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FACTORIZED GROUPS WITH max, min and min-p

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ABSTRACT. Let $\mathscr{X}$ be a class of groups which is closed under the forming of subgroups, epimorphic images and extensions. It is shown that every soluble product $G = AB$ of two $\mathscr{X}$-subgroups A and B, one of which satisfies max or min, is an $\mathscr{X}$-group (Theorem A). If X satisfies an additional requirement, then every soluble product $G = AB$ of two $\mathscr{X}$-subgroups A and B, one of which is a torsion group with min-p for every prime p, is an $\mathscr{X}$-group (Theorem B). Corollary: Every soluble product $G = AB$ of two π-subgroups A and B with min-p for every prime p in the set of primes π, is a π-group with min-p for every p.

Throughout the following let $\mathscr{X}$ be a class of groups which is closed under the forming of subgroups, epimorphic images and extensions. For which classes $\mathscr{X}$ is every soluble product $G = AB$ of two $\mathscr{X}$-subgroups A and B an $\mathscr{X}$-group? The following theorem generalizes Theorems A and B of [2] on this question.

THEOREM A. *If the soluble group $G = AB$ is the product of two $\mathscr{X}$-subgroups A and B, one of which satisfies* max *or* min, *then G is an $\mathscr{X}$-group.*

Here, a group satisfies max [min] if its subgroups satisfy the maximum [minimum] condition. A group satisfies min-p if its p-subgroups satisfy the minimum condition.

If the class $\mathscr{X}$ fulfills an additional requirement one can prove the following result.

THEOREM B. *Let every soluble torsion group with finite Sylow-p-subgroups for every prime p, which is factorized by two $\mathscr{X}$-subgroups, be an $\mathscr{X}$-group. If the soluble group $G = AB$ is the product of two $\mathscr{X}$-subgroups A and B, one of which is a torsion group with* min-p *for every prime p, then G is an $\mathscr{X}$-group.*

In this theorem the class $\mathscr{X}$ can for instance be the class of soluble torsion groups with min-p for every prime p or the class of soluble groups with finite sectional rank or the class of soluble groups with finite torsionfree rank. In particular Theorem B contains as a special case the following answer to a question of [3].

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COROLLARY. *If the soluble group $G = AB$ is the product of two π-subgroups A and B with* min-p *for every prime p in the set of primes π, then G is a π-group with* min-p *for every p.*

Among the many group classes $\mathfrak{X}$ that can be taken for the class $\mathfrak{X}$ in Theorem A are the class of groups with max, the class of groups with min, the class of polyminimax groups, the class of π-groups with min-p for every prime p, the class of groups with finite Prüfer rank, the class of soluble groups with finite sectional rank and the class of soluble groups with finite torsionfree rank. Using the arguments below and the results of [8] one can easily prove that for each of these particular group classes $\mathfrak{X}$ the following holds:

If the soluble group $G = AB$ is the product of two $\mathfrak{X}$-subgroups A and B, one of which is abelian, then G is an $\mathfrak{X}$-group.

This raises the following question:

Is for each of these special group classes $\mathfrak{X}$ every soluble product of two $\mathfrak{X}$-groups an $\mathfrak{X}$-group?

To prove the above theorems one has to show that an abelian normal subgroup K of a product $G = AB$ of two $\mathfrak{X}$-subgroups A and B is an $\mathfrak{X}$-group. For this the factorizer $X = \mathfrak{X}(K)$ of K in G is investigated, which is the smallest factorized subgroup of G which contains K; see [1], Theorem 1.7, p. 108. This is done in Theorem 1.1 and 2.3 below. Since this question seems to be of independent interest, we have formulated these results in a rather general form, to make them applicable for later use.

The notation of this paper is standard and may be found in [5] or [7]. Note that $\mathfrak{X}$ is always a class of groups inherited by subgroups, epimorphic images and extensions.

1. **Groups with max or min.** If K is an abelian normal subgroup of the group $G = AB$, which is the product of two $\mathfrak{X}$-subgroups A and B, and if $X = \mathfrak{X}(K)$ is the factorizer of K in G, then by [1], Theorem 1.7, p. 108,

$$X = AK \cap BK = KA^* = KB^* = A^*B^*$$

with $A^* = A \cap BK$ and $B^* = B \cap AK$. Since K is abelian, $C = (A^* \cap K)(B^* \cap K)$ is an abelian normal subgroup of X. As an abelian product of two $\mathfrak{X}$-subgroups C is an $\mathfrak{X}$-group. The factor group X/C has the following properties

$$X/C = (K/C)(A^*C/C) = (K/C)(B^*C/C) = (A^*C/C)(B^*C/C)$$

where the abelian normal subgroup K/C of X/C satisfies

$$(K/C) \cap (A^*C/C) = (K/C) \cap (B^*C/C) = 1.$$

In particular $A^*C/C \simeq B^*C/C$.

