BULL. AUSTRAL. MATH. SOC. MOS 55A25, 55A05, 55A10

VOL. 5 (1971), 127-130.

On prime noninvertible links

Wilbur Whitten

It is the purpose of this paper to exhibit an infinite collection of prime, noninvertible links; each link in the collection is the union of two invertible knots. In fact, we present two such families of links. Our examples heuristically indicate that collections of links of this type exist in many varieties.

An oriented, ordered link L imbedded in the oriented 3-sphere S is *invertible* provided that it is of the same (oriented) type as its inverse. The *inverse*, L^{-1}, of L is obtained by reversing the orientation of each component of L. Examples of noninvertible links (with more than one component) were first given in 1969 [*4*]; each link of [*4*] is the union of two invertible knots, and although it was not proved there, the links presented are also prime.

In [*3*], McPherson asks whether there exists an infinite family of such links: Does there exist an infinite family of prime, noninvertible, two-component links in S with invertible components? We answer this question affirmatively in this paper by exhibiting a disjoint pair of such families.

We denote by C_1 and C_2, respectively, our two families. A typical link, $\Lambda_n = \psi_n \cup \psi$ $(n \geq 2)$, in C_1 is shown in Figure 1 (page 128). In each Λ_n, ψ_n is of trivial knot type and ψ is of knot type 5_1. The linking number of Λ_n is $n - 1$, so that C_1 is an infinite class of distinct links.

Received 15 February 1971.

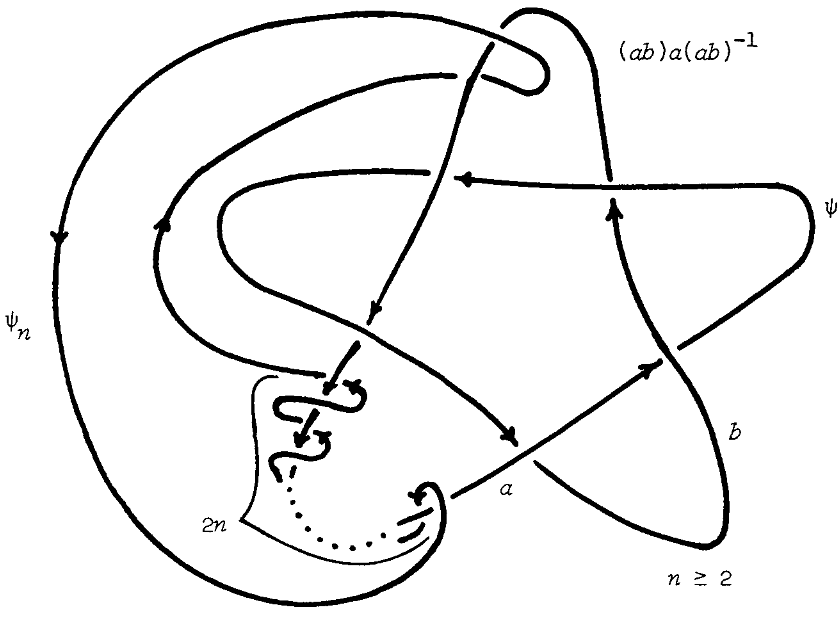

Figure 1.

In order to define the elements of C_2, we begin with the link $\Lambda_2 = \psi_2 \cup \psi$ of C_1. Let V denote a closed tubular neighborhood of ψ in $S - \psi_2$, and take as orientation of V that induced by S. Consider a twist knot K_ρ with twist ρ imbedded in the obvious nice way in the interior of a closed, oriented, unknotted solid torus T (see the figure at the bottom of page 143 of [1]). Orient K_ρ, and map T onto V by a faithful homeomorphism. We denote the (oriented) image of K_ρ under this map by Φ_ρ. Φ_ρ is simply an oriented double of ψ with a twist ρ. A typical link in C_2 is then $\Xi_\rho = \psi_2 \cup \Phi_\rho$. Evidently, the links of C_2 are distinct, C_1 and C_2 are disjoint families, and each link in both classes has invertible components.

