q_4 = 0$ and choose, without lost of generality, $q_4 = 0$. Condition (3.2) implies $q_1q_2q_3 = 0$. Therefore, a second wave must have a zero initial wave action, say $q_3 = 0$. From system (1.1) with $|b_i|^2 = q_i$, $i = 1, 2$, we get the steady bichromatic wave first considered by Phillips (1960) and Longuet-Higgins & Phillips (1962) (see also Zakharov 1967; Hogan, Gruman & Stiassnie 1988; Leblanc 2009):

$$\left.\begin{aligned} b_1(t) &= \sqrt{q_1}\mathrm{e}^{\mathrm{i}\varphi_1}\mathrm{e}^{-\mathrm{i}\Omega_1 t}, \quad \Omega_1 = \omega_1 + T_{11}q_1 + 2T_{12}q_2, \\ b_2(t) &= \sqrt{q_2}\mathrm{e}^{\mathrm{i}\varphi_2}\mathrm{e}^{-\mathrm{i}\Omega_2 t}, \quad \Omega_2 = \omega_2 + T_{22}q_2 + 2T_{21}q_1, \end{aligned}\right\} \tag{3.3}$$

where we recall that $\varphi_i = p_i(0)$. These solutions were given by Benney (1962) who also noticed that if in addition $q_2 = 0$ one recovers the Stokes wave:

$$b_1(t) = \sqrt{q_1}\mathrm{e}^{\mathrm{i}\varphi_1}\mathrm{e}^{-\mathrm{i}\Omega_1 t}, \quad \Omega_1 = \omega_1 + T_{11}q_1. \tag{3.4}$$

Turning now to the case $q_1q_2q_3q_4 \neq 0$ and $p_0 = 0$ or π, we get from (3.2):

$$\Delta\omega + B_1q_1 + B_2q_2 - B_3q_3 - B_4q_4 = 2T\frac{q_1q_2(q_3 + q_4) - (q_1 + q_2)q_3q_4}{\sqrt{q_1q_2q_3q_4}\cos p_0}. \tag{3.5}$$

Since from (2.7) $p(t) = p_0$ at equilibrium, we conclude that:

THEOREM 3.1. *The wave quartet defined by*

$$\left.\begin{aligned} b_i(t) &= \sqrt{q_i}\mathrm{e}^{\mathrm{i}\varphi_i}\mathrm{e}^{-\mathrm{i}\Omega_i t}, \quad q_i > 0, \quad i = 1, \ldots, 4, \\ \Omega_i &= \omega_i - T_{ii}q_i + 2\sum_{j=1}^{4} T_{ij}q_j + \frac{2T}{q_i}\sqrt{q_1q_2q_3q_4}\cos p_0, \\ p_0 &= \varphi_1 + \varphi_2 - \varphi_3 - \varphi_4 = 0 \textit{ or } \pi, \end{aligned}\right\} \tag{3.6}$$

and satisfying the compatibility condition (3.5) *is a steady solution of* (1.1) *with* (1.2).

3.2. *Linear stability*

Analogy with Newtonian dynamics allows us to formulate a simple criterion for linear stability of steady equilibria. Indeed, (3.1) yields, after linearization around $q = 0$,

$$\ddot{q} - \gamma^2 q = 0, \quad \gamma^2 = \tfrac{1}{2}f''(0) = -U''(0), \tag{3.7}$$

where from (2.20) with (2.21) and (2.22), we have $\gamma^2 = c$. Therefore, the null solution is linearly stable if $c < 0$ and unstable otherwise; growth is exponential if $c > 0$ or algebraic if $c = 0$.

Consider first the linear stability of the bichromatic solution (3.3) for which $q_3 = q_4 = 0$. In that case, $c = 16T^2q_1q_2 - B^2$, where B is defined in (2.9). Therefore, the bichromatic wavetrain (3.3) is exponentially unstable if

$$16T^2q_1q_2 > (\Delta\omega + B_1q_1 + B_2q_2)^2. \tag{3.8}$$

Recalling that $q_i = |b_i|^2$ and that B_i are defined in (2.9), we recover the criterion derived in Leblanc (2009, equation (15)) and recovered by Andrade & Stuhlmeier (2023*a*). We also recall that (3.8) characterizes respectively 'type B' and 'class-Ia' instabilities following the respective classifications of Okamura (1984) for one-dimensional standing waves and of Ioualalen & Kharif (1994) for steady bichromatic wavetrains.

Turning now to the stability of steady wave quartets with non-zero initial wave actions (q_1, q_2, q_3, q_4) satisfying (3.5) and $\cos p_0 = \pm 1$, the linear stability criterion becomes, after replacing B by the right-hand side of (3.5) and C by (2.19) onto the expression of c in (2.21):

THEOREM 3.2. *The steady wave quartet* (3.6) *satisfying* (3.5) *is exponentially unstable if*

$$\frac{A}{2T}\frac{\cos p_0}{\sqrt{q_1q_2q_3q_4}} > \frac{1}{4}\left(\frac{1}{q_1}+\frac{1}{q_2}-\frac{1}{q_3}-\frac{1}{q_4}\right)^2 + \frac{(q_1+q_2)(q_3+q_4)-q_1q_2-q_3q_4}{q_1q_2q_3q_4}, \tag{3.9}$$

algebraically unstable if equality holds, otherwise linearly stable.

3.3. *The particular case of equal wave actions*

Consider the particular case of the steady wave quartet with equal wave actions $q_1 = q_2 = q_3 = q_4 \equiv q_0 > 0$ and initial phase mismatch $p_0 = 0$ or π. We recall that the existence of such steady interactions is conditioned by (3.5) which reduces in the present case to

$$\Delta\omega + q_0\Delta B = 0, \tag{3.10}$$

where $\Delta B = B_1 + B_2 - B_3 - B_4$, i.e. from (2.9):

$$\Delta B = T_{11} + T_{22} - T_{33} - T_{44} + 4(T_{12} - T_{34}). \tag{3.11}$$

At resonance, (3.10) implies $\Delta B = 0$, while off resonance we get $q_0 = -\Delta\omega/\Delta B$. Since $q_0 > 0$, $\Delta\omega$ and ΔB must have opposite signs for the quartet to exist. The corresponding instability criterion may be easily deduced from (3.9). Therefore:

CRITERION 3.1. *Steady wave quartets* (3.6) *with* $q_1 = q_2 = q_3 = q_4 \equiv q_0$ *exist if* $q_0 = -\Delta\omega/\Delta B > 0$. *At resonance, they exist for any* $q_0 > 0$ *if* $\Delta B = 0$. *In both cases, they are exponentially unstable if*

$$(A/T)\cos p_0 > 4 \quad (p_0 = 0 \text{ or } \pi). \tag{3.12}$$

3.4. *Lagrange theorem and Lyapunov stability*

In his treatise on analytical mechanics published in 1788, Lagrange presented his famous principle on the stability of equilibrium positions, which was rigorously proved by Lejeune–Dirichlet in 1846 and generalized by Lyapunov in 1892 (see Loria & Panteley 2017). Lagrange theorem may be stated as (Gantmacher 1975, pp. 166–173; Arnold 1989, p. 99):

LAGRANGE THEOREM. *If* $U(0) = 0$ *is a strict local minimum of the potential* $U(q)$, *then the null solution of* (2.18) *is Lyapunov stable.*

For our purpose, stability in the sense of Lyapunov is defined by:

LYAPUNOV STABILITY. *The null solution is Lyapunov stable if for each* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that, if* $|\dot{q}(0)| < \delta$ *initially, then* $\sup(|q(t)|, |\dot{q}(t)|) < \varepsilon$ *for all* $t \geq 0$.

In the present case, $U(q) = -\frac{1}{2}(aq^4 + bq^3 + cq^2 + dq + e)$. But, as stated previously, $q = 0$ is a steady equilibrium if $U(0) = 0$ and $U'(0) = 0$, or equivalently $d = e = 0$.

Therefore, at equilibrium, $U(q) = -\frac{1}{2}q^2(aq^2 + bq + c)$. But since in that case $U''(0) = -c$, then $U(q)$ admits a strict local minimum at $q = 0$ if $c < 0$. Therefore in our case, a linearly stable equilibrium is also Lyapunov stable. This means that if $c < 0$, any sufficiently small disturbance will remain bounded.

Of course it would be necessary to be precise about what is meant by 'sufficiently small' but we shall not pursue that direction for at least two reasons: firstly, because my study is restricted to the interactions inside a single quartet while interactions with other waves may be destabilizing (see e.g. Okamura 1985); secondly, because higher-order interactions are not taken into account (see e.g. Andrade & Stuhlmeier 2023*a*). As a consequence, we have to keep in mind that stability is only indicative because limited to the discrete four-wave interactions considered in the present study.

By contrast, linear instability criteria are meaningful in the nonlinear regime as it is known that the existence of an exponentially growing solution of the linearized equation (3.7) implies instability of the null solution in the nonlinear equation (3.1) (see Verhulst 1996, p. 88). Furthermore, if other interactions were taken into account, various instability mechanisms would compete without mutual cancellation.

Therefore, the various instability criteria presented in the present study have to be considered as sufficient conditions for instability.

4. Exact unsteady solutions for a quartic potential

4.1. *General properties*

We have seen in § 2 that solutions of (2.18) with (2.5) are bounded. Furthermore, following Inoue (1975), we get from (2.18) the following inequalities:

$$\left.\begin{aligned} f(0) = 16T^2 q_1 q_2 q_3 q_4 \sin^2 p_0 \geq 0, \\ f(-q_1) \leq 0, \quad f(-q_2) \leq 0, \quad f(q_3) \leq 0, \quad f(q_4) \leq 0. \end{aligned}\right\} \tag{4.1}$$

Therefore we can conclude that, by continuity, f admits at least two real roots, say ξ_- and ξ_+, verifying $\xi_- \leq 0 \leq \xi_+$; f is therefore a polynomial function of degree at least equal to two: either quartic, cubic or quadratic.

Excluding the case $\xi_- = \xi_+ = 0$ corresponding to root 0 with multiplicity at least equal to 2 corresponding to a steady solution since $d = 0$ and $e = 0$ as explained in § 3.1, we restrict from now on our discussion to the cases where either $\xi_- \leq 0 < \xi_+$ or $\xi_- < 0 \leq \xi_+$. Without lost of generality, suppose that ξ_- and ξ_+ are the closest roots from 0. Equation (2.18) shows that unsteady solutions exist if $f(q) \geq 0$; since $q(0) = 0$ and $f(0) \geq 0$, Jeffreys & Jeffreys (1956, pp. 667–668), Bretherton (1964) and Pars (1965, pp. 4–6) showed that $\xi_- \leq q(t) \leq \xi_+$, $\forall t \geq 0$, and that the particle 'velocity' vanishes on the boundaries of this interval: $\dot{q}(\xi_\pm) = 0$. If ξ_- and ξ_+ are both simple roots, they are turning points, i.e. the sign of $\dot{q}(t)$ changes on turning points and the direction of the particle is reversed; therefore $q(t)$ is periodic. If the multiplicity of either ξ_- or ξ_+, say ξ_-, is strictly greater than one, then $\lim_{t\to\pm\infty} q(t) = \xi_-$ and the solution is a breather. Finally if ξ_- and ξ_+ are both double roots, then either $\lim_{t\to\pm\infty} q(t) = \xi_\pm$ or $\lim_{t\to\pm\infty} q(t) = \xi_\mp$ and the solution is a pump. Finally, from (2.6), we have also

$$-\min(q_1, q_2) \leq \xi_- \leq q(t) \leq \xi_+ \leq \min(q_3, q_4). \tag{4.2}$$

Equation (2.18) with (2.5) may be integrated (see e.g. Craik 1985, p. 138):

$$t \equiv t(q) = \pm \int_0^q \frac{\mathrm{d}\xi}{\sqrt{f(\xi)}} = \pm \int_0^q \frac{\mathrm{d}\xi}{\sqrt{-2U(\xi)}}, \tag{4.3}$$

from which $q \equiv q(t)$ may formally be obtained by inversion. The sign indeterminacy above shows that for each fixed values of the parameters, (2.18) admits two solutions, say $\{Q_-(t), Q_+(t)\}$, such that $Q_-(t) = Q_+(-t)$. Now, let Q_+ be the solution such that at the initial time $\dot{Q}_+(0) \geq 0$. Therefore, $\dot{Q}_-(0) = -\dot{Q}_+(0) \leq 0$. The solution to choose is determined because of (2.7) from which we get

$$\dot{q}(0) = -4T\sqrt{q_1 q_2 q_3 q_4}\,\sin p_0. \tag{4.4}$$

Thus, if $-4T\sqrt{q_1 q_2 q_3 q_4}\,\sin p_0 \geq 0$, then $q(t) = Q_+(t)$. Else $q(t) = Q_-(t) = Q_+(-t)$. Finally, if $q(t)$ is periodic, the period τ is

$$\tau = 2\int_{\xi_-}^{\xi_+} \frac{\mathrm{d}\xi}{\sqrt{f(\xi)}} = 2\int_{\xi_-}^{\xi_+} \frac{\mathrm{d}\xi}{\sqrt{-2U(\xi)}}. \tag{4.5}$$

In the quartic case, f in (2.20) with $a \neq 0$ may be factorized as

$$f(q) = a(q-\xi_1)(q-\xi_2)(q-\xi_3)(q-\xi_4), \tag{4.6}$$

where ξ_1, ξ_2, ξ_3 and ξ_4 are the four roots of f. (The labelling of the roots ξ_i is independent of the labelling of the initial wave actions q_i.) Since f is real-valued, roots are either real or complex conjugate by pairs. The roots of a quartic polynomial may formally be obtained by quadrature with Ferrari's method (published by Cardan in 1545) but formulae, which are too lengthy to be reported here, are implemented in computer algebra systems. Simple expressions are given in Appendix A in the case $q_1 = q_2 = q_3 = q_4$.

The nature of the solutions of the Bretherton equation (2.18) depends on the sign of a and on the nature of the roots, as illustrated in figure 1 (see also Turner 1980). If $a = 0$, the potential is either cubic or quadratic and the solutions are postponed to Appendix B.

