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A Steinberg Cross Section for Non-Connected Affine Kac–Moody Groups

Stephan Mohrdieck

Abstract. In this paper we generalise the concept of a Steinberg cross section to non-connected affine Kac–Moody groups. This Steinberg cross section is a section to the restriction of the adjoint quotient map to a given exterior connected component of the affine Kac–Moody group. (The adjoint quotient is only defined on a certain submonoid of the entire group, however, the intersection of this submonoid with each connected component is non-void.) The image of the Steinberg cross section consists of a "twisted Coxeter cell", a transversal slice to a twisted Coxeter element. A crucial point in the proof of the main result is that the image of the cross section can be endowed with a $\mathbb{C}^*$-action.

1 Introduction

The aim of this paper is the construction of a Steinberg cross section, a section to the adjoint quotient map for non-connected affine Kac–Moody groups. This work generalises previous results known for connected semisimple algebraic group which were proven by Steinberg [St]; see [Moh] for the non-connected case, [Br] for connected affine Kac–Moody groups and also [Mok2] for a weaker result in the indefinite case. The adjoint quotient is the quotient with respect to the conjugacy action. In general, for Kac–Moody groups the existence of an adjoint quotient, even in the affine setting, is a complicated issue; but, recently, there has been some progress in this direction, see [Mok1]. Applications of the Steinberg cross section are found in the theory of principal bundles over an elliptic curve as well as in singularity theory [HS].

An affine Kac–Moody group is a semidirect product $\widehat{\mathcal{L}G} := \widetilde{\mathcal{L}G} \rtimes \widetilde{\mathbb{C}^*}$ of $\mathbb{C}^*$ with the centrally extended holomorphic loop group corresponding to a simple algebraic group G which might be non connected. Here the loop group is the *group of open loops* introduced in [TL], *i.e.,* the group of paths from $\mathbb{C}$ to the universal cover $\tilde{G}$ of G whose endpoints differ by an element of the group of covering transformations. If we restrict ourselves to a cyclic subgroup Σ of the component group and consider only central extensions of this subgroup, then its central extensions are classified by a multiple of a certain fundamental level k_f (Theorem 3.2). Furthermore the translation action of $\mathbb{C}$ on the group of open loops descends to a $\mathbb{C}^*$-action on the central extension. Now, the adjoint quotient map χ is defined using the characters $\chi_{\Lambda_0}, \dots, \chi_{\Lambda_{s-1}}, \chi_\delta$ of the generators $\Lambda_0, \dots, \Lambda_{s-1}, \delta$ of Σ-fixed point weight lattice:

$$\widehat{\mathcal{L}G}_{<1,\tau} \ni g \mapsto (\chi_{\Lambda_0}(g), \dots, \chi_{\Lambda_{s-1}}(g), \chi_\delta(g)).$$

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Here, $\widehat{\mathcal{L}G}_{<1,\tau}$ is the intersection of the connected component of the Kac–Moody group corresponding to a fixed generator τ of Σ with the submonoid of all elements having a $\widetilde{\mathbb{C}^*}$ component smaller than 1.

We construct a local section $S\colon \mathbb{C}^s \times D^* \to \widehat{\mathcal{L}G}_{<1,\tau}$ to the adjoint quotient map whose image consists of a "twisted Coxeter cell" $U^{\tau}_{\mathrm{cox}} \operatorname{cox}^{\tau} \times D^* \subset \widehat{\mathcal{L}G}_{<1,\tau}$. Our main result (Theorem 4.2) states that the restriction $S|_{\mathbb{C}^s \times \{\tilde{q}\}}$ is a section to $\chi|_{\widetilde{\mathcal{L}G} \times \{\tilde{q}\}}$ for $|\tilde{q}|$ small. One of the key results for proving the theorem is the fact that the image of $S_{\tilde{q}} := S|_{\mathbb{C}^s \times \{\tilde{q}\}}$ can be endowed with a $\mathbb{C}^*$-action (Lemma 4.1) which has no analogue in the finite dimensional situation, however see [Br] for connected affine Kac–Moody groups.

Let us briefly explain how the section is used to give a new proof that the moduli space of semistable bundles over the elliptic curve $E_{\tilde{q}} := \mathbb{C}^*/\tilde{q}^{\mathbb{Z}}$ with $|\tilde{q}| < 1$ in each component is a weighted projective space (which was proven by different methods in [Lo, FM, Sc]): Assume G to be connected with cyclic fundamental group which is generated by τ. Set $\operatorname{im} S^*_{\tilde{q}} := \operatorname{im} S_{\tilde{q}} \backslash \{(\operatorname{cox}^{\tau}, _{\tilde{q}})\}$. Then there is a principal G-bundle on $E_{\tilde{q}} \times \operatorname{im} S^*_{\tilde{q}}$ with $\operatorname{im} S^*_{\tilde{q}}$, such that over every point of $\operatorname{im} S^*_{\tilde{q}}$ the bundle is semistable over $E_{\tilde{q}}$ and their isomorphism class is constant on $\mathbb{C}^*$-orbits. It can be shown that this amounts to an algebraic isomorphism from $\operatorname{im} S^*_{\tilde{q}}/\mathbb{C}^*$ to the moduli space of principal G-bundles over $E_{\tilde{q}}$ whose topological type corresponds to τ. The details will be published elsewhere [MW].

The paper is organised as follows: in Section 2 we fix our notation, introduce the notion of twisted Coxeter elements and investigate its basic properties. Section 3 deals with the classification of central extension for non-connected loop groups and their representation theory. The definition of the adjoint quotient map and construction recipe for the section will be contained in Section 4. In the Appendix we compile tables containing several data appearing in this paper.

2 Symmetries of Affine Dynkin Diagrams

2.1 Notations and Basic Definitions

Throughout the article we use the following symbols: Let Π be a Dynkin diagram of finite type of rank r with Cartan matrix C, Weyl group $\mathcal{W}$, root system Δ and its dual $\check{\Delta}$. The set of simple roots will also be denoted by $\Pi := \{\alpha_1, \dots, \alpha_r\}$ and the corresponding simple reflections by s_i, $i \in \{1, \dots, r\}$. We write $Q = \mathbb{Z}\Delta$ for the root lattice, $\check{Q}$ for the co-root lattice, $P := \operatorname{Hom}(\check{Q}, \mathbb{Z})$ for the weight lattice and $\check{P}$ for the co-weight lattice.

Now let us fix similar notations for the corresponding affine untwisted root systems, using the conventions of Kac's book [Ka]. Thus, we have the affine Dynkin diagram $\widehat{\Pi}$, the affine Cartan matrix $\widehat{C}$, which has rank $r+1$, the affine Weyl group $\widehat{\mathcal{W}} = \check{Q} \rtimes \mathcal{W}$, root system $\widehat{\Delta}$, the set of simple roots $\widehat{\Pi} = \{\alpha_0, \dots, \alpha_r\}$, the corresponding set of simple reflections by $\{s_0, \dots, s_r\}$, and the set of simple co-roots $\check{\widehat{\Pi}} := \{\check{\alpha}_0, \dots, \check{\alpha}_r\}$. The free complex vector space generated by $\check{\widehat{\Pi}}$ is denoted by $\mathfrak{h}'$, its dual by $\mathfrak{h}'^*$ and the minimal imaginary root by $\delta = a_0\alpha_0 + \cdots + a_r\alpha_r$. The natural numbers a_i appearing in this equality are called the Kac labels. Similarly, there are the

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dual Kac labels $\check{a}_i$. For any $\mathcal{W}$-stable lattice with $\check{Q} \subset \check{\chi} \subset \check{P}$ (*i.e.*, $\check{\chi}$ is the co-character lattice of a simple algebraic group with Dynkin diagram Π), we set $\widehat{\mathcal{W}}(\check{\chi}) := \check{\chi} \rtimes \mathcal{W}$. Furthermore, we introduce the vector space $\mathfrak{h} := \mathfrak{h}' \oplus \mathbb{C}d$, its dual $\mathfrak{h}^*$ and extend the roots to linear forms on $\mathfrak{h}$ via:

$$\alpha_i(d) := 0, \quad i \in \{1, \dots, r\} \tag{1}$$

$$\delta(d) := 1. \tag{2}$$

The $\widehat{\mathcal{W}}$-action extends to $\mathfrak{h}$ by imposing $s_i(d) := d - \alpha_i(d)\check{\alpha}_i$. Then we define the affine co-root lattice by $\check{\widehat{Q}} := \mathbb{Z}\langle \check{\alpha}_0, \dots, \check{\alpha}_r, d\rangle$, the weight lattice $\widehat{P} := \mathrm{Hom}(\check{\widehat{Q}}, \mathbb{Z})$ as its dual with dual basis $\{\lambda_0, \dots, \lambda_r, \delta\}$, and the cone of dominant weights $\widehat{P}^+ := \mathbb{N}_0\langle \lambda_0, \dots, \lambda_r\rangle \oplus \mathbb{Z}\delta$.

