Can. J. Math., Vol. XL, No. 4, 1988, pp. 1010-1024

REPRESENTATION FORMULAS FOR INTEGRABLE AND ENTIRE FUNCTIONS OF EXPONENTIAL TYPE I

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1. Introduction. Let B_τ denote the class of entire functions of exponential type τ (>0) bounded on the real axis. For the function $f \in B_\tau$ we have the interpolation formula [**1**, p. 143]

$$(1) \qquad \sin \gamma f'(t) - \cos \gamma \tilde{f}'(t) = 2\tau \sum_{k=-\infty}^{\infty} (-1)^k \left(\frac{\sin\left(\frac{k\pi + \gamma}{2}\right)}{k\pi + \gamma} \right)^2 f\left(t + \frac{(k\pi + \gamma)}{\tau}\right),$$

where t, γ are real numbers and $\tilde{f}$ is the so called conjugate function of f. Let us put

$$(2) \qquad G_{\gamma,f}(\alpha) := \sum_{k=-\infty}^{\infty} e^{-\alpha k\pi i} \left(\frac{\sin\left(\frac{k\pi + \gamma}{2}\right)}{k\pi + \gamma} \right)^2 f\left(\frac{k\pi + \gamma}{\tau}\right).$$

The function $G_{\gamma,f}$ is a periodic function of α, with period 2. For $t = 0$ (the general case is obtained by translation) the righthand member of (1) is $2\tau G_{\gamma,f}(1)$. In the following paper we suppose that f satisfies an additional hypothesis of the form $f(x) = O(|x|^{-\epsilon})$, for some $\epsilon > 0$, as $x \to \pm\infty$ and we give an integral representation of $G_{\gamma,f}(\alpha)$ which is valid for $0 \leqq \alpha \leqq 2$. More precisely, the Theorems 1, 2 and 3, below, contain formulas giving a representation of $G_{\gamma,f}(\alpha)$ valid respectively for $0 \leqq \alpha \leqq 1$, $1 \leqq \alpha \leqq 3/2$ and $3/2 \leqq \alpha \leqq 2$. Before we examine the method of proof we state explicitly the results in question.

THEOREM 1. *Let $f \in B_\tau$ such that $f(x) = O(|x|^{-\epsilon})$, ($\epsilon > 0$, $x \to \pm\infty$). For all reals γ and $0 \leqq \alpha \leqq 1$ we have*

$$(3) \qquad -4\pi\tau e^{-\alpha i\gamma} G_{\gamma,f}(\alpha) = \int_{-\infty}^{\infty} f(x) \frac{[e^{-i\tau x} - \alpha i\tau x e^{-i\tau x} - 2e^{-\alpha i\tau x} + e^{(1-\alpha)i\tau x}]}{x^2} dx$$

Received December 1, 1987.

https://doi.org/10.4153/CJM-1988-040-3 Published online by Cambridge University Press

$$+ e^{-2i\gamma} \int_{-\infty}^{\infty} f(x) \frac{[e^{(1-\alpha)i\tau x} - e^{i\tau x} + \alpha i\tau x e^{i\tau x}]}{x^2} dx.$$

THEOREM 2. *Let $f \in B_\tau$ such that $f(x) = O(|x|^{-\epsilon})$, ($\epsilon > 0$, $x \to \pm\infty$). For all reals γ and $1 \leqq \alpha \leqq 3/2$ we have*

$$(4) \quad -4\pi\tau e^{2i\gamma} \sum_{k=-\infty}^{\infty} e^{-\alpha(k\pi+\gamma)i} \left(\frac{\sin\left(\frac{k\pi + \gamma}{2}\right)}{k\pi + \gamma} \right)^2 f\left(\frac{\alpha(k\pi + \gamma)}{\tau}\right)$$

$$= \int_{-\infty}^{\infty} f(\alpha x) \frac{[e^{\alpha i\tau x} + e^{(1-\alpha)i\tau x} - 2e^{(2-\alpha)i\tau x} - (2\alpha - 3)i\tau x e^{\alpha i\tau x}]}{x^2} dx$$

$$+ e^{2i\gamma} \int_{-\infty}^{\infty} f(\alpha x) \frac{[e^{(1-\alpha)i\tau x} - e^{-\alpha i\tau x} - i\tau x e^{-\alpha i\tau x}]}{x^2} dx.$$

Formulas (3) and (4) coincide for $\alpha = 1$.

