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ABSTRACT

We propose a variation of the classical Hilbert scheme of points, the *double nested Hilbert scheme of points*, which parametrizes flags of zero-dimensional subschemes whose nesting is dictated by a Young diagram. Over a smooth quasi-projective curve, we compute the generating series of topological Euler characteristic of these spaces, by exploiting the combinatorics of reversed plane partitions. Moreover, we realize this moduli space as the zero locus of a section of a vector bundle over a smooth ambient space, which therefore admits a virtual fundamental class. We apply this construction to the stable pair theory of a local curve, that is the total space of the direct sum of two line bundles over a curve. We show that the invariants localize to virtual intersection numbers on double nested Hilbert scheme of points on the curve, and that the localized contributions to the invariants are controlled by three universal series for every Young diagram, which can be explicitly determined after the anti-diagonal restriction of the equivariant parameters. Under the anti-diagonal restriction, the invariants are matched with the Gromov–Witten invariants of local curves of Bryan–Pandharipande, as predicted by the Maulik–Nekrasov–Okounkov–Pandharipande (MNOP) correspondence. Finally, we discuss K -theoretic refinements à la Nekrasov–Okounkov.

1. Introduction

1.1 Double nested Hilbert scheme of points

Let X be a quasi-projective scheme over \mathbb{C} . We denote by $X^{[n]}$ the *Hilbert scheme of n points* on X , which parametrizes zero-dimensional closed subschemes $Z \subset X$ of length n . Given a tuple of non-decreasing integers $\mathbf{n} = (n_0 \leq \dots \leq n_d)$, the *nested Hilbert scheme of points* $X^{[\mathbf{n}]}$ parametrizes flags of zero-dimensional subschemes $(Z_0 \subset \dots \subset Z_d)$ of X , where each Z_i has length n_i . The scheme structure of these moduli spaces has been intensively studied in the literature; see, for example, [Che98].

We propose a variation of this moduli space, by parametrizing flags of subschemes nesting in two directions. Let λ be a Young diagram and $\mathbf{n}_\lambda = (n_\square)_{\square \in \lambda}$ a reversed plane partition, that is a labelling of λ by non-negative integers non-decreasing in rows and columns. We denote by $X^{[\mathbf{n}_\lambda]}$ the *double nested Hilbert scheme of points*,

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the moduli space parametrizing flags of zero-dimensional closed subschemes $(Z_{\square})_{\square \in \lambda} \subset X$

$$\begin{array}{cccccc}
 Z_{00} & \subset & Z_{01} & \subset & Z_{02} & \subset & Z_{03} & \subset & \dots \\
 \cap & & \cap & & \cap & & \cap & & \\
 Z_{10} & \subset & Z_{11} & \subset & Z_{12} & \subset & Z_{13} & \subset & \dots \\
 \cap & & \cap & & \cap & & \cap & & \\
 Z_{20} & \subset & Z_{21} & \subset & Z_{22} & \subset & \dots & & \\
 \cap & & \cap & & \cap & & & & \\
 \dots & & \dots & & \dots & & & &
 \end{array}$$

where each Z_{\square} has length n_{\square} . If λ is a horizontal or vertical Young diagram, the nesting is linear and we recover the usual nested Hilbert scheme of points.

The scheme structure of these moduli spaces is interesting already in dimension one, for a smooth curve C . Cheah proved [Che98] that the nested Hilbert scheme $C^{[n]}$ is smooth, being isomorphic to a product of symmetric powers of C via a Hilbert–Chow-type morphism. However, as soon as we allow double nestings, $C^{[n\lambda]}$ can have several irreducible components (see Example 2.6), therefore failing to be smooth.

Our first result is a closed formula for the generating series of topological Euler characteristic of $C^{[n\lambda]}$ in terms of the hook-lengths $h(\square)$ of λ .

THEOREM 1.1 (Theorem 2.11). *Let C be a smooth quasi-projective curve and λ a Young diagram. Then*

$$\sum_{n\lambda} e(C^{[n\lambda]})q^{|n\lambda|} = \prod_{\square \in \lambda} (1 - q^{h(\square)})^{-e(C)}.$$

This is achieved by exploiting the power structure on the Grothendieck ring of varieties $K_0(\text{Var}_{\mathbb{C}})$, by which we reduce to the combinatorial problem of counting the number of reversed plane partitions of a given Young diagram, which was solved by Stanley [Sta71] and Hillman and Grassl [HG76]. Motivic analogues of this formula are studied in [MR22].

1.2 Virtual fundamental class

The double nested Hilbert scheme $C^{[n\lambda]}$ is in general singular, making it hard to perform intersection theory. To remedy this, we show that $C^{[n\lambda]}$ admits a *perfect obstruction theory* in the sense of Behrend and Fantechi [BF97] and Li and Tian [LT98]. In fact, we can (globally!) realize $C^{[n\lambda]}$ as the zero locus of a section of a vector bundle over a smooth ambient space.

THEOREM 1.2 (Theorem 2.8). *Let C be an irreducible smooth quasi-projective curve. There exists a section s of a vector bundle \mathcal{E} over a smooth scheme $A_{C,n\lambda}$ such that*

$$C^{[n\lambda]} \cong Z(s) \hookrightarrow A_{C,n\lambda} \xleftarrow{\mathcal{E}}$$

By this construction $C^{[n\lambda]}$ naturally admits a perfect obstruction theory (see Example 2.7) and, in particular, carries a virtual fundamental class $[C^{[n\lambda]}]^{\text{vir}}$, which recovers the usual fundamental class in the case where the nesting is linear. We pause a moment to explain this construction in the easiest interesting example, that is for the reversed plane partition

n_{00}	n_{01}
n_{10}	n_{11}

The embedding in the smooth ambient space is

$$C^{[n\lambda]} \hookrightarrow A_{C, n\lambda} := C^{[n00]} \times C^{[n10-n00]} \times C^{[n11-n01]} \times C^{[n01-n00]} \times C^{[n11-n10]},$$

$$(Z_{00}, Z_{01}, Z_{10}, Z_{11}) \mapsto (Z_{00}, Z_{10} - Z_{00}, Z_{11} - Z_{01}, Z_{01} - Z_{00}, Z_{11} - Z_{10}).$$

In other words, $A_{C, n\lambda}$ records the subscheme in position $(0, 0)$ and all possible vertical and horizontal differences of subschemes, where sum and difference are well-defined by seeing the closed subschemes Z_{ij} as divisors on C . At the level of closed points, the image of the embedding is given by all $(Z_{00}, X_1, X_2, Y_1, Y_2) \in A_{C, n\lambda}$ such that $X_1 + Y_2 = Y_1 + X_2$, again as divisors. Note that $X_1 + Y_2$ and $Y_1 + X_2$ are effective divisors of the same degree, therefore they are equal if and only if one is contained into the other, say $X_1 + Y_2 \subset Y_1 + X_2$.

This relation is encoded into a section of a vector bundle \mathcal{E} , as we now explain. Denote by $\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}_1, \mathcal{Y}_2$ the universal divisors on $A_{C, n\lambda} \times C$ and set

$$\Gamma^1 = \mathcal{Y}_1 + \mathcal{X}_2,$$

$$\Gamma^2 = \mathcal{X}_1 + \mathcal{Y}_2.$$

The vector bundle \mathcal{E} is defined as

$$\mathcal{E} = \pi_* \mathcal{O}_{\Gamma^2}(\Gamma^1),$$

where $\pi : A_{C, n\lambda} \times C \rightarrow A_{C, n\lambda}$ is the projection. The section s of \mathcal{E} is that induced, via π_* , by the section of $\mathcal{O}_{A_{C, n\lambda} \times C}(\Gamma^1)$ which vanishes on Γ^1 and then restricted to Γ^2 .

1.3 Stable pair invariants of local curves

Let C be a smooth projective curve and L_1, L_2 two line bundles over C . We denote by *local curve* the total space $X = \text{Tot}_C(L_1 \oplus L_2)$ with its natural $\mathbf{T} = (\mathbb{C}^*)^2$ -action on the fibers.

For $d > 0$ and $n \in \mathbb{Z}$, we denote by $P_X = P_n(X, d[C])$ the moduli space of stable pairs $[\mathcal{O}_X \xrightarrow{s} F] \in D^b(X)$ with curve class $d[C]$ and $\chi(F) = n$. The moduli space P_X has a perfect obstruction theory [PT09a], but is, in general, non-proper. Still, the \mathbf{T} -action on X induces one on P_X with proper \mathbf{T} -fixed locus $P_X^{\mathbf{T}}$, therefore we can define invariants via Graber–Pandharipande virtual localization [GP99]

$$\text{PT}_{d,n}(X) := \int_{[P_X^{\mathbf{T}}]^{\text{vir}}} \frac{1}{e^{\mathbf{T}}(N^{\text{vir}})} \in \mathbb{Q}(s_1, s_2),$$

where s_1, s_2 are the generators of the \mathbf{T} -equivariant cohomology of a point and N^{vir} is the virtual normal bundle. We denote its generating series by

$$\text{PT}_d(X; q) := \sum_{n \in \mathbb{Z}} q^n \cdot \text{PT}_{d,n}(X) \in \mathbb{Q}(s_1, s_2)((q)).$$

Pandharipande and Pixton extensively studied stable pair theory on local curves [PP13, PP12] using degeneration techniques and relative invariants, focusing on the rationality of the generating series, including the case of descendent insertions. The novelty of this paper is the different approach which only relies on the Graber–Pandharipande localization, without degenerating the curve C , and the virtual structure constructed on the double nested Hilbert schemes $C^{[n\lambda]}$. This is in particular useful to address the K -theoretic generalizations of stable pair invariants (cf. § 1.8).