Now let us give a description of the automorphism group $\mathrm{Aut}(\widehat{\Pi})$, *i.e.*, the group of all those permutations τ of the index set $\{0, \dots, r\}$ satisfying $\widehat{C}_{\tau(i)\tau(j)} = \widehat{C}_{ij}$ for all $i, j \in \{0, \dots, r\}$. The following lemma can easily be derived from results in [Bo, VI, §2] or [Hu, §4.5]:

Lemma 2.1 *The automorphism group* $\mathrm{Aut}(\widehat{\Pi})$ *is a semidirect product:*

$$\mathrm{Aut}(\widehat{\Pi}) = \check{P}/\check{Q} \rtimes \mathrm{Aut}(\Pi) = \widehat{\mathcal{W}}(\check{P})_{\mathfrak{a}} \rtimes \mathrm{Aut}(\Pi). \tag{3}$$

Here $\mathrm{Aut}(\Pi)$ *is the group of symmetries of the Dynkin diagram of finite type* Π *and* $\mathfrak{a} := \{x \in \mathbb{R} \otimes \check{Q} \mid \alpha_i(x) \geq 0, i \in \{1, \dots, r\}$ *and* $\theta(x) \leq 1\}$ *the fundamental alcove.*

We can lift the $\mathrm{Aut}(\widehat{\Pi})$-action to $\mathfrak{h}$ by imposing $\alpha_{\tau(i)}(d) = \alpha_i(\tau(d))$ for any $\tau \in \mathrm{Aut}(\widehat{\Pi})$.

Note that all of the notions introduced above can also be defined for twisted affine root systems, twisted by an automorphism $\sigma \in \mathrm{Aut}(\Pi)$. If we want to stress the fact that we are dealing with the twisted case we shall write $\widehat{\Pi}(\sigma)$ for the twisted affine Dynkin diagram, $\widehat{C}(\sigma)$ for its Cartan matrix, and so on. For instance, as an analogue of Lemma 2.1 we have:

$$\mathrm{Aut}(\widehat{\Pi}(\sigma)) = P^\sigma/Q^\sigma = \widehat{\mathcal{W}}(P)(\sigma)_{\mathfrak{a}}, \tag{4}$$

the superscript σ denoting the σ-invariants.

In the sequel $\widehat{C}$ denotes any affine Cartan matrix, either twisted or untwisted.

An important tool for the understanding of the representation theory of the non-connected affine Kac–Moody groups is the folding of Dynkin diagrams. This was considered first by Jantzen [Ja] in the finite dimensional case and by Fuchs, Schweigert and Schellekens [FSS] for affine Kac–Moody algebras.

Let us summarise the construction recipe here. To a pair $(\widehat{C}, \tau)$ consisting of an affine (twisted or untwisted) Cartan matrix $\widehat{C}$ and a symmetry $\tau \in \mathrm{Aut}(\widehat{\Pi})$ thereof we associate another Cartan matrix ${}^{\tau}\widehat{C}$ according to the following rule: denote by Σ the cyclic group of $\mathrm{Aut}(\widehat{\Pi})$ generated by τ and assume τ to be of order N.

Using I for the index set $\{0, \dots, r\}$ set $t_i := 3 - \frac{1}{|\Sigma_i|}\sum_{k=0}^{N-1} \widehat{C}_{i\tau^k(i)}$. For all cases except A_2^1, t_i is the number of all elements in the Σ-orbit of i adjacent to i (including i itself). Taking a look at the Cartan matrices of affine type we see $t_i \leq 2$ except for the case A_n^1 with τ being the cyclic permutation of order $n+1$ where we have $t_i = 3$. Now we define the matrix $\widetilde{C}$ by taking for its columns the sum over all Σ-orbits of the columns of $\widehat{C}$ and multiplying with $t_i|\Sigma_i|$. (Here Σ_i is the stabiliser of $i \in I$ in Σ.)

$$\widetilde{C}_{ij} := \frac{t_j}{|\Sigma_j|}\sum_{k=0}^{N-1} \widehat{C}_{i\tau^k(j)}. \tag{5}$$

Definition 2.2 The square matrix obtained from $\widetilde{C}$ by removing redundant columns and rows is called the *folded Cartan matrix* and denoted by ${}^\tau\widehat{C}$. Its Dynkin diagram is called the *folded Dynkin diagram* ${}^\tau\widehat{\Pi}$.

The folded Dynkin diagrams associated to all pairs $(\widehat{C}, \tau)$ can be found in the Appendix, Table 1.

Here and for the rest of the paper we use the following symbols for the specific symmetries of the Dynkin diagrams (the vertices of the Dynkin diagrams are labelled as in [Br]): Except for the D^4-case the automorphism group of the Dynkin diagrams of finite type is trivial or $\mathbb{Z}/2\mathbb{Z}$. In case of existence the corresponding non-trivial symmetry lifts uniquely to an element $\sigma \in \mathrm{Aut}(\widehat{\Pi})$. In the remaining case D_4^1 we use σ for the order 2 symmetry interchanging the vertices with label 3 and 4 while fixing the remaining ones and ρ for the element of order 3 permuting the labels $1, 3, 4$ cyclically and fixing the other two. We denote by γ the only nontrivial automorphism in the cases of A_1^1, B_n^1, C_n^1, E_7^1, A_{2n+1}^2, D_n^2. In the A_n^1, $n \geq 2$-case, the symmetry γ raises the index by $+1$ and in the E_6^1-case γ is given by $\gamma(0) = 1$, $\gamma(1) = 5$, $\gamma(2) = 4$, $\gamma(3) = 3$, $\gamma(4) = 6$, $\gamma(5) = 1$ and $\gamma(6) = 0$. In the D_n^1-case γ is the element of order 4 acting by $\gamma(0) = n$, $\gamma(1) = n-1$, $\gamma(n-1) = 0$ and $\gamma(n) = 1$.

2.2 Twisted Coxeter Elements

In this subsection we generalise the notion of a twisted Coxeter element which was introduced by Springer [Sp] for finite Coxeter groups.

The twisted Coxeter element plays a key role in our investigation.

Fix any element $\tau \in \mathrm{Aut}(\widehat{\Pi})$ acting with s orbits on the Dynkin diagram. After relabelling the index set we can assume that the set $\{0, \dots, s-1\}$ is a set of representatives of τ-orbits on I.

Definition 2.3 The element $\mathrm{cox}^\tau := s_0 \cdots s_{s-1}\tau \in \widehat{W} \rtimes \mathrm{Aut}(\widehat{\Pi})$ is called a twisted Coxeter element of $\widehat{W} \rtimes \mathrm{Aut}(\Pi)$ corresponding to τ.

Remark

(i) Observe that we get back the usual definition of a Coxeter element for $\tau = e$.

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(ii) Twisted Coxeter elements can be defined for arbitrary, not necessarily affine, generalised Cartan matrices. Lemma 2.4, Proposition 2.5 and Corollary 2.6 still hold in this more general context.

Applying [Sp, Lemma 7.5] (see also [Bo, Lemma 1, p. 117]) to our situation yields:

Lemma 2.4 *If the Dynkin diagram $\widehat{\Pi}$ contains no cycles, the twisted Coxeter element* cox^τ *is unique up to conjugation with elements in $\widehat{\mathcal{W}}$.*

Remark In case of the Dynkin diagram A_r^1 and τ, the cyclic permutation of order $r+1$, it is easily verified that the corresponding twisted Coxeter element is also unique up to conjugation in $\widehat{\mathcal{W}}$, *e.g.*, $s_1\tau = s_1(s_0\tau)s_1$. But for $\sigma = \tau^l$ with $l|r+1$, we can indicate a counterexample:

Consider the case (A_5^1, τ^2) with $\tau \in \mathrm{Aut}(\widehat{\Pi})$, $\tau(i) = i + 1 \mod 5$. As twisted Coxeter elements we can choose $(s_0\tau)^2 = s_0 s_1 \tau^2$ or $s_0 s_3 \tau^2$. Since the characteristic polynomials for their action on $\mathfrak{h}'$ are $(t-1)(t^5-1)$, resp., $(t^2-1)(t^4-1)$, they cannot be conjugate.