4.2. *The case $a > 0$*

4.2.1. *Periodic solution*

If f admits four distinct real roots $\xi_1, \dots, \xi_4$ such that (figure 1*a*)

$$\xi_1 < \xi_2 \leq 0 < \xi_3 < \xi_4 \quad \text{or} \quad \xi_1 < \xi_2 < 0 \leq \xi_3 < \xi_4, \tag{4.7}$$

then (2.18) with (2.5) has the pair of periodic solutions $\{Q_I(t), Q_I(-t)\}$ with

$$\left.\begin{aligned} Q_I(t) &= \frac{\xi_2(\xi_3-\xi_1) - \xi_1(\xi_3-\xi_2)\,\mathrm{sn}^2(u(t),k)}{(\xi_3-\xi_1) - (\xi_3-\xi_2)\,\mathrm{sn}^2(u(t),k)}, \quad u(t) = \frac{\gamma t}{2} + \mathrm{sn}^{-1}(l,k), \\ \gamma &= \sqrt{a(\xi_4-\xi_2)(\xi_3-\xi_1)}, \quad k = \sqrt{\frac{(\xi_3-\xi_2)(\xi_4-\xi_1)}{(\xi_4-\xi_2)(\xi_3-\xi_1)}}, \quad l = \sqrt{\frac{\xi_2(\xi_3-\xi_1)}{\xi_1(\xi_3-\xi_2)}}, \end{aligned}\right\} \tag{4.8}$$

where sn and sn^{-1} are Jacobi elliptic functions defined here by (see e.g. Byrd & Friedman (1971, p. 18))

$$x = \mathrm{sn}^{-1}(y,k) = \int_0^y \frac{\mathrm{d}\xi}{\sqrt{(1-\xi^2)(1-k^2\xi^2)}}, \quad y = \mathrm{sn}(x,k). \tag{4.9}$$

Period is

$$\tau = \frac{4}{\gamma}\,\mathrm{K}(k), \quad \mathrm{K}(k) = \int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{1-k^2\sin^2\theta}}, \tag{4.10}$$

where K is the complete elliptic integral of the first kind. (Different conventions exist for the arguments of elliptic functions; we follow here the notations of Byrd & Friedman

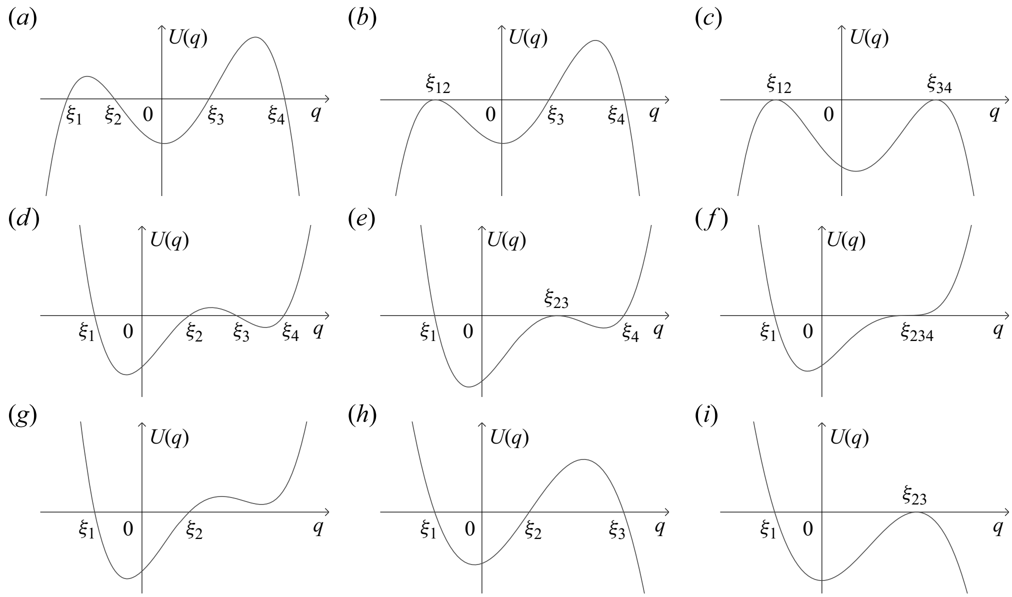

Figure 1. The possible configurations for bounded unsteady solutions of (2.18) satisfying (2.5). Motion occurs in the potential well defined by $q \in [\xi_-, \xi_+]$, where $\xi_\pm$ such that $\xi_- \leq 0 < \xi_+$ or $\xi_- < 0 \leq \xi_+$ are the nearest roots around zero between which $U(q) = -\frac{1}{2}f(q) \leq 0$. Quartic potential (4.6) with $a > 0$: (*a*) periodic solution (4.8); (*b*) breather solution (4.14); (*c*) pump solution (4.16). Quartic potential (4.6) with $a < 0$: (*d*) periodic solution (4.21); (*e*) breather solution (4.23); (*f*) rational breather solution (4.25); (*g*) periodic solution (4.27). Cubic potential (B1) with $b > 0$: (*h*) periodic solution (B2); (*i*) breather solution (B4). Any other possibility may be deduced by symmetry with respect to the vertical axis. The case of quadratic potential with periodic solution (B6) has been omitted.

(1971). By contrast, in Wolfram Mathematica the entry for $\mathrm{sn}(x, k)$ as defined in (4.9) is JacobiSN[x,m], where $m = k^2$; similarly for $\mathrm{K}(k)$ defined in (4.10) for which the entry is EllipticK[m].)

Note that at the initial time

$$\dot{Q}_I(0) = \frac{\gamma \xi_1 \xi_2}{l(\xi_2 - \xi_1)} \,\mathrm{cn}(\mathrm{sn}^{-1}(l, k), k)\,\mathrm{dn}(\mathrm{sn}^{-1}(l, k), k), \tag{4.11}$$

where at fixed modulus k: $\mathrm{cn}(x) = \sqrt{1 - \mathrm{sn}^2(x)}$ and $\mathrm{dn}(x) = \sqrt{1 - k^2\,\mathrm{sn}^2(x)}$ (Byrd & Friedman 1971, p. 19). Therefore, at fixed k:

$$\mathrm{cn}(\mathrm{sn}^{-1}(y))\,\mathrm{dn}(\mathrm{sn}^{-1}(y)) = \sqrt{1 - y^2}\sqrt{1 - k^2 y^2}. \tag{4.12}$$

Then

$$\dot{Q}_I(0) = \frac{\gamma \xi_1 \xi_2}{l(\xi_2 - \xi_1)} \sqrt{1 - l^2}\sqrt{1 - k^2 l^2} \geq 0. \tag{4.13}$$

Formula (254.00) in Byrd & Friedman (1971, p. 112) has been used to get (4.8), plotted in figure 2(*a*). Similar solutions were found by Inoue (1975), Boyd & Turner (1978), Shemer & Stiassnie (1985), Chen & Snyder (1989), Cappellini & Trillo (1991) and Stiassnie & Shemer (2005).

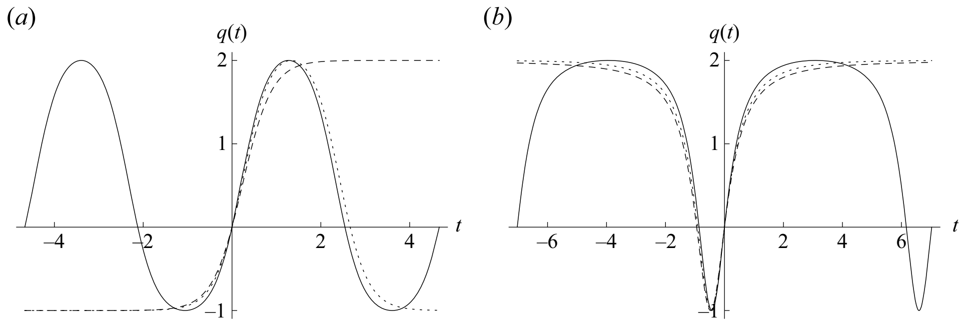

Figure 2. Unsteady solutions $q(t)$ in a quartic potential (4.6) with $a = 1$ (*a*) or $a = -1$ (*b*). (*a*) Periodic solution (4.8) with $\xi_1 = -1.2$, $\xi_2 = -1$, $\xi_3 = 2$, $\xi_4 = 2.4$ (solid line); breather solution (4.14) with $\xi_{12} = -1$, $\xi_3 = 2$, $\xi_4 = 2.4$ (dotted line); pump solution (4.16) with $\xi_{12} = -1$, $\xi_{34} = 2$ (dashed line). (*b*) Periodic solution (4.21) with $\xi_1 = -1$, $\xi_2 = 2$, $\xi_3 = 2.1$, $\xi_4 = 2.2$ (solid line); breather solution (4.23) with $\xi_1 = -1$, $\xi_{23} = 2$, $\xi_4 = 2.2$ (dotted line); rational breather solution (4.25) with $\xi_1 = -1$, $\xi_{234} = 2$ (dashed line).

4.2.2. *Breather solution*

If f admits a double real root ξ_{12} and two distinct real roots ξ_3 and ξ_4 such that $\xi_{12} < 0 \leq \xi_3 < \xi_4$ (figure 1*b*), then (2.18) with (2.5) has the pair of breather solutions $\{Q_{II}(t), Q_{II}(-t)\}$ with

$$\left.\begin{aligned} Q_{II}(t) &= \xi_{12}\frac{v^4(t) + 4n^2v^3(t) - 2sv^2(t) + 4n^2rv(t) + r^2}{v^4(t) - 2(8\xi_{12}^2n^2 + r)v^2(t) + r^2}, \\ v(t) &= (\xi_{12}(\xi_3 + \xi_4) - 2(\xi_3\xi_4 - n\sqrt{\xi_3\xi_4}))\exp(\sqrt{a}nt), \\ n &= \sqrt{(\xi_3 - \xi_{12})(\xi_4 - \xi_{12})}, \qquad r = \xi_{12}^2(\xi_4 - \xi_3)^2, \\ s &= 4\xi_{12}(\xi_{12}^2 + \xi_3\xi_4)(\xi_3 + \xi_4) - \xi_{12}^2(3\xi_3 + \xi_4)(\xi_3 + 3\xi_4). \end{aligned}\right\} \tag{4.14}$$

At the initial time: $\dot{Q}_{II}(0) = -\sqrt{a\xi_3\xi_4}\xi_{12} \geq 0$. Note that from (4.1): $\xi_{12} = -q_1 = -q_2$.

Derived with the assistance of a computer algebra system and plotted in figure 2(*a*), solution (4.14) has not been found in the literature in this general form. However, if $\xi_3 = 0$, we get

$$Q_{II}(t) = \xi_{12} - \frac{2n^2}{2\xi_{12} - \xi_4(1 + \cosh(\sqrt{a}nt))}, \qquad n = \sqrt{-\xi_{12}(\xi_4 - \xi_{12})}, \tag{4.15}$$

corresponding to expression (22*a*) in Inoue (1975). The 'pump-depletion' solutions (A6) and (A7) in Cappellini & Trillo (1991) are similar to (4.14), but correspond in their formulation to specific values of the roots ξ_i. Finally, connection between (4.14) and the 'discrete Akhmediev breathers' (5.4) and (5.15) in Andrade & Stuhlmeier (2023*b*) remains to be clarified since their solutions are given in implicit form, contrary to (4.14).

4.2.3. *Pump solution*

If f admits two double real roots ξ_{12} and ξ_{34} such that $\xi_{12} < 0 < \xi_{34}$ (figure 1*c*), then (2.18) with (2.5) has the pair of pump solutions $\{Q_{III}(t), Q_{III}(-t)\}$ with

$$Q_{III}(t) = \xi_{34}\frac{e^{\mu t} - 1}{e^{\mu t} - m}, \qquad \mu = \sqrt{a}(\xi_{34} - \xi_{12}), \qquad m = \frac{\xi_{34}}{\xi_{12}}. \tag{4.16}$$

At the initial time: $\dot{Q}_{III}(0) = -\sqrt{a}\xi_{12}\xi_{34} > 0$. The 'shock-like' solutions of Inoue (1975) and the 'multibreather' solution of Andrade & Stuhlmeier (2023*a*,*b*) are equivalent to (4.16), plotted in figure 2(*a*).

From (4.1): $\xi_{12} = -q_1 = -q_2 = -q_{12}$ and $\xi_{34} = q_3 = q_4 = q_{34}$; from (4.6):

$$f(q) = a(q - \xi_{12})^2(q - \xi_{34})^2 = (16T^2 - A^2)(q + q_{12})^2(q - q_{34})^2. \tag{4.17}$$

Identification with (2.18) yields, excluding the case $A = 0$, the following two compatibility conditions (Andrade & Stuhlmeier 2023*a*):

$$\cos p_0 = \frac{A}{4T} \in [-1, 1], \quad \Delta\omega + (B_1 + B_2 - A)q_{12} - (B_3 + B_4 - A)q_{34} = 0. \tag{4.18}$$

Finally, if $q_{12} = q_{34} = q_0 > 0$, the second condition in (4.18) gives $\Delta\omega + (\Delta B)q_0 = 0$, and (4.16) with $-\xi_{12} = \xi_{34} = q_0$ becomes simply

$$Q_{III}(t) = q_0 \tanh(\sqrt{a} q_0 t). \tag{4.19}$$

4.3. *The case $a < 0$*

4.3.1. *Periodic solution (four real roots)*

If f admits four distinct real roots $\xi_1, \ldots, \xi_4$ such that (figure 1*d*)

$$\xi_1 \le 0 < \xi_2 < \xi_3 \le \xi_4 \quad \text{or} \quad \xi_1 < 0 \le \xi_2 < \xi_3 \le \xi_4, \tag{4.20}$$

then (2.18) with (2.5) has the pair of periodic solutions $\{Q_{IV}(t),\ Q_{IV}(-t)\}$ with

$$\left.\begin{aligned} Q_{IV}(t) &= \frac{\xi_1(\xi_4 - \xi_2) + \xi_4(\xi_2 - \xi_1)\,\mathrm{sn}^2(u(t), k)}{(\xi_4 - \xi_2) + (\xi_2 - \xi_1)\,\mathrm{sn}^2(u(t), k)}, \quad u(t) = \frac{\gamma t}{2} + \mathrm{sn}^{-1}(l, k), \\ \gamma &= \sqrt{-a(\xi_4 - \xi_2)(\xi_3 - \xi_1)}, \quad k = \sqrt{\frac{(\xi_4 - \xi_3)(\xi_2 - \xi_1)}{(\xi_4 - \xi_2)(\xi_3 - \xi_1)}}, \quad l = \sqrt{-\frac{\xi_1(\xi_4 - \xi_2)}{\xi_4(\xi_2 - \xi_1)}}. \end{aligned}\right\} \tag{4.21}$$

Period is $\tau = 4\gamma^{-1}\mathrm{K}(k)$. At the initial time:

$$\dot{Q}_{IV}(0) = \frac{\gamma l(\xi_2 - \xi_1)\xi_4^2}{(\xi_4 - \xi_1)(\xi_4 - \xi_2)}\sqrt{1 - l^2}\sqrt{1 - k^2 l^2} \ge 0. \tag{4.22}$$

Formula (252.00) in Byrd & Friedman (1971, p. 103) has been used to get (4.21), plotted in figure 2(*b*). A similar solution is given by Chen (1989).