A proof that each Λ_n in C_1 is noninvertible can easily be constructed along the lines established in [4]. To see that each Ξ_ρ in C_2 is noninvertible, we note that $\pi_1(S-\Phi_\rho)$ is isomorphic to the free

https://doi.org/10.1017/S0004972700046943 Published online by Cambridge University Press

product of $\pi_1(S-\mathrm{Int}V)$ and $\pi_1(V-\Phi_\rho)$ amalgamated with respect to $\pi_1(\partial V)$. By methods similar to those in [5], it follows that Ξ_ρ is noninvertible.

It remains to show that each Λ_n $(n \geq 2)$ and each Ξ_ρ (ρ an integer) is prime. If Λ_n is composite, then it clearly follows (from the definition of primeness of links and Theorem 1 of [2]) that Λ_n is the product of a knot of type 5_1 and a (prime) link (the hub of Λ_n) which is the union of two trivially knotted components, one of which is ψ_n. This means that an element of $\pi_1(S-\psi)$ represented by the (oriented) knot ψ_n, considered as a loop, must be the $(n-1)$st power of a meridian of ψ. Since ψ_n represents

$$a^n(ab)a^{-1}(ab)^{-1}$$

in $\pi_1(S-\psi)$ (see Figure 1, page 128), there exists an element μ belonging to $\pi_1(S-\psi)$ such that

$$\mu a^{n-1}\mu^{-1} = a^n(ab)a^{-1}(ab)^{-1} .$$

It is easy to see that no such μ exists (see [4]); thus Λ_n $(n \geq 2)$ is prime.

Finally, we show that each Ξ_ρ is prime. Recall that $\pi_1(S-\Phi_\rho)$ is isomorphic to the free product of $\pi_1(S-\mathrm{Int}V)$ and $\pi_1(V-\Phi_\rho)$ amalgamated with respect to $\pi_1(\partial V)$. The component ψ_2 of Ξ_ρ, considered as a loop, represents the element

$$v = a^2(ab)a^{-1}(ab)^{-1}$$

of $\pi_1(S-\mathrm{Int}V) \subset \pi_1(S-\Phi_\rho)$. If Ξ_ρ were splittable, ψ_2 would also represent the identity element of $\pi_1(S-\Phi_\rho)$. But this is impossible, since v would then be the identity of $\pi_1(S-\Phi_\rho)$, and hence of $\pi_1(S-\mathrm{Int}V)$, which it obviously is not [4]. (In this connection, note that the linking number of Ξ_ρ is 0 .)

Since ψ_2 is a trivial knot, and Φ_ρ is prime (the genus of Φ_ρ is 1), it follows that if Ξ_ρ is composite, then it must be the product of a knot (of the same type as Φ_ρ), and the hub of Ξ_ρ . The hub is then the union of two trivial knots, one of which is ψ_2 . Since the linking number of Ξ_ρ is 0 , the element v (above) of $\pi_1(S-\Phi_\rho)$ represented by ψ_2 is, therefore, the 0-th power of a meridian of Φ_ρ . This implies, however, that $v = 1$ in $\pi_1(S-\Phi_\rho)$, which is not true. It follows that Ξ_ρ is prime, and that our link families C_1 and C_2 possess the desired properties.

References

[1] R.H. Fox, "A quick trip through knot theory", *Topology of 3-manifolds and related topics* (*Proc. Univ. Georgia Institute*, 1961), 120-167. (Prentice-Hall, Englewood Cliffs, New Jersey, 1962.)

[2] Yoko Hashizume, "On the uniqueness of the decomposition of a link", *Osaka J. Math.* 10 (1958), 283-300.

[3] James M. McPherson, "A family of non-invertible prime links", *Bull. Austral. Math. Soc.* 4 (1971), 105-108.

[4] W.C. Whitten, Jr, "A pair of non-invertible links", *Duke Math. J.* 36 (1969), 695-698.

[5] W.C. Whitten, Jr, "On noninvertible links with invertible proper sublinks", *Proc. Amer. Math. Soc.* 26 (1970), 341-346.

University of Southwestern Louisiana,
Lafayette,
Louisiana, USA.