We investigate the action of cox^τ on the vector spaces $\mathfrak{h}$ and $\mathfrak{h}'$. In particular we are interested in the multiplicity of eigenvalue 1. Our first result is a generalisation of [Co, Theorem 3.1]:

Proposition 2.5 *Let $\tau \in \mathrm{Aut}(\widehat{\Pi})$ be a diagram automorphism and $s_{i_1}, \dots, s_{i_p}$ simple reflections with the i_j lying in distinct Σ-orbits on I. Then the following is true for any $x \in \mathfrak{h}$ or $x \in \mathfrak{h}'$:*

$$s_{i_1} \cdots s_{i_p} \tau(x) = x \iff s_{i_j}(x) = x \text{ and } \tau(x) = x. \tag{6}$$

Proof Clearly, the right-hand side implies the left one. For the other direction, note that we have the following equivalence:

$$s_{i_1} \cdots s_{i_p} \tau(x) = x \iff \tau(x) - x = s_{i_p} \cdots s_{i_1}(x) - x = \sum_{j=1}^{p} c_j \check{\alpha}_{i_j}, \tag{7}$$

for certain $c_j \in \mathbb{Z}$. Taking the sum over the Σ-orbit of the terms on the left and right of this equation we obtain:

$$\sum_{j=1}^{p} d_j \sum (\Sigma\text{-orbit of } \check{\alpha}_{i_j}) = 0, \tag{8}$$

for $d_j = c_j |\Sigma_{\check{\alpha}_{i_j}}|$. The condition on $i_1, \dots, i_p$ yields $c_j = 0$. This implies:

$$\mathbb{C}\check{\alpha}_{i_p} + x \ni s_{i_p}(x) = s_{i_{p-1}} \cdots s_{i_1}(x) \in x + \bigoplus_{j=1}^{p-1} \mathbb{C}\check{\alpha}_{i_j}. \tag{9}$$

Since the affine spaces on the right and left intersect only in x, we get $s_{i_p}(x) = x$. Now the statement follows by induction. ■

Note that the restrictions $\alpha_0|_{\mathfrak{h}^\tau}, \dots, \alpha_{s-1}|_{\mathfrak{h}^\tau}$ are linearly independent while the restrictions $\alpha_0|_{\mathfrak{h}'^\tau}, \dots, \alpha_{s-1}|_{\mathfrak{h}'^\tau}$ satisfy the one equation (the a_i being the Kac labels):

$$a_0|\Sigma_{\alpha_0}|\alpha_0|_{\mathfrak{h}'^\tau} + \cdots + a_{s-1}|\Sigma_{\alpha_{s-1}}|\alpha_{s-1}|_{\mathfrak{h}'^\tau} = 0. \tag{10}$$

Therefore, the proposition implies:

Corollary 2.6 *Denote by $\mathfrak{h}^\tau$ the fixed point subspace of $\mathfrak{h}$. Then the following result holds:*

$$\{x \in \mathfrak{h}, \operatorname{cox}^\tau(x) = x\} = \mathfrak{h}^\tau \cap \bigcap_{i=0}^{s} (\ker \alpha_i|_{\mathfrak{h}^\tau}) = \mathbb{C}c = \mathbb{C}\sum_{i=0}^{r} \check{a}_i \check{\alpha}_i. \tag{11}$$

Similarly, for $\mathfrak{h}'$ we have:

$$\{x \in \mathfrak{h}', \operatorname{cox}^\tau(x) = x\} = \mathbb{C}c. \tag{12}$$

Our next aim is to calculate the multiplicity of (t-1) as factor of the characteristic polynomial of cox^τ. Since the element δ is fixed by the s_i and by τ we have:

$$\tau(d) = d + h, \text{with } h \in \mathfrak{h}' \tag{13}$$

$$s_i(d) = d + h_i, \text{with } h_i \in \mathfrak{h}'. \tag{14}$$

Hence, cox^τ has upper triangular shape with respect to the direct sum decomposition $\mathfrak{h} = \mathbb{C}d \bigoplus \mathfrak{h}'$. Thus, 1 is a zero of the characteristic polynomial of cox^τ on $\mathfrak{h}$ with multiplicity at least 2. In fact, even more is true:

Proposition 2.7 *The multiplicity of* 1 *as zero of the characteristic polynomial of* cox^τ *on $\mathfrak{h}'$ is two and hence, is three on $\mathfrak{h}$.*

For $\tau = \mathrm{id}$ this statement follows from the list of the characteristic polynomials which were computed by Coleman [Co, Table 3]. For non-trivial τ the characteristic polynomials of the twisted Coxeter elements have been calculated by the author and are compiled in Table 1 of the Appendix. Here, we have made the following choices for $\operatorname{cox}^\tau = s_{i_1} \cdots s_{i_p}\tau$ in the $\mathrm{A}_n^1, n \geq 2$-case: $\operatorname{cox}^\tau = s_{i_1} \cdots s_{i_p}\tau$, where the i_j have smallest index among all indices in its orbits and they are in increasing order $i_1 < i_2 < \cdots < i_p$.

For the construction of the $\mathbb{C}^*$-action on the Steinberg cross section in Section 4 we need a solution $b \in \mathfrak{h}'$ to the equation $(\operatorname{cox}^\tau - 1)(b) = c = \sum_{i=0}^{r} \check{a}_i \check{\alpha}_i$. For deriving an explicit solution we make the following choices for the twisted Coxeter elements. In the A_n^1 case we use cox^τ as above. For the cases B_n^1, A_{2n-1}^2 we take $\operatorname{cox}^\tau = s_1 \cdots s_n\tau$. In the case of E_6^1 and the automorphism $\tau = \gamma$, we choose $\operatorname{cox}^\tau = s_1 s_2 s_3 \tau$. In the D_n^1 we make the following choices: if 0 and 1 lie in the same orbit of the exterior

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automorphism we take representatives of Σ-orbit of minimal possible index in $I\backslash\{0\}$ and multiply them in the order of increasing index. Finally, we multiply the result with the automorphism τ from the right. In all the other cases, *i.e.*, C_n^1, D_{n+1}^2, E_7^1, E_6^1 and σ and all the other automorphisms of D_n^1 we choose representatives of the Σ orbits of least possible index which we multiply in increasing order and finally we multiply the result with the automorphism τ from the right. A calculation yields the following generalisation of [Br, Proposition 10] for $\tau \neq \mathrm{id}$:

Proposition 2.8 *Let $\widehat{\Pi}$ be an affine Dynkin diagram, $\widehat{W}$ its Weyl group, $\tau \in \mathrm{Aut}(\widehat{\Pi})$ an exterior automorphism, and cox^τ the twisted Coxeter element. Then there are an element $b \in \mathbb{N}\langle\check{\widehat{\Pi}}\rangle$ and a number $k \in \mathbb{N}$ such that $(\mathrm{cox}^\tau - 1)(b) = kc$, which are uniquely determined by requiring them to be minimal and $b - c \notin \mathbb{N}\langle\check{\widehat{\Pi}}\rangle$. The number k as well as the element b expressed in the basis $\check{\widehat{\Pi}}$ are listed in the Table 2 in the Appendix for $\tau \neq \mathrm{id}$.*

Now consider those positive roots $\beta_i = s_0 \cdots s_{i-1}\alpha_i$, $i \in \{0, \dots, s-1\}$ that are mapped to negative ones by $\mathrm{cox}^{\tau\,-1} = \tau^{-1}s_{s-1}\cdots s_0$.