4.3.2. *Breather solution*

If f admits a double real root ξ_{23} and two distinct real roots ξ_1 and ξ_4 such that $\xi_1 \le 0 < \xi_{23} < \xi_4$ (figure 1*e*), then (2.18) with (2.5) has the pair of breather solutions

https://doi.org/10.1017/jfm.2024.126 Published online by Cambridge University Press

$\{Q_V(t), Q_V(-t)\}$ with

$$\left.\begin{aligned} Q_V(t) &= \xi_{23}\frac{v^4(t)+4n^2v^3(t)-2sv^2(t)+4n^2rv(t)+r^2}{v^4(t)+2(8\xi_{23}^2n^2-r)v^2(t)+r^2}, \\ v(t) &= -(n^2+2n\sqrt{-\xi_1\xi_4}+\xi_{23}^2-\xi_1\xi_4)\exp(\sqrt{-a}nt), \\ n &= \sqrt{(\xi_{23}-\xi_1)(\xi_4-\xi_{23})}, \quad r=\xi_{23}^2(\xi_4-\xi_1)^2, \\ s &= 4\xi_{23}(\xi_{23}^2+\xi_1\xi_4)(\xi_1+\xi_4)-\xi_{23}^2(3\xi_1+\xi_4)(\xi_1+3\xi_4). \end{aligned}\right\} \tag{4.23}$$

At the initial time: $\dot{Q}_V(0)=\sqrt{a\xi_1\xi_4}\xi_{23}\geq 0$. Plotted in figure 2(*b*), solution (4.23) has not been found in the literature. If $\xi_1=0$, we get

$$Q_V(t)=\xi_{23}+\frac{2n^2}{2\xi_{23}-\xi_4(1+\cosh(\sqrt{-a}nt))}, \quad n=\sqrt{\xi_{23}(\xi_4-\xi_{23})}. \tag{4.24}$$

4.3.3. *Rational breather solution*

If f admits a triple real root ξ_{234} and a single real root ξ_1 such that $\xi_1\leq 0<\xi_{234}$ (figure 1*f*), then (2.18) with (2.5) has the pair of rational breather solutions $\{Q_{VI}(t), Q_{VI}(-t)\}$ with

$$Q_{VI}(t)=\frac{\xi_{234}\mu t(\mu t+4m)}{\mu t(\mu t+4m)+4(1+m^2)}, \quad \mu=\sqrt{-a}(\xi_{234}-\xi_1), \quad m=\sqrt{-\xi_1/\xi_{234}}. \tag{4.25}$$

At the initial time: $\dot{Q}_{VI}(0)=\sqrt{a\xi_1\xi_{234}^3}\geq 0$. Plotted in figure 2(*b*), a solution equivalent to (4.25) is given by Turner (1980).

Finally, if $\xi_1=0$, (4.25) may be written as

$$Q_{VI}(t)=\xi_{234}\tilde{Q}(\tilde{t}), \quad \tilde{Q}(\tilde{t})=\tilde{t}^2/(\tilde{t}^2+4), \quad \tilde{t}=\sqrt{-a}\xi_{234}t. \tag{4.26}$$

4.3.4. *Periodic solution (two real roots and two complex conjugate roots)*

Finally, if f admits two distinct real roots ξ_1 and ξ_2 such that $\xi_1\leq 0<\xi_2$ or $\xi_1<0\leq\xi_2$, and two complex conjugate roots ξ_3 and $\xi_4=\xi_3^*$ (figure 1*g*), then (2.18) with (2.5) has the pair of periodic solutions $\{Q_{VII}(t), Q_{VII}(-t)\}$ with

$$\left.\begin{aligned} Q_{VII}(t) &= \frac{(\beta_1\xi_2+\beta_2\xi_1)-(\beta_1\xi_2-\beta_2\xi_1)\,\mathrm{cn}(w(t),k)}{(\beta_1+\beta_2)-(\beta_1-\beta_2)\,\mathrm{cn}(w(t),k)}, \quad w(t)=\gamma t+\mathrm{cn}^{-1}(l,k), \\ \gamma &= \sqrt{-a\beta_1\beta_2}, \quad k=\sqrt{\frac{(\xi_2-\xi_1)^2-(\beta_2-\beta_1)^2}{4\beta_1\beta_2}}, \quad l=\frac{\beta_1\xi_2+\beta_2\xi_1}{\beta_1\xi_2-\beta_2\xi_1}, \\ \beta_1 &= \sqrt{(\xi_1-x_3)^2+y_3^2}, \quad \beta_2=\sqrt{(\xi_2-x_3)^2+y_3^2}, \quad x_3=\mathrm{Re}(\xi_3), \quad y_3=\mathrm{Im}(\xi_3). \end{aligned}\right\} \tag{4.27}$$

Period is $\tau=4\gamma^{-1}\mathrm{K}(k)$. At the initial time:

$$\dot{Q}_{VII}(0)=-\frac{a(\beta_1\xi_2-\beta_2\xi_1)^2}{2\gamma(\xi_2-\xi_1)}\sqrt{1-l^2}\sqrt{1+k^2(l^2-1)}\geq 0, \tag{4.28}$$

since at fixed k, $\mathrm{sn}(\mathrm{cn}^{-1}(y))\,\mathrm{dn}(\mathrm{cn}^{-1}(y))=\sqrt{1-y^2}\sqrt{1+k^2(y^2-1)}$.

Equation (259.00) in Byrd & Friedman (1971, p. 133) has been used to get (4.27). A similar solution is given by Turner (1980).

5. Deep-water gravity waves

5.1. *The truncated quartet model*

Deep-water irrotational gravity waves propagating in an inviscid incompressible fluid are governed in spectral space at third order in amplitude by the Zakharov equation (Zakharov 1966, 1968; Krasitskii 1990, 1994):

$$\mathrm{i}\frac{\partial \mathcal{B}(\boldsymbol{k})}{\partial t} = \omega(\boldsymbol{k})\mathcal{B}(\boldsymbol{k}) + \int_{\mathbb{R}^6} T(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q},\boldsymbol{r})\mathcal{B}^*(\boldsymbol{p})\mathcal{B}(\boldsymbol{q})\mathcal{B}(\boldsymbol{r})\delta(\boldsymbol{k}+\boldsymbol{p}-\boldsymbol{q}-\boldsymbol{r})\,\mathrm{d}\boldsymbol{p}\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}\boldsymbol{r}, \tag{5.1}$$

where $\boldsymbol{k} \in \mathbb{R}^2$, $\omega(\boldsymbol{k}) = \sqrt{g|\boldsymbol{k}|}$, $\delta(\boldsymbol{k})$ is Dirac delta function and the real-valued function $T(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q},\boldsymbol{r})$ is Krasitskii's kernel given in Appendix C. At leading order, $\mathcal{B}(\boldsymbol{k},t)$ is related to the free-surface elevation $z = \eta(\boldsymbol{x},t)$, $\boldsymbol{x} \in \mathbb{R}^2$, by (we follow Janssen's (2004, p. 132) convention for the Fourier transform, so that expressions given here for T (and T_{ij}) must be divided by $(2\pi)^2$ to recover those given in Krasitskii (1994))

$$\eta(\boldsymbol{x},t) = \int_{\mathbb{R}^2} \hat{\eta}(\boldsymbol{k},t)\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\mathrm{d}\boldsymbol{k}, \quad \hat{\eta}(\boldsymbol{k},t) = \sqrt{\frac{|\boldsymbol{k}|}{2\omega(\boldsymbol{k})}}(\mathcal{B}(\boldsymbol{k},t)+\mathcal{B}^*(-\boldsymbol{k},t)). \tag{5.2}$$

According to Zakharov (1966, 1968), an exact solution of (5.1) is the Stokes wave:

$$\mathcal{B}(\boldsymbol{k},t) = b(t)\delta(\boldsymbol{k}-\boldsymbol{k}_0), \quad b(t) = b_0\mathrm{e}^{-\mathrm{i}\varOmega_0 t}, \quad \varOmega_0 = \omega(\boldsymbol{k}_0) + T(\boldsymbol{k}_0,\boldsymbol{k}_0,\boldsymbol{k}_0,\boldsymbol{k}_0)|b_0|^2. \tag{5.3}$$

If we now consider a linear combination of waves $\mathcal{B}(\boldsymbol{k},t) = \sum_{i=1}^N b_i(t)\delta(\boldsymbol{k}-\boldsymbol{k}_i)$ with $N > 1$, the Zakharov equation (5.1) yields a system of ordinary differential equations which is not closed, as noticed by Okamura (1985); it leads indeed to the generation of higher harmonics on time scales of order $(|T||b|^2)^{-1}$. The mathematical validity of such an ansatz is therefore an open question. (This was pointed to me out by an anonymous reviewer even for $N = 2$; see also discussion in Badulin *et al.* (1995). Zakharov (1967) already noticed that the bichromatic wave (3.3) is an approximate solution.)

If the terms corresponding to higher harmonics are neglected, one gets a truncated model consisting of a closed system of ordinary differential equations considered in various textbooks and review articles (e.g. Yuen & Lake 1982; Craik 1985; Shemer & Stiassnie 1991; Janssen 2004; Kartashova 2010) and implicitly or explicitly used in a number of articles, including those by Saffman & Yuen (1980), Caponi, Saffman & Yuen (1982), Okamura (1984, 1985), Shemer & Stiassnie (1985), Hogan *et al.* (1988), Badulin *et al.* (1995), Stiassnie & Shemer (2005), Leblanc (2009) and Andrade & Stuhlmeier (2023*a*,*b*):

$$\mathrm{i}\frac{\mathrm{d}b_i}{\mathrm{d}t} = \omega_i b_i + \sum_{j,m,n=1}^{N} \hat{T}_{ijmn} b_j^* b_m b_n, \quad i = 1,\ldots,N, \tag{5.4}$$

where $\omega_i = \sqrt{g|\boldsymbol{k}_i|}$ and $\hat{T}_{ijmn} = T(\boldsymbol{k}_i,\boldsymbol{k}_j,\boldsymbol{k}_m,\boldsymbol{k}_n)$ if $\boldsymbol{k}_i+\boldsymbol{k}_j = \boldsymbol{k}_m+\boldsymbol{k}_n$ and 0 otherwise. In the case of four waves satisfying (1.3), (5.4) reduces to (1.1) where $T \equiv T(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{k}_4)$

and $T_{ij} = T(\boldsymbol{k}_i, \boldsymbol{k}_j, \boldsymbol{k}_i, \boldsymbol{k}_j)$. The 'free-surface elevation' of this truncated low-order model corresponding to an 'isolated' quartet would be

$$\eta_{quartet}(\boldsymbol{x}, t) = \sum_{i=1}^{4} \sqrt{\frac{|\boldsymbol{k}_i|}{2\omega_i}} \left(b_i(t) \mathrm{e}^{\mathrm{i}\boldsymbol{k}_i\cdot\boldsymbol{x}} + b_i^*(t) \mathrm{e}^{-\mathrm{i}\boldsymbol{k}_i\cdot\boldsymbol{x}} \right). \tag{5.5}$$

Although the formal validity of such a model with respect to actual solutions of the Zakharov equation (5.1) is an open question, numerical and experimental results support its usefulness (Liu & Liao 2014; Liu *et al.* 2015; Liao *et al.* 2016). If the quartet is resonant, Benney's equations (1.1) obtained with the method of multiple scales are recovered.

5.2. *Bidirectional standing waves and their stability*

A particular case of interaction (1.3) concerns bidirectional standing waves for which

$$(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3, \boldsymbol{k}_4) = (\boldsymbol{k}_1, -\boldsymbol{k}_1, \boldsymbol{k}_3, -\boldsymbol{k}_3), \quad \boldsymbol{k}_1 \neq \boldsymbol{k}_3. \tag{5.6}$$

For simplicity, we consider the case where initially $q_1 = q_2 = q_3 = q_4 \equiv q_0 > 0$. From criterion 3.1 (§ 3.3), the wave quartet is steady providing that $q_0 = -\Delta\omega/\Delta B > 0$, where ΔB is given in (3.11). For deep water, we have (see Appendix C)

$$T_{11} = T_{22} = k_1^3, \quad T_{33} = T_{44} = k_3^3, \quad T_{12} = -k_1^3, \quad T_{34} = -k_3^3, \tag{5.7}$$

so that $\Delta B = -2(k_1^3 - k_3^3)$ and $-\Delta\omega/\Delta B = (\omega_1/k_1^3)\rho(k_3/k_1)$, where the function $\rho(\kappa) = (1-\sqrt{\kappa})/(1-\kappa^3)$ is strictly positive for any $\kappa > 0$ (the discontinuity at $\kappa = 1$ may be removed since $\rho(\kappa) \to 1/6$ when $\kappa \to 1$). Therefore $q_0 = (\omega_1 - \omega_3)/(k_1^3 - k_3^3)$ is defined and strictly positive if $k_1 \neq k_3$. If $k_1 = k_3$, then both $\Delta\omega = 0$ and $\Delta B = 0$ so that, from criterion 3.1, $q_0 > 0$ may be chosen arbitrarily.

Now, from (3.6):

$$\left.\begin{aligned} \varOmega_1 = \varOmega_2 = \omega_1 + (2\bar{T} - k_1^3)q_0, \quad \varOmega_3 = \varOmega_4 = \omega_3 + (2\bar{T} - k_3^3)q_0, \\ \bar{T} = T(\boldsymbol{k}_1, \boldsymbol{k}_3, \boldsymbol{k}_1, \boldsymbol{k}_3) + T(\boldsymbol{k}_1, -\boldsymbol{k}_3, \boldsymbol{k}_1, -\boldsymbol{k}_3) + T(\boldsymbol{k}_1, -\boldsymbol{k}_1, \boldsymbol{k}_3, -\boldsymbol{k}_3)\cos p_0, \end{aligned}\right\} \tag{5.8}$$

where $\bar{T}$ may be evaluated explicitly thanks to the expressions given in Appendix C. Choosing $\varphi_1 = \varphi_2 = 0$ and $\varphi_3 = \varphi_4 = -p_0/2$, it may be shown from (5.5) that the free-surface elevation is, at leading order,

$$\left.\begin{aligned} \eta_{quartet}(\boldsymbol{x}, t) = 2\mathcal{A}_1 \cos(\boldsymbol{k}_1\cdot\boldsymbol{x})\cos(\varOmega_1 t) + 2\mathcal{A}_3\cos(\boldsymbol{k}_3\cdot\boldsymbol{x})\cos\left(\varOmega_3 t + \frac{p_0}{2}\right), \\ \mathcal{A}_i = \sqrt{\frac{2k_i q_0}{\omega_i}}, \quad i = 1, 3, \quad q_0 = \frac{\omega_1 - \omega_3}{k_1^3 - k_3^3}, \quad p_0 = 0 \text{ or } \pi. \end{aligned}\right\} \tag{5.9}$$

Free surface (5.9) is periodic if $\varOmega_1$ and $\varOmega_3$ are commensurate. If $k_1 = k_3 \equiv k_0$ and $p_0 = 0$, we recover the bidirectional standing waves studied by Okamura (1985), since in that case $\varOmega_1 = \varOmega_3 = \omega_0 + (2\bar{T} - k_0^3)q_0$, where $q_0 > 0$.

Let us now apply the instability criterion 3.1 (§ 3.3). Using homogeneity property (C4), we can define, without lost of generality, a function $\chi(\kappa, \psi)$ such that

$$\left.\begin{aligned} \frac{A(\boldsymbol{k}_1, -\boldsymbol{k}_1, \boldsymbol{k}_3, -\boldsymbol{k}_3)}{T(\boldsymbol{k}_1, -\boldsymbol{k}_1, \boldsymbol{k}_3, -\boldsymbol{k}_3)} = \frac{A(\boldsymbol{i}, -\boldsymbol{i}, \boldsymbol{\kappa}, -\boldsymbol{\kappa})}{T(\boldsymbol{i}, -\boldsymbol{i}, \boldsymbol{\kappa}, -\boldsymbol{\kappa})} \equiv 4\chi(\kappa, \psi), \\ \boldsymbol{i} = (1, 0), \quad \boldsymbol{\kappa} = (\kappa\cos\psi, \kappa\sin\psi), \quad \kappa = k_3/k_1. \end{aligned}\right\} \tag{5.10}$$

Since $\psi = \text{angle}(\boldsymbol{i}, \boldsymbol{\kappa}) = \text{angle}(\boldsymbol{k}_1, \boldsymbol{k}_3)$, it is sufficient to consider $\psi \in [0, \pi/2]$. Function $\chi(\kappa, \psi)$ is plotted on figure 3; since $\chi(\kappa, \psi) > 0$, there is no instability according to

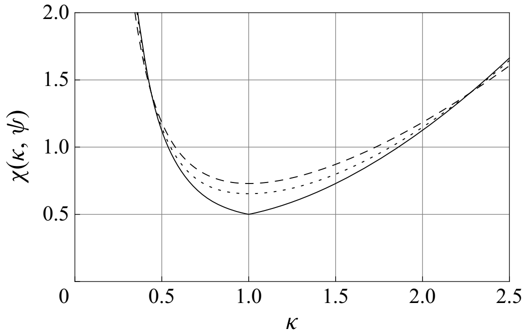

Figure 3. Graph of $\chi(\kappa, \psi)$ defined in (5.10) for $\psi = 0$ (solid line); $\psi = \pi/8$ (dotted line); $\psi = \pi/2$ (dashed line). Steady bidirectional standing waves (5.9) with $p_0 = 0$ are exponentially unstable if $\chi(\kappa, \psi) > 1$, where $\kappa = k_3/k_1$ and $\psi = \text{angle}(\boldsymbol{k}_1, \boldsymbol{k}_3)$.