THEOREM 3. *Let $f \in B_\tau$ such that $f(x) = O(|x|^{-\epsilon})$, ($\epsilon > 0$, $x \to \pm\infty$). For all reals γ and $3/2 \leqq \alpha \leqq 2$ we have*

$$(5) \quad -4\pi\tau e^{4i\gamma} \sum_{k=-\infty}^{\infty} e^{-\alpha(k\pi+\gamma)i} \left(\frac{\sin\left(\frac{k\pi + \gamma}{2}\right)}{k\pi + \gamma} \right)^2 f\left(\frac{\alpha(k\pi + \gamma)}{\tau}\right)$$

$$= \int_{-\infty}^{\infty} f(\alpha x) \frac{[(2\alpha - 3)i\tau x e^{\alpha i\tau x} + e^{(3-\alpha)i\tau x} - e^{\alpha i\tau x}]}{x^2} dx$$

$$+ e^{2i\gamma} \int_{-\infty}^{\infty} f(\alpha x) \frac{[e^{(1-\alpha)i\tau x} - 2e^{(2-\alpha)i\tau x} + e^{(3-\alpha)i\tau x}]}{x^2} dx$$

$$+ e^{4i\gamma} \int_{-\infty}^{\infty} f(\alpha x) \frac{[e^{(1-\alpha)i\tau x} - e^{-\alpha i\tau x} - i\tau x e^{-\alpha i\tau x}]}{x^2} dx.$$

Formulas (4) and (5) coincide for $\alpha = 3/2$.

In the statements of Theorems 2 and 3 we suppose $\alpha \geqq 1$. We can therefore see the function f as being an element of $B_{\tau\alpha}$. If we change τ to $\tau\alpha$ in formulas (4) and (5) then their lefthand members are equal to

$$-4\pi\tau e^{(2-\alpha)i\gamma} G_{\gamma,f}(\alpha) \quad \text{and} \quad -4\pi\tau e^{(4-\alpha)i\gamma} G_{\gamma,f}(\alpha)$$

respectively. This gives us a precise representation of $G_{\gamma,f}(\alpha)$ valid for $1 \leqq \alpha \leqq 2$.

We observe that the distance between two consecutive interpolation points is

$$\frac{(k+1)\pi+\gamma}{\tau}-\frac{k\pi+\gamma}{\tau}=\frac{\pi}{\tau}$$

in Theorem 1 and

$$\frac{\alpha\pi}{\tau} \geqq \frac{\pi}{\tau}$$

in Theorems 2 and 3. If we apply formula (1) to a function of the form $e^{i\beta z}f(z) \in B_{\tau+|\beta|}$ then the corresponding distance is only

$$\frac{\pi}{\tau+|\beta|} \leqq \frac{\pi}{\tau}.$$

2. The method of proof. We consider the Levitan's polynomials f_h (with the notation of [**6**]) defined, for $f \in B_\tau$ and $h > 0$, by

$$(6) \quad f_h(x) = \sum_{k=-\infty}^{\infty} \varphi(hx+k) f\left(x+\frac{k}{h}\right),$$

where

$$\varphi(x) := \left(\frac{\sin \pi x}{\pi x}\right)^2.$$

LEMMA 1. ([**6**, p. 23]) *The functions f_h defined by* (6) *are trigonometric polynomials with period $1/h$ and order $\leqq N := [\tau/2\pi h]$. When x is real we have*

$$|f_h(x)| \leqq \max_{-\infty<t<\infty} |f(t)|,$$

and $f_h(z) \to f(z)$ uniformly in every bounded set of the complex plane as $h \to 0$.

In view of Lemma 1 we may write

$$(7) \quad f_h(x) = \sum_{j=-N}^{N} C_j(h) e^{2\pi h i j x}.$$

Let

$$h_f(\theta) := \overline{\lim_{r\to\infty}} \frac{\ln|f(re^{i\theta})|}{r}$$

be the Phragmén-Lindelof indicator function. We shall need subsequently

LEMMA 2. ([**5**, p. 982] or [**2**, p. 465]) *If, in addition, $h_f(\pi/2) \leqq 0$ then*

$$C_{-m}(h) = 0 \quad \textit{for } m = 1, 2, 3 \ldots.$$

Let us illustrate now our method of proof. We have, from (7),

$$f_h(t) = \sum_{j=-N}^{N} C_j(h) e^{2\pi h i j t},$$

where the Fourier coefficients $C_j(h)$ are equal to [**6**, p. 22]

$$(8) \qquad C_j(h) = h \int_{-\infty}^{\infty} \varphi(hx) f(x) e^{-2\pi h i j x} dx,$$

whence

$$f_h(t) = h \int_{-\infty}^{\infty} \varphi(hx) f(x) \sum_{j=-N}^{N} e^{2\pi h i j (t-x)} dx$$

$$= \int_{-\infty}^{\infty} h\varphi(hx) f(x) \frac{[e^{2\pi h(N+1)i(t-x)} - e^{-2\pi h N i (t-x)}]}{(e^{2\pi h i (t-x)} - 1)} dx.$$