Our main result is that the generating series $\text{PT}_d(X; q)$ of such invariants is controlled by some universal series and determine them under the anti-diagonal restriction $s_1 + s_2 = 0$.

THEOREM 1.3 (Theorems 8.2 and 8.1). *There are universal series $A_\lambda(q), B_\lambda(q), C_\lambda(q) \in \mathbb{Q}(s_1, s_2)[[q]]$ such that*

$$PT_d(X; q) = \sum_{\lambda \vdash d} (q^{-|\lambda|} A_\lambda(q))^{g-1} \cdot (q^{-n(\lambda)} B_\lambda(q))^{\deg L_1} \cdot (q^{-n(\bar{\lambda})} C_\lambda(q))^{\deg L_2},$$

where $\bar{\lambda}$ is the conjugate partition of λ , $n(\lambda) = \sum_{i=0}^{l(\lambda)} i \cdot \lambda_i$ and $g = g(C)$. Moreover, under the anti-diagonal restriction $s_1 + s_2 = 0$

$$\begin{aligned} A_\lambda(q, s_1, -s_1) &= (-s_1^2)^{|\lambda|} \cdot \prod_{\square \in \lambda} h(\square)^2, \\ B_\lambda(-q, s_1, -s_1) &= (-1)^{n(\lambda)} \cdot s_1^{-|\lambda|} \cdot \prod_{\square \in \lambda} h(\square)^{-1} \cdot \prod_{\square \in \lambda} (1 - q^{h(\square)}), \\ C_\lambda(-q, s_1, -s_1) &= (-1)^{n(\bar{\lambda})} \cdot (-s_1)^{-|\lambda|} \cdot \prod_{\square \in \lambda} h(\square)^{-1} \cdot \prod_{\square \in \lambda} (1 - q^{h(\square)}). \end{aligned}$$

We now sketch the main steps required in proving Theorem 1.3.

1.4 Proof of the main theorem

The connected components of the \mathbf{T} -fixed locus $P_n(X, d[C])^{\mathbf{T}}$ are double nested Hilbert schemes of points $C^{[\mathbf{n}\lambda]}$, for suitable reversed plane partitions $\mathbf{n}\lambda$ and Young diagram λ . In fact, pushing forward via $X \rightarrow C$ a \mathbf{T} -fixed stable pair $[\mathcal{O}_X \xrightarrow{s} F]$, corresponds a decomposition $\bigoplus_{(i,j) \in \mathbb{Z}^2} [\mathcal{O}_C \xrightarrow{s_{ij}} F_{ij}]$ on C , where every F_{ij} is a line bundle with section s_{ij} . These data produce divisors $Z_{ij} \subset C$ satisfying the nesting conditions dictated by λ , in other words an element of $C^{[\mathbf{n}\lambda]}$.

On each connected component, there is an induced virtual fundamental class $[C^{[\mathbf{n}\lambda]}]_{PT}^{\text{vir}}$, coming from the deformation of stable pairs. This virtual cycle coincides with that constructed by the zero-locus construction of Theorem 1.2. By determining the class in K -theory of the virtual normal bundle, stable pair invariants on X are reduced to (\mathbf{T} -equivariant) virtual intersection numbers on $C^{[\mathbf{n}\lambda]}$, namely

$$\int_{[C^{[\mathbf{n}\lambda]}]_{\text{vir}}} e^{\mathbf{T}}(-N_{C, L_1, L_2}^{\text{vir}}) \in \mathbb{Q}(s_1, s_2). \tag{1.1}$$

The generating series of these invariants, for every fixed Young diagram λ , is controlled by three universal series (Theorem 5.1)

$$\sum_{\mathbf{n}\lambda} q^{|\mathbf{n}\lambda|} \int_{[C^{[\mathbf{n}\lambda]}]_{\text{vir}}} e^{\mathbf{T}}(-N_{C, L_1, L_2}^{\text{vir}}) = A_\lambda^{g-1} \cdot B_\lambda^{\deg L_1} \cdot C_\lambda^{\deg L_2} \in \mathbb{Q}(s_1, s_2)[[q]].$$

This universal structure is proven by following the strategy of [EGL01]. In fact, these invariants are *multiplicative* on triples of the form $(C, L_1, L_2) = (C' \sqcup C'', L'_1 \oplus L''_1, L'_2 \oplus L''_2)$ and are polynomial in the Chern numbers of (C, L_1, L_2) . The latter is obtained by pushing the virtual intersection number to $C^{[\mathbf{n}\lambda]}$ on the smooth ambient space $A_{C, \mathbf{n}\lambda}$, a product of symmetric powers of C , and later to a product of Jacobians $\text{Pic}^{n_i}(C)$, where the integrand is a polynomial on well-behaved cohomology classes.

By the universal structure any computation is reduced to a basis of the three-dimensional \mathbb{Q} -vector space of Chern numbers of triples (C, L_1, L_2) . A simple basis consists of the Chern numbers of $(\mathbb{P}^1, \mathcal{O}, \mathcal{O})$ and any two (\mathbb{P}^1, L_1, L_2) with $L_1 \otimes L_2 = K_{\mathbb{P}^1}$. In both cases, the invariants are explicitly determined under the anti-diagonal restriction $s_1 + s_2 = 0$ by further applying the virtual localization formula.

1.5 Toric computations

The \mathbb{C}^* -action on \mathbb{P}^1 canonically lifts to the double nested Hilbert scheme $\mathbb{P}^{1[\mathbf{n}_\lambda]}$, with only finitely many \mathbb{C}^* -fixed points, therefore we can further \mathbb{C}^* -localize the invariants (1.1) to obtain

$$\int_{[\mathbb{P}^{1[\mathbf{n}_\lambda]}]_{\text{vir}}} e^{\mathbf{T}}(-N_{\mathbb{P}^1, L_1, L_2}^{\text{vir}}) = \left(\sum_{Z \in \mathbb{P}^{1[\mathbf{n}_\lambda], \mathbb{C}^*}} e^{\mathbf{T} \times \mathbb{C}^*}(-T_Z^{\text{vir}} - N_{\mathbb{P}^1, L_1, L_2, Z}^{\text{vir}}) \right) \Big|_{s_3=0},$$

where s_3 is the generator of the \mathbb{C}^* -equivariant cohomology of a point and T_Z^{vir} is the virtual tangent bundle of $\mathbb{P}^{1[\mathbf{n}_\lambda]}$ at the fixed point Z .

Under the anti-diagonal restriction $s_1 + s_2 = 0$, this translates the computation of the invariants into a purely combinatorial problem, which we explicitly solve in the trivial vector bundle case $L_1 = L_2 = \mathcal{O}_{\mathbb{P}^1}$ and in the Calabi–Yau case $L_1 \otimes L_2 = K_{\mathbb{P}^1}$. A few remarks are in order. In the trivial vector bundle case, the solution is equivalent to the vanishing

$$\int_{[\mathbb{P}^{1[\mathbf{n}_\lambda]}]_{\text{vir}}} e^{\mathbf{T}}(-N_{\mathbb{P}^1, \mathcal{O}, \mathcal{O}}^{\text{vir}}) \Big|_{s_1+s_2=0} = 0,$$

for every reversed plane partition of positive size $|\mathbf{n}_\lambda| > 0$. This relies on the vanishing $e^{\mathbf{T} \times \mathbb{C}^*}(-T_Z^{\text{vir}} - N_{\mathbb{P}^1, L_1, L_2, Z}^{\text{vir}}) = 0$, which comes from a simple vanishing property of the topological vertex in stable pair theory proved in [MPT10].

In the Calabi–Yau case, the invariants turn out to be *topological*, under the anti-diagonal restriction.

THEOREM 1.4 (Theorem 7.3). *Let X be Calabi–Yau. Then the generating series of the invariants (1.1) after the restriction $s_1 + s_2 = 0$ coincides, up to a sign, with the generating series of the topological Euler characteristic*

$$\sum_{\mathbf{n}_\lambda} q^{|\mathbf{n}_\lambda|} \cdot \left(\int_{[\mathbb{P}^{1[\mathbf{n}_\lambda]}]_{\text{vir}}} e^{\mathbf{T}}(-N_{\mathbb{P}^1, L_1, L_2}^{\text{vir}}) \right) \Big|_{s_1+s_2=0} = (-1)^{\deg L_1(c_\lambda + |\lambda|) + |\lambda|} \cdot \sum_{\mathbf{n}_\lambda} (-q)^{|\mathbf{n}_\lambda|} e(\mathbb{P}^{1[\mathbf{n}_\lambda]}),$$

where $c_\lambda = \sum_{(i,j) \in \lambda} (j - i)$.

This happens as, under the anti-diagonal restriction, each \mathbb{C}^* -fixed point Z contributes with a sign

$$e^{\mathbf{T} \times \mathbb{C}^*}(-T_Z^{\text{vir}} - N_{\mathbb{P}^1, L_1, L_2, Z}^{\text{vir}}) \Big|_{s_1+s_2=0} = (-1)^{\deg L_1(c_\lambda + |\lambda|) + |\lambda| + |\mathbf{n}_\lambda|}, \tag{1.2}$$

which is independent of Z and the invariants amount to a (signed) count of the \mathbb{C}^* -fixed points. It is not *a priori* clear how to obtain the same sign through the vertex formalism for stable pairs developed by Pandharipande and Thomas (PT) [PT09b].

