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SOME THEOREMS ON A VOLTERRA EQUATION OF THE SECOND KIND

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In this paper we state and prove three theorems on positive solutions of a Volterra equation of the second kind. The equation considered is

$$u(x) = f(x)+\int_0^x K(x, t)u(t)\,dt \tag{1}$$

where $K(x, t)$ is a Volterra type kernel, that is, $K(x, t)=0$ for $t>x$. Unless otherwise stated, we will assume that $f\in L_2^+(I)$ and $K\in L_2^+(I\times I)$, where $I=\{x: 0\le x\le\infty\}$ and $L_2^+(I)=\{f: f\in L_2(I) \text{ and } f(x)\ge 0 \text{ for } x\in I\}$.

Define $K_1(x, t)=K(x, t)$ and for $n\ge 2$,

$$K_n(x, t)=\int_t^x K(x, s)K_{n-1}(s, t)\,ds.$$

By the *resolvent kernel* we mean $H(x, t)=\sum_{i=1}^{\infty} K_i(x, t)$. This series converges in the $L_2(I\times I)$ norm.

For completeness we state without proof the principal theorem for Volterra equations.

THEOREM 0. *Let $f\in L_2(I)$ and $K\in L_2(I\times I)$ then the equation*

$$u(x) = f(x)+\int_0^x K(x, t)u(t)\,dt$$

has a unique L_2-solution given by

$$u(x) = f(x)+\int_0^x H(x, t)f(t)\,dt \quad \text{a.e. on } I.$$

For the proof of this theorem the reader is referred to [2].

We will need the following lemmas. The reader is referred to [1] for the proofs.

LEMMA 1. *Let $f\in L_2^+(I)$ and $K\in L_2^+(I\times I)$ in* (1), *then $u(x)\ge 0$* a.e. *on I.*

LEMMA 2. *Let $v\in L_2(I)$, and let $v(x)\le f(x)+\int_0^x K(x, t)v(t)\,dt$* a.e. *on I. If u is the unique L_2-solution of* (1), *then $v(x)\le u(x)$* a.e. *on I.*

THEOREM 1. *Let $f_i \in L_2^+(I)$ and let $K_i \in L_2^+(I\times I)$, and let u_i be the unique L_2-solution of*

$$u_i(x) = f_i(x)+\int_0^x K_i(x, t)u_i(t)\, dt$$

where $i=1,\ldots,n$. If

$$F(x) = \sum_{i=1}^n f_i(x) \quad \text{for } x\in I \qquad \textit{and} \qquad K(x, t) = \sum_{i=1}^n K_i(x, t) \quad \text{for } (x, t)\in I\times I,$$

then the unique L_2-solution u of $u(x)=F(x)+\int_0^x K(x, t)u(t)\, dt$ exists, and $\sum_{i=1}^n u_i(x)\le u(x)$ a.e. *on I.*

Proof. Since $f_i\in L_2(I)$ and $K_i\in L_2(I\times I)$ then $F\in L_2(I)$ and $K\in L_2(I\times I)$, and so, from Theorem 0, $u(x)$ exists. The proof that $\sum_{i=1}^n u_i(x)\le u(x)$ a.e. on I is by induction on n.

The theorem is obvious for $n=1$. Assume its truth for $i=1,\ldots,n-1$. Let

$$v(x) = \sum_{i=1}^{n-1} f_i(x)+\int_0^x \sum_{i=1}^{n-1} K_i(x, t)v(t)\, dt.$$

Therefore

$$\begin{aligned} u_n(x)+v(x) &= f_n(x)+\sum_{i=1}^{n-1} f_i(x)+\int_0^x K_n(x, t)u_n(t)\, dt \\ &\quad +\int_0^x \sum_{i=1}^{n-1} K_i(x, t)v(t)\, dt \quad \text{a.e. on } I. \end{aligned}$$

From Lemma 1, $u_i(x)\ge 0$ a.e. on I for $i=1,\ldots,n$. Therefore

$$u_n(x)+v(x) \le \sum_{i=1}^n f_i(x)+\int_0^x \sum_{i=1}^n K_i(x, t)[u_n(t)+v(t)]\, dt$$

or

$$u_n(x)+v(x) \le F(x)+\int_0^x K(x, t)[u_n(t)+v(t)]\, dt \quad \text{a.e. on } I.$$

Therefore, from Lemma 2 we have $u_n(x)+v(x)\le u(x)$. But by our inductive assumption

$$\sum_{i=1}^{n-1} u_i(x) \le v(x) \quad \text{a.e. on } I.$$

Therefore

$$\sum_{i=1}^{n} u_i(x) \le u(x) \quad \text{a.e. on } I.$$

This completes the proof.

We need the following definition before we state and prove the second theorem.