Let us denote the integrand by $F_h(x)$. If $f \in L(-\infty, \infty)$ then, for $x \in \mathbf{R}$:

$$|F_h(x)| = \left| h\varphi(hx) f(x) \sum_{j=-N}^{N} e^{2\pi h i j (t-x)} \right|$$

$$\leqq (2N + 1) h |f(x)|$$

$$\leqq \tau |f(x)|, \quad (h \to 0),$$

that is $F_h(x)$ is dominated by an integrable function. Thus, using Lebesgue dominated convergence theorem and Lemma 1, we obtain

$$(9) \qquad f(t) = \lim_{h \to 0} f_h(t)$$

$$= \int_{-\infty}^{\infty} f(x) \lim_{h \to 0} h\varphi(hx) \frac{[e^{2\pi h(N+1)i(t-x)} - e^{-2\pi h N i (t-x)}]}{(e^{2\pi h i (t-x)} - 1)} dx$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \frac{\sin \tau(t-x)}{(t-x)} dx.$$

If f is not necessarily in $L(-\infty, \infty)$ but satisfies a condition of the form

$$(10) \qquad f(x) = O(|x|^{-\epsilon}), \quad \epsilon > 0, x \to \pm\infty,$$

then the functions

$$g_\delta(z) := \frac{\sin \delta z}{\delta z} f(z)$$

are in $B_{\tau+\delta}$ $(\delta > 0)$ and

$$g_\delta(x) = O(|x|^{-1-\epsilon});$$

these functions are thus integrable so that, from (9):

$$(11)\quad g_\delta(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} g_\delta(x)\frac{\sin((\tau + \delta)(t - x))}{(t - x)}dx,$$

with

$$\left|g_\delta(x)\frac{\sin((\tau + \delta)(t - x))}{(t - x)}\right| \leqq |f(x)|\left|\frac{\sin((\tau + \delta)(t - x))}{(t - x)}\right|$$

$$\leqq \frac{K}{|x|^{1+\epsilon}} \quad \text{if } x \to \pm\infty.$$

Thus a passage to the limit is again justified, in (11), by the Lebesgue dominated convergence theorem and we have proved the following result: if $f \in B_\tau$ satisfies (10) then for all real t,

$$(12)\quad f(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\frac{\sin \tau(t - x)}{(t - x)}dx.$$

Remark. Formula (12) is more easily proved with a quadrature formula; if $g \in B_\sigma$ satisfied the condition

$$g(x) = O(|x|^{-\delta}), \quad (\delta > 1, x \to \pm\infty),$$

then [**4**]

$$(13)\quad \int_{-\infty}^{\infty} g(x)dx = \frac{2\pi}{\sigma}\sum_{k=-\infty}^{\infty} g\left(\frac{2k\pi}{\sigma}\right).$$

Applying (13) to the function $g \in B_{2\tau}$,

$$g(z) := f(z)\frac{\sin \tau z}{z},$$

where f satisfies (10), we obtain

$$\int_{-\infty}^{\infty} f(x)\frac{\sin \tau x}{x}dx = \frac{\pi}{\tau}\sum_{k=-\infty}^{\infty} g\left(\frac{k\pi}{\tau}\right)$$

$$= \frac{\pi}{\tau}g(0) = \pi f(0)$$

and (12) follows by translation. However, it is not clear if a similar remark can be applied to prove Theorems 2 and 3.

3. Other lemmas. We will use several times the basic formulas

$$(14) \quad \sum_{j=m}^{n} z^j = \frac{z^{n+1} - z^m}{z - 1}$$

and

$$(15) \quad \sum_{j=m}^{n} jz^j = \frac{nz^{n+2} - (m - 1)z^{m+1} - (n + 1)z^{n+1} + mz^m}{(z - 1)^2}.$$

In order to obtain our formulas we prove the corresponding (and more precise) formulas for trigonometric polynomials and extend it, using the method described in the preceding section, to integrable and entire functions of exponential type. The following lemmas are used for that purpose.