Nevertheless, the topological nature of the invariants in the Calabi–Yau case is not surprising also for a non-toric curve C . If X is Calabi–Yau and $P_n(X, d[C])$ is proper, which happens only in rare cases, the anti-diagonal restriction would compute its virtual Euler characteristic and Behrend’s weighted Euler characteristic, which is a purely motivic invariant of a scheme with a symmetric perfect obstruction theory [Beh09].

1.6 Gromov–Witten/stable pairs correspondence

In the seminal work [MNOP06a], a conjectural correspondence, known as the Maulik–Nekrasov–Okounkov–Pandharipande (MNOP) conjecture, between Gromov–Witten (GW) invariants and Donaldson–Thomas (DT) invariants of projective threefolds is formulated, proven for toric varieties in [MNOP06a, MNOP06b, MOOP11] for primary insertions. By defining the GW/DT

invariants via equivariant residues, the conjecture has been extended to local curves in [BP08] and proven by combining the results of [BP08, OP10].

Stable pair invariants were later introduced by Pandharipande and Thomas [PT09a] to give a more natural geometric interpretation of the MNOP conjecture through the DT/PT correspondence proved by Toda and Bridgeland in [Tod10, Bri11] using wall-crossing and Hall algebra techniques. The GW/stable pairs correspondence has been subsequently extended to include descendent insertions and to quasi-projective varieties whenever invariants can be defined through virtual localization. The correspondence had been confirmed by Pandharipande and Pixton for Calabi–Yau and Fano complete intersections in product of projective spaces and toric varieties [PP17, PP14] and had been recently addressed in [OOP20]. See [Pan18] for a complete survey on the subject.

1.7 The local GW theory of curves

For $X = \text{Tot}_C(L_1 \oplus L_2)$ a local curve, let $\overline{M}_h^\bullet(X, d[C])$ denote the moduli space of stable maps (with possibly disconnected domain) of genus h and degree $d[C]$. Define the partition function of GW invariants of X (with a shifted exponent)

$$\begin{aligned} & \text{GW}_d(g | \deg L_1, \deg L_2; u) \\ &= u^{2-2g+\deg L_1+\deg L_2} \sum_{h \in \mathbb{Z}} u^{2h-2} \int_{[\overline{M}_h^\bullet(X, d[C])^{\text{T}}]_{\text{vir}}} \frac{1}{e^{\mathbf{T}(N^{\text{vir}})}} \in \mathbb{Q}(s_1, s_2)((u)), \end{aligned}$$

where the dependence is only on the genus $g = g(C)$, the degrees of the line bundles and the degree d . The GW theory of local curves had been solved by Bryan and Pandharipande [BP08, Theorem 7.1] using a topological quantum field theory approach. Moreover they deduced an explicit closed formula for the partition function under the anti-diagonal restriction $s_1 + s_2 = 0$.

THEOREM 1.5 (Bryan–Pandharipande). *The partition function of GW invariants satisfies*

$$\begin{aligned} \text{GW}_d(g | \deg L_1, \deg L_2; u)|_{s_1+s_2=0} &= (-1)^{d(g-1-\deg L_2)} s_1^{d(2g-2-\deg L_1-\deg L_2)} \\ &\cdot \sum_{\lambda \vdash d} Q^{(1/2)c_\lambda(\deg L_1-\deg L_2)} \prod_{\square \in \lambda} h(\square)^{2g-2-\deg L_1-\deg L_2} \\ &\cdot i^{-\deg L_1-\deg L_2} (Q^{h(\square)/2} - Q^{-h(\square)/2})^{\deg L_1+\deg L_2}, \end{aligned}$$

where we set $Q = e^{iu}$ and $i = \sqrt{-1}$.

With this explicit expression it is immediate to check the GW/stable pairs correspondence under the anti-diagonal restriction.

COROLLARY 1.6 (Corollary 8.3). *Let X be a local curve. Under the anti-diagonal restriction $s_1 + s_2 = 0$ the GW/stable pair correspondence holds*

$$(-i)^{d(2-2g+\deg L_1+\deg L_2)} \cdot \text{GW}_d(g | \deg L_1, \deg L_2; u) = (-q)^{-(1/2) \cdot d(2-2g+\deg L_1+\deg L_2)} \text{PT}_d(X, q),$$

after the change of variable $q = -e^{iu}$.

1.8 K-theoretic refinement

The K -theoretic refinement of DT theory and stable pair theory have attracted much attention recently, both in mathematics and string theory: see, for example, [Tho20, Afg20, Arb21, FMR21] for Calabi–Yau threefolds, [Nek20, NP19, CKM22a] for Calabi–Yau fourfolds and [NO16, Oko17, KOO21, Oko19] for local curves.

A scheme X with a perfect obstruction theory is endowed not only with a virtual fundamental class, but also with a *virtual structure sheaf* $\mathcal{O}_X^{\text{vir}} \in K_0(X)$. If X is proper, K -theoretic invariants are simply of the form

$$\chi(X, \mathcal{O}_X^{\text{vir}} \otimes V) \in \mathbb{Z},$$

where $V \in K_0(X)$. If X is a local curve the moduli space of stable pairs P_X is, in general, not proper and K -theoretic stable pair invariants are defined by virtual localization [FG10] on the proper \mathbf{T} -fixed locus $P_X^{\mathbf{T}}$, that is one set

$$\chi(P_X, \mathcal{O}_{P_X}^{\text{vir}} \otimes V) := \chi\left(P_X^{\mathbf{T}}, \frac{\mathcal{O}_{P_X^{\mathbf{T}}}^{\text{vir}} \otimes V|_{P_X^{\mathbf{T}}}}{\Lambda \bullet N^{\text{vir},*}}\right) \in \mathbb{Q}(t_1, t_2).$$

In §9 we show that, also in the K -theoretic setting, the invariants are controlled by universal series.

The naive generalization of cohomological invariants is for $V = \mathcal{O}_X$, that is no insertions. However, we learn from Nekrasov and Okounkov [NO16] that it is more natural to consider the *twisted virtual structure sheaf*

$$\widehat{\mathcal{O}}^{\text{vir}} = \mathcal{O}^{\text{vir}} \otimes K_{\text{vir}}^{1/2},$$

where $K_{\text{vir}}^{1/2}$ is a square root¹ of the *virtual canonical bundle*. Denote by $\text{PT}_d^{\widehat{K}}(X; q)$ the generating series of K -theoretic invariants with $V = K_{\text{vir}}^{1/2}$.

THEOREM 1.7 (Corollary 9.5). *There exist universal series*

$$A_{\widehat{K},\lambda}(q), \quad B_{\widehat{K},\lambda}(q), \quad C_{\widehat{K},\lambda}(q) \in \mathbb{Q}(t_1^{1/2}, t_2^{1/2})[[q]]$$

such that

$$\text{PT}_d^{\widehat{K}}(X; q) = \sum_{\lambda \vdash d} (q^{-|\lambda|} A_{\widehat{K},\lambda}(q))^{g-1} \cdot (q^{-n(\lambda)} B_{\widehat{K},\lambda}(q))^{\deg L_1} \cdot (q^{-n(\lambda)} C_{\widehat{K},\lambda}(q))^{\deg L_2}.$$

Moreover, the universal series are explicitly computed under $t_1 t_2 = 1$.

We are not aware of a K -theoretic GW refinement for which a *refined* GW/stable pairs correspondence holds.

Conventions

All *schemes* are separated and of finite type over \mathbb{C} . We let $K^0(X)$ be the K -group of vector bundles on X . When X carries an action by an algebraic torus \mathbf{T} , we let $K_{\mathbf{T}}^0(X)$ be the K -group of \mathbf{T} -equivariant vector bundles on X . Similarly, we let $K_0(X)$ denote the K -group of *coherent sheaves* on X , and we let $K_0^{\mathbf{T}}(X)$ be the K -group of (the abelian category of) \mathbf{T} -equivariant coherent sheaves on X . When X is smooth, the natural \mathbb{Z} -linear map $K^0(X) \rightarrow K_0(X)$, respectively $K_{\mathbf{T}}^0(\text{pt})$ -linear map $K_{\mathbf{T}}^0(X) \rightarrow K_0^{\mathbf{T}}(X)$, is an isomorphism. Chow groups $A^*(X)$ and cohomology groups $H^*(X)$ are taken with rational coefficients. We use $(\cdot)^{\vee}$ for the derived dual of complexes and $(\cdot)^*$ for the (underived) dual of coherent sheaves. For clarity of exposition, we suppress various pullback maps, whenever they are clear from the context.

2. Double nested Hilbert schemes

2.1 Young diagrams

By definition, a *partition* λ of $d \in \mathbb{Z}_{\geq 0}$ is a finite sequence of positive integers

$$\lambda = (\lambda_0 \geq \lambda_1 \geq \lambda_2 \geq \dots),$$

¹ This square root may not exist as a line bundle, but it does exist as a class in K -theory after inverting 2.

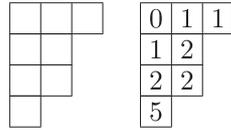


FIGURE 1. On the left, a Young diagram of size 8. On the right, a reversed plane partition of size 14.

where

$$|\lambda| = \sum_i \lambda_i = d.$$

The number of parts of λ is called the *length* of λ and is denoted by $l(\lambda)$. A partition λ can be equivalently described by its associated *Young diagram*, which is the collection of d boxes in \mathbb{Z}^2 located at (i, j) where $0 \leq j < \lambda_i$.²

Given a partition λ , a *reversed plane partition* $\mathbf{n}_\lambda = (n_\square)_{\square \in \lambda} \in \mathbb{Z}_{\geq 0}$ is a collection of non-negative integers such that $n_\square \leq n_{\square'}$ for any $\square, \square' \in \lambda$ such that $\square \leq \square'$.³ In other words, a reversed plane partition is a Young diagram labelled with non-negative integers which are non-decreasing in rows and columns (Figure 1). The *size* of a reversed plane partition is

$$|\mathbf{n}_\lambda| = \sum_{\square \in \lambda} n_\square.$$

The *conjugate partition* $\bar{\lambda}$ is obtained by reflecting the Young diagram of λ about the $i = j$ line.