DEFINITION 1. y is a δ-approximate solution of (1) iff

$$|y(x)-f(x)-\int_0^x K(x,t)y(t)\,dt| \le \delta(x) \quad \text{a.e. on } I.$$

THEOREM 2. *Let $G\in L_2^+(I\times I)$, and let $G(x,t)\ge K(x,t)\ge 0$ on $I\times I$, and let y be a δ-approximate solution of* (1). *If $\delta\in L_2^+(I)$, u is the unique L_2-solution of* (1), *and v is the unique L_2-solution of*

$$v(x) = \delta(x)+\int_0^x G(x,t)v(t)\,dt,$$

then

$$|y(x)-u(x)| \le v(x) \quad \text{a.e. on } I.$$

Proof. Now

$$y(x) \le \delta(x)+f(x)+\int_0^x K(x,t)y(t)\,dt \quad \text{a.e. on } I.$$

Since u is the unique L_2 solution of (1),

$$y(x)-u(x) \le \delta(x)+\int_0^x K(x,t)[y(t)-u(t)]\,dt \quad \text{a.e. on } I.$$

Let

$$w(x) = \delta(x)+\int_0^x K(x,t)w(t)\,dt.$$

Then

$$w(x) \le \delta(x)+\int_0^x G(x,t)w(t)\,dt$$

by Lemma 1. Therefore from Lemma 2 we have

$$y(x)-u(x) \le w(x) \le v(x), \quad \text{or} \quad y(x)-u(x) \le v(x) \quad \text{a.e. on } I.$$

Similarly a.e. on I

$$y(x) \ge -\delta(x)+f(x)+\int_0^x K(x,t)y(t)\,dt,$$

and hence by Lemmas 1 and 2 we have

$$u(x)-y(x) \le v(x) \quad \text{a.e. on } I.$$

Therefore $|y(x)-u(x)|\le v(x)$ a.e. on I. This completes the proof.

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We can see from Lemma 1 that it is impossible to obtain a solution u of (1) such that $0 \le u(x) \le f(x)$ on I when $K(x, t) \ge 0$ on $I \times I$. The natural question therefore arises: Under what conditions will (1) have a solution u such that $0 \le u(x) \le f(x)$ on I? Clearly $K(x, t)$ must be negative – at least for x near zero and $0 \le t \le x$.

In our next theorem we give conditions for such a solution. Instead of (1) with $K(x, t) \le 0$ on $I \times I$ we will consider

$$u(x) = f(x) - \int_0^x K(x, t)u(t)\,dt \tag{2}$$

with $K(x, t) \ge 0$.

THEOREM 3. *Let $f \in C(I)$ with $f(x) > 0$ on I, and let $K \in C(T)$ with $K(x, t) > 0$ on*

$$T = \{(x, t): 0 \le t \le x \le \infty\}.$$

If

$$\frac{f(x)}{f(y)} \le \frac{K(x, t)}{K(y, t)} \quad \text{for } 0 \le t \le x \le y,$$

then the unique solution u of (2) *satisfies $f(x) \ge u(x) \ge 0$ on I.*

Proof. If $u(x) \ge 0$ then it follows from (2) that $f(x) \ge u(x) \ge 0$. It therefore remains to show that $u(x)$ cannot be negative anywhere on I.

Suppose the theorem is false. Then since $u(0) = f(0) > 0$, by continuity there exist $x_1 > 0$ and $\delta > 0$ such that

$$\begin{aligned} u(x) &\ge 0, & 0 < x < x_1 \\ &= 0, & x = x_1 \\ &< 0, & x_1 < x \le x_1 + \delta. \end{aligned}$$

Therefore for $x_1 < x \le x_1 + \delta$ we have

$$\begin{aligned} 0 > u(x) &> f(x) - \int_0^{x_1} K(x, t)u(t)\,dt \\ &= \frac{f(x)}{f(x_1)} \left\{ f(x_1) - \frac{f(x_1)}{f(x)} \int_0^{x_1} K(x, t)u(t)\,dt \right\} \\ &\ge \frac{f(x)}{f(x_1)} \left\{ f(x_1) - \int_0^{x_1} K(x_1, t)u(t)\,dt \right\} \\ &= \frac{f(x)}{f(x_1)} u(x_1) \\ &= 0 \end{aligned}$$

or $0 > u(x) \ge 0$ for $x_1 < x \le x_1 + \delta$, which is absurd.

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Therefore the theorem is true, i.e.

$$f(x) \geq u(x) \geq 0 \text{ on } I.$$

COROLLARY 1. *If $K(x, t)$ is monotone decreasing in x and $f(x)$ is monotone increasing, the hypotheses of the above theorem are satisfied. In particular, if $K(x, t) = k(x-t)$, then it suffices to have k monotone decreasing and f monotone increasing.*

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