Lemma 2.9 *The s-tuple $(\beta_0(b), \dots, \beta_{s-1}(b))$ coincides up to a factor p (listed in Table 2 of the Appendix) with the s-tuple of the dual Kac labels of the folded Dynkin diagram ${}^\tau\widehat{\Pi}$. (In the cases where there is no folded Dynkin diagram,* i.e., *the cases A_n^1, τ with $\tau = \gamma$, we set $\check{a}_0^\tau = 1$.)*

Proof For any $\lambda \in \mathfrak{h}^*$ and reduced expression $w = s_{i_1}\cdots s_{i_t} \in \widehat{W}$ the following formula can be proven by induction:

$$\lambda - w(\lambda) = \sum_{j=1}^{t} \lambda(\check{\alpha}_{i_j})\, s_{i_1}\cdots s_{i_{j-1}}(\alpha_{i_j}). \tag{15}$$

Now set $\tilde{\alpha}_i := \frac{1}{|\Sigma_i|}\sum_{l=0}^{N-1}\tau^l(\alpha_i)$. Applying the formula above to the case $w = s_0\cdots s_{s-1} \in \widehat{W}$ and $\lambda = \tau^l(\alpha_i)$ yields:

$$\tilde{\alpha}_i - \mathrm{cox}^\tau(\tilde{\alpha}_i) = \tilde{\alpha}_i - s_0\cdots s_{s-1}(\tilde{\alpha}_i) = \frac{1}{|\Sigma_i|}\sum_{l=0}^{N-1}\sum_{j=0}^{s-1}\widehat{C}_{j\tau^l(i)}\beta_j. \tag{16}$$

Evaluating the result on the element b yields:

$$\begin{aligned}\frac{1}{|\Sigma_i|}\sum_{l=0}^{N-1}\sum_{j=0}^{s-1}\widehat{C}_{j\tau^l(i)}\beta_j(b) &= \tilde{\alpha}_i(b) - \mathrm{cox}^\tau(\tilde{\alpha}_i)(b)\\ &= \tilde{\alpha}_i(b - \mathrm{cox}^{\tau\,-1}(b)) = \tilde{\alpha}_i(kc) = 0.\end{aligned} \tag{17}$$

Thus the s-tuple $(\beta_0(b), \dots, \beta_{s-1}(b))$ is mapped to zero by the matrix

$$\mathrm{diag}(t_0^{-1}, \dots, t_{s-1}^{-1})\,{}^\tau\widehat{C}.$$

Recall that the t_i are the numbers used in (5). This clearly implies the result. ∎

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3 Affine Kac–Moody Groups

3.1 Affine Kac–Moody Groups

Let us briefly describe the construction of affine Kac–Moody groups for non-connected and non-simply-connected underlying Lie group G. Dealing with underlying groups in this generality we obtain new results. The line of reasoning presented is an adaption of the one suggested by Toledano Laredo [TL], where the connected but non-simply-connected case is dealt with.

Assume G to be an algebraic group over $\mathbb{C}$ with simple identity component G_0 and Lie algebra $\mathfrak{g}$. If G is not connected itself we assume that G is a semidirect product $G_0 \rtimes \Gamma$ with $\Gamma < \mathrm{Aut}(\Pi)$ being a subgroup of the automorphism group of the Dynkin diagram Π of G_0. Denote by $\tilde{G}$ the universal cover of G_0 and by $Z \cong \pi_1(G_0)$ the kernel of the covering map.

Instead of working with the loop group itself we use the group of "open loops" with "boundary values" in Z:

$$\mathcal{L}_{Z,\Gamma}\tilde{G} : \{X\colon \mathbb{C} \to \tilde{G} \rtimes \Gamma, X \text{ holomorphic}, X(t)X(t+1)^{-1} \in Z\}. \tag{18}$$

Endowed with the compact-open topology it becomes a topological group. Note that its identity component is the usual loop group associated to the universal cover $\tilde{G}$ of G_0 and that its component group is given by $\pi_1(G_0) \rtimes \Gamma$. Its Lie algebra is the loop algebra $\mathcal{L}\mathfrak{g}$.

Furthermore, there is an action of $\mathbb{C}$ on $\mathcal{L}_{Z,\Gamma}\tilde{G}$ by translations:

$$(s.X)(t) := X(t-s), \quad \forall s, t \in \mathbb{C} \text{ and } X \in \mathcal{L}_{Z,\Gamma}\tilde{G}. \tag{19}$$

For the construction of central extensions of this group it turns out to be convenient to consider elements of the co-character lattice $\check{\chi}(T) := \mathrm{Hom}_{\mathrm{alg\,gr}}(\mathbb{C}^*, T) \subset \mathrm{Hom}_{\mathrm{alg\,gr}}(\mathbb{C}^*, T) \otimes \mathbb{R}$ as open loops. To $\check{\beta} \in \check{\chi}(T)$ we associate the open loop, also denoted by $\check{\beta}$:

$$\check{\beta}(t) := \exp(2\pi i t\check{\beta}). \tag{20}$$

By this identification, we obtain $\check{Q} = \check{\chi}(\tilde{T}) \subset \mathcal{L}\tilde{G}$. This implies:

$$\mathcal{L}_{Z,\Gamma}\tilde{G} \cong (\mathcal{L}\tilde{G} \rtimes \check{\chi}(T))/\check{\chi}(\tilde{T}) \rtimes \Gamma \cong \mathcal{L}\tilde{G} \rtimes (\check{\chi}(T) \rtimes \Gamma)/\check{\chi}(\tilde{T}). \tag{21}$$

Since we are only interested in a single exterior component of $\mathcal{L}_{Z,\Gamma}\tilde{G}$ we restrict ourselves to a cyclic subgroup $\Sigma \subset \pi_0(\mathcal{L}_{Z,\Gamma}\tilde{G}) = Z \rtimes \Gamma$ generated by $\tau = \rho\sigma$ with $\rho \in Z$ and $\sigma \in \Gamma$. Denote by $\mathcal{L}_\Sigma\tilde{G}$ the subgroup of $\mathcal{L}_{Z,\Gamma}\tilde{G}$ having component group Σ.

There is a fundamental co-root $\check{\lambda}_\rho$ (respectively, $\check{\lambda}_\rho = 0$, for $\rho = \mathrm{id}$) with $\check{\lambda}_\rho + \check{\chi}(\tilde{T}) = \rho$. Considering $\check{\lambda}_\rho$ as an open loop the element $\check{\lambda}_\rho\sigma\mathcal{L}\tilde{G}$ generates the component group of $\mathcal{L}_\Sigma\tilde{G}$. Indeed, if p is the order of the diagram automorphism $\rho\sigma$ we see that $(\check{\lambda}_\rho\sigma)^p \in \chi(\tilde{T})$. Furthermore, we have $1 = (\rho\sigma)^p = \rho\sigma(\rho)\cdots\sigma^{p-1}(\rho)\sigma^p$ which implies $\sigma^p = 1$. Thus, $(\check{\lambda}_\rho\sigma)^p = \check{\lambda}_\rho + \sigma(\check{\lambda}_\rho) + \cdots + \sigma^{p-1}(\check{\lambda}_\rho)$ has even to be σ-invariant. Hence, we get an identification $\mathcal{L}_\Sigma\tilde{G} = (\mathcal{L}\tilde{G} \rtimes \tilde{\Sigma})/\tilde{\Sigma}^p$, where $\tilde{\Sigma} :=< \check{\lambda}_\rho\sigma >$. Note, that the group $\tilde{\Sigma}$ is $\mathbb{Z}$ unless $\rho = \mathrm{id}$ in which case it is finite.

For the translation action we calculate:

$$s.\check{\lambda}_\rho\sigma = \exp(-2\pi i s\check{\lambda}_\rho)\,\check{\lambda}_\rho\sigma. \tag{22}$$

The Kac–Moody group will be a central extension of the following semidirect product:

$$\mathcal{L}_\Sigma\tilde{G} \rtimes \mathbb{C} = ((\mathcal{L}\tilde{G} \rtimes \mathbb{C}) \rtimes \tilde{\Sigma})/\tilde{\Sigma}^p. \tag{23}$$

Using [TL, Lemma 3.1.1], we see that any central extension of $\mathcal{L}\tilde{G} \rtimes \mathbb{C}$ is uniquely determined by its restriction $\widetilde{\mathcal{L}\tilde{G}_0}$ to $\mathcal{L}\tilde{G}$, up to a character of $\mathbb{C}$ which we fix to be the identity, and is a semidirect product

$$\widetilde{\mathcal{L}\tilde{G} \rtimes \mathbb{C}} = \widetilde{\mathcal{L}\tilde{G}} \rtimes \mathbb{C}.$$

For lifting the action of $\check{\lambda}_\rho\sigma$ to this centrally extended group we consider the action on its Lie algebra

$$\mathrm{Lie}\,\widetilde{\mathcal{L}\tilde{G} \rtimes \mathbb{C}}^{k} = \widetilde{\mathcal{L}\mathfrak{g}}^{k} \oplus \mathbb{C}d,$$

which has the following form (here $d = \frac{1}{2\pi i}\frac{d}{dt}$ is the derivation and k the level of the central extension):

$$\begin{aligned}\widetilde{\mathrm{Ad}(\check{\lambda}_\rho\sigma)}(x + bd + ac) = \mathrm{Ad}\,\check{\lambda}_\rho(x) &+ \frac{b}{2\pi i}\check{\lambda}_\rho\frac{d}{dt}\check{\lambda}_\rho^{-1} + bd \\ &+ \Big(a + \frac{k}{2\pi i}\int_{t=0}^{1}\big\langle \check{\lambda}_\rho^{-1}\frac{d}{-}dt\check{\lambda}_\rho,\, \sigma(x)\big\rangle\, dt \\ &\quad + b\frac{k}{8\pi^2}\int_{t=0}^{1}\big\langle \check{\lambda}_\rho^{-1}\frac{d}{dt}\check{\lambda}_\rho\,\check{\lambda}_\rho^{-1}\frac{d}{dt}\check{\lambda}_\rho\big\rangle\, dt\Big)\,c.\end{aligned} \tag{24}$$

This allows us to consider the semidirect product $\widetilde{\mathcal{L}\tilde{G} \rtimes \mathbb{C}}^{k} \rtimes \widetilde{\Sigma}$ for the central extension of $\mathcal{L}\tilde{G}$ of level k.