These observations may be used to obtain criteria for X to be an $\mathfrak{X}$-group.

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THEOREM 1.1. *Let the group $G = AB$ be the product of two $\mathscr{X}$-subgroups A and B. The factorizer $X = \mathscr{X}(K)$ of the abelian normal subgroup K of G is an $\mathscr{X}$-group if at least one of the following conditions holds*:

(a) *X is hyperabelian and A or B satisfies* max,

(b) *X is hyperabelian or locally soluble and A or B satisfies* min.

Proof. Consider the reduction above with the same notations. Then $A^*C/C \simeq B^*C/C$.

In case (b) the hyperabelian or locally soluble group $X/C = (A^*C/C)(B^*C/C)$ is the product of two $\mathscr{X}$-subgroups A^*C/C and B^*C/C with min. By [5], Theorem 5.8, p. 172, A^*C/C and B^*C/C are Černikov groups. Hence by [4], Theorem 4, X/C is a soluble Černikov group. Since X/C is the product of two $\mathscr{X}$-subgroups, it is an $\mathscr{X}$-group by [2], Lemma 3.1, p. 9. Since C is also an $\mathscr{X}$-group, X is an $\mathscr{X}$-group.

In case (a) the hyperabelian group X/C is the product of two $\mathscr{X}$-subgroups A^*C/C and B^*C/C with max. By [1], Corollary 3.3, p. 112, X/C satisfies the maximum condition on normal subgroups, so that X/C is soluble. As a soluble product of two subgroups with max also X/C satisfies max and therefore is polycyclic; see [6] or [9]. By the following Lemma 1.2 X/C is an $\mathscr{X}$-group as a polycyclic product of two $\mathscr{X}$-groups. Since C is an $\mathscr{X}$-group, X is an $\mathscr{X}$-group. This proves the theorem.

LEMMA 1.2. *If the polycyclic group $G = AB$ is the product of two $\mathscr{X}$-subgroups A and B, then G is an $\mathscr{X}$-group.*

Proof. Every polycyclic group G has a finite series leading from 1 to G with cyclic factors in which the first factors are all infinite; see [7], 5.4.15., p. 148. If G is finite, then G is an $\mathscr{X}$-group as a finite soluble product of two $\mathscr{X}$-groups. If G is infinite, A or B is infinite and the infinite cyclic group is an $\mathscr{X}$-group. Since $\mathscr{X}$ is closed under the forming of extensions, it is now easy to see that G is an $\mathscr{X}$-group.

REMARK 1.3. If every finite product of two $\mathscr{X}$-subgroups is an $\mathscr{X}$-group, then Theorem 1.1 also holds if $\mathscr{X}$ is locally finite and A or B satisfies min.

2. **Groups with min-*p*.** The following reduction lemma is useful in the investigation of groups factorized by two subgroups with min-p for the prime p.

LEMMA 2.1. *Let the group $G = AB$ be the product of two subgroups A and B with min-p for the prime p. If the locally finite-soluble factorizer $X = \mathscr{X}(K)$ of the abelian normal subgroup K of G does not satisfy min-p, then there exists an epimorphic image $\bar{X}$ of X which does not satisfy* min-p *with the following properties*:

$$\text{(i)} \qquad X = \overline{KA} = \overline{KB} = \overline{AB} \quad \textit{and} \quad \bar{K} \cap \bar{A} = \bar{K} \cap \bar{B} = 1$$

where the normal subgroup $\bar{K}$ of $\bar{X}$ is an epimorphic image of K and the subgroups $\bar{A}$ and $\bar{B}$ of $\bar{X}$ are factors of A and B and therefore satisfy min-p,
(ii) *If the p-subgroups of A and B are finite, the p-subgroups of $\bar{A}$ and $\bar{B}$ are finite, if X has infinite p-subgroups then so does $\bar{X}$,*

