EQUIVALENT CONDITIONS FOR A RING TO BE A MULTIPLICATION RING

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In this paper a ring will always mean a commutative ring with identity element. Furthermore, a ring R is called a multiplication ring if, whenever A and B are ideals of R and A is contained in B, there is an ideal C such that $A = BC$. Noetherian multiplication rings have been studied by Asano **(1)**, Krull **(4, 5)**, and Mori **(6, 7)**. Krull also studied non-Noetherian multiplication rings **(3)**. In **(8, 9)**, Mori studied non-Noetherian multiplication rings which did not necessarily contain an identity element.

The notation and terminology used will be in general that of **(10)**. In particular, the symbol $\subset$ will mean "contained in or equal," $<$ will denote proper containment, and $\not\subset$ will mean "not contained in or equal." If A is an ideal of R and P is a minimal prime ideal of A, then the intersection of all P-primary ideals containing A is called an isolated P-primary component of A **(2**, p. 737**)**. The intersection of all isolated primary components of A is called the kernel of A **(2**, p. 738**)**.

This paper is concerned with equivalent conditions for a ring to be a multiplication ring. The conditions are contained in the following theorem.

THEOREM. *The following statements are equivalent:*

(I) *A ring R is a multiplication ring.*

(II) *If P is a prime ideal of R containing an ideal A, then there is an ideal C such that $A = PC$.*

(III) *R is a ring in which the following three conditions are valid:*

(*a*) *every ideal is equal to its kernel,*

(*b*) *every primary ideal is a power of its radical, and*

(*c*) *if P is a minimal prime ideal of an ideal B and n is the least positive integer such that P^n is an isolated primary component of B and if $P^n \neq P^{n+1}$, then P does not contain the intersection of the remaining isolated primary components of B.*

Proof. If R is a multiplication ring, then II follows. Therefore, suppose II is valid in R. The following properties (i) through (x) are consequences of II:

(i) For any ideal A of R, R/A satisfies II.

(ii) If R is an integral domain, then R is a Dedekind domain.

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(iii) There are no ideals between a maximal ideal M and its square (**1**, p. 85). Furthermore, there are no ideals between M and M^n except powers of M, and R/M^n is a special primary ring (**1**, p. 83).

(iv) There is no prime ideal chain $P_1 < P_2 < P_3 < R$.

If P_1, P_2, and P_3 are prime ideals such that $P_1 \subset P_2 < P_3 < R$, then in the Dedekind domain R/P_1, $P_2/P_1 < P_3/P_1$, and therefore $P_2/P_1 = P_1/P_1$. Consequently, $P_1 = P_2$.

(v) If M is a proper maximal ideal properly containing the prime ideal P, then

$$P = \bigcap_{n=1}^{\infty} M^n$$

and $MP = P$.

In $\bar{R} = R/P$,

$$(0) = \bar{P} = \bigcap_{n=1}^{\infty} M^n,$$

and, consequently,

$$P \supset \bigcap_{n=1}^{\infty} M^n.$$

Since $P \subset M$, there is an ideal C such that $P = MC$. Using the fact that P is a prime ideal and $M \not\subset P$, it follows that $C \subset P$, and $P = MP$. Therefore $P = MP = M^2P$, etc., so that

$$P \subset \bigcap_{n=1}^{\infty} M^n.$$

Hence

$$P = \bigcap_{n=1}^{\infty} M^n.$$

(vi) Every ideal is equal to its kernel.

If A is an ideal of R, suppose $A \neq A^*$, where A^* denotes the kernel of A. Let $a \in A^* \setminus A$, and consider the ideal $A' = A:(a)$. Let M be a minimal prime ideal of A'; then by a theorem of Krull (**2**, p. 738), M properly contains a minimal prime ideal P of A. Thus M is a maximal ideal,

$$P = \bigcap_{n=1}^{\infty} M^n,$$

and $P = MP$. Since $A' \subset M$, there is an ideal C such that $A' = MC$. If $C \subset A'$, then $A' = MA' = M^2A'$, etc., so that

$$A' \subset \bigcap_{n=1}^{\infty} M^n = P.$$

This would imply that M is not a minimal prime ideal of A'. Therefore, $C \not\subset A'$, and hence $(a)C \not\subset A$. On the other hand, $(a)C \subset (a) \subset P$ since $a \in A^*$. As a consequence, there is an ideal S such that

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$$(a)C = PS = MPS = M(a)C = (a)A' \subset A.$$

This contradiction proves $A = A^*$.