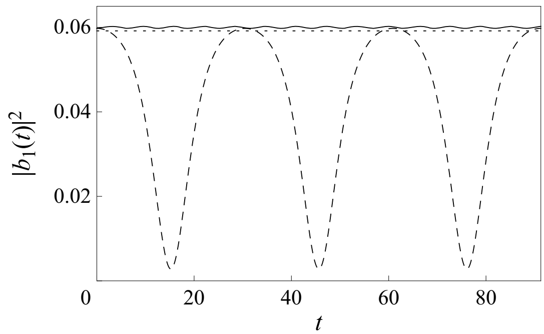

Figure 4. Numerical integration of system (1.1) with $g = 1$, $\boldsymbol{k}_1 = -\boldsymbol{k}_2 = (1, 0)$, $\boldsymbol{k}_3 = -\boldsymbol{k}_4 = (0, 2)$, $q_i = q_0(1 + \varepsilon)$ with $q_0 = (\omega_1 - \omega_3)/(k_1^3 - k_3^3)$ and $\varphi_1 = \varphi_2 = 0$, $\varphi_3 = \varphi_4 = -p_0/2$. Dotted line ($\varepsilon = 0$, $p_0 = \pi$): steady bidirectional standing waves (5.9). Dashed line ($\varepsilon = 0.01$, $p_0 = 0$): unstable disturbance. Solid line ($\varepsilon = 0.01$, $p_0 = \pi$): stable disturbance.

(3.12) for $p_0 = \pi$. On the contrary, if $p_0 = 0$, standing waves are unstable in regions where $\chi(\kappa, \psi) > 1$, that is, when $0 < \kappa < 1/\kappa_c$ or $\kappa > \kappa_c$, where $\kappa_c \equiv \kappa_c(\psi) > 1$ is such that $\chi(\kappa_c(\psi), \psi) = 1$. It is found numerically that $1.730 < \kappa_c(\psi) < 1.861$, $\forall \psi \in [0, \pi/2]$. Therefore, our results support the following statement:

PROPOSITION. *On deep water, bidirectional standing waves* (5.9) *with* (5.8) *and* $p_0 = 0$ *are exponentially unstable when* k_3/k_1 *(or* k_1/k_3*)* ≥ 1.861.

Theoretical predictions have been compared with numerical integration of (1.1) and results are in excellent agreement. A representative example is plotted in figure 4. Of course, the unsteady perturbed solutions that appear on this plot may be expressed with one of the explicit formulas given in § 4. To this aim, we first determine in each case the function f in (2.20). From the data given in the caption of figure 4, we get $a < 0$ so that f may be factorized as (4.6), where the four roots ξ_i, $i = 1, \ldots, 4$, may be explicitly evaluated thanks to the expressions given in Appendix A since in both cases $q_1 = q_2 = q_3 = q_4 = q_0(1 + \varepsilon)$; their numerical values are reported in table 1. Therefore, because of the Manley–Rowe relations (2.4), the unsteady periodic solutions plotted in figure 4 are $|b_1(t)|^2 = q_1 + q(t)$, where the functions $q(t)$ are given by (4.21) for $p_0 = 0$ (dashed line) and by (4.27) for $p_0 = \pi$ (solid line). Their respective periods are given in

p_0	ξ_1	ξ_2	ξ_3	ξ_4	τ
0	−0.05696	0	0.005644	0.05743	30.43
π	0	0.0004755	0.002822 + 0.1970i	0.002822 − 0.1970i	6.306

Table 1. Truncated values of the roots ξ_i of the quartic function (4.6) with $a = -25.57$ corresponding to the parameters of the unsteady periodic solutions represented in figure 4 for $\varepsilon = 0.01$. Their respective exact solutions $q(t)$ are (4.21) for $p_0 = 0$ and (4.27) for $p_0 = \pi$; their crest–trough amplitude is $\xi_2 - \xi_1$ and their period is τ.

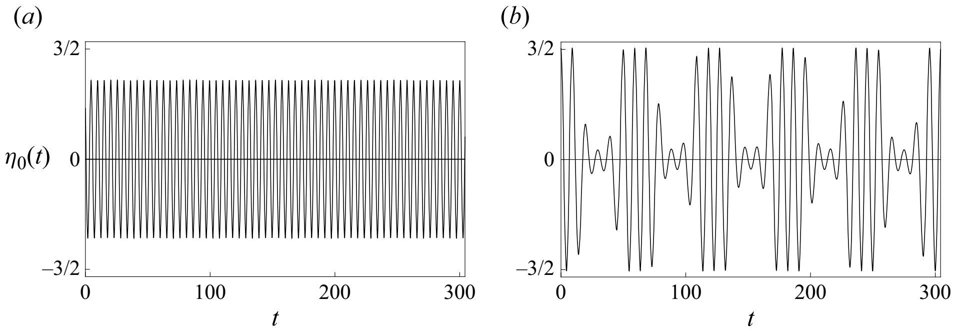

Figure 5. Free-surface elevation (5.11) at $\boldsymbol{x} = (0, 0)$ for $\varepsilon = 0.01$. Stable disturbance $p_0 = \pi$ (*a*); unstable disturbance $p_0 = 0$ (*b*). Data are those of figure 4 and table 1.

table 1. Their graphs cannot be distinguished from their numerical counterparts, as the relative errors between numerical and exact solutions is of the order of 0.1 % at the final time of the computations.

Finally, the free-surface elevation of these bifurcated solutions may be written as

$$\left.\begin{aligned} \eta_{quartet}(\boldsymbol{x}, t) &= 2\mathcal{A}_1\eta_1(t)\cos(\boldsymbol{k}_1\boldsymbol{\cdot}\boldsymbol{x})\cos p_1(t) + 2\mathcal{A}_3\eta_3(t)\cos(\boldsymbol{k}_3\boldsymbol{\cdot}\boldsymbol{x})\cos p_3(t),\\ \eta_1(t) &= \sqrt{1+\varepsilon+q(t)/q_0}, \quad \eta_3(t) = \sqrt{1+\varepsilon-q(t)/q_0}, \end{aligned}\right\} \tag{5.11}$$

where $\mathcal{A}_i$ and q_0 are defined in (5.9) and where the phases $p_i(t)$ are integrated numerically from (2.3). Results are plotted in figure 5: they show in the unstable case (figure 5*b*) five large-scale beats over ten periods τ of the corresponding function $q(t)$. This is due to the fact that the amplitudes $|b_i(t)|$ involve the square root of $q(t)$; see (2.4).

5.3. *Steady wave quartets and their stability*

We now turn to the general case (1.3), on or off resonance. Consider first Kartashova's resonant quartet (Lvov, Nazarenko & Pokorni 2006; Kartashova 2010, p. 78):

$$\boldsymbol{K}_1 = (495, 90), \quad \boldsymbol{K}_2 = (64, 128), \quad \boldsymbol{K}_3 = (359, 118), \quad \boldsymbol{K}_4 = (200, 100), \tag{5.12}$$

useful for symbolic computation. Since $\Delta\omega = 0$ and $\Delta B \neq 0$ in that case, criterion 3.1 (§ 3.3) is of no use to construct a steady solution; we need another strategy. Testing $q_1 = q_2 \equiv q_{12}$ and $q_3 = q_4 \equiv q_{34}$, the compatibility condition (3.5) with $\Delta\omega = 0$ becomes

$$\frac{q_{34}}{q_{12}} = \frac{4T - (B_1 + B_2)\cos p_0}{4T - (B_3 + B_4)\cos p_0} \approx \begin{cases} 0.8546, & \text{if } p_0 = 0, \\ 0.5857, & \text{if } p_0 = \pi. \end{cases} \tag{5.13}$$

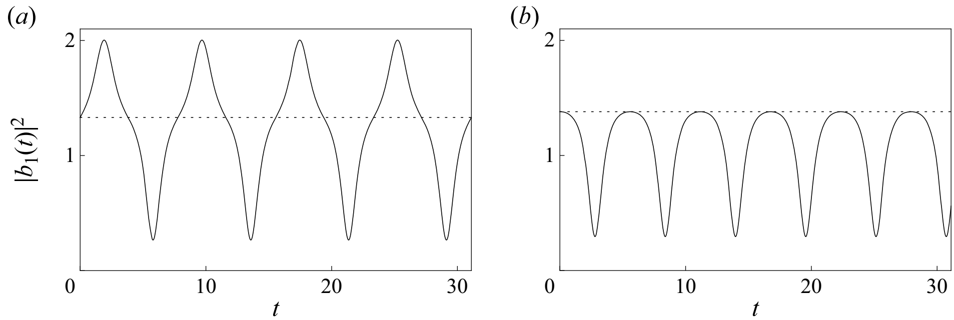

Figure 6. Wave action of steady Kartashova's resonant quartet ($\lambda \boldsymbol{K}_1, \lambda \boldsymbol{K}_2, \lambda \boldsymbol{K}_3, \lambda \boldsymbol{K}_4$) given by (5.12), $\lambda = 1/300$ and $p_0 = \pi$ (dotted lines) versus unstable solutions according to criterion 5.1 (solid lines). Initial wave actions are in each case $q_{12} \approx 1.380$ and $q_{34} \approx 0.8084$. (*a*) Resonant solution with perturbed initial phase mismatch $p_0 = 1.05\pi$ and same wave vectors. (*b*) Non-resonant solution with same phase mismatch $p_0 = \pi$ and perturbed wave vectors ($\lambda \boldsymbol{k}_1, \lambda \boldsymbol{k}_2, \lambda \boldsymbol{k}_3, \lambda \boldsymbol{k}_4$) given by (5.15) and $\lambda = 1/300$. Solutions are $|b_1(t)|^2 = q_1 + q(t)$ with $q(t)$ given by (*a*) (4.27) and (*b*) (4.21).

Since $q_{34}/q_{12} > 0$ in both cases, steady states exist. It may be checked that the instability condition (3.9) becomes also (3.12) when $q_1 = q_2 \equiv q_{12}$ and $q_3 = q_4 \equiv q_{34}$. Since $A/T \approx -7.430 < -4$ for Kartashova's resonant quartet (5.12) or for any rescaled non-trivial resonant quartet ($\lambda \boldsymbol{K}_1, \lambda \boldsymbol{K}_2, \lambda \boldsymbol{K}_3, \lambda \boldsymbol{K}_4$) according to (C4), we conclude immediately that the corresponding steady wave quartet is stable if $p_0 = 0$ and unstable if $p_0 = \pi$. This method may be employed for any other resonant quartet provided that the ratio q_{34}/q_{12} is positive. Therefore in summary, we have proved:

CRITERION 5.1. *Steady resonant wave quartets* (3.6) *with* $q_1 = q_2 \equiv q_{12}$ *and* $q_3 = q_4 \equiv q_{34}$ *exist if*

$$\frac{q_{34}}{q_{12}} = \frac{4T - (B_1 + B_2)\cos p_0}{4T - (B_3 + B_4)\cos p_0} > 0 \quad (p_0 = 0 \text{ or } \pi). \tag{5.14}$$

They are exponentially unstable if $(A/T)\cos p_0 > 4$.

(A companion criterion is given in Appendix D for $q_1 = q_3$ and $q_2 = q_4$.)

Unstable disturbed solutions can lie on the same resonant surface (figure 6*a*), or not. For instance, if we consider the following non-resonant (but nearly resonant) quartet:

$$\boldsymbol{k}_1 = (500, 90), \quad \boldsymbol{k}_2 = (60, 130), \quad \boldsymbol{k}_3 = (360, 120), \quad \boldsymbol{k}_4 = (200, 100), \tag{5.15}$$

the corresponding unsteady solution is plotted in figure 6(*b*), showing again instability.

Let us now construct a steady solution for the non-resonant quartet (5.15). We first test criterion 3.1 (§ 3.3), but since in that case $\Delta\omega/\Delta B > 0$, a steady solution with $q_1 = q_2 = q_3 = q_4$ does not exist; we need one more time another strategy. We are, however, lucky with (5.15) because all the B_i defined in (2.9) have the same sign: negative. Thus we can set $q_i = \alpha/B_i$ expecting $\alpha < 0$ for existence. From (3.5) we get

$$\alpha = -\frac{\sqrt{B_1 B_2 B_3 B_4}\,\Delta\omega}{2T\Delta B}\cos p_0 \quad (p_0 = 0 \text{ or } \pi). \tag{5.16}$$

Since $T > 0$ for (5.15), we conclude that the only possibility to have $\alpha < 0$ is $p_0 = 0$. Concerning stability, condition (3.9) with $q_i = \alpha/B_i$ becomes

$$\frac{A}{2T}\sqrt{B_1 B_2 B_3 B_4}\cos p_0 > \frac{(\Delta B)^2}{4} + (B_1 + B_2)(B_3 + B_4) - B_1 B_2 - B_3 B_4. \tag{5.17}$$

Applying this condition for quartet (5.15) and $p_0 = 0$ gives stability.

The method presented here may be applied to any non-resonant quartet providing that all the B_i have the same sign:

CRITERION 5.2. *Steady non-resonant wave quartets* (3.6) *with* $B_i > 0$ *(respectively* < 0*) and* $q_i = \alpha/B_i$, $i = 1, \ldots, 4$, *where* α *is defined in* (5.16)*, exist if* $\alpha > 0$ *(respectively* < 0*). They are exponentially unstable if* (5.17) *holds.*

Other strategies might be used to construct steady non-resonant quartets. For instance we can choose $q_1 = q_2$ and $q_3 = q_4$, or $q_1 = q_3$ and $q_2 = q_4$. In these cases, compatibility condition (3.5) would yield affine relations between q_i, but conditions for exponential instability resulting from (3.9) would involve q_i, contrary to the instability conditions given in criteria 1 to 4. We shall not pursue this issue here.