LEMMA 3. *For all trigonometric polynomials*

$$S(\theta) := \sum_{j=-n}^{n} C_j e^{ij\theta}$$

we have

$$(16) \quad \frac{1}{2n} \sum_{k=1}^{2n} (-1)^k e^{((m/n)(k\pi+\gamma))i} A_k(R, \gamma) S\left(\theta + \frac{(k\pi + \gamma)}{n}\right)$$

$$\equiv e^{i\gamma}\left[\sum_{j=-m}^{n-m} (R^{j+m} - 1)C_j e^{ij\theta} + \sum_{j=n-m+1}^{n} (R^{2n-m-j} - 1)C_j e^{ij\theta}\right]$$

$$+ e^{-i\gamma} \sum_{j=-n}^{-m} (R^{-j-m} - 1)C_j e^{ij\theta},$$

where

$$A_k(R, \gamma) := R^n - 1 + 2 \sum_{\nu=1}^{n} (R^{n-\nu} - 1) \cos \frac{\nu(k\pi + \gamma)}{n}$$

and $0 \leqq m \leqq n$ *is an integer.*

LEMMA 4. ([**3**, Lemma 3]) *For all trigonometric polynomials*

$$S(\theta) := \sum_{j=-n}^{n} C_j e^{ij\theta}, \quad n \geqq 3,$$

we have

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$$(17)\quad \frac{e^{i\gamma}}{2(n-m)} \sum_{k=1}^{2(n-m)} (-1)^k e^{-m(k\pi+\gamma)i/(n-m)} A_{k,m}(R,\gamma) S\left(\theta + \frac{(k\pi+\gamma)}{(n-m)}\right)$$

$$\equiv \sum_{j=-n}^{2m-n} (R^{2n-3m+j} - 1) C_j e^{ij\theta} + \sum_{j=2m-n+1}^{m} (R^{m-j} - 1) C_j e^{ij\theta}$$

$$+ e^{2i\gamma} \sum_{j=m}^{n} (R^{j-m} - 1) C_j e^{ij\theta}$$

for $0 \leqq m \leqq n/3$, *whereas*

$$(18)\quad \frac{e^{3i\gamma}}{2(n-m)} \sum_{k=1}^{2(n-m)} (-1)^k e^{-m(k\pi+\gamma)i/(n-m)} A_{k,m}(R,\gamma) S\left(\theta + \frac{(k\pi+\gamma)}{(n-m)}\right)$$

$$\equiv \sum_{j=-n}^{3m-2n} (R^{3m-2n-j} - 1) C_j e^{ij\theta}$$

$$+ e^{2i\gamma}\left[\sum_{j=3m-2n}^{2m-n} (R^{2n-3m+j} - 1) C_j e^{ij\theta} \right.$$

$$\left. + \sum_{j=2m-n+1}^{m} (R^{m-j} - 1) C_j e^{ij\theta}\right]$$

$$+ e^{4i\gamma} \sum_{j=m}^{n} (R^{j-m} - 1) C_j e^{ij\theta}$$

for $n/3 \leqq m \leqq n/2$. *The coefficients* $A_{k,m}(R, \gamma)$ *are*

$$A_{k,m}(R,\gamma) := R^{n-m} - 1 + 2\sum_{\nu=1}^{n-m} (R^{n-m-\nu} - 1)\cos\frac{\nu(k\pi+\gamma)}{(n-m)}.$$

We do not include a proof of Lemma 3 since it is similar (and easier) than that of Lemma 4.

4. Proofs of the theorems. We observe that both members of each of the formulas (3), (4), (5) are periodic functions of γ, with period π. Also, we may suppose that

$$\max_{-\infty<t<\infty} |f(t)| \leqq 1;$$

in view of Lemma 1 this implies that $|f_h(x)| \leqq 1$, $-\infty < x < \infty$.

Hence, it is sufficient to prove the theorems for $0 \leqq \gamma \leqq \pi$ and we may assume that $|f_h(x)| \leqq 1$ for all reals x.

Proof of Theorem 1. Dividing both members of (16) by $(R - 1)$ and letting $R \to 1$ we obtain

$$(19)\quad \frac{e^{i\gamma}}{2n} \sum_{k=1}^{2n} (-1)^k e^{((m/n)(k\pi+\gamma))i} \left(\frac{\sin\left(\frac{k\pi + \gamma}{2}\right)}{\sin\left(\frac{k\pi + \gamma}{2n}\right)} \right)^2 S\left(\theta + \frac{(k\pi + \gamma)}{n}\right)$$

$$= e^{2i\gamma}\left[\sum_{j=-m}^{n-m} (j + m)C_j e^{ij\theta} + \sum_{j=n-m+1}^{n} (2n-m-j)C_j e^{ij\theta} \right]$$

$$- \sum_{j=-n}^{-m} (j + m)C_j e^{ij\theta}.$$

We apply (19), with $\theta = 0$, to the trigonometric polynomials (7). We choose $n = N$ and $m = (p/q)N$ where p and q are positive integers such that $0 \leqq p \leqq q$. In order that $N \equiv 0 \pmod q$ we need to take h of the form