(vii) If M is a proper maximal ideal, and if A is an ideal contained in M^n, then there is an ideal C such that $A = M^n C$. Furthermore, if $A \not\subset M^{n+1}$, then $C \not\subset M$.

The proof of the above statement will be by induction. The statement is obviously true for $n = 1$. Suppose $A \subset M^k$ implies $A = M^k C$. Then if $A \subset M^{k+1}$, $A = M^k C$ since $M^{k+1} \subset M^k$. If $M^{k+1} = M^k$, obviously $A = M^{k+1}C$. Suppose that $M^{k+1} \neq M^k$. Since M^{k+1} is an M-primary ideal containing $A = M^k C$ and $M^k \not\subset M^{k+1}$, it follows that $C \subset M$. Hence $C = MC'$ and $A = M^{k+1}C'$.

If $A \subset M^n$ and $A \not\subset M^{n+1}$, then $A = M^n C$ by the above, but $C \not\subset M$ because if $C \subset M$, then $C = MB$ and this would imply that $A = M^{n+1}B \subset M^{n+1}$.

(viii) If M is a maximal ideal and $M^n \neq M^{n+1}$ for each positive integer n, then

$$P = \bigcap_{n=1}^{\infty} M^n$$

is a prime ideal.

Suppose $x \notin P$ and $y \notin P$. Then there are positive integers k and n such that $x \in M^k$ and $y \in M^n$, but $x \notin M^{k+1}$ and $y \notin M^{n+1}$. Consequently, there are ideals B and C, not contained in M, such that $(x) = M^k B$ and $(y) = M^n C$. Therefore, $(xy) = M^{n+k}BC$, where $BC \not\subset M$. As a result, $xy \notin P$ and P is a prime ideal.

(ix) If Q is a P-primary ideal, then Q is a power of P.

It is well known that if P is a non-maximal prime ideal in a ring in which every ideal is equal to its kernel, then $P = P^2$ and $Q = P$ (**9**, p. 99). Assume P is a maximal ideal. The following two cases will be considered: (a) $P^n \neq P^{n+1}$ for every positive integer n and (b) $P^n = P^{n+1}$ for some positive integer n.

If $P^n \neq P^{n+1}$ for each positive integer n, then Q is not contained in every power of P since Q is not contained in the prime ideal

$$P' = \bigcap_{n=1}^{\infty} P^n.$$

Therefore, there is an integer k such that $Q \subset P^k$ but $Q \not\subset P^{k+1}$. This implies $Q = P^k C$, where $C \not\subset P$. If C is a proper ideal of R, any proper prime divisor P of C must contain Q and hence must contain the maximal ideal P. This would imply $P = P'$ and therefore $C \subset P$. This contradiction shows that $C = R$ and $Q = P^k$.

If $P^n = P^{n+1}$ for some integer n, suppose k is the least positive integer such that $P^k = P^{k+1}$. There are two cases to consider here. Either $Q \subset P^k$ or $Q \not\subset P^k$.

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If $Q \subset P^k = P^{2k}$, then for each $a \in P^k$ there is an ideal C such that $(a) = P^kC = P^{2k}C = P^k(a)$. Therefore, there is an element $p \in P$ such that $a = pa = p^2a$, etc. Consequently, $a \in Q$ since $p^s \in Q$ for some integer s. Hence $P^k \subset Q$ and, as a result, $Q = P^k$.

If $Q \neq P^k$, then $Q + P^k$ is a P-primary ideal properly containing P^k (**10**, p. 154). Therefore, by (iii) $Q + P^k = P^t$ for some integer $t > k$. Thus, there is an integer m such that $t \geqslant m > k$ and $Q \subset P^m$ but $Q \not\subset P^{m+1}$. There is an ideal C such that $Q = P^mC$ and $C \not\subset P$. As before, it will follow that $C = R$ and $Q = P^m$.

(x) If P is a minimal prime ideal of an ideal B and n is the least positive integer such that P^n is an isolated primary component of B andi f $P^n \neq P^{n+1}$, then P does not contain the intersection of the remaining isolated primary components of B.

Since B is equal to its kernel, let $B = P^n \cap B'$, where

$$B' = \bigcap_{\alpha} P_{\alpha}^{n_{\alpha}}$$

is the intersection of all the isolated primary components of B except P^n, Since $B \subset P^n$ and $B \not\subset P^{n+1}$, there is an ideal C such that $B = P^nC$, where $C \not\subset P$. It follows that $C \subset P_{\alpha}^{n_{\alpha}}$ for each α since $B \subset P_{\alpha}^{n_{\alpha}}$ and $P^n \not\subset P_{\alpha}$. Therefore $C \subset B'$ and $B' \not\subset P$ since $C \not\subset P$.