6. The X-quartet: pump and beats

After steady and periodic solutions, let us now turn to the 'exceptional cases' mentioned by Bretherton (1964). We start with the pump solution (4.16). Although an example has already been given by Andrade & Stuhlmeier (2023*a*), we purpose here another specimen based on the following orthogonal resonant quartet, the 'X-quartet':

$$\boldsymbol{k}_1^0 = (k_0, 0), \quad \boldsymbol{k}_2^0 = (-k_0, 0), \quad \boldsymbol{k}_3^0 = (0, k_0), \quad \boldsymbol{k}_4^0 = (0, -k_0), \quad k_0 > 0, \tag{6.1}$$

with initially $q_1 = q_2 = q_3 = q_4 \equiv q_0 > 0$. We have, from Appendix C:

$$\frac{T}{k_0^3} = -\frac{3}{4}, \quad \left[\frac{T_{ij}}{k_0^3}\right] = \begin{bmatrix} 1 & -1 & \nu & \nu \\ -1 & 1 & \nu & \nu \\ \nu & \nu & 1 & -1 \\ \nu & \nu & -1 & 1 \end{bmatrix}, \quad \nu = \frac{4\sqrt{2}-5}{28}, \tag{6.2}$$

so that, from (2.9) and (2.21):

$$\frac{A}{k_0^3} = -\frac{4}{7}(1+2\sqrt{2}), \quad \frac{A}{4T} = \frac{4}{21}(1+2\sqrt{2}) \in [-1, 1], \quad \frac{a}{k_0^6} = \frac{1}{49}(297 - 64\sqrt{2}) > 0. \tag{6.3}$$

Furthermore, since $\Delta\omega = 0$ and $\Delta B = 0$ in that case, both conditions in (4.18) are fulfilled for the existence of the pump solution (4.19). It remains to set $p_0 = P_0$, where

$$P_0 = \arccos\left(\frac{A}{4T}\right) = \arccos\left(\frac{4}{21}(1+2\sqrt{2})\right). \tag{6.4}$$

Once $q(t)$ is replaced by (4.19), it may be checked, after squaring (2.7), that $p(t) = P_0$. Furthermore, the individual phases may be integrated since, from (2.3), we have simply

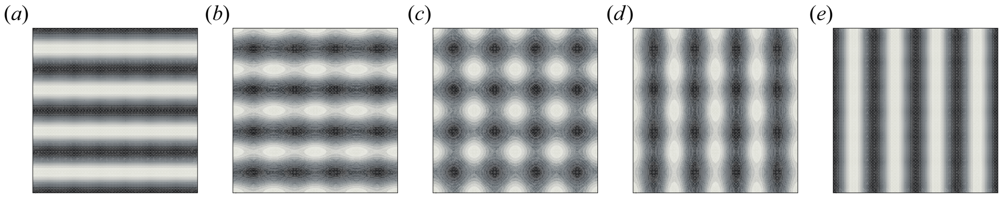

Figure 7. Snapshots of the free-surface elevation (6.6) of the X-pump in the region $k_0x \in [-4\pi, 4\pi]$, $k_0y \in [-4\pi, 4\pi]$ $(g = 1)$: (a) $\Omega_0 t = -2$, (b) $\Omega_0 t = -1$, (c) $\Omega_0 t = 0$, (d) $\Omega_0 t = 1$, (e) $\Omega_0 t = 2$.

$dp_i/dt = -\omega_0 + 2k_0^3 q_0$, $i = 1, \ldots, 4$. Therefore, the solution of (1.1) is the 'X-pump':

$$b_1(t) = b_2(t) = \sqrt{q_0 + q(t)}\,\mathrm{e}^{-\mathrm{i}\Omega_0 t}, \quad b_3(t) = b_4(t) = \sqrt{q_0 - q(t)}\,\mathrm{e}^{-\mathrm{i}(\Omega_0 t + \frac{1}{2}P_0)}, \tag{6.5}$$

where $q(t) = q_0 \tanh(\sqrt{a} q_0 t)$, $\Omega_0 = \omega_0 - 2k_0^3 q_0$ and $\omega_0 = \sqrt{gk_0}$. Free surface is

$$\left.\begin{aligned} &\eta_{quartet}(\boldsymbol{x}, t) = 2\mathcal{A}_0 \eta_+(t)\cos(k_0 x)\cos(\Omega_0 t) + 2\mathcal{A}_0\eta_-(t)\cos(k_0 y)\cos\left(\Omega_0 t + \frac{P_0}{2}\right), \\ &\mathcal{A}_0 = \sqrt{\frac{2k_0 q_0}{\omega_0}}, \quad \eta_\pm(t) = \sqrt{1 \pm \tanh(\sqrt{a} q_0 t)}, \quad \Omega_0 = \omega_0 - 2k_0^3 q_0, \quad q_0 > 0, \end{aligned}\right\} \tag{6.6}$$

and where a and P_0 are given respectively in (6.3) and (6.4). Figure 7 illustrates the fact that the energy of components $(\boldsymbol{k}_3^0, \boldsymbol{k}_4^0)$ is pumped and totally transferred to $(\boldsymbol{k}_1^0, \boldsymbol{k}_2^0)$, that is, from the y standing wave to the x standing wave.

One might expect that this pump solution could be reproduced experimentally; however, both standing waves constituting the complete solution (6.6) are linearly unstable with respect to infinitesimal disturbances constituted by the other couple of waves according to instability condition (3.8) applied for instance to the bichromatic wave $(\boldsymbol{k}_1^0, \boldsymbol{k}_2^0)$ with $q_1 = q_2$ disturbed by $(\boldsymbol{k}_3^0, \boldsymbol{k}_4^0)$ yielding $16T^2 > 4(1 + 4\nu)^2 k_0^6$, true from (6.2). The converse being true, both instabilities may feed each other.

This is illustrated in figure 8 in which is represented the X-pump (dashed line) and a perturbed solution where a relative error of 0.1 % has been made on the initial phase mismatch p_0: the perturbed solution is periodic (solid line). In fact, the roots of (4.6) are, from (A2) with $B = 0$ and T and A given by (6.2) and (6.3):

$$\frac{\xi_i}{q_0} \in \left\{\pm\sqrt{\frac{21(1 + \cos p_0)}{25 + 8\sqrt{2}}}, \pm\sqrt{\frac{21(1 - \cos p_0)}{17 - 8\sqrt{2}}}\right\}. \tag{6.7}$$

They are plotted in figure 9(a). They coincide by pairs for $p_0 = P_0$ and are distinct for any $p_0 \notin \{0, P_0, \pi\}$, two being positive, two negative. The solution is in that case periodic and completely determined by (4.8) together with a in (6.3) and ξ_i (6.7); their period is (4.10) (figure 9b), and their crest–trough amplitude at fixed p_0 is: $2\min|\xi_i|$, $i = 1, \ldots, 4$. These are the 'X-beats' in which energy is periodically exchanged between components $(\boldsymbol{k}_1^0, \boldsymbol{k}_2^0)$ and $(\boldsymbol{k}_3^0, \boldsymbol{k}_4^0)$. Note finally that 0 is a double root if $p_0 = 0$ or π, corresponding to steady standing waves discussed in § 5.2.

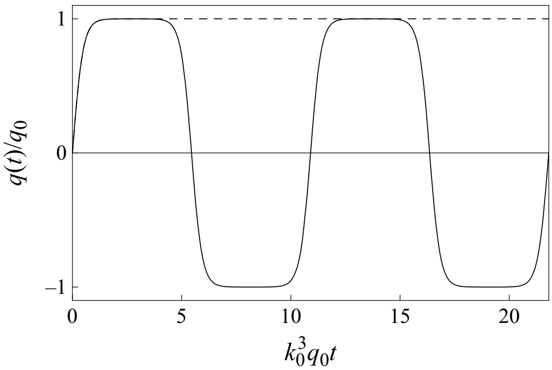

Figure 8. Solutions of (2.18) for (6.1) with $q_1 = q_2 = q_3 = q_4 \equiv q_0$. X-pump with $p_0 = P_0$ (dashed line); X-beats with $p_0 = 1.0001P_0$ (solid line). P_0 is given by (6.4).

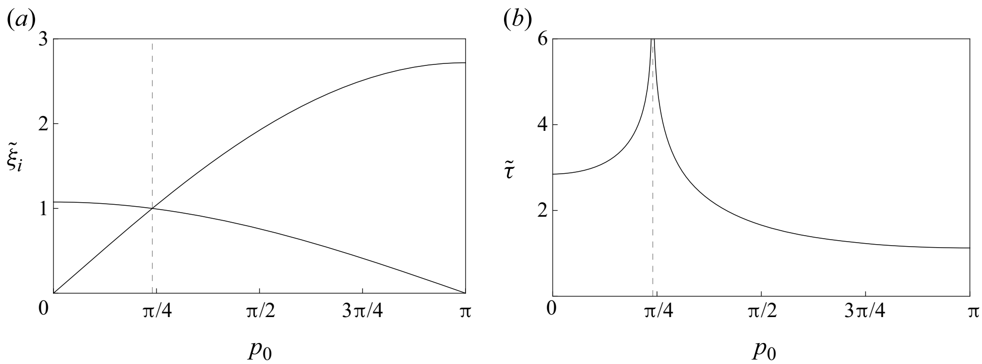

Figure 9. (*a*) Dimensionless roots $\tilde{\xi}_i = |\xi_i/q_0|$ given by (6.7) of the quartic (4.6) for the X-quartet (6.1). The dashed line corresponds to $p_0 = P_0$ (6.4) of the X-pump solution (6.6). (*b*) If $p_0 \notin \{0, P_0, \pi\}$, solutions are periodic X-beats given by (4.8) with dimensionless period $\tilde{\tau} = k_0^3 q_0 \tau$ given by (4.10).

7. Breathers

7.1. *The Ψ-breathers*

Looking for breathers is a difficult task because two or three roots of the quartic function (4.6) have to coincide in a non-symmetric way, as compared with the X-pump solution (4.19) for which the roots were equal by pairs and opposite.

Fortunately, one branch of such breathers bifurcates from the X-pump solution studied previously. Indeed, consider the one-parameter family of resonant quartet:

$$\left.\begin{aligned} &\boldsymbol{k}_1^{\varepsilon} = (\alpha_+, 0), \quad \boldsymbol{k}_2^{\varepsilon} = (-\alpha_-, 0), \quad \boldsymbol{k}_3^{\varepsilon} = (\beta, \gamma), \quad \boldsymbol{k}_4^{\varepsilon} = (\beta, -\gamma), \\ &\alpha_{\pm} = k_0\left(1 \pm \tfrac{1}{2}\varepsilon + \tfrac{1}{4}\varepsilon^2\right)^2, \quad \beta = k_0\varepsilon\left(1 + \tfrac{1}{4}\varepsilon^2\right), \quad \gamma = k_0\left(1 - \tfrac{1}{16}\varepsilon^4\right), \end{aligned}\right\} \tag{7.1}$$

where $\varepsilon \geq 0$, with $\varepsilon \neq 2$ to avoid $\boldsymbol{k}_3^{\varepsilon} = \boldsymbol{k}_4^{\varepsilon}$, is the main parameter ($k_0 > 0$ is just a scaling parameter). Quartet (7.1) is a member of the 'tridents' (D2) proposed by Lvov *et al.* (2006), with $(m, n) = (1, \varepsilon/2)$. The X-quartet (6.1) is matched for $\varepsilon = 0$.

With (7.1), it is found numerically that $T < 0, A < 0$ and $\Delta B > 0$ for any positive $\varepsilon \neq 2$. Furthermore, $4T - A > 0$ (then $a < 0$) for any $\varepsilon \neq 2$ such that $\varepsilon_1 < \varepsilon < \varepsilon_2$, and $4T - A < 0$ (then $a > 0$) otherwise, where $\varepsilon_1 \approx 0.4672$ and $\varepsilon_2 \approx 8.562$. Therefore, we may expect

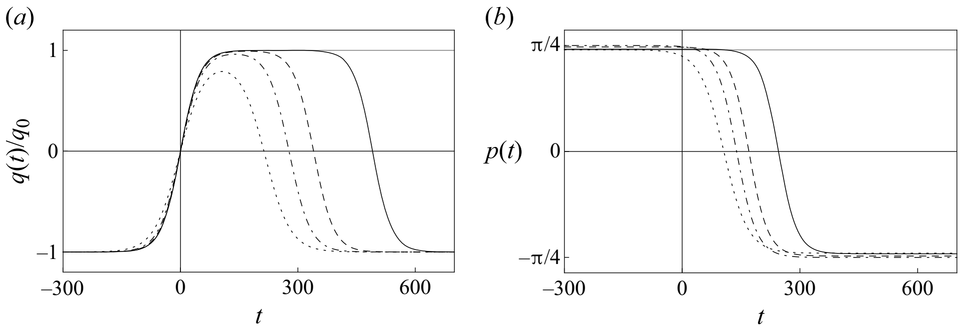

Figure 10. The Ψ-breathers corresponding to quartets (7.1) with $k_0 = 1$, $q_0 = 0.01$ and p_0 given by (7.4), for $\varepsilon = 0.2$ (dotted line), $\varepsilon = 0.1$ (dot-dashed line), $\varepsilon = 0.05$ (dashed line) and $\varepsilon = 0.01$ (solid line); the grey line corresponds to the X-pump (4.19) for which $\varepsilon = 0$. (*a*) Relative action $q(t)$. (*b*) Relative phase $p(t)$.

to find breathers of the kind (4.14) for $0 < \varepsilon < \varepsilon_1$ (because $a > 0$), and those of the kind (4.23) or (4.25) for $\varepsilon_1 < \varepsilon < \varepsilon_2$, $\varepsilon \neq 2$ (because $a < 0$). The case $\varepsilon = \varepsilon_1$ (then $a = 0$) will be considered later.

To construct breathers (4.14) bifurcating from the X-pump (6.6), we consider $0 < \varepsilon < \varepsilon_1$ (so that $a > 0$) and choose again initially $q_1 = q_2 = q_3 = q_4 \equiv q_0 > 0$. To do so, two roots have to coincide, so we impose arbitrarily the following constraint for the negative roots: $\xi_{12} = \xi_1 = \xi_2 = -q_0 < 0$, while $0 \leq \xi_3 < \xi_4$. Since the initial wave actions are equal, the four roots are given explicitly by (A2), here with

$$\Delta\omega = 0, \quad \Delta B > 0, \quad T < 0, \quad 4T \pm A < 0, \tag{7.2}$$

since $0 < \varepsilon < \varepsilon_1$. Roots are therefore real and such that $\xi_-^{(-)} < 0 < \xi_-^{(+)}$. Since the two negative roots must coincide with $-q_0$, we get a first compatibility condition:

$$\sqrt{(\Delta B)^2 + 16T(4T - A)(1 - \cos p_0)} = 2(A - 4T) - \Delta B. \tag{7.3}$$

Positiveness of the right-hand side is ensured when $0 < \varepsilon \leq \varepsilon_3$, with $\varepsilon_3 \approx 0.3705$. In that range, on squaring the above equality, we get $\cos p_0 = (A - \Delta B)/(4T) \geq 0$. Furthermore, it may be checked numerically that the constraint $\cos p_0 \leq 1$ is ensured providing that $0 < \varepsilon \leq \varepsilon_4$, with $\varepsilon_4 \approx 0.3213$. Therefore, in that range, we may choose

$$p_0 = \arccos\left(\frac{A - \Delta B}{4T}\right). \tag{7.4}$$

With (7.4), the four roots are therefore $\xi_{12} = \xi_\pm^{(\pm)}$, $\xi_3 = \xi_+^{(-)}$, $\xi_4 = \xi_-^{(+)}$:

$$\xi_{12} = -q_0, \quad \xi_3 = \left(1 - \frac{\Delta B}{A - 4T}\right) q_0, \quad \xi_4 = \left(1 - \frac{\Delta B}{A + 4T}\right) q_0. \tag{7.5}$$

Note that $p_0 = 0$, $\xi_3 = 0$ and $\xi_4/q_0 = 8T/(A + 4T)$ when $\varepsilon = \varepsilon_4 \approx 0.3213$, which corresponds to the end point of this branch of solutions.