$$h = \frac{\tau}{2\pi S q}$$

where S is an integer which tends to ∞ if and only if h tends to 0. That choice is permitted since all the limits under consideration will exist. We have thus

$$(20)\quad e^{i\gamma} S_1(h) = e^{2i\gamma} S_2(h) - S_3(h),$$

where

$$S_1(h) := \frac{1}{2N} \sum_{k=1}^{2N} (-1)^k e^{((p/q)(k\pi+\gamma))i} \left(\frac{\sin\left(\frac{k\pi + \gamma}{2}\right)}{\sin\left(\frac{k\pi + \gamma}{2N}\right)} \right)^2 f_h\left(\frac{k\pi + \gamma}{2\pi h N}\right),$$

$$S_2(h) := \sum_{j=-(pN/q)}^{(1-(p/q))N} \left(j + \frac{pN}{q}\right) C_j(h)$$

$$+ \sum_{j=(1-(p/q))N+1}^{N} \left(\left(2 - \frac{p}{q}\right)N - j\right) C_j(h)$$

and

$$S_3(h) := \sum_{j=-N}^{-(p/q)N} \left(j + \frac{p}{q}N\right) C_j(h).$$

Now,

$$S_1(h) = \frac{1}{2N} \sum_{k=1}^{N-1} (-1)^k e^{((p/q)(k\pi+\gamma))i} \left(\frac{\sin\left(\frac{k\pi + \gamma}{2}\right)}{\sin\left(\frac{k\pi + \gamma}{2N}\right)} \right)^2 f_h\left(\frac{k\pi + \gamma}{2\pi hN}\right)$$

$$+ \frac{1}{2N} \sum_{k=N}^{2N} (-1)^k e^{((p/q)(k\pi+\gamma))i} \left(\frac{\sin\left(\frac{k\pi + \gamma}{2}\right)}{\sin\left(\frac{k\pi + \gamma}{2N}\right)} \right)^2 f_h\left(\frac{k\pi + \gamma}{2\pi hN}\right).$$

In the second summation we change k to $(2N + k)$. Using the periodicity of the function f_h (Lemma 1) we obtain

(21) $hS_1(h)$

$$= \frac{h}{2N} \sum_{k=-N}^{N-1} (-1)^k e^{((p/q)(k\pi+\gamma))i} \left(\frac{\sin\left(\frac{k\pi + \gamma}{2}\right)}{\sin\left(\frac{k\pi + \gamma}{2N}\right)} \right)^2 f_h\left(\frac{k\pi + \gamma}{2\pi hN}\right).$$

If $0 \leqq \gamma \leqq \pi$ then

$$\left|\frac{k\pi + \gamma}{2N}\right| \leqq \frac{\pi}{2} \quad (\text{for } -N \leqq k < N)$$

so that

$$\left|\sin\left(\frac{k\pi + \gamma}{2N}\right)\right| \geqq \frac{2}{\pi}\left|\frac{k\pi + \gamma}{2N}\right|.$$

Also,

$$\left|f_h\left(\frac{k\pi + \gamma}{2\pi hN}\right)\right| \leqq 1$$

(recall that we may assume

$$\max_{-\infty<t<\infty} |f(t)| \leqq 1).$$

The summand in (21) is thus

$$\leqq \frac{hN}{2\left(N \sin\left(\frac{k\pi + \gamma}{2N}\right)\right)^2} \leqq \frac{\tau}{(k\pi + \gamma)^2}, \quad (h \to 0),$$

and the dominated convergence theorem implies that

(22) $$\lim_{h\to 0} hS_1(h) = \frac{\tau}{\pi}\sum_{k=-\infty}^{\infty}(-1)^k e^{((p/q)(k\pi+\gamma))i}\left(\frac{\sin\left(\frac{k\pi+\gamma}{2}\right)}{k\pi+\gamma}\right)^2 f\left(\frac{k\pi+\gamma}{\tau}\right).$$

Here we must observe that

$$\lim_{h\to 0} f_h\left(\frac{k\pi+\gamma}{2\pi hN}\right) = f\left(\frac{k\pi+\gamma}{\tau}\right)$$

which follows from the inequalities

$$\left|f_h\left(\frac{k\pi+\gamma}{2\pi hN}\right) - f\left(\frac{k\pi+\gamma}{\tau}\right)\right|$$
$$\leqq \left|f_h\left(\frac{k\pi+\gamma}{2\pi hN}\right) - f\left(\frac{k\pi+\gamma}{2\pi hN}\right)\right| + \left|f\left(\frac{k\pi+\gamma}{2\pi hN}\right) - f\left(\frac{k\pi+\gamma}{\tau}\right)\right|$$

and [**6**, p. 22]

$$\left|f_h\left(\frac{k\pi+\gamma}{2\pi hN}\right) - f\left(\frac{k\pi+\gamma}{2\pi hN}\right)\right| \leqq 2\left(1 - \varphi\left(\frac{k\pi+\gamma}{2\pi N}\right)\right).$$