Properties (vi), (ix), and (x) show that II implies III.

Assume III is valid and A and B are ideals such that $A < B$. Since A and B are equal to their kernels, let

$$B = \Big(\bigcap_{\alpha} P_{\alpha}^{\lambda_{\alpha}}\Big) \cap \Big(\bigcap_{\beta} P_{\beta}'^{\sigma_{\beta}}\Big)$$

and

$$A = \Big(\bigcap_{\alpha} P_{\alpha}^{\mu_{\alpha}}\Big) \cap \Big(\bigcap_{\tau} P_{\tau}''^{\nu_{\tau}}\Big)$$

where P'' is a minimal prime ideal of A but not of B, P' is a minimal prime ideal of B but not of A, and P is a minimal prime ideal of both A and B. Also, the exponents λ_{α}, μ_{α}, σ_{β}, and ν_{τ} denote the least positive integers such that $P_{\alpha}^{\lambda_{\alpha}}$, $P_{\beta}'^{\sigma_{\beta}}$ are isolated primary components of B and $P_{\alpha}^{\mu_{\alpha}}$, $P_{\tau}''^{\nu_{\tau}}$ are isolated primary components of A. Clearly $\lambda_{\alpha} \leqslant \mu_{\alpha}$ for each α since $P_{\alpha}^{\mu_{\alpha}} \subset P_{\alpha}^{\sigma_{\alpha}}$. Let

$$C = \Big(\bigcap_{\alpha} P_{\alpha}^{\mu_{\alpha}-\lambda_{\alpha}}\Big) \cap \Big(\bigcap_{\tau} P_{\tau}''^{\nu_{\tau}}\Big).$$

Then for $x \in BC$,

$$x = \sum_{i=1}^{n} b_i c_i,$$

where $b_i \in B$ and $c_i \in C$ for each i. Therefore $b_i \in P_{\alpha}^{\lambda_{\alpha}}$, $c_i \in P_{\alpha}^{\mu_{\alpha}-\lambda_{\alpha}}$, and $c_i \in P_{\tau}''^{\nu_{\tau}}$ for each α, τ, and consequently $b_ic_i \in P_{\alpha}^{\mu_{\alpha}}$ and $b_ic_i \in P_{\tau}''^{\nu_{\tau}}$. Hence $x \in P_{\alpha}^{\mu_{\alpha}}$ and $x \in P_{\tau}''^{\nu_{\tau}}$ for each α, τ, and, as a result, $BC \subset A$. It is obvious that $A \subset C$.

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Any minimal prime ideal P of BC must contain B or C. If $B \subset P$, then P is a minimal prime of B and also of A. Hence $P = P_\alpha$ for some α. If $B \not\subset P$, then $C \subset P$ and P is a minimal prime ideal of A and also of C. In this case $P = P_\tau''$ for some τ. In particular, any minimal prime ideal of BC must be a minimal prime ideal of A. Therefore, let

$$BC = \Big(\bigcap_\alpha P_\alpha^{\mu_\alpha'}\Big) \cap \Big(\bigcap_\tau P_\tau''^{\nu_\tau'}\Big)$$

be the kernel of BC and let μ_α' and ν_τ' be the minimal exponents such that $P_\alpha^{\mu_\alpha'}$ and $P''^{\nu_\tau'}$ are isolated primary components of BC. Clearly, $\mu_\alpha \leqslant \mu_\alpha'$ and $\nu_\tau \leqslant \nu_\tau'$. Furthermore, P''^{ν_τ} is an isolated primary component of C, since $A \subset C \subset P''^{\nu_\tau}$ and P''^{ν_τ} is an isolated primary component of A. Thus, since $B \not\subset P_\tau''$ and $BC \subset P_\tau''^{\nu_\tau'}$, it follows that $C \subset P''^{\nu_\tau'}$. This being the case, one concludes that $\nu_\tau' \leqslant \nu_\tau$ and hence $\nu_\tau' = \nu_\tau$. If $P_\alpha^{\mu_\alpha} = P_\alpha^{\mu_\alpha+1}$, then clearly $\mu_\alpha = \mu_\alpha'$. Suppose that $P_\alpha^{\mu_\alpha} \neq P_\alpha^{\mu_\alpha+1}$. Since every ideal is equal to its kernel, every non-maximal prime is idempotent. Thus, one sees that P_α is a maximal ideal. Let