In the paper we require the following standard quantities. Given a box in the Young diagram λ , define the *content* $c(\square) = j - i$ and the *hooklength* $h(\square) = \lambda_i + \bar{\lambda}_j - i - j - 1$. The total content

$$c_\lambda = \sum_{\square \in \lambda} c(\square)$$

satisfies the following identities (cf. [Mac95, p. 11]):

$$\sum_{\square \in \lambda} h(\square) = n(\lambda) + n(\bar{\lambda}) + |\lambda|, \quad c_\lambda = n(\bar{\lambda}) - n(\lambda), \tag{2.1}$$

where

$$n(\lambda) = \sum_{i=0}^{l(\lambda)} i \cdot \lambda_i.$$

For any Young diagram λ there is an associated graph, where any box of λ corresponds to a vertex and any face common to two boxes correspond to an edge connecting the corresponding vertices (Figure 2). A *square* of this graph is a circuit made of four different edges.

LEMMA 2.1. *Let λ be a Young diagram and denote by V, E and Q the number of vertices, edges and squares of the associated graph, respectively. Then*

$$V - E + Q - 1 = 0.$$

² This notation was borrowed from [BP08, § 3.1]; however, in our conventions, (i, j) labels the box's corner closest to the origin. In displaying Young diagrams, we denote by i the vertical (negative) axis and by j the horizontal (positive) axis.

³ This means that if $\square = (i, j), \square' = (i', j')$, then $i \leq i'$ and $j \leq j'$.

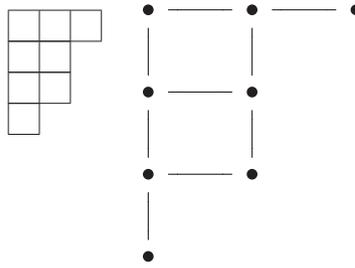


FIGURE 2. A Young diagram and its associated graph, with eight vertices, nine edges and two squares.

Proof. We prove the claim by induction on the size of λ . If $|\lambda| = 1$, this is clear. Suppose it holds for all λ with $|\lambda| \leq n - 1$. Then we construct λ of size n by adding a box with lattice coordinates (i, j) to a Young diagram $\tilde{\lambda}$ of size $n - 1$. There are two possibilities: either one of i, j is zero, so we added one vertex and one edge, or both i, j are non-zero, so we added one vertex, one square and two edges. In both cases the claim is proved. \square

2.2 Double nested Hilbert schemes

Let X be a projective scheme and $\mathcal{O}(1)$ a fixed ample line bundle. The *Hilbert polynomial* of a closed subscheme $Y \subset X$ is defined by

$$m \mapsto \chi(\mathcal{O}_Y \otimes \mathcal{O}(m)).$$

Given a polynomial $p(m)$, the *Hilbert scheme* is the moduli space parametrizing closed subschemes $Y \subset X$ with Hilbert polynomial $p(m)$, which is representable by a projective scheme (e.g. by [Gro61]). We consider here a more general situation, where we replace closed subschemes by flags of closed subschemes, satisfying certain nesting conditions dictated by Young diagrams.

Let λ be a Young diagram and $\mathbf{p}_\lambda = (p_\square)_{\square \in \lambda} \in \mathbb{Z}[x]$ be a collection of polynomials indexed by λ . If all p_\square are non-negative integers which are non-decreasing in rows and columns, $\mathbf{p}_\lambda = \mathbf{n}_\lambda$ is a reversed plane partition.

DEFINITION 2.2. Let X be a projective scheme and \mathbf{p}_λ as previously. The *double nested Hilbert functor* of X of type \mathbf{p}_λ is the moduli functor

$$\underline{\text{Hilb}}^{\mathbf{p}_\lambda}(X) : \text{Sch}^{\text{op}} \rightarrow \text{Sets},$$

$$T \mapsto \left\{ (\mathcal{Z}_\square)_{\square \in \lambda} \subset X \times T \mid \begin{array}{l} \mathcal{Z}_\square \text{ a } T\text{-flat closed subscheme with Hilbert polynomial } p_\square, \\ \text{such that } \mathcal{Z}_\square \subset \mathcal{Z}_{\square'} \text{ for } \square \leq \square' \end{array} \right\}.$$

Remark 2.3. If $|\lambda| = 1$ we recover the classical Hilbert scheme, whereas if λ is a horizontal (or vertical) Young diagram we recover the *nested Hilbert scheme*, already widely studied in the literature.

PROPOSITION 2.4. Let X be a projective scheme and \mathbf{p}_λ as previously. Then $\underline{\text{Hilb}}^{\mathbf{p}_\lambda}(X)$ is representable by a projective scheme $\text{Hilb}^{\mathbf{p}_\lambda}(X)$, which we call the *double nested Hilbert scheme*.

Proof. We prove our claim in the case \mathbf{p}_λ is

p_{00}	p_{01}
p_{10}	p_{11}

as the general case will follow by an analogous reasoning. There are forgetful maps between nested Hilbert functors

$$\begin{aligned} \underline{\text{Hilb}}^{[p_{00}, p_{01}, p_{11}]}(X) &\rightarrow \underline{\text{Hilb}}^{[p_{00}, p_{11}]}(X), \\ \underline{\text{Hilb}}^{[p_{00}, p_{10}, p_{11}]}(X) &\rightarrow \underline{\text{Hilb}}^{[p_{00}, p_{11}]}(X), \end{aligned}$$

which forget the second subscheme of the corresponding flag. Consider their fiber product

$$\begin{array}{ccc} \underline{\text{Hilb}}^{[p_{00}, p_{01}, p_{11}]}(X) \times_{\underline{\text{Hilb}}^{[p_{00}, p_{11}]}(X)} \underline{\text{Hilb}}^{[p_{00}, p_{10}, p_{11}]}(X) &\longrightarrow & \underline{\text{Hilb}}^{[p_{00}, p_{01}, p_{11}]}(X) \\ \downarrow & & \downarrow \\ \underline{\text{Hilb}}^{[p_{00}, p_{01}, p_{11}]}(X) &\longrightarrow & \underline{\text{Hilb}}^{[p_{00}, p_{11}]}(X). \end{array}$$

There is an obvious morphism of functors

$$\underline{\text{Hilb}}^{\mathbf{p}_\lambda}(X) \rightarrow \underline{\text{Hilb}}^{[p_{00}, p_{01}, p_{11}]}(X) \times_{\underline{\text{Hilb}}^{[p_{00}, p_{11}]}(X)} \underline{\text{Hilb}}^{[p_{00}, p_{10}, p_{11}]}(X),$$

which is easily checked to be an isomorphism by comparing each flat family of flags over every scheme T . We conclude by the fact that the nested Hilbert functors (and their fiber products) are representable by a projective scheme by [Ser06, Theorem 4.5.1]. \square

Thanks to representability, double nested Hilbert schemes are equipped with universal subschemes, for any $\square \in \lambda$,

$$\mathcal{Z}_\square \subset X \times \text{Hilb}^{\mathbf{p}_\lambda}(X),$$

such that the fiber over a point $\underline{Z} = (Z_\square)_{\square \in \lambda} \in \text{Hilb}^{\mathbf{p}_\lambda}(X)$ is

$$\mathcal{Z}_\square|_{\underline{Z}} = Z_\square \subset X.$$

Remark 2.5. If $\mathbf{p}_\lambda = \mathbf{n}_\lambda$, Definition 2.2 generalizes to X quasi-projective. In fact, let $X \subset \bar{X}$ be any compactification of X . We define the *double nested Hilbert scheme points* as the open subscheme

$$X^{[\mathbf{n}_\lambda]} := \text{Hilb}^{\mathbf{n}_\lambda}(X) \subset \text{Hilb}^{\mathbf{n}_\lambda}(\bar{X})$$

consisting of the zero-dimensional subschemes supported on $X \subset \bar{X}$.

Double nested Hilbert schemes of points are rarely smooth varieties. Some smooth examples consist of:

- (i) $|\lambda| = 1$, X a smooth quasi-projective curve or surface (see, e.g., [Nak99]);
- (ii) λ a vertical/horizontal Young diagram, X a smooth quasi-projective curve (see, e.g., [Che98]).

In general, $X^{[\mathbf{n}_\lambda]}$ is singular even for X a smooth quasi-projective curve.

Example 2.6. Let C be a smooth curve and consider the reversed plane partition \mathbf{n}_λ

$$\begin{array}{|c|c|} \hline 0 & 1 \\ \hline 1 & 2 \\ \hline \end{array}$$

There are two types of flags of divisors, of the form

$$\begin{array}{ccc} \emptyset & \subset & P \\ \cap & & \cap \\ Q & \subset & P + Q, \end{array} \quad \begin{array}{ccc} \emptyset & \subset & P \\ \cap & & \cap \\ P & \subset & P + Q, \end{array}$$

where $P, Q \in C$. Therefore, its reduced scheme structure consists of two irreducible components $C \times C \cup C \times C$, intersecting at the diagonals of $C \times C$.

Singularities make it hard to perform intersection theory on $X^{[n_\lambda]}$. To remedy this we construct, in special cases, *virtual fundamental classes* in $A_*(X^{[n_\lambda]})$. We briefly recall the language of perfect obstruction theories of Behrend and Fantechi [BF97] and Li and Tian [LT98].