On the other hand,

(23) $$hS_2(h) = h^2\sum_{j=-(p/q)N}^{(1-(p/q))N}\left(j+\frac{p}{q}N\right)\int_{-\infty}^{\infty}\varphi(hx)f(x)e^{-2\pi hijx}dx$$
$$+ h^2\sum_{j=(1-(p/q))N+1}^{N}\left(\left(2-\frac{p}{q}\right)N - j\right)\int_{-\infty}^{\infty}\varphi(hx)f(x)e^{-2\pi hijx}dx$$
$$= \int_{-\infty}^{\infty} h^2\varphi(hx)f(x)$$
$$\times \frac{\left[\left(1-\frac{p}{q}\right)Ne^{-2\pi hix((1-(p/q))N+2)} + \left(\frac{p}{q}N+1\right)e^{2\pi hix((p/q)N-1)}\right.}{(e^{-2\pi hix}-1)^2}$$
$$+ \frac{\left.-\left(\left(1-\frac{p}{q}\right)N+1\right)e^{-2\pi hix((1-(p/q))N+1)} - \frac{pN}{q}e^{2\pi hix(pN/q)}\right]}{(e^{-2\pi hix}-1)^2}dx$$
$$+ \int_{-\infty}^{\infty}\frac{p}{q}Nh^2\varphi(hx)f(x)$$

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$$\times \frac{[e^{-2\pi hix((1-(p/q))N+1)} - e^{2\pi hix(pN/q)}]}{(e^{-2\pi hix} - 1)} dx$$

$$+ \int_{-\infty}^{\infty} \left(2 - \frac{p}{q}\right) Nh^2 \varphi(hx) f(x)$$

$$\times \frac{[e^{-2\pi hix(N+1)} - e^{-2\pi hix((1-(p/q))N+1)}]}{(e^{-2\pi hix} - 1)} dx$$

$$- \int_{-\infty}^{\infty} h^2 \varphi(hx) f(x)$$

$$\times \frac{\left[Ne^{-2\pi hix(N+2)} - \left(1 - \frac{p}{q}\right) Ne^{-2\pi hix((1-(p/q))N+2)}\right.}{(e^{-2\pi hix} - 1)^2}$$

$$+ \frac{\left. -(N+1)e^{-2\pi hix(N+1)} + \left(\left(1 - \frac{p}{q}\right)N + 1\right) e^{-2\pi hix((1-(p/q))N+1)}\right]}{(e^{-2\pi hix} - 1)^2} dx.$$

In each of the four integrals of (23) we can justify a passage to the limit under the integral sign; the integrands are uniformly bounded by $\mathscr{C}(\tau)|f(x)|$ where $\mathscr{C}(\tau)$ is independent of h (<1). Thus, assuming for the moment that $f \in L(-\infty, \infty)$ we obtain

(24) $$\lim_{h \to 0} hS_2(h)$$

$$= -\frac{1}{4\pi^2} \int_{-\infty}^{\infty} f(x)$$

$$\times \frac{\left[e^{-i\tau x} - \left(1 - \frac{p}{q}\right) i\tau x e^{-i\tau x} + e^{i\tau x(p/q)} - 2e^{-(1-(p/q))i\tau x}\right]}{x^2} dx.$$

Similarly,

(25) $$\lim_{h \to 0} hS_3(h)$$

$$= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} f(x) \frac{\left[e^{i\tau x(p/q)} - e^{i\tau x} + \left(1 - \frac{p}{q}\right) i\tau x e^{i\tau x}\right]}{x^2} dx.$$

Using (20), (22), (24) and (25) we see that the following result is established: if $f \in B_\tau$ is integrable then, for all real γ and all rational numbers p/q such that $0 \leqq p/q \leqq 1$, we have

$$(26)\quad -4\pi\tau e^{i\gamma}\sum_{k=-\infty}^{\infty}(-1)^k e^{((p/q)(k\pi+\gamma))i}\left(\frac{\sin\left(\frac{k\pi+\gamma}{2}\right)}{k\pi+\gamma}\right)^2 f\left(\frac{k\pi+\gamma}{\tau}\right)$$

$$= e^{2i\gamma}\int_{-\infty}^{\infty} f(x)$$

$$\times\frac{\left[e^{-i\tau x}-\left(1-\frac{p}{q}\right)i\tau xe^{-i\tau x}+e^{i\tau x(p/q)}-2e^{-(1-(p/q))i\tau x}\right]}{x^2}dx$$

$$+\int_{-\infty}^{\infty} f(x)\frac{\left[e^{i\tau x(p/q)}-e^{i\tau x}+\left(1-\frac{p}{q}\right)i\tau xe^{i\tau x}\right]}{x^2}dx.$$