(iii) $$\mathcal{O}_{p'}(\bar{X})=1.$$

Proof. Since $X=\mathscr{X}(K)$ is the factorizer of K,

$$X=KA^*=KB^*=A^*B^*$$

with $A^*=A\cap BK$ and $B^*=B\cap AK$. If $Q=\mathcal{O}_{p'}(X)$, then

$$X/Q=(KQ/Q)(A^*Q/Q)=(KQ/Q)(B^*Q/Q)=(A^*Q/Q)(B^*Q/Q)$$

where $KQ/Q\simeq K/(K\cap Q)\simeq K_p$ is a normal abelian p-subgroup of X/Q and $A^*Q/Q\simeq A^*/(A^*\cap Q)$ and $B^*Q/Q\simeq B^*/(B^*\cap Q)$ satisfy min-p (and have finite p-subgroups in case (ii)). Then

$$C/Q=((KQ/Q)\cap(A^*Q/Q))((KQ/Q)\cap(B^*Q/Q))$$

is a normal p-subgroup of X/C which satisfies min-p and hence is a Černikov p-group (and even finite in case (ii)). Consider

$$\begin{aligned}\bar{X}&=(X/Q)/(C/Q)\simeq X/C\\ \bar{K}&=(KQ/Q)(C/Q)/(C/Q)\simeq(KC/C)\simeq K/(K\cap C)\\ \bar{A}&=(A^*Q/Q)(C/Q)/(C/Q)\simeq A^*C/C\simeq A^*/(A^*\cap C)\\ \bar{B}&=(B^*Q/Q)(C/Q)/(C/Q)\simeq B^*C/C\simeq B^*/(B^*\cap C).\end{aligned}$$

Since C satisfies min-p (has finite p-subgroups) and X does not, $\bar{X}$ does not satisfy min-p (has infinite p-subgroups). Then

$$\bar{X}=\overline{KA}=\overline{KB}=\overline{AB},\qquad \bar{K}\cap\bar{A}=\bar{K}\cap\bar{B}=1,$$

where $\bar{K}$ is a normal abelian p-subgroup of $\bar{X}$ and the subgroups $\bar{A}$ and $\bar{B}$ of $\bar{X}$ satisfy min-p (have finite p-subgroups).

The maximal normal p'-subgroup of X/C has the form $\mathcal{O}_{p'}(X/C)=S/C$ with a normal subgroup S of X containing C. Since S is a normal subgroup of X containing Q,

$$\mathcal{O}_{p'}(S)=\mathcal{O}_{p'}(X)=Q\subseteq C.$$

Since C/Q is a Černikov (finite) p-group and S/C and Q are p'-groups, S satisfies min-p (has finite p-subgroups). Since S is locally finite-soluble, by [5], Theorem 3.17, p. 94, $S/\mathcal{O}_{p'}(S)=S/Q$ is an almost-p group and hence a Černikov group (is finite). As a p'-factor of such a group S/C is finite. Hence $\mathcal{O}_{p'}(\bar{X})$ is finite. This proves the lemma.

In any group G let $\mathscr{J}(G)$ be the intersection of all subgroups of finite index

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in G. If G is locally finite-soluble with min-p for every prime p, then by [5], Corollary 3.18, p. 95, $\mathcal{J}(G)$ is a divisible abelian torsion normal subgroup of G whose primary components have finite rank and the Sylow-p-subgroups of $G/\mathcal{J}(G)$ are finite for every p.

LEMMA 2.2. *If the radical torsion group $G = AB = KA = KB$ is the product of an abelian normal p-subgroup K and subgroups A and B with $K \cap A = K \cap B = 1$ such that $A \simeq B$ satisfies min-q for every prime q and $\mathcal{O}_{p'}(G)$ is finite, then G is an almost-p Černikov group. In particular, if the p-subgroups of A and B are finite, then G is finite.*

Proof. Let q be any prime. By hypothesis the maximal normal q-subgroup $\mathcal{O}_q(A)$ of A is a Černikov group. If $\mathcal{J}_q(A) = \mathcal{J}(\mathcal{O}_q(A))$, then $\mathcal{J}_q(A)$ is a divisible abelian normal q-subgroup of A of finite rank and $\mathcal{O}_q(A)/\mathcal{J}_q(A)$ is finite. The structure of $\mathcal{O}_q(B)$ is similar.