2.3 Perfect obstruction theories

A *perfect obstruction theory* on a scheme X is the datum of a morphism

$$\phi: \mathbb{E} \rightarrow \mathbb{L}_X$$

in $\mathbf{D}^{[-1,0]}(X)$, where \mathbb{E} is a perfect complex of perfect amplitude contained in $[-1, 0]$, such that $h^0(\phi)$ is an isomorphism and $h^{-1}(\phi)$ is surjective. Here, $\mathbb{L}_X = \tau_{\geq -1} L_X^\bullet$ is the cut-off at -1 of the full cotangent complex $L_X^\bullet \in \mathbf{D}^{[-\infty, 0]}(X)$. A perfect obstruction theory is called *symmetric* (see [BF08]) if there exists an isomorphism $\theta: \mathbb{E} \xrightarrow{\sim} \mathbb{E}^\vee[1]$ such that $\theta = \theta^\vee[1]$. The *virtual dimension* of X with respect to (\mathbb{E}, ϕ) is the integer $\text{vd} = \text{rk } \mathbb{E}$. This is just $\text{rk } E^0 - \text{rk } E^{-1}$ if one can write $\mathbb{E} = [E^{-1} \rightarrow E^0]$.

A perfect obstruction theory determines a cone

$$\mathfrak{C} \hookrightarrow E_1 = (E^{-1})^*.$$

Letting $\iota: X \hookrightarrow E_1$ be the zero section of the vector bundle E_1 , the induced *virtual fundamental class* on X is the refined intersection

$$[X]^{\text{vir}} = \iota^![\mathfrak{C}] \in A_{\text{vd}}(X).$$

By a result of Siebert [Sie04, Theorem 4.6], the virtual fundamental class depends only on the K -theory class of \mathbb{E} .

Example 2.7. Let $\iota: Z = Z(s) \hookrightarrow A$ be the zero locus of a section $s \in \Gamma(A, \mathcal{E})$, where \mathcal{E} is a vector bundle over a smooth quasi-projective variety A . Then there exists an induced perfect obstruction theory on Z

$$\begin{array}{ccc} \mathbb{E} & = & [\mathcal{E}^*|_Z \xrightarrow{ds^*} \Omega_A|_Z] \\ \phi \downarrow & & \downarrow s^* \qquad \qquad \downarrow \text{id} \\ \mathbb{L}_Z & = & [\mathcal{I}/\mathcal{I}^2 \xrightarrow{d} \Omega_A|_Z] \end{array}$$

in $\mathbf{D}^{[-1,0]}(Z)$, where we represented the truncated cotangent complex by means of the exterior derivative d constructed out of the ideal sheaf $\mathcal{I} \subset \mathcal{O}_Z$ of the inclusion $Z \hookrightarrow A$. Moreover,

$$\iota_*[Z]^{\text{vir}} = e(\mathcal{E}) \cap [A] \in A_*(A),$$

where $e(\cdot)$ denotes the Euler class.

2.4 Points on curves

Let C be an irreducible smooth quasi-projective curve and \mathbf{n}_λ a reversed plane partition. In this section we show that $C^{[n_\lambda]}$ is the zero locus of a section of a vector bundle over a smooth ambient space, and therefore admits a perfect obstruction theory as in Example 2.7.

We define

$$A_{C, \mathbf{n}_\lambda} = C^{[n_{00}]} \times \prod_{\substack{(i,j) \in \lambda \\ i \geq 1}} C^{[n_{ij} - n_{i-1,j}]} \times \prod_{\substack{(l,k) \in \lambda \\ k \geq 1}} C^{[n_{lk} - n_{l,k-1}]}.$$

As $C^{[n]} \cong C^{(n)}$ is a symmetric product via the Hilbert–Chow morphism, $A_{C, \mathbf{n}_\lambda}$ is a smooth quasi-projective variety of dimension

$$\dim(A_{C, \mathbf{n}_\lambda}) = n_{00} + \sum_{\substack{(i,j) \in \lambda \\ i \geq 1}} (n_{ij} - n_{i-1,j}) + \sum_{\substack{(l,k) \in \lambda \\ k \geq 1}} (n_{lk} - n_{l,k-1}).$$

To ease the notation, we denote its elements by $\underline{Z} = ((Z_{00}, X_{ij}, Y_{lk}))_{i,j,l,k} \in A_{C, \mathbf{n}_\lambda}$, where $Z_{00} \subset C$ is a divisor of length n_{00} and $X_{ij} \subset C$ (respectively, $Y_{lk} \subset C$) is a divisor of length $n_{ij} - n_{i-1,j}$ (respectively, $n_{lk} - n_{l,k-1}$).

Here $A_{C, \mathbf{n}_\lambda}$ comes equipped with universal divisors, which we denote by

$$\mathcal{Z}_{00}, \mathcal{X}_{ij}, \mathcal{Y}_{lk} \subset C \times A_{C, \mathbf{n}_\lambda},$$

with fibers are

$$\mathcal{Z}_{00}|_{\underline{Z}} = Z_{00}, \quad \mathcal{X}_{ij}|_{\underline{Z}} = X_{ij}, \quad i \geq 1, \quad \mathcal{Y}_{lk}|_{\underline{Z}} = Y_{lk}, \quad k \geq 1.$$

For every $(i, j) \in \lambda$ with $i, j \geq 1$ define the universal effective divisors

$$\Gamma_{ij}^1 = \mathcal{X}_{i,j} + \mathcal{Y}_{i-1,j}, \quad \Gamma_{ij}^2 = \mathcal{Y}_{i,j} + \mathcal{X}_{i,j-1}.$$

THEOREM 2.8. *Let C be an irreducible smooth quasi-projective curve, $\pi : C \times A_{C, \mathbf{n}_\lambda} \rightarrow A_{C, \mathbf{n}_\lambda}$ be the natural projection and define the vector bundle*

$$\mathcal{E} = \bigoplus_{\substack{(i,j) \in \lambda \\ i,j \geq 1}} \pi_* \mathcal{O}_{\Gamma_{ij}^2}(\Gamma_{ij}^1).$$

Then there exists a section s of \mathcal{E} whose zero set is isomorphic to $C^{[\mathbf{n}_\lambda]}$

$$C^{[\mathbf{n}_\lambda]} \cong Z(s) \hookrightarrow A_{C, \mathbf{n}_\lambda} \xrightarrow{\mathcal{E}} C^{[\mathbf{n}_\lambda]}.$$

Proof. Note that \mathcal{E} is a vector bundle, as by cohomology and base change all higher direct images vanish

$$\mathbf{R}^k \pi_* \mathcal{O}_{\Gamma_{ij}^2}(\Gamma_{ij}^1) = 0, \quad k > 0.$$

There is a closed immersion

$$C^{[\mathbf{n}_\lambda]} \hookrightarrow A_{C, \mathbf{n}_\lambda},$$

given on closed points $(Z_{ij})_{(i,j) \in \lambda} \in C^{[\mathbf{n}_\lambda]}$ as

$$(Z_{ij})_{(i,j) \in \lambda} \mapsto (Z_{00}, (Z_{ij} - Z_{i-1,j}), (Z_{ij} - Z_{i,j-1}))_{(i,j) \in \lambda} \in A_{C, \mathbf{n}_\lambda}.$$

Define sections $\tilde{s}_{ij} \in H^0(C \times A_{C, \mathbf{n}_\lambda}, \mathcal{O}_{\Gamma_{ij}^2}(\Gamma_{ij}^1))$ as the composition

$$\tilde{s}_{ij} : \mathcal{O}_{C \times A_{C, \mathbf{n}_\lambda}} \xrightarrow{s'_{ij}} \mathcal{O}_{C \times A_{C, \mathbf{n}_\lambda}}(\Gamma_{ij}^1) \rightarrow \mathcal{O}_{\Gamma_{ij}^2}(\Gamma_{ij}^1),$$

where s'_{ij} is the section vanishing on Γ_{ij}^1 whereas the second morphism is the restriction along $j : \Gamma_{ij}^2 \rightarrow A_{C, \mathbf{n}_\lambda} \times C$; in other words, $\tilde{s}_{ij} = j_* j^* s'_{ij}$. The sections \tilde{s}_{ij} induce sections $s_{ij} = \pi_* \tilde{s}_{ij}$ of $\pi_* \mathcal{O}_{\Gamma_{ij}^2}(\Gamma_{ij}^1)$ and we set $s = (s_{ij})_{ij} \in H^0(A_{C, \mathbf{n}_\lambda}, \mathcal{E})$. We claim that

$$C^{[\mathbf{n}_\lambda]} \cong Z(s).$$

To prove it, we follow the strategy of [CK18, Proposition 2.4]. For $(i, j) \in \lambda$ with $i, j \geq 1$, consider the universal divisors

$$\begin{array}{ccc}
 & \Gamma_{ij}^2 & \xleftarrow{j} A_{C, \mathbf{n}_\lambda} \times C \\
 & \swarrow & \searrow \tilde{\pi} \quad \downarrow \pi \\
 C & & A_{C, \mathbf{n}_\lambda}
 \end{array}$$

Let $\underline{Z}_T = (Z_{00}^T, X_{ij}^T, Y_{lk}^T)_{ij, lk}$ be any T -flat family with corresponding classifying morphism $f : T \rightarrow A_{C, \mathbf{n}_\lambda}$, where $Z_{00}^T, X_{ij}^T, Y_{lk}^T \subset T \times C$ have zero-dimensional fibers of appropriate length. Consider the commutative diagram

$$\begin{array}{ccc}
 & \Gamma_{ij}^{2, T} & \xleftarrow{j_T} T \times C \\
 & \swarrow & \searrow \tilde{\pi}_T \quad \downarrow \pi_T \\
 C & & T
 \end{array}$$

where $\Gamma_{ij}^{1, T}, \Gamma_{ij}^{2, T}$ are the pullbacks of the universal divisors $\Gamma_{ij}^1, \Gamma_{ij}^2$ along $f \times \text{id}_C$. To prove the claim it suffices to show that \underline{Z}_T is a T -point of $C^{[\mathbf{n}_\lambda]}$ if and only if f factors through $Z(s)$.