Set $\check{\beta} := (\check{\lambda}_\rho\sigma)^p$, a generator of $\widetilde{\Sigma}^p$. A calculation similar to the proof of [TL, Proposition 3.2.2], yields:

$$\widetilde{\mathrm{Ad}(\check{\lambda}_\rho\sigma)}(\tilde{\check{\beta}}) = (-1)^{k\langle\check{\lambda}_\rho,\check{\beta}\rangle}. \tag{25}$$

Now the proof of [TL, Proposition 3.3.1] implies that $N := \{(\check{\beta}^n, 1, \check{\beta}^{-n}),\ n \in \mathbb{Z}\} \subset (\widetilde{\mathcal{L}\tilde{G}}^{k} \rtimes \mathbb{C}) \rtimes \widetilde{\Sigma}$ is a normal subgroup if and only if this expression is equal to 1. This motivates the following definition:

Definition 3.1 The smallest positive integer k such that $k\langle\check{\lambda}_\rho, \check{\beta}\rangle \in 2\mathbb{Z}$ is called the fundamental level of $\mathcal{L}\tilde{G}_0 \rtimes \Sigma$ and will be denoted by k_f.

Remark the fundamental levels are compiled in Table 1 of the Appendix.

As a summary we get:

Theorem 3.2 *Every central extension of $\mathcal{L}_\Sigma\tilde{G}\rtimes\mathbb{C}$ is uniquely determined by the level k of its restriction to $\mathcal{L}\tilde{G}$ which has to be a multiple of k_f and a character of $\mathbb{C}$.*

Definition 3.3 The quotient group $\widehat{\mathcal{L}_\Sigma\tilde{G}}^k := ((\widetilde{\mathcal{L}\tilde{G}}^k \rtimes\mathbb{C})\rtimes\widetilde{\Sigma})/N$ is called the affine Kac–Moody group corresponding to G and Σ.

Remark Let q be the smallest positive integer such that $q\check{\lambda}_\rho \in \check{Q}$. It follows from (22) that $\check{\lambda}_\rho\sigma$ fixes the translation by q. Thus we effectively obtain an action of $\widetilde{\mathbb{C}^*} := \mathbb{C}/q\mathbb{Z}$.

The case of twisted Kac–Moody groups can be treated along the same line of reasoning. We restrict ourselves to defining the twisted "group of open loops" with "boundary values" in Z^σ having $|\pi_1(G)^\sigma|$ many connected components:

$$\mathcal{L}_Z\tilde{G}(\sigma) := \{X \in \mathcal{L}\tilde{G}, \sigma(X(e^{2\pi i/s}z))X(z)^{-1} \in Z^\sigma\}. \tag{26}$$

The corresponding Kac–Moody group is denoted by $\widehat{\mathcal{L}_{Z^\sigma}\tilde{G}}^k(\sigma)$.

3.2 Representation Theory

Here we shall give a brief account on the representation theory of the Kac–Moody groups $\widehat{\mathcal{L}_\Sigma\tilde{G}}^k$, respectively, $\widehat{\mathcal{L}_{Z^\sigma}\tilde{G}}^k(\sigma)$. For the simplicity of the exposition we will formulate the results only for the non-twisted case $\widehat{\mathcal{L}_\Sigma\tilde{G}}^k$.

The representation theory has already been investigated, see [FSS, We2, TL]; see also [Ja] for the finite dimensional situation.

Denote by $V(\lambda)^{an}$ the irreducible highest weight module of the identity component $\widehat{\mathcal{L}_{Z^\sigma}\tilde{G}}^k(\sigma)_0$ with highest weight $\lambda \in \widehat{P}^+$ and by $V(\lambda)^{ss}$ its Hilbert space completion with respect to the hermitian form introduced by Garland [Ga]. (Note, that we require $\lambda(c)$ to be a multiple of k.)

The representation theory for the non-connected Kac–Moody groups $\widehat{\mathcal{L}_\Sigma\tilde{G}}^{k_f}$ can be obtained by a kind of Mackey induction with respect to the component group $\Sigma \cong \widehat{\mathcal{L}_\Sigma\tilde{G}}^{k_f}/\widehat{\mathcal{L}\tilde{G}}^{k_f}$. We denote by $\widehat{P}^{+k}$ the set of dominant weights of level k. the following theorem was proven by [We2, Theorem 2.8] and [TL, Theorem 6.1]. (Note that [TL] considers only projective representations.)

Theorem 3.4 *Let k be multiple of k_f. To each Σ-orbit $I \subset \widehat{P}^+$ there are $|\Sigma|/|I|$ many non-isomorphic irreducible representations of $\widehat{\mathcal{L}_\Sigma\tilde{G}}^{k_f}$ whose restrictions to $\widehat{\mathcal{L}\tilde{G}}^{k_f}$ decompose into a direct sum:*

$$V(I)^{an} = \bigoplus_{\lambda\in I\subset\widehat{P}^+} V(\lambda)^{an}. \tag{27}$$

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We shall give some details of the induction procedure. Recall that Σ is generated by the element $\tau = \rho\sigma$ with $\rho \in Z$ and $\sigma \in \Gamma$. Using Lemma 2.1 we can uniquely associate an element $\check{\beta}_\rho w_\rho \in \widehat{W}(\check{P})_{\mathfrak{a}}$ representing the element $\rho \in \mathrm{Aut}(\widehat{\Pi})$. Interpreting $\check{\beta}_\rho$ as open loop $t \mapsto \exp(2\pi i t\check{\beta}_\rho)$ and choosing a representative $n'_{w_\rho} \in N_{\mathcal{L}\tilde{G}}(T)$ of minimal possible order, here $T \subset \tilde{G}$ is a maximal torus of $\mathcal{L}\tilde{G}$, we define n_τ as a lift of $\check{\beta}_\rho n'_{w_\rho}\sigma$ in $\widetilde{\mathcal{L}_\Sigma\tilde{G}}^{k_f}$ preserving its order.

Now consider the irreducible representation $V(I)^{an}$ where $n_\tau^{|I|}$ acts as the identity on $V(I)^{an}_\lambda$ for every highest weight vector $\lambda \in \widehat{P}^{+k}$. This condition determines the module $V(I)^{an}$ uniquely. It is easy to see that we obtain a unitary action of n_τ on $V(\lambda)$. Thus, there is a $\widehat{\mathcal{L}_\Sigma\tilde{G}}^{k_f} = \widetilde{\mathcal{L}_\Sigma\tilde{G}}^{k_f} \rtimes \widetilde{\mathbb{C}^*}$-action by requiring d to act trivially on the highest weight spaces $V(I)^{an}$. (Recall that we have a q-fold cover $\widetilde{\mathbb{C}^*} \to \mathbb{C}^*$ which acts on $\widetilde{\mathcal{L}\tilde{G}}^{k_f}$.) Denote the elements of $\widetilde{\mathbb{C}^*}$ by $\tilde{q}$.

We define the sets

$$\widehat{\mathcal{L}_\Sigma\tilde{G}}^{k_f}_{\tilde{q}} := \widetilde{\mathcal{L}_\Sigma\tilde{G}}^{k_f} \times \{\tilde{q}\} \quad \text{and} \quad \widehat{\mathcal{L}_\Sigma\tilde{G}}^{k_f}_{<1} := \bigcup_{\tilde{q},\, |\tilde{q}|<1} \widehat{\mathcal{L}_\Sigma\tilde{G}}^{k_f}_{\tilde{q}}.$$

Then we have the following result, see also [We2, Corollary 2.9]:

Corollary 3.5 *Let k be a multiple of k_f and $I \subset \widehat{P}^{+k}$ a Σ-orbit as above. For any $\tilde{q} \in \tilde{\mathbb{C}}^*$ with $|\tilde{q}| < 1$ and any $g \in \widetilde{\mathcal{L}_\Sigma\tilde{G}}^{k_f}$ the operator $g\tilde{q}^{-d}\colon V(I)^{an} \to V(I)^{an}$ uniquely extends to a trace class operator on $V(I)^{ss} := \bigoplus_{\lambda\in I} V(\lambda)^{ss}$.*

Let us define the character $\chi_I\colon \widehat{\mathcal{L}_\Sigma\tilde{G}}^{k_f}_{<1} \to \mathbb{C}$ for $(g,\tilde{q}) \in \widehat{\mathcal{L}_\Sigma\tilde{G}}^{k_f}_{<1}$ by:

$$\chi_I(g,\tilde{q}) = \mathrm{Tr}_{V(I)^{ss}}(g\tilde{q}^{-d}). \tag{28}$$

For $g \in \widetilde{\mathcal{L}\tilde{G}}^{k_f} = \widetilde{\mathcal{L}_\Sigma\tilde{G}}^{k_f}_0$ we clearly obtain $\chi_I(g,\tilde{q}) = \sum_{\lambda\in I}\chi_\lambda(g,\tilde{q})$. Since n_τ acts as unitary operator, the following result holds, see [We2, Corollary 2.10:].