By [3], Lemma 1.2, $\mathcal{J}_q(A)K = \mathcal{J}_q(B)K$. By [3], Lemma 1.3, $\langle \mathcal{J}_q(A), \mathcal{J}_q(B)\rangle$ is a q-group. Let $q \neq p$. If $a \in \mathcal{J}_q(A)$, then $a = bx$ with $b \in \mathcal{J}_q(B)$ and $x \in K$. Hence $ab^{-1} = x$ is a q-element and a p-element, so that $a = b$. This show $\mathcal{J}_q(A) \subseteq \mathcal{J}_q(B)$. Similarly $\mathcal{J}_q(B) \subseteq \mathcal{J}_q(A)$, so that $\mathcal{J}_q(A) = \mathcal{J}_q(B)$ is a normal q-subgroup of G. Since $\mathcal{O}_{p'}(G)$ is finite, $\mathcal{J}_q(A) = \mathcal{J}_q(B) = 1$. Hence $\mathcal{O}_q(A)$ and $\mathcal{O}_q(B)$ are finite.

By [3], Lemma 1.2, $\mathcal{O}_q(A)K = \mathcal{O}_q(B)K$. By [3], Lemma 1.3, $\langle \mathcal{O}_q(A), \mathcal{O}_q(B)\rangle$ is a finite q-group. If $a_1 \in \mathcal{O}_q(A)$, then $a_1 = b_1x_1$ with $b_1 \in \mathcal{O}_q(A)$ and $x_1 \in K$. Hence $a_1b_1^{-1} = x_1$ is a q-element and a p-element, so that $a_1 = b_1$. This shows that $\mathcal{O}_q(A) \subseteq \mathcal{O}_q(B)$ is a normal q-subgroup of G.

Since G is locally finite, $\mathcal{O}_q(A) = \mathcal{R}_q(A)$ is the q-component of the Hirsch–Plotkin radical $\mathcal{R}(A)$ of A. Hence

$$\mathcal{R}_{p'}(A) = \prod_{q \neq p} \mathcal{O}_q(A) \subseteq \mathcal{R}_{p'}(G) \subseteq \mathcal{O}_{p'}(G),$$

so that $\mathcal{R}_{p'}(A)$ is finite. Since $\mathfrak{R}_p(A) = \mathcal{O}_p(A)$ is a Černikov p-group, $\mathcal{R}(A) = \mathcal{R}_{p'}(A) \times \mathcal{R}_p(A)$ is a Černikov group. Since A is radical, $\mathcal{R}(A)$ contains its centralizer in A. Thus $A/\mathcal{R}(A)$ is a torsion group of automorphism of the Černikov group $\mathcal{R}(A)$ and hence is a Černikov group; see [5], Theorem 1.F.3, p. 35. It follows that A is a Černikov group, so that also $B \simeq A$ is a Černikov group. By [4], Theorem 4, G is a Černikov group. Since $\mathcal{O}_{p'}(G)$ is finite, G is an almost-p group. By [1], Lemma 5.6, p. 113,

$$\mathcal{J}(G) = \mathcal{J}(A)\mathcal{J}(B).$$

If the p-subgroups of A and B are finite, then $\mathcal{J}(G) = \mathcal{J}(A)\mathcal{J}(B) = 1$ and G is finite.

The following theorem contains information on the factorizer of an abelian normal torsion subgroup of a group which is the product of two subgroups with min-p for every prime p.

THEOREM 2.3. *If the group $G = AB$ is the product of two subgroups A and B with* min-p *for every prime p, then the radical torsion factorizer $X = \mathscr{X}(K)$ of an abelian normal subgroup K of G satisfies* min-p *for every prime p. If the p-subgroups of A and B are finite, then the p-subgroups of X are finite.*

Proof. Assume that X does not satisfy min-p for the prime p. By Lemma 2.1 there exists an epimorphic image $\bar{X}$ of X which does not satisfy min-p and which has certain properties. These imply that by Lemma 2.2 $\bar{X}$ is an almost-p Černikov group. If the p-subgroups of A and B are finite, then $\bar{X}$ is finite. This proves the theorem.

The following lemma contains further information on the structure of the subgroup X in Theorem 2.3.