Together with $a = (4T + A)(4T - A)$, roots (7.5) completely determine the solution (4.14), referred to as the 'Ψ-breathers' represented in figure 10 for various values of ε. Note that $\Delta B = 0$ for $\varepsilon = 0$, so that (7.4) matches with (6.4) obtained for the X-pump

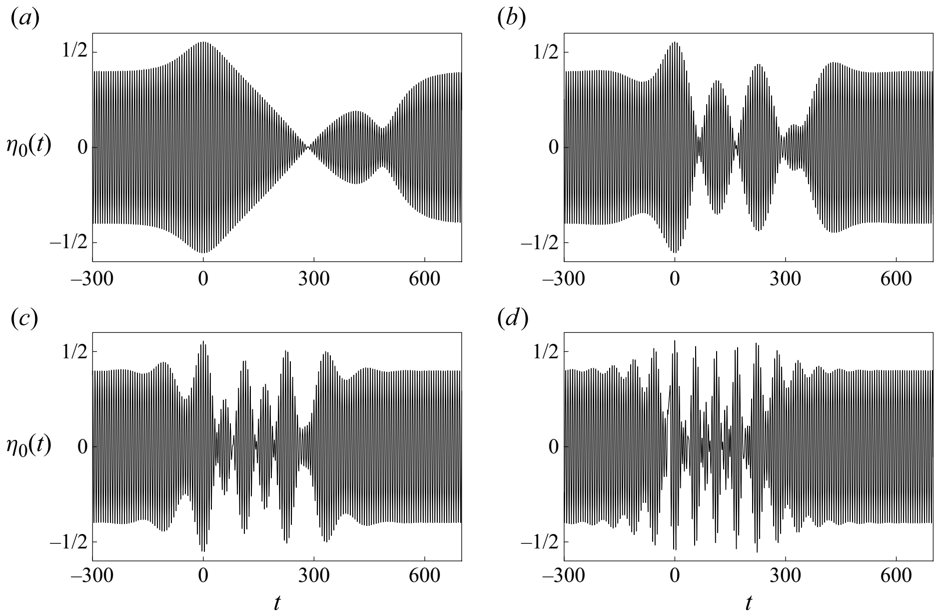

Figure 11. Free-surface elevation (5.5) at $\boldsymbol{x} = (0, 0)$ for the $\varPsi$-breathers (7.1) with (7.4) for $\varepsilon = 0.01$ (*a*), $\varepsilon = 0.05$ (*b*), $\varepsilon = 0.1$ (*c*) and $\varepsilon = 0.2$ (*d*). Computations have been carried out backward and forward from $t = 0$, with $g = 1$, $k_0 = 1$ and $q_0 = 0.01$.

solution. Expansion near $\varepsilon = 0$ gives, up to the second order:

$$\left.\begin{aligned} \frac{\xi_{12}}{q_0} &= -1, \quad \frac{\xi_3}{q_0} = 1 - \frac{5}{46}(17 + 8\sqrt{2})\varepsilon^2, \quad \frac{\xi_4}{q_0} = 1 + \frac{5}{142}(25 - 8\sqrt{2})\varepsilon^2, \\ \frac{a}{k_0^6} &= \frac{1}{49}(297 - 64\sqrt{2}) - \frac{8}{7}(17 - 8\sqrt{2})\varepsilon + \frac{1}{9604}(35\,017 - 86\,052\sqrt{2})\varepsilon^2. \end{aligned}\right\} \tag{7.6}$$

It is interesting for the $\varPsi$-breathers to reconstruct the free-surface elevation. Since the relative $q(t)$ is given explicitly by (4.14), the relative phase $p(t)$ defined in (2.2) may be deduced from (2.7) and plotted in figure 10. Then, the four individual moduli $|b_i(t)|$ are deduced from the Manley–Rowe equation (2.4), while numerical integration of (2.3) is necessary to obtain the individual phases $p_i(t) = \arg b_i(t)$ in order reconstruct the free-surface elevation given by (5.5). This is achieved in figure 11 for the $\varPsi$-breathers defined by (7.1), (7.4) and (7.5) for different values of the parameter ε that quantifies the departure from the X-pump solution (6.6). Is is clear from figure 11 that the sole considerations of $q(t)$ and $p(t)$ represented in figure 10 cannot explain the local behaviour of $\eta_{quartet}(0, t)$ and that phase mixing must be invoked to explain such destructive/constructive interferences (the $\pi/2$ relative phase jump is also worth noting). Whether this kind of behaviour may be related or not to the occurrence of extreme events is left for future investigation.

Finally, if $\varepsilon = \varepsilon_1$ or $\varepsilon = \varepsilon_2$, then $A = 4T$ and $a = 0$, so that f in (2.18) becomes cubic and reads as (B1), where here $b = -2A(\Delta B)q_0 > 0$. It may be deduced from the considerations of Appendix B in that case that if we constrain the positive roots to coincide, i.e. $\xi_{23} \equiv \xi_2 = \xi_3 = q_0$, then we get the following compatibility conditions:

$$\cos p_0 = 1 + \frac{\Delta B}{A} \in [-1, 1], \quad \xi_1 = -\left(1 + \frac{\Delta B}{2A}\right) q_0 \leq 0. \tag{7.7}$$

For both $\varepsilon = \varepsilon_1$ and $\varepsilon = \varepsilon_2$ we have the same numerical value for $\Delta B/A \approx -0.8487$; both conditions above are fulfilled and the corresponding solution is breather (B4).

7.2. *Looking for rational breathers*

Since $a < 0$ when $\varepsilon \neq 2$ lies in the range $\varepsilon_1 < \varepsilon < \varepsilon_2$, we expect to find rational breathers (4.25). Suppose that the triple root which characterizes this solution is positive, $\xi_2 = \xi_3 = \xi_4 \equiv \xi_{234} > 0$, so that $\xi_1 \leq 0$. Now, contrary to the case $a > 0$ for which the multiple roots must coincide (in absolute value) with the initial wave actions because of (4.1), roots satisfy in the present case the following inequalities:

$$-q_0 \leq \xi_1 \leq 0 < \xi_{234} \leq q_0. \tag{7.8}$$

However, no rational breather has been found for the resonant trident (7.1).

Thus, constraints imposed by the coincidence of three roots of the quartic function f make the search for rational breathers even more difficult than for other breathers. For reasons that will be made clear later, still for the quartet (7.1), we consider the point in parameter space where $A = 12T$. It is found numerically that it corresponds to $\varepsilon = \varepsilon_0$, where $\varepsilon_0 \approx 0.7475$. Now, from this point of bifurcation, we consider the following family of non-resonant quartet:

$$\boldsymbol{k}_1 = \boldsymbol{k}_1^{\varepsilon_0} + \boldsymbol{\kappa}, \quad \boldsymbol{k}_2 = \boldsymbol{k}_2^{\varepsilon_0} - \boldsymbol{\kappa}, \quad \boldsymbol{k}_3 = \boldsymbol{k}_3^{\varepsilon_0}, \quad \boldsymbol{k}_4 = \boldsymbol{k}_4^{\varepsilon_0}, \quad \boldsymbol{\kappa} = (k_0\kappa_x, k_0\kappa_y), \tag{7.9}$$

where the wave vectors $\boldsymbol{k}_i^{\varepsilon}$ are defined in (7.1) and where (κ_x, κ_y) are real numbers.

Now, if we impose on the roots in (7.8) the constraints $\xi_1 = 0$ and $\xi_{234} = q_0 > 0$, and if we assume that $A = 12T$ and $\cos p_0 = -1$, then it may be deduced from (A2) that q_0 must satisfy the following compatibility condition:

$$q_0 = -\frac{\Delta\omega}{\Delta B + 16T} > 0, \tag{7.10}$$

where all parameters are to be evaluated on the non-resonant manifold (7.9). In figure 12 is represented in the (κ_x, κ_y) plane the curve $(\varGamma)$ defined by $A = 12T$ that bifurcates from the origin O corresponding to $\boldsymbol{\kappa} = 0$. This curve passes through points P_1, P_2 and P_3 and forms a loop which crosses light-grey regions in which condition (7.10) is violated, so that rational breathers of the kind considered here exist on any point of $(\varGamma)$, except in these shadowed regions.

Truncated values of $a = -128T^2$ (since $A = 12T$) and q_0 in (7.10) are reported in table 2. Since $\xi_1 = 0$ and $\xi_{234} = q_0$, the corresponding rational breathers are (4.26). The values of the frequency mismatch show that for $|\Delta\omega|/(k_0^3 q_0)$ to stay of order 1, one must stay on curve $(\varGamma)$ between O and approximately P_1 to keep physical insight.

8. Summary and perspectives

The complete solutions of the cubic nonlinear equations (1.1) first derived by Armstrong *et al.* (1962) and Benney (1962) have been presented in order to determine, paraphrasing Phillips (1960), the 'elementary interactions' inside an isolated quartet of waves

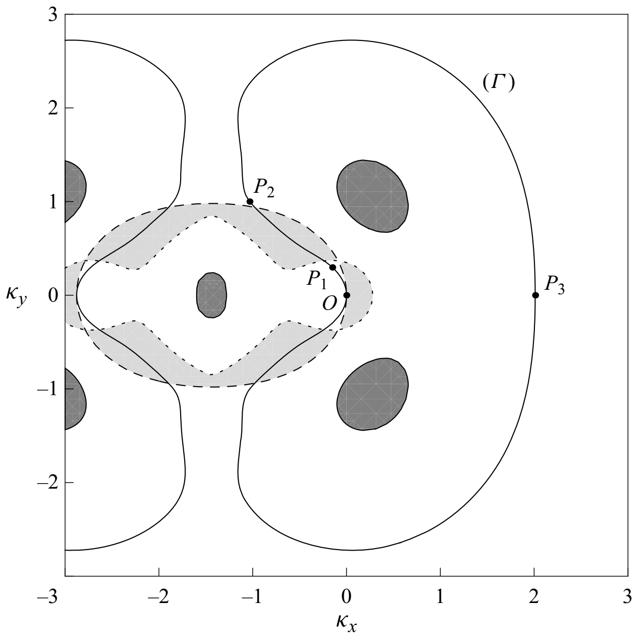

Figure 12. Curve (Γ) defined by $A = 12T$ in the (κ_x, κ_y) plane defined in (7.9). The origin O corresponds to $(\kappa_x, \kappa_y) = (0, 0)$ for which (7.9) matches (7.1). The oval (dashed line) crossing (Γ) in O is the resonant curve $\Delta\omega = 0$. The dashed curve is defined by $\Delta B + 16T = 0$. The light-grey region delimited by these two curves is defined by $\Delta\omega/(\Delta B + 16T) > 0$. The dark-grey regions are defined by $a = 16T^2 - A^2 > 0$. The coordinates of points P_1, P_2 and P_3 are reported in table 2.

Point	κ_x	κ_y	a/k_0^6	$(k_0^3/\sqrt{g})q_0$	$\lvert\Delta\omega\rvert/\sqrt{g}$	$\lvert\Delta\omega\rvert/(k_0^3 q_0)$
P_1	−0.15	0.2971	−7.010	0.1344	0.08258	0.6145
P_2	−1.032	1	−13.96	0.007141	0.03479	4.872
P_3	2.013	0	−282.8	0.03019	1.407	46.62

Table 2. Coordinates and values of a, q_0 and $|\Delta\omega|$ for P_1, P_2 and P_3 of figure 12.

satisfying (1.3). Following the works of Bretherton (1964), Inoue (1975), Turner (1980), Shemer & Stiassnie (1985), Cappellini & Trillo (1991), Stiassnie & Shemer (2005) and Andrade & Stuhlmeier (2023*a*,*b*), the system has been reduced in § 2 to a single ordinary differential equation – the Bretherton equation (2.18) – governing the evolution of an auxiliary variable: the relative action $q(t)$ vanishing initially and related to the modulus $|b_i(t)|$ by the Manley–Rowe relations (2.4).

To summarize, the nature of a given non-degenerate wave quartet with wave vectors $\boldsymbol{k}_i$ satisfying (1.3) and prescribed initial amplitudes $b_i(0) \neq 0$, $i = 1, \ldots, 4$, may be determined through the following steps:

(*a*) Compute the initial wave actions $q_i = |b_i(0)|^2$, the initial phase mismatch p_0 from (2.12), the linear frequencies $\omega_i = \omega(\boldsymbol{k}_i)$ from the dispersion law of the medium under consideration and the frequency mismatch $\Delta\omega$ from (1.4).

(*b*) Compute the nonlinear coupling coefficients T_{ij} and T as given in Appendix C for deep-water gravity waves, and the coefficients A, B, C and a, b, c, d, e from (2.9), (2.19) and (2.21). Form the polynomial function f in (2.20).

(*c*) Nature of the quartet.

(i) If $p_0 = 0$ or $p_0 = \pi$, and if condition (3.5) holds, then the wave quartet is steady (theorem 3.1). In that case, the quartet is exponentially unstable if inequality (3.9) holds (theorem 3.2), algebraically unstable if equality holds, otherwise the quartet is nonlinearly stable in the sense of Lyapunov (Lagrange theorem).

(ii) If $p_0 \neq 0$ and $p_0 \neq \pi$, or if condition (3.5) is violated, then the wave quartet is unsteady. In that case, we have to compute the roots ξ_i of f and determine the corresponding solution $q(t)$ according to the classification of either § 4 if $a \neq 0$, or Appendix B if $a = 0$.

As anticipated by Bretherton (1964), periodic solutions $q(t)$ are the rule (the full field is, however, generally not periodic, as discussed by Stiassnie & Shemer (2005)), steady or unsteady non-periodic solutions are the exceptions. Indeed, steady solutions are constrained by the conditions recalled above, while unsteady non-periodic solutions (breathers or pump) are constrained by the equality of two or three roots ξ_i of the function f.

The reverse problem which consists of finding *a priori* steady or non-periodic solutions is much more delicate, because of the complexity of the coefficients T_{ij} and T of the dispersive medium under consideration. In the steady case, one has to solve the algebraic compatibility relation (3.5), while in the unsteady case one has to elaborate strategies to make two or three roots ξ_i of f coincide. This reverse problem was the object of § 5 (steady solutions), and §§ 6 and 7 (pump and breather solutions).

By steady solutions, we mean those for which the relative wave action $q(t)$ vanishes identically, the corresponding waves of the quartet having therefore constant amplitudes $|b_i|^2 = q_i$. In addition to theorems 3.1 and 3.2 giving general conditions for existence and stability of non-degenerate steady quartets, four criteria have been deduced for particular choices of the initial amplitudes q_i. As pointed out in § 3.4, the various stability criteria derived through the present paper are also formally valid in the nonlinear regime because of a famous theorem by Lagrange, stability being defined in the sense of Lyapunov. This formal result is, however, of restricted utility for reasons explained at the end of § 3.4, essentially because our study is restricted to a single isolated quartet. However, it might be useful for experimental purposes to observe the steady stable solutions under controlled conditions.