Corollary 3.6 *The functions χ_I are holomorphic and invariant under conjugation.*

4 The Quotient Map and the Cross Section

This section contains our main result. For the readability of the exposition the reasoning will be carried out for the Kac–Moody groups $\widehat{\mathcal{L}_\Sigma\tilde{G}}^{k_f}$, being still valid for the twisted groups $\widehat{\mathcal{L}_{Z^\sigma}\tilde{G}}^{k_f}(\sigma)$. As before, we write $\tau = \rho\sigma$ for the generator of its component group Σ.

A set of dominant generators of the fixed point weight lattice $\widehat{P}^\Sigma$ is given by $\{\Lambda_0,\ldots,\Lambda_{s-1},\delta\}$ where we use $\Lambda_i := \frac{1}{|\Sigma_i|}\sum_{j=1}^{\mathrm{ord}\,\tau}\lambda_{\tau^j(i)}$.

Note that in general the Λ_i do not coincide with the sum over Σ-orbit of λ_i; a certain rational multiple of δ night have to be added.

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Taking a look at Table 1, one observes that the level of the irreducible $\widehat{\mathcal{L}\mathfrak{g}}^k$-module $V(\Lambda_i)^{an}$ is a multiple of the fundamental level k_f of $\mathcal{L}_\Sigma\tilde{G}$ and thus gives rise to an irreducible representation of $\widehat{\mathcal{L}_\Sigma\tilde{G}}^{k_f}$ on $V(\Lambda_i)^{an}$, by Theorem 3.4. Corollary 3.6 then implies that the characters of these representations are holomorphic and conjugacy invariant functions on $\widehat{\mathcal{L}_\Sigma\tilde{G}}^{k_f}_{<1}$.

Motivated by the finite dimensional situation, see [St, Moh], we define the "quotient map" (recall that D^* is the punctured unit disk):

$$\begin{aligned} \chi\colon \widehat{\mathcal{L}_\Sigma\tilde{G}}^{k_f}_{<1,\tau} &\to \mathbb{C}^s \times D^* \\ (g,\tilde{q}) &\mapsto (\chi_{\Lambda_0}(g,\tilde{q}),\dots,\chi_{\Lambda_{s-1}}(g,\tilde{q}),\chi_\delta(g,\tilde{q})=\tilde{q}). \end{aligned} \tag{29}$$

Its restriction of χ to the set $\widehat{\mathcal{L}_\Sigma\tilde{G}}^{k_f}_{\tilde{q},\tau}$ will be denoted by $\chi_{\tilde{q}}$.

We proceed by indicating the construction recipe for S. Let $\{\alpha_0,\dots,\alpha_{s-1}\}$ be a set of representatives of the s Σ-orbits on $\widehat{\Pi}$. For each index i consider the corresponding canonical embedding $\phi_i\colon SL_2 \to \widehat{\mathcal{L}_\Sigma G}^{k_f}$, the corresponding root group $X_{\alpha_i}\colon \mathbb{C} \to \widehat{\mathcal{L}_\Sigma\tilde{G}_0}^{k_f}$, *i.e.*, $\operatorname{im} X_{\alpha_i} = \phi_i((\begin{smallmatrix}1 & \mathbb{C}\\ 0 & 1\end{smallmatrix}))$, and $n_i := \phi_i((\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix})) \in N_{\widehat{\mathcal{L}_\Sigma\tilde{G}_0}^{k_f}}(\widehat{T})$ a representative of the simple reflection $s_i \in \widehat{\mathcal{W}}$. Recall the definition of $n_\tau \in N_{\widehat{\mathcal{L}_\Sigma\tilde{G}}^{k_f}}(\widehat{T})$ from the previous section. We introduce the section S:

$$S\colon \mathbb{C}^s \times \mathbb{C}^* \to \widehat{\mathcal{L}_\Sigma\tilde{G}}^{k_f}_{\tau,\tilde{q}} \tag{30}$$

$$(c_0,\dots,c_{s-1},\tilde{q}) \mapsto (X_0(c_0)n_0\cdots X_{s-1}(c_{s-1})n_{s-1}n_\tau,\ \tilde{q}). \tag{31}$$

Remark Note that $S(0,\dots,0,1) = n_0\cdots n_{s-1}n_\tau$ is a representative of the twisted Coxeter element.

Set $S_{\tilde{q}} := S|_{\mathbb{C}^s\times\{\tilde{q}\}}$.

The image $\operatorname{im} S_{\tilde{q}}$ of the section can be endowed with a $\mathbb{C}^*$-action: Recall the definition of the parametrisation of the centre $\iota\colon \mathbb{C}^* \to \widehat{\mathcal{L}_\Sigma\tilde{G}_0}^{k_f}$, the element $b \in \check{\widehat{Q}}$, and the number $k\,b \in \check{\widehat{Q}}$ appearing in Proposition 2.8. (The pedantic reader might object that the $\{\alpha_0,\dots,\alpha_{s-1}\}$ must be chosen as in the paragraph preceding Proposition 2.8.) We define the $\mathbb{C}^*$-action by:

$$\mathbb{C}^* \times \operatorname{im} S_{\tilde{q}} \to \operatorname{im} S_{\tilde{q}} \tag{32}$$

$$(u, S_{\tilde{q}}(c_0,\dots,c_{s-1})) \mapsto \iota(u)^k\mu_b(u)S_{\tilde{q}}(c_0,\dots,c_{s-1})\mu_b(u)^{-1}. \tag{33}$$

A simple calculation using Lemma 2.9 and Table 2 yields:

Lemma 4.1 *For the $\mathbb{C}^*$-action we obtain explicitly:*

$$\iota(u)^k\mu_b(u)S_{\tilde{q}}(c_0,\dots,c_{s-1})\mu_b(u)^{-1} = S_{\tilde{q}}(u^{k\Lambda_0(c)}c_0,\dots,u^{k\Lambda_{s-1}(c)}c_{s-1}). \tag{34}$$

Here comes our main result:

Theorem 4.2 *For small $|\tilde{q}|$ the map $S_{\tilde{q}}$ is a section to $\chi_{\tilde{q}}$,* i.e., *$\chi_{\tilde{q}} \circ S_{\tilde{q}} \colon \mathbb{C}^s \to \mathbb{C}^s$ is an isomorphism.*

The proof will proceed in the following way: First, we will show that the Jacobian of the map $\chi \circ S$ is a unit in the ring $\mathbb{C}\{\tilde{q}\}$, the ring of convergent power series. Then the isomorphism follows by exploitation of the $\mathbb{C}^*$-action on the section and the quotient space.

The first step is carried out using representation theory. Fix some dominant and Σ-invariant Λ. Denote by $V(\Lambda)^{an}_{\mu}$ the weight space of $V(\Lambda)^{an}$ of weight μ and by i_μ and p_μ the corresponding canonical inclusion and projection.

The following immediate lemma describes the action of certain group elements on the weight spaces:

Lemma 4.3 *Let us keep the notation as above. Then, for all $v \in V(\lambda)^{an}_{\mu}$ and $t \in \widehat{T}$ one obtains:*

(i) $t.v = \mu(t)v$,
(ii) $n_w.v \in V(\Lambda)^{an}_{w(\mu)}$, *for all* $w \in \widehat{\mathcal{W}}$,
(iii) $n_\tau.v \in V(\Lambda)^{an}_{\tau(\mu)}$, *if* $\tau(\Lambda) = \Lambda$.
(iv) $X_\alpha(c).v = v + \sum_{j=1}^{\infty} c^j v_j$ *with* $v_j \in V(\Lambda)^{an}_{\mu+j\alpha}$.