Now \underline{Z}_T is a T -point of $C^{[\mathbf{n}_\lambda]}$ if and only if $\Gamma_{ij}^{2, T} = \Gamma_{ij}^{1, T}$ for all $(i, j) \in \lambda$ such that $(i, j) \geq 1$. Note that the inclusion $\Gamma_{ij}^{2, T} \subset \Gamma_{ij}^{1, T}$ is enough to have the equality, as all fibers are divisors in C of the same degree.

On the other hand, we have that f factors through $Z(s)$ if and only if f^*s is the zero section of $f^*\mathcal{E}$, i.e. if $f^*s_{ij} = 0$ for all $(i, j) \in \lambda$ such that $(i, j) \geq 1$. Repeatedly applying flat base change, we obtain

$$f^*s_{ij} = \tilde{\pi}_* j_T^*(f \times \text{id}_C)^* s'_{ij}.$$

Therefore, f^*s_{ij} is the zero section if and only if $\Gamma_{ij}^{2, T} \subset \Gamma_{ij}^{1, T}$ as required. □

Thanks to Theorem 2.8, $C^{[\mathbf{n}_\lambda]}$ falls in the situation of Example 2.7 and we obtain a virtual fundamental class.

COROLLARY 2.9. *Let C be a smooth quasi-projective curve and \mathbf{n}_λ a reversed plane partition. Then $C^{[\mathbf{n}_\lambda]}$ has a perfect obstruction theory*

$$[\mathcal{E}^*|_{C^{[\mathbf{n}_\lambda]}} \rightarrow \Omega_{A_{C, \mathbf{n}_\lambda}}^1|_{C^{[\mathbf{n}_\lambda]}}] \rightarrow \mathbb{L}_{C^{[\mathbf{n}_\lambda]}}. \tag{2.2}$$

In particular, there exists a virtual fundamental class

$$[C^{[\mathbf{n}_\lambda]}]^{\text{vir}} \in A_*(C^{[\mathbf{n}_\lambda]}).$$

2.5 Topological Euler characteristic

Recall that we can view Euler characteristic weighted by a constructible function as a Lebesgue integral, where the measurable sets are constructible sets, measurable functions are constructible functions and the measure of a set is given by its Euler characteristic (cf. [BK19, § 2]). In this language we have

$$e(X) = \int_X 1 \cdot de,$$

for any constructible set X . The following lemma is reminiscent of the existence of a power structure on the Grothendieck ring of varieties.

LEMMA 2.10 [BK19, Lemma 32]. *Let B be a scheme of finite type over \mathbb{C} and $e(B)$ its topological Euler characteristic. Let $g : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}((p))$ be any function with $g(0) = 1$.*

Let $G : \text{Sym}^n B \rightarrow \mathbb{Z}((p))$ be the constructible function defined by

$$G(\mathbf{ax}) = \prod_i g(a_i),$$

for all $\mathbf{ax} = \sum_i a_i x_i \in \text{Sym}^n B$, where $x_i \in B$ are distinct closed points. Then

$$\sum_{n=0}^{\infty} q^n \int_{\text{Sym}^n B} G \cdot de = \left(\sum_{a=0}^{\infty} g(a)q^a \right)^{e(B)}.$$

Using Lemma 2.10 we compute the topological Euler characteristic of double nested Hilbert schemes of points of any quasi-projective smooth curve.

THEOREM 2.11. *Let C be a smooth quasi-projective curve and λ a Young diagram. Then*

$$\sum_{\mathbf{n}_\lambda} e(C^{[\mathbf{n}_\lambda]})q^{|\mathbf{n}_\lambda|} = \prod_{\square \in \lambda} (1 - q^{h(\square)})^{-e(C)}.$$

Proof. Consider the constructible map

$$\rho_n : \bigsqcup_{|\mathbf{n}_\lambda|=n} C^{[\mathbf{n}_\lambda]} \rightarrow \text{Sym}^n C,$$

defined, for $\underline{Z} = (Z_\square)_{\square \in \lambda} \in C^{[\mathbf{n}_\lambda]}$, by

$$\rho_n(\underline{Z}) = \sum_{\square \in \lambda} Z_\square \in \text{Sym}^n C.$$

In other words, ρ just forgets the distribution and the nesting of the divisor $\sum_{\square \in \lambda} Z_\square$ among all $\square \in \lambda$.

Let $\mathbf{ax} = \sum_i a_i x_i \in \text{Sym}^n C$, with x_i different to each other. The fiber $\rho_n^{-1}(\mathbf{ax})$ is clearly zero-dimensional and satisfies

$$\rho_n^{-1}(\mathbf{ax}) \cong \prod_i \rho_{a_i}^{-1}(a_i x_i). \tag{2.3}$$

In particular, the Euler characteristic of the fiber $\rho_n(nx)$ does not depend on the point $x \in C$ and counts the number of reversed plane partition of size n and underlying Young diagram λ

$$e(\rho_n^{-1}(nx)) = \sum_{|\mathbf{n}_\lambda|=n} 1. \tag{2.4}$$

Now consider

$$\int_{\bigsqcup_{|\mathbf{n}_\lambda|=n} C^{[\mathbf{n}_\lambda]}} 1 \cdot de = \int_{\text{Sym}^n C} \rho_{n*} 1 \cdot de,$$

where for any $\mathbf{ax} \in \text{Sym}^n C$ with x_i different to each other, using (2.3) and (2.4)

$$\begin{aligned} \rho_{n*} 1(\mathbf{ax}) &= e(\rho_n^{-1}(\mathbf{ax})) \\ &= \prod_i \sum_{|\mathbf{n}_\lambda|=a_i} 1. \end{aligned}$$

Now, $g(a) = \sum_{|\mathbf{n}_\lambda|=a} 1$ and $G(\mathbf{ax}) = \rho_{n*} 1(\mathbf{ax})$ satisfy the hypotheses of Lemma 2.10 and therefore

$$\sum_{n=0}^{\infty} \sum_{|\mathbf{n}_\lambda|=n} e(C^{[\mathbf{n}_\lambda]})q^n = \sum_{n=0}^{\infty} q^n \int_{\bigsqcup_{|\mathbf{n}_\lambda|=n} C^{[\mathbf{n}_\lambda]}} 1 \cdot de$$

$$\begin{aligned}
 &= \sum_{n=0}^{\infty} q^n \int_{\text{Sym}^n C} \rho_{n*} 1 \cdot de \\
 &= \left(\sum_{n=0}^{\infty} \sum_{|\mathbf{n}_\lambda|=n} q^{|\mathbf{n}_\lambda|} \right)^{e(C)}.
 \end{aligned}$$

A closed formula for the generating series of reversed plane partitions was given by Stanley [Sta71, Proposition 18.3] and by Hillman and Grassl [HG76, Theorem 1] using hook-lengths

$$\sum_{n=0}^{\infty} \sum_{|\mathbf{n}_\lambda|=n} q^{|\mathbf{n}_\lambda|} = \prod_{\square \in \lambda} (1 - q^{h(\square)})^{-1},$$

by which we conclude the proof. □

2.6 Double nesting of divisors

We conclude this section with a generalization of the zero-locus construction of Theorem 2.8.

Let X be a smooth projective variety of dimension d and $\beta_\lambda = (\beta_\square)_{\square \in \lambda}$ be a collection of homology classes $\beta_\square \in H_{n-2}(X, \mathbb{Z})$. Denote by H_{β_λ} the double nested Hilbert scheme of effective divisors on X , which parametrizes flags of divisors $(Z_\square)_{\square \in \lambda} \subset X$ satisfying the nesting condition dictated by β_λ . Denote by

$$A_{X, \beta_\lambda} = H_{\beta_{00}} \times \prod_{\substack{(i,j) \in \lambda \\ i \geq 1}} H_{\beta_{ij} - \beta_{i-1,j}} \times \prod_{\substack{(l,k) \in \lambda \\ k \geq 1}} H_{\beta_{lk} - \beta_{l,k-1}},$$

where H_β is the usual Hilbert scheme of divisors on X of class β . Analogously to §2.4, A_{X, β_λ} comes equipped with universal (Cartier) divisors $\mathcal{Z}_{00}, \mathcal{X}_{ij}, \mathcal{Y}_{lk} \subset X \times A_{X, \beta_\lambda}$ and for every $(i, j) \in \lambda$ with $i, j \geq 1$ we define the universal effective divisors

$$\Gamma_{ij}^1 = \mathcal{X}_{i,j} + \mathcal{Y}_{i-1,j}, \quad \Gamma_{ij}^2 = \mathcal{Y}_{i,j} + \mathcal{X}_{i,j-1}.$$

Define the coherent sheaf

$$\mathcal{E} = \bigoplus_{\substack{(i,j) \in \lambda \\ i, j \geq 1}} \pi_* \mathcal{O}_{\Gamma_{ij}^2}(\Gamma_{ij}^1),$$

where $\pi : X \times A_{X, \beta_\lambda} \rightarrow A_{X, \beta_\lambda}$ is the natural projection. Under some extra assumptions on X and β_λ Theorem 2.8 generalizes.