Most of the unsteady solutions of the Bretherton equation (2.18) discussed in § 4 and Appendix B have already been published in the literature, but spread in different fields of physics including essentially plasma physics, nonlinear optics and hydrodynamics, with more or less permeability between the various fields. Chronologically, these include the works by Inoue (1975), Boyd & Turner (1978), Turner (1980), Shemer & Stiassnie (1985), Chen & Snyder (1989), Chen (1989), Cappellini & Trillo (1991), Stiassnie & Shemer (2005) and recently Andrade & Stuhlmeier (2023*a*,*b*). Some solutions were known only in particular cases, such as the breathers (4.14) and (4.23), and, to our point of view, a complete and unified classification was lacking.

Various examples of isolated deep-water gravity wave quartets have then been elaborated, such as among others, the X-pump and X-beats (§ 6) which are bifurcated solutions from the bidirectional standing wave built on the X-quartet (6.1), and the Ψ-breathers (§ 7) built from the trident quartets (7.1) of Lvov *et al.* (2006). A number of these solutions have been found to bifurcate to periodic large-amplitude solutions (the X-beats for instance) in a way similar to the Fermi–Pasta–Ulam recurrence for the modulational instability (Yuen & Lake 1982; Shemer & Stiassnie 1985, 1991; Trillo & Wabnitz 1991; Janssen 2004; Leblanc 2009). This behaviour is also illustrated in the phase portraits in figure 2 of Andrade & Stuhlmeier (2023*a*): in this figure, the dots correspond to steady solutions and are connected by a heteroclinic orbit in phase space which corresponds to a breather; one sees that any departures from these saddle points or from heteroclinic orbits bifurcate to closed orbits corresponding to periodic large-amplitude solutions. Similar observations were made by Cappellini & Trillo (1991) and Andrade & Stuhlmeier (2023*b*) for the three-wave cubic interaction.

What was surprising to me is that stability or instability of non-degenerate quartets has to be understood as structural since no external disturbance is necessary for a solution to be unstable, in contrast for instance to the decay instability or to the modulational instability of Stokes wave or bichromatic wavetrains, where disturbances assimilated to interacting waves with initially infinitesimal amplitudes have to be superimposed to the initial states. Here, in the unstable case, any small departure from the initial conditions leads to a solution that diverges initially from the steady state.

I conclude this paper with possible related issues:

(*a*) Compute the response of the Zakharov equation (5.1) to initial data of the form $\mathcal{B}(\boldsymbol{k},0)=\sum_{i=1}^{4} b_i(0)\delta(\boldsymbol{k}-\boldsymbol{k}_i)$ and compare with solutions of the truncated model (1.1), on or off resonance. Alternatively, is the Stokes wave the unique exact solution of (5.1) in the form $\mathcal{B}(\boldsymbol{k},t)=\sum_{i=1}^{N} b_i(t)\delta(\boldsymbol{k}-\boldsymbol{k}_i)$, $N\geq 1$?

(*b*) Can isolated quartet models such as the bidirectional standing wave (5.9) and the X-pump (6.6), the Ψ-breather or their bifurcated periodic states be reproduced numerically or experimentally, as for steady states (Liu *et al.* 2015)?

(*c*) Is there a connection between the large variations observed in figure 11 and the occurrence of rogue waves (Onorato *et al.* 2013)?

(*d*) Would instabilities grow super-exponentially if a linear forcing term were added to (1.1) or (1.6), as predicted from cubic forced nonlinear Schrödinger and Davey–Stewartson equations (Leblanc 2007, 2008)?

Acknowledgments. I would like to acknowledge the anonymous referees for constructive comments and relevant additional references, and Michael Le Bars and Miguel Onorato for their encouragements. Symbolic and numerical computations have been carried out with the assistance of Wolfram Mathematica.

Declaration of interests. The author reports no conflict of interest.

Author ORCIDs.
Stéphane Leblanc https://orcid.org/0009-0002-6924-7055.

Appendix A. Simple expressions for the roots of the quartic

If $a\neq 0$ and $q_1=q_2\equiv q_{12}$ and $q_3=q_4\equiv q_{34}$, then (2.18) is

$$f(q)=f^{(-)}(q)f^{(+)}(q),\quad f^{(\pm)}(q)=4T(q_{12}+q)(q_{34}-q)\pm(Aq^2+Bq+C), \tag{A1}$$

from which the roots may be easily deduced. When $q_1 = q_2 = q_3 = q_4 \equiv q_0$, they are

$$\left.\begin{aligned} \xi_\pm^{(-)} &= \frac{-\Delta\omega - q_0\Delta B \pm \sqrt{(\Delta\omega + q_0\Delta B)^2 + 16T(4T-A)(1-\cos p_0)q_0^2}}{2(A-4T)}, \\ \xi_\pm^{(+)} &= \frac{-\Delta\omega - q_0\Delta B \pm \sqrt{(\Delta\omega + q_0\Delta B)^2 + 16T(4T+A)(1+\cos p_0)q_0^2}}{2(A+4T)}. \end{aligned}\right\} \tag{A2}$$

Appendix B. Exact solutions for cubic or quadratic potentials

If $a = 0$ and $b \neq 0$, f is a cubic polynomial the factorization of which is

$$f(q) = b(q-\xi_1)(q-\xi_2)(q-\xi_3). \tag{B1}$$

Roots ξ_1, ξ_2, ξ_3 are real. Unsteady solutions of (2.18) with (2.5) for $b > 0$ are given below (see Craik 1985, pp. 137–139). Those for $b < 0$ may be deduced by symmetry.

If f admits three distinct roots such that $\xi_1 \leq 0 < \xi_2 < \xi_3$ or $\xi_1 < 0 \leq \xi_2 < \xi_3$ (figure 1*h*), then (2.18) with (2.5) admits the pair of periodic solutions $\{Q_{VIII}(t),\ Q_{VIII}(-t)\}$ with

$$\left.\begin{aligned} Q_{VIII}(t) = \xi_1 + (\xi_2-\xi_1)\,\mathrm{sn}^2(u(t),k), &\quad u(t) = \frac{\gamma t}{2} + \mathrm{sn}^{-1}(l,k), \\ \gamma = \sqrt{b(\xi_3-\xi_1)}, \quad k = \sqrt{(\xi_2-\xi_1)/(\xi_3-\xi_1)}, &\quad l = \sqrt{-\xi_1/(\xi_2-\xi_1)}. \end{aligned}\right\} \tag{B2}$$

Period is $\tau = 4\gamma^{-1}\mathrm{K}(k)$. At the initial time:

$$\dot{Q}_{VIII}(0) = \gamma l(\xi_2-\xi_1)\sqrt{1-l^2}\sqrt{1-k^2l^2} \geq 0. \tag{B3}$$

A solution similar to (B2) was first found by Armstrong *et al.* (1962).

If f admits a single root ξ_1 and a double root ξ_{23} such that $\xi_1 \leq 0 < \xi_{23}$ (figure 1*i*), then (2.18) with (2.5) has the pair of breather solutions $\{Q_{IX}(t),\ Q_{IX}(-t)\}$ with

$$\left.\begin{aligned} Q_{IX}(t) = \xi_{23}\frac{\cosh(\mu t) + 2r\sinh(\mu t) - 1}{\cosh(\mu t) + 2r\sinh(\mu t) + s}, &\quad \mu = \sqrt{b(\xi_{23}-\xi_1)}, \\ r = \sqrt{-\xi_1(\xi_{23}-\xi_1)}/(\xi_{23}-2\xi_1), &\quad s = \xi_{23}/(\xi_{23}-2\xi_1). \end{aligned}\right\} \tag{B4}$$

At the initial time: $\dot{Q}_{IX}(0) = \sqrt{-b\xi_1}\xi_{23} \geq 0$. Solution (B4) is not mentioned by Craik (1985) and has curiously not been found elsewhere. Note that from (4.1): $\xi_{23} = q_3 = q_4 \equiv q_{34}$. Therefore, expanding (B1) and (2.18) with $4T = \pm A$ yields the compatibility conditions $\xi_1 \leq 0$ and $q_{34} > 0$ with

$$\xi_1 = \frac{q_1q_2\sin^2 p_0}{\pm 2\sqrt{q_1q_2}\cos p_0 - q_1 - q_2}, \quad q_{34} = \frac{\pm A\sqrt{q_1q_2}\cos p_0 - \Delta\omega - B_1q_1 - B_2q_2}{A - B_3 - B_4}. \tag{B5a,b}$$

Finally, if $a = b = 0$, then f is quadratic with factorization $f(q) = c(q-\xi_1)(q-\xi_2)$. Since $q(t)$ is bounded, then necessarily $c < 0$ and the two roots ξ_1 and ξ_2 are real and such that $\xi_1 \leq 0 < \xi_2$ or $\xi_1 < 0 \leq \xi_2$. Therefore, (2.18) with (2.5) has the pair of sinusoidal solutions $\{Q_X(t),\ Q_X(-t)\}$ with

$$Q_X(t) = \tfrac{1}{2}(\xi_1+\xi_2)(1-\cos(\sqrt{-c}t)) + \sqrt{-\xi_1\xi_2}\sin(\sqrt{-c}t). \tag{B6}$$

Period is $\tau = 2\pi/\sqrt{-c}$. At the initial time: $\dot{Q}_X(0) = \sqrt{c\xi_1\xi_2} \geq 0$. An equivalent solution is given by Chen (1989). The two roots ξ_1, ξ_2 may be given explicitly.

Appendix C. Coupling coefficients

Krasitskii's kernel $T \equiv T(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3, \boldsymbol{k}_4)$ as given by Janssen (2004, p. 206) is:

$$
\begin{aligned}
T = {} & U_{-1,-2,3,4} + U_{3,4,-1,-2} - U_{3,-2,-1,4} - U_{-1,3,-2,4} - U_{-1,4,3,-2} - U_{4,-2,3,-1} \\
& - V^{(-)}_{1,3,1-3} V^{(-)}_{4,2,4-2} \left((\omega_3 + \omega_{1-3} - \omega_1)^{-1} + (\omega_2 + \omega_{4-2} - \omega_4)^{-1} \right) \\
& - V^{(-)}_{2,3,2-3} V^{(-)}_{4,1,4-1} \left((\omega_3 + \omega_{2-3} - \omega_2)^{-1} + (\omega_1 + \omega_{4-1} - \omega_4)^{-1} \right) \\
& - V^{(-)}_{1,4,1-4} V^{(-)}_{3,2,3-2} \left((\omega_4 + \omega_{1-4} - \omega_1)^{-1} + (\omega_2 + \omega_{3-2} - \omega_3)^{-1} \right) \\
& - V^{(-)}_{2,4,2-4} V^{(-)}_{3,1,3-1} \left((\omega_4 + \omega_{2-4} - \omega_2)^{-1} + (\omega_1 + \omega_{3-1} - \omega_3)^{-1} \right) \\
& - V^{(-)}_{1+2,1,2} V^{(-)}_{3+4,3,4} \left((\omega_{1+2} - \omega_1 - \omega_2)^{-1} + (\omega_{3+4} - \omega_3 - \omega_4)^{-1} \right) \\
& - V^{(+)}_{-1-2,1,2} V^{(+)}_{-3-4,3,4} \left((\omega_{1+2} + \omega_1 + \omega_2)^{-1} + (\omega_{3+4} + \omega_3 + \omega_4)^{-1} \right),
\end{aligned} \tag{C1}
$$

where $\omega_i = \omega(\boldsymbol{k}_i) = \sqrt{g k_i}$, $k_i = |\boldsymbol{k}_i|$ and $\omega_{i\pm j} = \omega(\boldsymbol{k}_i \pm \boldsymbol{k}_j)$, and where for instance

$$
\left.
\begin{aligned}
U_{1,2,3,4} &= \frac{k_1 k_2}{16} \sqrt{\frac{\omega_3 \omega_4}{\omega_1 \omega_2}} (2k_1 + 2k_2 - k_{1+3} - k_{1+4} - k_{2+3} - k_{2+4}), \\
V^{(\pm)}_{1,2,3} &= \frac{\sqrt{2g}}{8} \left(X^{(\pm)}_{1,2} \sqrt{\frac{\omega_3}{\omega_1 \omega_2}} + X^{(\pm)}_{1,3} \sqrt{\frac{\omega_2}{\omega_1 \omega_3}} + X^{(+)}_{2,3} \sqrt{\frac{\omega_1}{\omega_2 \omega_3}} \right), \quad \text{etc.,} \\
X^{(\pm)}_{i,j} &= \boldsymbol{k}_i \cdot \boldsymbol{k}_j \pm k_i k_j, \quad k_{i\pm j} = |\boldsymbol{k}_i \pm \boldsymbol{k}_i|, \quad i,j = 1, \ldots, 4.
\end{aligned}
\right\} \tag{C2}
$$

According to Krasitskii (1990, 1994), T obeys the symmetries

$$
T(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3, \boldsymbol{k}_4) = T(\boldsymbol{k}_2, \boldsymbol{k}_1, \boldsymbol{k}_3, \boldsymbol{k}_4) = T(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_4, \boldsymbol{k}_3) = T(\boldsymbol{k}_3, \boldsymbol{k}_4, \boldsymbol{k}_1, \boldsymbol{k}_2). \tag{C3}
$$

Another important property of T is positive homogeneity:

$$
T(\lambda \boldsymbol{k}_1, \lambda \boldsymbol{k}_2, \lambda \boldsymbol{k}_3, \lambda \boldsymbol{k}_4) = |\lambda|^3 T(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3, \boldsymbol{k}_4), \quad \forall \lambda \in \mathbb{R}. \tag{C4}
$$

In particular, $T(-\boldsymbol{k}_1, -\boldsymbol{k}_2, -\boldsymbol{k}_3, -\boldsymbol{k}_4) = T(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3, \boldsymbol{k}_4)$. Note also that, on deep water, T does not depend explicitly on gravity acceleration: $\partial T / \partial g = 0$.

Kernel T is singular for $\boldsymbol{k}_2 = -\boldsymbol{k}_1$, or $\boldsymbol{k}_3 = \boldsymbol{k}_1$, or $\boldsymbol{k}_4 = \boldsymbol{k}_1$, etc., but singularities can be removed (see Zakharov 1999). For instance, $T_{ij} = T(\boldsymbol{k}_i, \boldsymbol{k}_j, \boldsymbol{k}_i, \boldsymbol{k}_j)$ may be written as (Zakharov 1999; Leblanc 2009)

$$
\begin{aligned}
T_{ij} = {} & (k_i + k_j) \boldsymbol{k}_i \cdot \boldsymbol{k}_j - \frac{3k_i^2 k_j^2 + (\boldsymbol{k}_i \cdot \boldsymbol{k}_j)^2}{4\sqrt{k_i k_j}} \\
& - \frac{1}{2\sqrt{k_i k_j}} \left(\frac{(\omega_i - \omega_j)^2 (X^{(+)}_{i,j})^2}{(\omega_{i-j})^2 - (\omega_i - \omega_j)^2} + \frac{(\omega_i + \omega_j)^2 (X^{(-)}_{i,j})^2}{(\omega_{i+j})^2 - (\omega_i + \omega_j)^2} \right),
\end{aligned} \tag{C5}
$$

from which can be deduced $T_{ii} = T(\boldsymbol{k}_i, \boldsymbol{k}_i, \boldsymbol{k}_i, \boldsymbol{k}_i) = k_i^3$, in agreement with Benney (1962).