Given any real number α in the interval $[0, 1]$ let us choose a sequence $\{p/q\}$ of rational numbers in $[0, 1]$ such that $\{p/q\} \to \alpha$. The summand in (26) is bounded by $4\pi\tau/(k\pi + \gamma)^2$ and the integrands by $D(\tau)|f(x)|$ for some constant $D(\tau)$ independent of the sequence $\{p/q\}$. Taking the appropriate limits in (26) we get:

$$(27)\quad -4\pi\tau e^{i\gamma}\sum_{k=-\infty}^{\infty}(-1)^k e^{\alpha(k\pi+\gamma)i}\left(\frac{\sin\left(\frac{k\pi+\gamma}{2}\right)}{k\pi+\gamma}\right)^2 f\left(\frac{k\pi+\gamma}{\tau}\right)$$

$$= e^{2i\gamma}\int_{-\infty}^{\infty} f(x)$$

$$\times\frac{[e^{-i\tau x}-(1-\alpha)i\tau xe^{-i\tau x}+e^{\alpha i\tau x}-2e^{-(1-\alpha)i\tau x}]}{x^2}dx$$

$$+\int_{-\infty}^{\infty} f(x)\frac{[e^{\alpha i\tau x}-e^{i\tau x}+(1-\alpha)i\tau xe^{i\tau x}]}{x^2}dx.$$

If f is not necessarily in $L(-\infty, \infty)$ but satisfies the condition (10) then the functions

$$g_\delta(z) := \frac{\sin\delta z}{\delta z}f(z),\quad \delta > 0,$$

are integrable functions of $B_{\tau+\delta}$ such that

$$|g_\delta(x)| \leqq |f(x)| \quad\text{and}\quad \lim_{\delta\to 0} g_\delta(z) = f(z).$$

Applying (27) to the functions g_δ we obtain

https://doi.org/10.4153/CJM-1988-040-3 Published online by Cambridge University Press

$$(28)\quad -4\pi(\tau+\delta)e^{i\gamma}\sum_{k=-\infty}^{\infty}(-1)^k e^{\alpha(k\pi+\gamma)i}\left(\frac{\sin\left(\frac{k\pi+\gamma}{2}\right)}{k\pi+\gamma}\right)^2 g_\delta\left(\frac{k\pi+\gamma}{\tau+\delta}\right)$$

$$= e^{2i\gamma}\int_{-\infty}^{\infty} g_\delta(x)$$

$$\times \frac{[e^{-i(\tau+\delta)x} - (1-\alpha)i(\tau+\delta)xe^{-i(\tau+\delta)x}}{x^2}$$

$$+ \frac{e^{\alpha i(\tau+\delta)x} - 2e^{-(1-\alpha)i(\tau+\delta)x}]}{x^2}dx$$

$$+ \int_{-\infty}^{\infty} g_\delta(x)$$

$$\times \frac{[e^{\alpha i(\tau+\delta)x} - e^{i(\tau+\delta)x} + (1-\alpha)i(\tau+\delta)xe^{i(\tau+\delta)x}]}{x^2}dx.$$

The two terms in brackets, in (28), are bounded, for large $|x|$ and small δ, by $E(\tau)|x|$; the integrands are thus bounded by

$$\frac{E(\tau)|x|\,|f(x)|}{x^2} \leqq \frac{K(\tau)}{|x|^{1+\epsilon}}, \quad (x \to \pm\infty).$$

The summand in the lefthand member of (28) being bounded by

$$\frac{8\pi\tau}{|k\pi+\gamma|^{2+\epsilon}} \quad (\text{for } \delta < \tau)$$

we may invoke the dominated convergence theorem and after the passage to the limit we see that formula (27) is valid with the less restrictive condition (10).

Finally, formula (3) is the same as (27) where we change α to $(1-\alpha)$.

Proof of Theorems 2 *and* 3. Lack of space does not permit us to include a detailed proof of Theorems 2 and 3. However, we point out that they are very similar to that of Theorem 1. We use respectively (17) and (18) to get the formulas corresponding to (19). The lefthand members of the resulting formulas are examined like the term $S_1(h)$ in (20); for the righthand members we use (14) and (15) to obtain explicit integrands whose limits, as $h \to 0$, are evaluated. The passages to the limit are justified exactly as in the proof of Theorem 1. We obtain (4) and (5) after possibly some obvious change of variables.