Our interest is to determine those weights μ fulfilling:

$$p_\mu S(c_0, \dots, c_{s-1}, \tilde{q}) i_\mu \neq 0. \tag{35}$$

Obviously, the following identity holds:

$$p_\mu S(c_0, \dots, c_{s-1}, \tilde{q}) i_\mu = \tilde{q}^{\mu(d)} p_\mu S(c_0, \dots, c_{s-1}, 1) i_\mu. \tag{36}$$

There is a first result:

Lemma 4.4 *For any weight μ of $V(\Lambda)^{an}$ with Λ Σ-invariant the following holds:*

$$p_\mu S(c_0, \dots, c_{s-1}, \tilde{q}) i_\mu = \tilde{q}^{\mu(d)} p_\mu X_{\alpha_0}(c_0) n_0 i_\mu \cdots p_\mu X_{\alpha_{s-1}}(c_{s-1}) n_{s-1} i_\mu p_\mu n_\tau i_\mu. \tag{37}$$

Furthermore, this is only non-vanishing if $\mu \prec \Lambda$, $\tau(\mu) = \mu$ and μ dominant.

Proof The statement follows by straightforward computation using Lemma 4.3, compare also [Br, (35)] and [Moh, (27)]. ∎

Let us introduce the set $\mathcal{D}(\Lambda_i) := \{\mu \in \widehat{P}^{\Sigma+} \mid V(\Lambda_i)^{an}_{\mu} \neq \{0\}\}$. Consider some $\mu = \sum_{j=0}^{s-1} m_j \Lambda_j + n\delta \in \mathcal{D}(\Lambda_i)$. For the central variable c, the equality $\Lambda_i(c) = \mu(c)$ implies (for the stabiliser Σ_i of α_i in Σ):

$$\sum_{l=1}^{\operatorname{ord}\tau} \frac{\check{a}_{\tau^l(i)}}{|\Sigma_i|} = \sum_{j=0}^{s-1} m_j \sum_{l=1}^{\operatorname{ord}\tau} \frac{\check{a}_{\tau^l(j)}}{|\Sigma_j|}, \tag{38}$$

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Using the relation ${}^\tau\check{a}_i = \sum_{l=1}^{\mathrm{ord}\,\tau} \frac{\check{a}_{\tau^l(i)}}{|\Sigma_i|}$ (recall that we set ${}^\tau\check{a}_0 = 1$ in the case A_n^1, γ with γ a generator of $\check{P}/\check{Q}$) and setting $n_i(\mu) := \max\{n \in \mathbb{Z} \mid \mu + n\delta \in \mathcal{D}(\Lambda_i)$ for $\mu \in \mathcal{D}(\Lambda_i)$ we have shown the following statement:

Lemma 4.5 *Any element $\mu \in \mathcal{D}(\Lambda_i)$ has one of the following forms:*

(i) $\mu = \Lambda_i + n\delta$, $n \leq n_i(\Lambda_i) = 0$,
(ii) $\mu = \Lambda_j + n\delta$ *with* $i \neq j$ *and* ${}^\tau\check{a}_i = {}^\tau\check{a}_j$, $n \leq n_i(\Lambda_i) \leq 0$,
(iii) $\mu = \sum_j m_j\Lambda_j + n\delta$ *with* $\sum_j m_j \geq 2$ *and* $m_j = 0$, *if* ${}^\tau\check{a}_i \leq {}^\tau\check{a}_j$ *and* $n \leq n_i(\mu) \leq 0$.

The set $\mathcal{D}(\Lambda_i) \bmod \mathbb{Z}\delta$ *is finite.*

Proof By assumption there are non-negative integers k_l, $0 \leq l \leq r$ satisfying:

$$\Lambda_i - \mu = \Lambda_i - \Big(\sum_j m_j\Lambda_j + n\delta\Big) = \sum_{l=0}^{r} k_l\alpha_l. \tag{39}$$

Evaluating this formula on the derivation d yields $-n = k_0$, whence the lemma. ∎

Choose some Σ-invariant dominant weight $\mu = \sum_{i=0}^{s-1} m_i\Lambda_i + n\delta$ of the representation $V(\Lambda_i)_\mu^{an}$. By Lemma 4.3 there exits a linear map $\Phi_{\Lambda_i,\mu}$ satisfying

$$p\mu S(c_0, \dots, c_{s-1}, \tilde{q})i_\mu = \tilde{q}^{-n} c_0^{m_0} \cdots c_{s-1}^{m_{s-1}} \Phi_{\Lambda_i,\mu}.$$

This implies:

$$\chi_i(S(c_0, \dots, c_{s-1}, \tilde{q})) = \sum_{\substack{\mu = \sum_{i=0}^{s-1} m_i\Lambda_i + n\delta \\ \mu \prec \Lambda_i}} \tilde{q}^{-n} c_0^{m_0} \cdots c_{s-1}^{m_{s-1}} \operatorname{Tr}|_{V(\Lambda_i)_\mu} \Phi_{\Lambda_i,\mu}. \tag{40}$$

Application of Lemma 4.5 yields the existence of holomorphic functions a_i, $a_{ji} \in \mathbb{C}\{\tilde{q}\}$ and of a polynomial $P \in \mathbb{C}[c_0, \dots, c_{s-1}]\{\tilde{q}\}$ with vanishing constant and linear terms and independent of those c_j with ${}^\tau\check{a}_j \leq {}^\tau\check{a}_i$, such that:

$$\chi_i(S(c_0, \dots, c_{s-1}, \tilde{q})) = a_i(\tilde{q})c_i + \sum_{j \neq i, {}^\tau\check{a}_j = {}^\tau\check{a}_i} a_{ji}(\tilde{q})c_j + P(c_0, \dots, c_{s-1}, \tilde{q}). \tag{41}$$

Using $\chi_\delta \circ S(c_0, \dots, c_{s-1}, \tilde{q}) = \tilde{q}$, the determinant of the Jacobian $\mathcal{J}$ of $\chi \circ S \colon \mathbb{C}^s \times \mathbb{C}^* \to \mathbb{C}^s \times \mathbb{C}^*$ has the shape:

$$\det \mathcal{J}\chi \circ S = \det \begin{pmatrix} \frac{\partial\chi_0 \circ S}{\partial c_0} & \dots & \frac{\partial\chi_0 \circ S}{\partial c_{s-1}} \\ \vdots & \vdots & \vdots \\ \frac{\partial\chi_{s-1} \circ S}{\partial c_0} & \dots & \frac{\partial\chi_{s-1} \circ S}{\partial c_{s-1}} \end{pmatrix}. \tag{42}$$

After reordering the index set $\{0, \dots, s-1\}$ in increasing order of the ${}^\tau\check{a}_j$ the matrix $(\frac{\partial\chi_i \circ S}{\partial c_j})$ will have upper block triangular shape with blocks $A_1, \dots, A_p$ on the diagonal

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implying: $\det \mathcal{J}\chi \circ S = \prod_{i=1}^{p} \det A_i$. Using (41) it only depends on $\tilde{q}$: $\det \mathcal{J}\chi \circ S \in \mathbb{C}\{q\}$.

Concerning the blocks, the following statement is true:

Proposition 4.6 *For all i the determinant* $\det A_i$ *is a unit in the ring* $\mathbb{C}\{\tilde{q}\}$.

Before proving this statement we show the following auxiliary result:

Lemma 4.7 *Let us keep the notation as above. Then*

$$\frac{\partial \chi_i \circ S}{\partial c_i}(c_0, \dots, c_{s-1}, \tilde{q}) = a_i(\tilde{q}), \quad \textit{and} \quad a_i(0) \neq 0. \tag{43}$$

Proof The first part of the statement follows immediately from (41). The second part is proven in [Br, Lemma 8]. ■

Proof of the Proposition First, note the following statement which can be proven as the Claim in [Br, p. 997]

Let $\Lambda_{j_1}, \dots, \Lambda_{j_l}$, $l > 1$ be distinct fundamental roots of the invariant lattice $\widehat{P}^{\Sigma+}$ having the same dual Kac labels, then $n_{j_i}(\Lambda_{j_{i+1}}) \leq 0$ (with $l+1 = 1$) and this inequality is strict for at least one $i \in \{1, \dots, l\}$.