For standing waves with $\boldsymbol{k}_2 = -\boldsymbol{k}_1$ and $\boldsymbol{k}_4 = -\boldsymbol{k}_3$, T may be written as

$$\left.\begin{aligned}
T(\boldsymbol{k}_1, -\boldsymbol{k}_1, \boldsymbol{k}_3, -\boldsymbol{k}_3) &= \frac{k_1^2 + k_3^2}{4}\sqrt{k_1 k_3} + \frac{k_{1+3} + k_{1-3}}{8}\left(2k_1 k_3 - \sqrt{k_1 k_3}(k_1 + k_3)\right) \\
&\quad - \frac{k_1 + k_3}{2} k_1 k_3 - \frac{1}{8\sqrt{k_1 k_3}}\left(\frac{Y_{1,3}^{(-)}}{(\omega_{1-3})^2 - (\omega_1 - \omega_3)^2} + \frac{Y_{1,3}^{(+)}}{(\omega_{1+3})^2 - (\omega_1 - \omega_3)^2}\right), \\
Y_{1,3}^{(-)} &= \left(\omega_{1-3} X_{1,3}^{(-)}\right)^2 - \left(\omega_1 X_{1-3,3}^{(+)} - \omega_3 X_{1,3-1}^{(+)}\right)^2, \\
Y_{1,3}^{(+)} &= \left(\omega_{1+3} X_{1,3}^{(+)}\right)^2 - \left(\omega_1 X_{1+3,3}^{(-)} - \omega_3 X_{1,3+1}^{(-)}\right)^2.
\end{aligned}\right\} \tag{C6}$$

If $k_1 = k_3 \equiv k_0$, we get $T(\boldsymbol{k}_1, -\boldsymbol{k}_1, \boldsymbol{k}_3, -\boldsymbol{k}_3) = -\frac{1}{8}k_0^3(7 + \cos(2\psi))$ with $\psi =$ angle $(\boldsymbol{k}_1, \boldsymbol{k}_3)$. In particular, $T(\boldsymbol{k}_0, -\boldsymbol{k}_0, \boldsymbol{k}_0, -\boldsymbol{k}_0) = -k_0^3$, in agreement with Okamura (1984).

Appendix D. An alternative criterion for resonant quartets

CRITERION D.1. *Steady resonant wave quartets* (3.6) *with* $q_1 = q_3 \equiv q_{13}$ *and* $q_2 = q_4 \equiv q_{24}$ *exist if* $q_{24}/q_{13} = (B_3 - B_1)/(B_2 - B_4) > 0$. *They are exponentially unstable if*

$$\frac{A}{2T}\cos p_0 > \frac{B_2 - B_4}{B_3 - B_1} + \frac{B_3 - B_1}{B_2 - B_4} \quad (p_0 = 0 \text{ or } \pi). \tag{D1}$$

Kartashova's quartet (5.12) is disqualified since $(B_3 - B_1)/(B_2 - B_4) < 0$. Instead, we consider the resonant 'tridents' proposed by Lvov *et al.* (2006):

$$\left.\begin{aligned}
&\boldsymbol{k}_1 = (\alpha_+, 0), \quad \boldsymbol{k}_2 = (-\alpha_-, 0), \quad \boldsymbol{k}_3 = (\beta, \gamma), \quad \boldsymbol{k}_4 = (\beta, -\gamma), \\
&\alpha_\pm = (m^2 \pm mn + n^2)^2, \quad \beta = 2mn(m^2 + n^2), \quad \gamma = m^4 - n^4,
\end{aligned}\right\} \tag{D2}$$

where (m, n) are positive integers (or real numbers) and $m \neq n$ to avoid $\boldsymbol{k}_3 = \boldsymbol{k}_4$. The first qualified candidates for criterion D.1 are couples $(m, n) = (1, 17)$ and $(2, 33)$: both exist for $p_0 = 0$ or π and are stable. No unstable configuration has been found for $m = 1$ and $n \geq 17$, nor for $m = 2$ and $n \geq 33$.

By contrast, according to criterion 5.1 (§ 5.3), a steady quartet corresponding to $(m, n) = (1, 3)$ exists for $p_0 = \pi$ but is unstable; $(1, 4)$ and $(1, 5)$ exist for $p_0 = \pi$ and are stable; $(1, 6)$ to $(1, 100)$ exist for $p_0 = 0$ or π and are stable; $(2, 5)$ to $(2, 8)$ exist for $p_0 = \pi$ but are unstable; $(2, 9)$ and $(2, 10)$ exist for $p_0 = \pi$ and are stable; $(2, 11)$ to $(2, 100)$ exist for $p_0 = 0$ or π and are stable.

REFERENCES

ANDRADE, D. & STUHLMEIER, R. 2023*a* Instability of waves on deep water – a discrete Hamiltonian approach. *Eur. J. Mech. (B/Fluids)* **101**, 320–336.

ANDRADE, D. & STUHLMEIER, R. 2023*b* The nonlinear Benjamin–Feir instability – Hamiltonian dynamics, discrete breathers and steady solutions. *J. Fluid Mech.* **958**, A17.

ARMSTRONG, J.A., BLOEMBERGEN, N., DUCUING, J. & PERSHAN, P.S. 1962 Interactions between light waves in a nonlinear dielectric. *Phys. Rev.* **127** (6), 1918–1939.

ARNOLD, V.I. 1989 *Mathematical Methods of Classical Mechanics*, 2nd edn. Springer.

BADULIN, S.I., SHRIRA, V.I., KHARIF, C. & IOUALALEN, M. 1995 On two approaches to the problem of instability of short-crested water waves. *J. Fluid Mech.* **303**, 297–326.

BENJAMIN, T.B. & FEIR, F.E. 1967 The disintegration of wave trains on deep water. Part 1. Theory. *J. Fluid Mech.* **27**, 417–430.

BENNEY, D.J. 1962 Non-linear gravity wave interactions. *J. Fluid Mech.* **14**, 577–584.

BOYD, T.J.M. & TURNER, J.G. 1978 Three- and four-wave interactions in plasmas. *J. Math. Phys.* **19** (6), 1403–1413.

BRETHERTON, F.P. 1964 Resonant interactions between waves. The case of discrete oscillations. *J. Fluid Mech.* **20**, 457–480.

BYRD, P.F. & FRIEDMAN, M.D. 1971 *Handbook of Elliptic Integrals for Engineers and Scientits*, 2nd edn. Springer.

CAPONI, E.A., SAFFMAN, P.G. & YUEN, H.C. 1982 Instability and confined chaos in a nonlinear dispersive wave system. *Phys. Fluids* **25** (12), 2159–2166.

CAPPELLINI, G. & TRILLO, S. 1991 Third-order three-wave mixing in single-mode fibers: exact solutions and spatial instability effects. *J. Opt. Soc. Am.* B **8** (4), 824–838.

CHEN, Y. 1989 Four-wave mixing in optical fibers: exact solution. *J. Opt. Soc. Am.* B **6** (11), 1986–1993.

CHEN, Y. & SNYDER, A.W. 1989 Four-photon parametric mixing in optical fibers: effect of pump depletion. *Opt. Lett.* **14** (1), 87–89.

CRAIK, A.D.D. 1985 *Wave Interactions and Fluid Flows*. Cambridge University Press.

GALEEV, A.A. & KARPMAN, V.I. 1963 Turbulence theory of a weakly nonequilibrium low-density plasma and structure of shock waves. *Zh. Eksp. Teor. Fiz.* **44**, 592–602. English transl.: *Sov. Phys. JETP* **17** (1963), 403–409.

GANTMACHER, F. 1975 *Lectures in Analytical Mechanics*. Mir Publishers.

HASSELMANN, K. 1962 On the non-linear energy transfer in a gravity-wave spectrum. Part 1. General theory. *J. Fluid Mech.* **12**, 481–500.

HASSELMANN, K. 1967 A criterion for nonlinear wave stability. *J. Fluid Mech.* **30**, 737–739.

HOGAN, S.J., GRUMAN, I. & STIASSNIE, M. 1988 On the changes in phase speed of one train of water waves in the presence of another. *J. Fluid Mech.* **192**, 97–114.

INOUE, Y. 1975 Resonant four-wave interaction in a dispersive medium. *J. Phys. Soc. Japan* **39** (4), 1092–1099.

IOUALALEN, M. & KHARIF, C. 1994 On the subharmonic instabilities of steady three-dimensional deep water waves. *J. Fluid Mech.* **262**, 265–291.

JANSSEN, P. 2004 *The Interaction of Ocean Waves and Wind*. Cambridge University Press.

JEFFREYS, H. & JEFFREYS, B. 1956 *Methods of Mathematical Physics*, 3rd edn. Cambridge University Press.

KARTASHOVA, E. 2010 *Nonlinear Resonance Analysis: Theory, Computation, Applications*. Cambridge University Press.

KRASITSKII, V.P. 1990 Canonical transformations in a theory of weakly nonlinear waves with a nondecay dispersion law. *Zh. Eksp. Teor. Fiz.* **98**, 1644–1655. English transl.: *Sov. Phys. JETP* **71**(5) (1990), 921–927.

KRASITSKII, V.P. 1994 On reduced equations in the Hamiltonian theory of weakly nonlinear surface waves. *J. Fluid Mech.* **272**, 1–20.

LEBLANC, S. 2007 Amplification of nonlinear surface waves by wind. *Phys. Fluids* **19**, 101705.

LEBLANC, S. 2008 Wind-forced modulations of finite-depth gravity waves. *Phys. Fluids* **20**, 116603.

LEBLANC, S. 2009 Stability of bichromatic gravity waves on deep water. *Eur. J. Mech. (B/Fluids)* **28** (5), 605–612.

LIAO, S., XU, D. & STIASSNIE, M. 2016 On the steady-state nearly resonant waves. *J. Fluid Mech.* **794**, 175–199.

LIU, Z. & LIAO, S. 2014 Steady-state resonance of multiple wave interactions in deep water. *J. Fluid Mech.* **742**, 664–700.

LIU, Z., XU, D., LI, J., PENG, T., ALSAEDI, A. & LIAO, S. 2015 On the existence of steady-state resonant waves in experiments. *J. Fluid Mech.* **763**, 1–23.

LONGUET-HIGGINS, M.S. 1962 Resonant interactions between two trains of gravity waves. *J. Fluid Mech.* **12**, 321–332.

LONGUET-HIGGINS, M.S. & PHILLIPS, O.M. 1962 Phase velocity effects in tertiary wave interactions. *J. Fluid Mech.* **12**, 323–336.

LORIA, A. & PANTELEY, E. 2017 Stability, as told by its developers. *IFAC PapersOnLine* **50** (1), 5219–5230.

LVOV, Y.V., NAZARENKO, S. & POKORNI, B. 2006 Discreteness and its effect on water-wave turbulence. *Physica* D **218**, 24–35.

MANLEY, J. & ROWE, H. 1956 Some general properties of nonlinear elements. Part I. General energy relations. *Proc. IRE* **44** (7), 904–913.

NAZARENKO, S. & LUKASCHUK, S. 2016 Wave turbulence on water surface. *Annu. Rev. Condens. Matt. Phys.* **7**, 61–88.

OKAMURA, M. 1984 Instabilities of weakly nonlinear standing gravity waves. *J. Phys. Soc. Japan* **53** (11), 3788–3796.

OKAMURA, M. 1985 On the instability of weakly nonlinear three-dimensional standing waves. *J. Phys. Soc. Japan* **54** (9), 3313–3320.

ONORATO, M., RESIDORI, S., BORTOLOZZO, U., MONLINA, A. & ARRECHI, F.T. 2013 Rogue waves and their generating mechanisms in different physical contexts. *Phys. Rep.* **528** (2), 47–89.

PARS, L.A. 1965 *A Treatise on Analytical Dynamics*. Heinemann.

PHILLIPS, O.M. 1960 On the dynamics of unsteady gravity waves of finite amplitude. Part 1. The elementary interactions. *J. Fluid Mech.* **9**, 193–217.

PHILLIPS, O.M. 1967 Theoretical and experimental studies of gravity wave interactions. *Proc. R. Soc. Lond.* A **299**, 104–119.

PHILLIPS, O.M. 1981 Wave interactions – the evolution of an idea. *J. Fluid Mech.* **106**, 215–227.

SAFDI, B.R. & SEGUR, H. 2007 Explosive instability due to four-wave mixing. *Phys. Rev. Lett.* **99** (24), 245004.

SAFFMAN, P.G. & YUEN, H.C. 1980 A new type of three-dimensional deep-water wave of permanent form. *J. Fluid Mech.* **101**, 797–808.

SHEMER, L. & STIASSNIE, M. 1985 Initial instability and long-time evolution of Stokes waves. In *The Ocean Surface* (ed. Y. Toba & H. Mitsuyasu), pp. 51–57. Springer.

SHEMER, L. & STIASSNIE, M. 1991 The Zakharov and modified Zakharov equations and their applications. In *Nonlinear Topics in Ocean Physics* (ed. R.A. Osborne), pp. 581–620. Elsevier.

STIASSNIE, M. & SHEMER, L. 2005 On the interaction of four water waves. *Wave Motion* **41**, 307–328.

TRILLO, S. & WABNITZ, S. 1991 Dynamics of the nonlinear modulational instability in optical fibers. *Opt. Lett.* **16** (13), 986–988.

TURNER, J.G. 1980 Four-wave interaction of positive and negative energy waves in plasmas. *Phys. Scr.* **21** (2), 185–190.

VERHEEST, F. 1982 Four-wave interactions in plasmas and other nonlinear media. *J. Phys.* A*: Math. Gen.* **15** (3), 1041–1050.

VERHULST, F. 1996 *Nonlinear Differential Equations and Dynamical Systems*, 2nd edn. Springer.

YUEN, H.C. & LAKE, B.M. 1982 Nonlinear dynamics of deep-water gravity waves. *Adv. Appl. Mech.* **22**, 67–229.

ZAKHAROV, V.E. 1966 The instability of waves in nonlinear dispersive media. *Zh. Eksp. Teor. Fiz.* **51**, 1107–1114. English transl.: *Sov. Phys. JETP* **24** (1967), 994–998.

ZAKHAROV, V.E. 1967 Instability of self-focusing of light. *Zh. Eksp. Teor. Fiz.* **53**, 1735–1743. English transl.: *Sov. Phys. JETP* **26** (1967), 740–744

ZAKHAROV, V.E. 1968 Stability of periodic waves of finite amplitude on the surface of deep fluid. *Zh. Prikl. Mekh. Tekh. Fiz.* **9**, 86–94. English transl.: *J. Appl. Mech. Tech. Phys.* **2** (1968), 190–194.

ZAKHAROV, V.E. 1999 Statistical theory of gravity and capillary waves on the surface of a finite-depth fluid. *Eur. J. Mech. (B/Fluids)* **18** (3), 327–344.

ZAKHAROV, V.E., L'VOV, V.S. & FALKOVICH, G. 1992 *Kolmogorov Spectra of Turbulence I. Wave Turbulence*. Springer.

ZAKHAROV, V.E. & OSTROVSKY, L.A. 2009 Modulation instability: the beginning. *Physica* D **238**, 540–548.