LEMMA 2.4. *If the radical torsion group $G = AB$* with min-p *for every prime p is the product of two subgroups A and B, then*

$$\mathscr{J}(G) = \mathscr{J}(A)\mathscr{J}(B)$$

Proof. For every prime p the p-component $J_p = \mathscr{J}_p(G)$ of $\mathscr{J}(G)$ is a normal p-subgroup of G. If $X = \mathscr{X}(J_p)$ is the factorizer of $\mathscr{J}_p$, then

$$X = \mathscr{X}(J_p) = J_pA^* = J_pB^* = A^*B^*$$

with $A^* = A \cap BJ_P$ and $B^* = B \cap AJ_p$. Since the p-component $\mathscr{J}_p(A)$ of $J(A)$ is contained in the abelian group J_p, $\mathscr{J}_p(A)$ is a normal subgroup of $A^*J_p = X$. Similarly $\mathscr{J}_p(B)$ is a normal subgroup of $B^*J_p = X$. Hence

$$N = \mathscr{J}_p(A)\mathscr{J}_p(B)$$

is a normal subgroup of X contained in J_p. Since the p-subgroups of $A/\mathscr{J}_p(A)$ and $B/\mathscr{J}_p(B)$ are finite, the p-subgroups of A^*N/N and B^*N/N are finite. It is easy to see that X/N is the factorizer of J_p/N in $X/N = (A^*N/N)(B^*N/N)$. By Theorem 2.3 the p-subgroups of X/N are finite. In particular the p-subgroup J_p/N of X/N is finite and hence must be trivial. Therefore $J_p = N = \mathscr{J}_p(A)\mathscr{J}_p(A)\mathscr{J}_p(B)$. It follows that

$$\mathscr{J}(G) = \prod_p \mathscr{J}_p(A)\mathscr{J}_p(B) = \prod_p \mathscr{J}_p(A) \prod_p \mathscr{J}_p(B) = \mathscr{J}(A)\mathscr{J}(B)$$

This proves the lemma.

We specialize now these results for soluble groups.

COROLLARY 2.5. *If the soluble group $G = AB$ is the product of two π-subgroups A and B with* min-p *for every prime p in the set of prime π, then G is a*

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π-group with min-p *for every p and*

$$\mathscr{J}(G)=\mathscr{J}(A)\mathscr{J}(B).$$

In particular, if the p-subgroups of A and B are finite for some prime p, then the p-subgroups of G are finite.

Proof. By [3], Corollary 2.2, G is a π-group. Assume that $G=AB$ is a counterexample of minimal derived length which does not satisfy min-p for the prime p. The last nontrivial term K of the derived series of G is an abelian normal subgroup of G. By Theorem 2.3 the factorizer $X=\mathscr{X}(K)$ of K satisfies min-p. Since G/K satisfies min-p for every prime p, G satisfies min-p. The remaining statements follow from Lemma 2.4.

3. **Proof of Theorems A and B.** Assume that not every soluble product of two $\mathscr{X}$-groups is an $\mathscr{X}$-group. Then there exists a counterexample $G=AB$ of minimal derived length. The last nontrivial term K of the derived series of G is an abelian normal subgroup of G such that G/K is an $\mathscr{X}$-group. Consider the factorizer

$$X=\mathscr{X}(K)=A^*K=B^*K=A^*B^*$$

with $A^*=A\cap BK$ and $B^*=B\cap AK$. In the case of Theorem A it follows from Theorem 1.1 that X is an $\mathscr{X}$-group, so that K is an $\mathscr{X}$-group, a contradiction.

In the case of Theorem B observe that by the reduction in the beginning of section 1 it may be assumed that $A^*\simeq B^*$ is a torsion group with min-p for every prime p. By Corollary 2.5 X is a torsion group with min-p for every p and $\mathscr{J}(X)=\mathscr{J}(A^*)\mathscr{J}(B^*)$. Since $\mathscr{J}(A^*)$ and $\mathscr{J}(B^*)$ are $\mathscr{X}$-groups, also the abelian group $\mathscr{J}(X)$ is an $\mathscr{X}$-group. The group $X/\mathscr{J}(X)=(A^*\mathscr{J}(X)/\mathscr{J}(X))(B^*\mathscr{J}(X)/\mathscr{J}(X))$ is the product of two $\mathscr{X}$-subgroups and has finite Sylow-p-subgroups for every prime p. By hypothesis $X/\mathscr{J}(X)$ is therefore an $\mathscr{X}$-group, so that $\mathscr{X}$ and hence K are $\mathscr{X}$-groups. It follows that G is an $\mathscr{X}$-group. This contradiction proves Theorem B.

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