$$C' = \Big(\bigcap_{\delta\neq\alpha} P_\delta^{\mu_\delta-\lambda_\delta}\Big) \cap \Big(\bigcap_\tau P_\tau''^{\nu_\tau}\Big),$$

$$B' = \Big(\bigcap_{\delta\neq\alpha} P_\delta^{\lambda_\delta}\Big) \cap \Big(\bigcap_\beta P_\beta'^{\sigma_\beta}\Big),$$

and

$$A' = \Big(\bigcap_{\delta\neq\alpha} P_\delta^{\mu_\delta}\Big) \cap \Big(\bigcap_\tau P_\tau''^{\nu_\tau}\Big).$$

Then by III(c), $P_\alpha \not\supset A'$, and since P_α is maximal, $P_\alpha^{\mu_\alpha} + A' = R$. Thus $A = P_\alpha^{\mu_\alpha} \cap A' = P_\alpha^{\mu_\alpha} \cdot A'$. Similarly $B' \not\subset P$, $P_\alpha^{\lambda_\alpha} + B' = R$, and $B = P_\alpha^{\lambda_\alpha} \cap B' = P_\alpha^{\lambda_\alpha} \cdot B'$. One sees that $C' \not\subset P_\alpha$ since $A' \subset C'$ and $A' \not\subset P_\alpha$. Therefore,

$$P_\alpha^{\mu_\alpha-\lambda_\alpha} + C' = R \quad\text{and}\quad C = P_\alpha^{\mu_\alpha-\lambda_\alpha} \cap C' = P_\alpha^{\mu_\alpha-\lambda_\alpha} \cdot C'.$$

As a consequence, $BC = P_\alpha^{\mu_\alpha} \cdot B'C'$ where $B'C' \not\subset P_\alpha$. Thus $P_\alpha^{\mu_\alpha}$ is an isolated primary component of BC and $\mu_\alpha = \mu_\alpha'$. We have shown that $\mu_\alpha = \mu_\alpha'$ and $\nu_\tau = \nu_\tau'$ for each α, τ. Thus the kernels of BC and A are equal and hence $BC =$ A. Then I follows from III and the proof of the theorem is complete.

As a corollary to this theorem, a generalization of a theorem due to Asano (**1**, p. 85) can be given.

Corollary. *If R is a ring in which*

(1) *to every ideal A contained in a prime ideal P there is an ideal C such that $A = PC$, and*

(2) $(0) = Q_1 \cap Q_2 \ldots \cap Q_n$, *where Q_i is P_i-primary for each i,*
then R is a direct sum of finitely many Dedekind domains and special primary rings. Consequently every ideal is a product of prime ideals (**1**, p. 83).

Proof. Suppose the representation of the 0-ideal is an irredundant representation and $P_i \neq P_j$ for $i \neq j$. By the theorem above, R is a multiplication ring,

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and from the properties of a multiplication ring, one sees that $Q_i + Q_j = R$ for $i \neq j$. Therefore, R is a direct sum, $R = R_1 \oplus R_2 \ldots \oplus R_n$, where R_i is isomorphic to R/Q_i. If P_i is non-maximal, then $Q_i = P_i$ and R/Q_i is a Dedekind domain. If P_i is maximal, then Q_i is a power of P_i and R/Q_i is a primary ring in which there are no ideals between the unique maximal ideal and its square. In this case, R/Q_i is a special primary ring.

It is well known that a multiplication ring is a subring of a cartesian product of Dedekind domains and special primary rings (**3**, p. 323). The following example, suggested to the author by Professor L. I. Wade, is an example of a multiplication ring which is not equal to a cartesian product of Dedekind domains and special primary rings.

Let R denote the set of all sequences $a = \{a_i\}$ where the a_i are taken from the field of two elements and $a_n = a_{n+1} = a_{n+2} = \ldots$ for some n. For $a = \{a_i\}$ and $b = \{b_i\}$, define $a + b = \{a_i + b_i\}$ and $a \cdot b = \{a_i b_i\}$. Thus R is a ring in which every element is idempotent. Consequently if A is an ideal of R contained in the ideal B, $A = BA$. It is clear that R is a subring of the cartesian product R' of countably many copies of the field of integers modulo 2. However, $R \neq R'$, since R' contains uncountably many elements and R contains only countably many elements.

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