Assume A_i to be a square matrix of size l. Then, we get for its determinant:

$$\det A_i = (A_i)_{11} \cdots (A_i)_{ll} + \sum_{\sigma \in \mathfrak{S}_l \setminus \{\mathrm{id}\}} \operatorname{sgn} \sigma \prod_{j=1}^{l} (A_i)_{j\sigma(j)}. \tag{44}$$

By Lemma 4.7 the first term on the right-hand side is a unit in $\mathbb{C}\{\tilde{q}\}$. Combining the claim above and Lemma 4.5 implies that the second sum is contained in $\tilde{q}\mathbb{C}\{\tilde{q}\}$. ■

Proof of Theorem 4.2 Proposition 4.6 proves that $S_{\tilde{q}}$ is a local isomorphism for small $|\tilde{q}|$. Since $\chi_{\tilde{q}} \circ S_{\tilde{q}} \colon \mathbb{C}^s \to \mathbb{C}^s$ is equivariant with respect to the $\mathbb{C}^*$ actions on the image and the preimage space keeping the positive weights and their multiplicities the statement follows from [Sl, §8.1, Lemma 1]. ■

A Appendix

In this appendix we compile the tables referred to in the text.

type	τ	folded type	k_f	$\chi(t)$
A_1^1	γ	0	2	$(t-1)^2$
A_n^1	γ	0	1	$(t-1)(t^n-1)$
n even	$\gamma^l, l \mid n+1, l>1$	A_{l-1}^1	1	$(t-1)(t^n-1)$
	σ	A_n^2	1	$(t-1)(t^n-1)$
A_n^1	γ	0	2	$(t-1)(t^n-1)$
n odd	$\gamma^l, l \mid n+1, l>1, l\,\mathit{even}$	A_{l-1}^1	1	$(t-1)(t^n-1)$
n odd	$\gamma^l, l \mid n+1, l>1, l\,\mathit{odd}$	A_{l-1}^1	2	$(t-1)(t^n-1)$
	σ	$D_{\frac{n+3}{2}}^2$	1	$(t^2-1)(t^{n-1}-1)$
	$\sigma\gamma$	$C_{\frac{n-1}{2}}^1$	1	$(t-1)(t^{n+1}-1)/(t+1)$
B_n^1	γ	A_{2n-2}^2	1	$(t-1)(t^n-1)$
C_n^1, n even	γ	A_{2n}^2	1	$(t^2-1)(t^{n-1}-1)$
C_n^1, n odd	γ	$C_{\frac{n-1}{2}}^1$	2	$(t-1)(t^n-1)$
D_4^1	γ	A_2^2	2	$(t^2-1)(t^3-1)$
	γ^2	C_2^1	1	$(t-1)(t^4-1)$
	σ	A_5^2	1	$(t^2-1)(t^3-1)$
	ρ	D_4^3	1	$(t^2-1)(t^3-1)$
D_n^1	γ	A_{n-2}^2	2	$(t^2-1)(t^{n-1}-1)$
n even	γ^2	C_{n-2}^1	1	$(t-1)(t^n-1)$
	σ	A_{2n-3}^2	1	$(t^2-1)(t^{n-1}-1)$
$n \equiv 0(4)$	$\gamma\sigma$	$B_{\frac{n}{2}}^1$	1	$(t^4-1)(t^{n-3}-1)$
$n \equiv 2(4)$	$\gamma\sigma$	$B_{\frac{n}{2}}^1$	2	$(t^4-1)(t^{n-3}-1)$
D_n^1	γ	$C_{\frac{n-3}{2}}^1$	2	$(t-1)(t^n-1)$
n odd	γ^2	C_{n-2}^1	1	$(t-1)(t^n-1)$
	σ	A_{2n-3}^2	1	$(t^2-1)(t^{n-1}-1)$
	$\gamma\sigma$	A_{n-2}^2	1	$(t-1)(t^2+1)(t^{n-2}-1)$
E_6^1	γ	G_2^1	1	$(t^2-1)(t^5-1)$
	σ	E_6^2	1	$(t^3-1)(t^4-1)$
E_7^1	γ	F_4^1	2	$(t^3-1)(t^5-1)$
A_{2n-1}^2	γ	C_{n-1}^1	1	$(t-1)(t^n-1)$
D_3^2	γ	A_1^1	1	$(t^2-1)^2$
D_{n+1}^2, n even	γ	$D_{\frac{n}{2}+1}^2$	1	$(t^2-1)(t^{n-1}-1)$
D_{n+1}^2, n odd	γ	A_{n-1}^2	1	$(t^2-1)(t^{n-1}-1)$

Table 1: Folded Dynkin diagrams, fundamental levels and characteristic polynomials of the twisted Coxeter elements

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type	τ	p	k	b
A^1_1	γ	2	1	$(1, 0)$
A^1_n	$\gamma^l, l \mid n+1$	$n+1$	l	$(n, n-1, \dots, 1, 0)$
n even	σ	1	1	$(\frac{n}{2}, \frac{n}{2}, \frac{n}{2}-1, \dots, 1, 0, 1, \dots, \frac{n}{2}-1)$
A^1_n	$\gamma^l, l \mid n+1$	$n+1$	l	$(n, n-1, \dots, 1, 0)$
n odd	σ	1	1	$(\frac{n-1}{2}, \frac{n-1}{2}, \frac{n-3}{2}, \dots, 1, 0, 1, \dots, \frac{n-3}{2})$
	$\sigma\gamma$	2	1	$(\frac{n+1}{2}, \frac{n-1}{2}, \frac{n-3}{2}, \dots, 1, 0, 1, \dots, \frac{n-1}{2})$
B^1_n, n even	γ	1	1	$(\frac{n}{2}-1, \frac{n}{2}, n-2, n-3, \dots, 1, 0)$
B^1_n, n odd	γ	2	2	$(n-2, n, 2n-4, 2n-6, \dots, 2, 0)$
C^1_n, n even	γ	1	1	$(\frac{n}{2}, \frac{n}{2}-1, \frac{n}{2}-2, \dots, 1, 0, 0, 1, \dots, \frac{n}{2}-1)$
C^1_n, n odd	γ	2	1	$(\frac{n+1}{2}, \frac{n-1}{2}, \frac{n-3}{2}, \dots, 1, 0, 1, \dots, \frac{n-1}{2})$
D^1_4	γ	2	1	$(3, 1, 2, 0, 2)$
	γ^2	2	1	$(1, 2, 2, 1, 0)$
	σ	2	2	$(3, 3, 4, 2, 0)$
	ρ	1	1	$(1, 2, 1, 1, 0)$
D^1_n	γ	2	1	$(\frac{n}{2}-2, \frac{n}{2}, n-4, \dots, 2, 0, 2, \dots, n-6, \frac{n}{2}-1, \frac{n}{2}-3)$
n even	γ^2	4	1	$(\frac{n}{2}-1, \frac{n}{2}, n-2, \dots, 2, 1, 0)$
	σ	2	2	$(n-1, n-1, 2n-4, 2n-6, \dots, 4, 2, 0)$
	$\gamma\sigma$	2	1	$(\frac{n}{2}-1, \frac{n}{2}-1, n-4, \dots, 2, 0, 2, \dots, \frac{n}{2}-2, \frac{n}{2}-2)$
D^1_n	γ	4	1	$(\frac{n-3}{2}, \frac{n+1}{2}, n-3, \dots, 2, 0, 2, \dots, n-5, \frac{n-1}{2}, \frac{n-5}{2})$
n odd	γ^2	4	2	$(n-2, n, 2n-4, \dots, 4, 2, 0)$
	σ	1	1	$(\frac{n-1}{2}, \frac{n-1}{2}, n-2, n, n-3, \dots, 2, 1, 0)$
D^1_n, n odd	$\gamma\sigma$	2	1	$(\frac{n-1}{2}, \frac{n-1}{2}, n-3, \dots, 2, 0, 2, \dots, \frac{n-3}{2}, \frac{n-3}{2})$
E^1_6	γ	6	2	$(3, 7, 8, 3, 4, 5, 0)$
	σ	2	2	$(1, 5, 6, 3, 2, 3, 0)$
E^1_7	γ	4	2	$(7, 10, 9, 4, 3, 6, 5, 0)$
A^2_{2n-1}, n even	γ	2	1	$(\frac{n}{2}-1, \frac{n}{2}, n-2, \dots, 1, 0)$
A^2_{2n-1}, n odd	γ	4	2	$(n-2, n, 2n-4, \dots, 2, 0)$
D^2_{n+1}, n even	γ	2	1	$(\frac{n}{2}, n-2, \dots, 2, 0, 0, 2, \dots, n-4, \frac{n}{2}-1)$
D^2_{n+1}, n odd	γ	2	1	$(\frac{n+1}{2}, n-1, \dots, 2, 0, 2, \dots, n-3, \frac{n-1}{2})$

Table 2: Values for p, k and b from Lemma 2.9 and Proposition 2.8

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Fachbereich Mathematik
Universität Hamburg
Bundesstraße 55
20146 Hamburg
e-mail: stephan.mohrdieck@math.uni-hamburg.de