PROPOSITION 2.12. *Assume that A_{X, β_λ} is smooth and \mathcal{E} is a vector bundle. Then there exists a section s of \mathcal{E} such that*

$$\begin{array}{ccc}
 & & \mathcal{E} \\
 & & \downarrow \wr^s \\
 H_{\beta_\lambda} \cong Z(s) & \hookrightarrow & A_{X, \beta_\lambda}.
 \end{array}$$

In particular, H_{β_λ} has a perfect obstruction theory.

COROLLARY 2.13. *Let $X = \mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_s}$. Then there exists a virtual fundamental class $[H_{\beta_\lambda}]^{\text{vir}} \in A_*(H_{\beta_\lambda})$.*

Proof. The smoothness of A_{X, β_λ} follows by the smoothness of $H_\beta \cong \mathbb{P}(H^0(X, \mathcal{O}_X(\beta)))$ for every $\beta \in H_{n-2}(X, \mathbb{Z})$. Let D_1, D_2 be two effective divisors on X such that $[D_1] = [D_2] \in H_{n-2}(X, \mathbb{Z})$;

in particular, D_1, D_2 are linearly equivalent. Combining the long exact sequence in cohomology of the short exact sequence

$$0 \rightarrow \mathcal{O}(D_1 - D_2) \rightarrow \mathcal{O}(D_1) \rightarrow \mathcal{O}_{D_2}(D_1) \rightarrow 0$$

and the vanishings

$$\begin{aligned} H^k(X, \mathcal{O}(D_1)) &= 0, \quad k = 1, \dots, \dim X - 1, \\ H^k(X, \mathcal{O}(D_1 - D_2)) &= 0, \quad k = 2, \dots, \dim X, \end{aligned}$$

yields that $H^k(X, \mathcal{O}_{D_2}(D_1)) = 0$ for $k \geq 1$; cohomology and base change implies that $\mathbf{R}^k \pi_* \mathcal{O}_{\Gamma_{ij}^2}(\Gamma_{ij}^1) = 0$ for $k \geq 1$. Finally, We have that, if $\dim X \geq 2$,

$$\dim H^0(X, \mathcal{O}_{D_2}(D_1)) = \dim H^0(X, \mathcal{O}_X(D_1)) - 1,$$

which depends only on the degree $[D_1] = [D_2] \in H_2(X, \mathbb{Z})$ and implies that the dimension of the fibers of \mathcal{E} is constant, thus \mathcal{E} is a vector bundle. \square

3. Moduli space of stable pairs

3.1 Moduli space of stable pairs

Moduli spaces of stable pairs were introduced by Pandharipande and Thomas [PT09a] in order to give a geometric interpretation of the MNOP conjectures [MNOP06a], through the DT/PT correspondence proved by Toda (for Euler characteristic) and Bridgeland in [Tod10, Bri11] using wall-crossing and Hall algebra techniques.

For a smooth quasi-projective threefold X , a curve class $\beta \in H_2(X, \mathbb{Z})$ and $n \in \mathbb{Z}$, we define $P_n(X, \beta)$ to be the moduli space of pairs

$$I^\bullet = [\mathcal{O}_X \xrightarrow{s} F] \in \mathbf{D}^b(X)$$

in the derived category of X where F is a pure one-dimensional sheaf with proper support $[\text{supp}(F)] = \beta$ with $\chi(F) = n$ and s is a section with zero-dimensional cokernel.

By the work of Huybrechts and Thomas [HT10], the Atiyah class gives a *perfect obstruction theory* on $P_n(X, \beta)$

$$\mathbb{E} = \mathbf{R}\mathcal{H}om_\pi(\mathbb{I}, \mathbb{I})_0^\vee[-1] \rightarrow \mathbb{L}_{P_n(X, \beta)}, \tag{3.1}$$

where $(\cdot)_0$ denotes the trace-free part, $\pi : X \times P_n(X, \beta) \rightarrow P_n(X, \beta)$ is the canonical projection and $\mathbb{I}^\bullet = [\mathcal{O} \rightarrow \mathbb{F}]$ is the universal stable pair on $X \times P_n(X, \beta)$.

If X is projective, the perfect obstruction theory induces a virtual fundamental class $[P_n(X, \beta)]^{\text{vir}} \in A_*(P_n(X, \beta))$ and one defines *stable pair* (or *PT*) *invariants* by integrating cohomology classes $\gamma \in H^*(P_n(X, \beta), \mathbb{Z})$ against the virtual fundamental class

$$\text{PT}_{\beta, n}(X, \gamma) = \int_{[P_n(X, \beta)]^{\text{vir}}} \gamma \in \mathbb{Z}. \tag{3.2}$$

We focus here in the case of X a *local curve*, i.e. $X = \text{Tot}_C(L_1 \oplus L_2)$ the total space of the direct sum of two line bundles L_1, L_2 on a smooth projective curve C and $\beta = d[C] \in H_2(X, \mathbb{Z}) \cong H_2(C, \mathbb{Z})$ a multiple of the zero section of $X \rightarrow C$.

Here X is a smooth *quasi-projective* threefold, therefore the moduli space of stable pairs $P_n(X, \beta)$ is hardly ever a proper scheme and one cannot define invariants as in (3.2). Nevertheless, the algebraic torus $\mathbf{T} = (\mathbb{C}^*)^2$ acts on X by scaling the fibers and the action naturally lifts to $P_n(X, d[C])$, making the perfect obstruction theory naturally \mathbf{T} -equivariant by [Ric21, Example 4.6]. Moreover, the \mathbf{T} -fixed locus $P_n(X, d[C])^\mathbf{T}$ is proper (cf. Proposition 3.1), therefore by

Graber and Pandharipande [GP99] there is naturally an induced perfect obstruction theory on $P_n(X, d[C])^{\mathbf{T}}$ and a virtual fundamental class $[P_n(X, d[C])^{\mathbf{T}}]^{\text{vir}} \in A_*(P_n(X, d[C])^{\mathbf{T}})$. We define \mathbf{T} -equivariant stable pair invariants by the Graber–Pandharipande virtual localization formula

$$\text{PT}_{d,n}(X) = \int_{[P_n(X, d[C])^{\mathbf{T}}]^{\text{vir}}} \frac{1}{e^{\mathbf{T}}(N^{\text{vir}})} \in \mathbb{Q}(s_1, s_2), \tag{3.3}$$

where s_1, s_2 are the generators of \mathbf{T} -equivariant cohomology and the virtual normal bundle

$$N^{\text{vir}} = (\mathbb{E}|_{P_n(X, d[C])^{\mathbf{T}}})^{\text{mov}} \in K_{\mathbf{T}}^0(P_n(X, d[C])^{\mathbf{T}}) \tag{3.4}$$

is the \mathbf{T} -moving part of the restriction of the dual of the perfect obstruction theory. Stable pair invariants with descendent insertions on local curves have been studied in [PP12, PP13, Obl19].

3.2 Torus representations and their weights

Let $\mathbf{T} = (\mathbb{C}^*)^g$ be an algebraic torus, with character lattice $\hat{\mathbf{T}} = \text{Hom}(\mathbf{T}, \mathbb{C}^*) \cong \mathbb{Z}^g$. Let $K_{\mathbf{T}}^0(\text{pt})$ be the K -group of the category of \mathbf{T} -representations. Any finite-dimensional \mathbf{T} -representation V splits as a sum of one-dimensional representations called the *weights* of V . Each weight corresponds to a character $\mu \in \hat{\mathbf{T}}$ and, in turn, each character corresponds to a monomial $t^\mu = t_1^{\mu_1} \cdots t_g^{\mu_g}$ in the coordinates of \mathbf{T} . The map

$$\text{tr}: K_{\mathbf{T}}^0(\text{pt}) \rightarrow \mathbb{Z}[t^\mu \mid \mu \in \hat{\mathbf{T}}], \quad V \mapsto \text{tr}_V,$$

sending the class of a \mathbf{T} -representation to its decomposition into weight spaces is a ring isomorphism, where tensor product on the left corresponds to the natural multiplication on the right. We therefore sometimes identify a (virtual) \mathbf{T} -representation with its character.

If X is a scheme with a trivial \mathbf{T} -action, every \mathbf{T} -equivariant coherent sheaf on X decomposes as $F = \bigoplus_{\mu \in \hat{\mathbf{T}}} F_\mu \otimes t^\mu$.

3.3 The fixed locus

In this section we prove that the \mathbf{T} -fixed locus $P_n(X, d[C])^{\mathbf{T}}$ is a disjoint union of double nested Hilbert schemes of points $C^{[\mathbf{n}, \lambda]}$, for suitable reversed plane partitions \mathbf{n}, λ , where λ are Young diagram of size $|\lambda| = d$. Our strategy is similar to Kool and Thomas [KT17, § 4] for local surfaces.