5. Complementary results.

5.1. If $h_f(\pi/2) \leqq 0$ then, in view of Lemma 2, we may write

$$f_h(x) = P_h(e^{2\pi hix})$$

where P_h is an algebraic polynomial of degree $\leqq N$ $(:= [\tau/2\pi h])$. All the coefficients C_j, $j = -1, -2, \ldots$, in (19), are thus equal to 0. Taking that observation into account we may follow the same line of reasoning as in the proof of Theorem 1 and we obtain

THEOREM 1′. *If we suppose, in addition to the hypothesis of Theorem* 1, *that* $h_f(\pi/2) \leqq 0$ *then*:

$$(3') \quad -4\pi\tau G_{\gamma,f}(\alpha) = \int_{-\infty}^{\infty} f(x)\frac{[e^{-i\tau x} - \alpha i\tau x e^{-i\tau x} - 2e^{-\alpha i\tau x} + 1 + (1-\alpha)i\tau x]}{x^2}dx.$$

An analogous observation can be made for Theorems 2 and 3. It is interesting to observe here that formulas (17) and (18) coincide when the trigonometric polynomial is of the form $S(\theta) = P(e^{i\theta})$ where P is an algebraic polynomial of degree $\leqq n$.

5.2. Under suitable restrictions we can differentiate both members of formula (3). In fact, if we differentiate two times then we obtain

COROLLARY 1. *Let* $f \in B_\tau$ *such that* $f(x) = O(|x|^{-\delta})$, $(\delta > 1$, $x \to \pm\infty)$. *For all reals* γ *and* $0 \leqq \alpha \leqq 1$ *we have*

$$(29) \quad \frac{4\pi}{\tau}\sum_{k=-\infty}^{\infty} e^{-\alpha(k\pi+\gamma)i}\sin^2\left(\frac{k\pi+\gamma}{2}\right)f\left(\frac{k\pi+\gamma}{\tau}\right) = \int_{-\infty}^{\infty} f(x)(2e^{-\alpha i\tau x} - e^{(1-\alpha)i\tau x})dx - e^{-2i\gamma}\int_{-\infty}^{\infty} f(x)e^{(1-\alpha)i\tau x}dx.$$

Here, the functions $f(z)e^{-\alpha i\tau z}$ and $f(z)e^{(1-\alpha)i\tau z}$ are respectively elements of $B_{(1+\alpha)\tau}$ and $B_{(2-\alpha)\tau}$. Using (13) we readily obtain, under the same hypothesis as in Corollary 1:

$$(30) \quad \sum_{k=-\infty}^{\infty} e^{-\alpha(k\pi+\gamma)i}\sin^2\left(\frac{k\pi+\gamma}{2}\right)f\left(\frac{k\pi+\gamma}{\tau}\right) = \frac{1}{(1+\alpha)}\sum_{k=-\infty}^{\infty} e^{2k\pi i/(1+\alpha)}f\left(\frac{2k\pi}{(1+\alpha)\tau}\right) - \frac{\cos\gamma e^{-i\gamma}}{(2-\alpha)}\sum_{k=-\infty}^{\infty} e^{-2k\pi i/(2-\alpha)}f\left(\frac{2k\pi}{(2-\alpha)\tau}\right).$$

As a particular case of (30) let us take $\alpha = 1/2$ and $\gamma = 0$; we obtain that if $f \in B_\tau$ is such that $f(x) = O(|x|^{-\delta})$, $\delta > 1$, $x \to \pm\infty$, then

$$(31)\quad 3\sum_{k=-\infty}^{\infty}(-1)^{k+1}f\left(\frac{(2k+1)\pi}{\tau}\right) = 4\sum_{k=-\infty}^{\infty}\sin\left(\frac{4k\pi}{3}\right)f\left(\frac{4k\pi}{3\tau}\right).$$

We have also

COROLLARY 1′. *If we suppose, in addition to the hypothesis of Corollary* 1, *that* $h_f(\pi/2) \leqq 0$ *then*:

$$(29')\quad \frac{2\pi}{\tau}\sum_{k=-\infty}^{\infty} e^{-\alpha(k\pi+\gamma)i}\sin^2\left(\frac{k\pi+\gamma}{2}\right)f\left(\frac{k\pi+\gamma}{\tau}\right)$$
$$= \int_{-\infty}^{\infty} f(x)e^{-\alpha i\tau x}dx.$$

Of course we can also integrate all the functions of α under consideration. The resulting formulas become more complicated.

Acknowledgment. This research was supported by the Natural Sciences and Engineering Research Council of Canada Grant No. OGP0009331.

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