Given a \mathbf{T} -equivariant coherent sheaf on X , its pushdown along $p : X \rightarrow C$ decomposes into weight spaces (e.g. by [Har77, Exercises II.5.17 and II.5.18])

$$p_* F = \bigoplus_{(i,j) \in \mathbb{Z}^2} F_{ij} \otimes t_1^i t_2^j,$$

where F_{ij} is a coherent sheaf on C . For example,

$$p_* \mathcal{O}_X = \bigoplus_{i,j \geq 0} L_1^{-i} \otimes L_2^{-j} \otimes t_1^{-i} t_2^{-j}.$$

As p is affine, the pushdown does not lose any information, and we recover the \mathcal{O}_X -module structure of F by the $p_* \mathcal{O}_X$ -action that $p_* F$ carries. This is generated by the action of the -1 pieces $L_1^{-1} \otimes t_1^{-1}, L_2^{-1} \otimes t_2^{-1}$, so we find that the \mathcal{O}_X -module structure is determined by the maps

$$\begin{aligned} \left(\bigoplus_{i,j} F_{ij} \otimes t_1^i t_2^j \right) \otimes L_1^{-1} \otimes t_1^{-1} &\rightarrow \bigoplus_{i,j} F_{ij} \otimes t_1^i t_2^j, \\ \left(\bigoplus_{i,j} F_{ij} \otimes t_1^i t_2^j \right) \otimes L_2^{-1} \otimes t_2^{-1} &\rightarrow \bigoplus_{i,j} F_{ij} \otimes t_1^i t_2^j, \end{aligned} \tag{3.5}$$

which commute with both the actions of \mathcal{O}_C and \mathbf{T} . In other words, (3.5) are \mathbf{T} -equivariant maps of \mathcal{O}_C -modules. By \mathbf{T} -equivariance, they are sums of maps

$$\begin{aligned} F_{ij} \otimes L_1^{-1} &\rightarrow F_{i-1,j}, \\ F_{ij} \otimes L_2^{-1} &\rightarrow F_{i,j-1}. \end{aligned} \tag{3.6}$$

Now let $(F, s) \in P_n(X, d[C])^{\mathbf{T}}$ be a \mathbf{T} -fixed stable pair. Then s is a \mathbf{T} -equivariant section of a \mathbf{T} -equivariant coherent sheaf F on X . Applying p_* to $\mathcal{O}_X \xrightarrow{s} F$ gives a graded map which commutes with the maps (3.6). Writing

$$G_{ij} = F_{-i,-j} \otimes L_1^i \otimes L_2^j,$$

we find that the \mathbf{T} -fixed stable pair (F, s) on X is equivalent to the following data of sheaves and commuting maps on C .

(3.7)

By the purity of F , each G_{ij} is either zero or a pure one-dimensional coherent sheaf on C , and the ‘vertical’ maps are generically isomorphisms. In particular, for every (i, j) such that either $i < 0$ or $j < 0$, it follows that G_{ij} is zero-dimensional and therefore vanishes by the purity assumption. Moreover, if G_{ij} is non-zero, it is a rank-one torsion-free sheaf on a smooth curve, that is a line bundle on C (with a section). Finally, any \mathbf{T} -equivariant stable pair on X is set-theoretically supported on C , thus is properly supported on X and only finitely many G_{ij} can be non-zero. This results in a diagram of the following shape.

(3.8)

where Z_{ij} are divisors on C and all ‘horizontal’ maps are injections of line bundles. Therefore, a \mathbf{T} -fixed stable pair (F, s) is equivalent to a nesting of divisors

$$\begin{array}{cccccc}
 Z_{00} & \subset & Z_{01} & \subset & Z_{02} & \subset & Z_{03} & \subset & \dots \\
 \cap & & \cap & & \cap & & \cap & & \\
 Z_{10} & \subset & Z_{11} & \subset & Z_{12} & \subset & Z_{13} & \subset & \dots \\
 \cap & & \cap & & \cap & & \cap & & \\
 Z_{20} & \subset & Z_{21} & \subset & Z_{22} & \subset & \dots & & \\
 \cap & & \cap & & \cap & & & & \\
 \dots & & \dots & & \dots & & & &
 \end{array} \tag{3.9}$$

where the nesting is dictated by a Young diagram λ . This results into a point of the double nested Hilbert scheme $C^{[\mathbf{n}_\lambda]}$, where by Riemann–Roch

$$\chi(F) = |\mathbf{n}_\lambda| + \mathbf{f}_{\lambda,g}(\deg L_1, \deg L_2),$$

where for a Young diagram λ and $g, k_1, k_2 \in \mathbb{Z}$ we define

$$\begin{aligned}
 \mathbf{f}_{\lambda,g}(k_1, k_2) &= \sum_{(i,j) \in \lambda} (1 - g - i \cdot k_1 - j \cdot k_2) \\
 &= |\lambda|(1 - g) - k_1 \cdot n(\lambda) - k_2 \cdot n(\bar{\lambda}).
 \end{aligned} \tag{3.10}$$

Conversely, any nesting of divisors as in (3.9) corresponds to a diagram of sheaves as in (3.8), which corresponds to a \mathbf{T} -fixed stable pair on X . Therefore, we have a bijection of sets

$$P_n(X, d[C])^{\mathbf{T}} = \bigsqcup_{\lambda \vdash d} \bigsqcup_{\mathbf{n}_\lambda} C^{[\mathbf{n}_\lambda]}, \tag{3.11}$$

where the disjoint union is over all Young diagrams λ of size d and all reversed plane partitions \mathbf{n}_λ satisfying $n = |\mathbf{n}_\lambda| + \mathbf{f}_{\lambda,g}(\deg L_1, \deg L_2)$. We mimic [KT17, Proposition 4.1] to prove that the above bijection on sets is an isomorphism of schemes.

PROPOSITION 3.1. *There exists an isomorphism of schemes*

$$P_n(X, d[C])^{\mathbf{T}} = \bigsqcup_{\lambda \vdash d} \bigsqcup_{\mathbf{n}_\lambda} C^{[\mathbf{n}_\lambda]},$$

where the disjoint union is over all Young diagrams λ of size d and all reversed plane partitions \mathbf{n}_λ satisfying

$$n = |\mathbf{n}_\lambda| + \mathbf{f}_{\lambda,g}(\deg L_1, \deg L_2).$$

In particular, $P_n(X, d[C])^{\mathbf{T}}$ is proper.

Proof. Let B be any (connected) scheme over \mathbb{C} . We need to adapt the construction of this section to a \mathbf{T} -fixed stable pair on $X \times B$, flat over B . Pushing down by the affine map $p : X \times B \rightarrow C \times B$ gives a graded sheaf $\bigoplus_{i,j} F_{ij}$ on $C \times B$, flat over B (therefore so are all its weight spaces F_{ij}). The original sheaf F on $X \times B$ can be reconstructed from the maps (3.6). Therefore, a \mathbf{T} -fixed pair (F, s) on $X \times B$, flat over B , is equivalent to the data (3.7), with each G_{ij} on $C \times B$, flat over B .

If (F, s) is a *stable pair*, over each closed fiber $C \times \{b\}$, where $b \in B$, we showed that each (non-zero) G_{ij} is a line bundle. By [HL10, Lemma 2.1.7], this shows that each (non-zero) G_{ij} is a line bundle on $C \times B$. Together with its non-zero section, this defines divisors $Z_{ij} \subset C \times B$, flat over B , satisfying the nesting condition of (3.9), which yields a B -point $B \rightarrow \bigsqcup_{\mathbf{n}_\lambda} C^{[\mathbf{n}_\lambda]}$.

Conversely, any B -point $B \rightarrow \bigsqcup_{\mathbf{n}\lambda} C^{[\mathbf{n}\lambda]}$ defines a diagram (3.8), equivalent to a \mathbf{T} -fixed stable pair (F, s) on $X \times B$, flat over B . □

As a corollary, we compute the generating series of the topological Euler characteristic of the moduli space of stable pairs on a local curve.

COROLLARY 3.2. *Let C be a smooth projective curve of genus g , L_1, L_2 line bundle on C and set $X = \text{Tot}_C(L_1 \oplus L_2)$. Then, for any $d > 0$, we have*

$$\sum_{n \in \mathbb{Z}} e(P_n(X, d[C])) \cdot q^n = \sum_{|\lambda|=d} q^{|\lambda|(1-g) - \deg L_1 n(\lambda) - \deg L_2 n(\bar{\lambda})} \prod_{\square \in \lambda} (1 - q^{h(\square)})^{2g-2}.$$

Proof. The topological Euler characteristic of a \mathbf{T} -scheme is the same of its \mathbf{T} -fixed locus, therefore

$$\begin{aligned} \sum_{n \in \mathbb{Z}} e(P_n(X, d[C])) \cdot q^n &= \sum_{n \in \mathbb{Z}} e(P_n(X, d[C])^{\mathbf{T}}) \cdot q^n \\ &= \sum_{n \in \mathbb{Z}} \sum_{|\lambda|=d} \sum_{\substack{|\mathbf{n}\lambda|=n- \\ \mathbf{f}_{\lambda, g}(\deg L_1, \deg L_2)}} q^n \cdot e(C^{[\mathbf{n}\lambda]}) \\ &= \sum_{|\lambda|=d} q^{\mathbf{f}_{\lambda, g}(\deg L_1, \deg L_2)} \sum_{\mathbf{n}\lambda} q^{|\mathbf{n}\lambda|} \cdot e(C^{[\mathbf{n}\lambda]}) \\ &= \sum_{|\lambda|=d} q^{\mathbf{f}_{\lambda, g}(\deg L_1, \deg L_2)} \prod_{\square \in \lambda} (1 - q^{h(\square)})^{2g-2}, \end{aligned}$$

where in the second line we applied Proposition 3.1 and in the last line we applied Theorem 2.11. □

4. K -theory class of the perfect obstruction theory

Let C be an irreducible smooth projective curve. On each connected component $C^{[\mathbf{n}\lambda]} \subset P_n(X, \beta)^{\mathbf{T}}$ of the \mathbf{T} -fixed locus there exists an induced virtual fundamental class $[C^{[\mathbf{n}\lambda]}]_{\text{PT}}^{\text{vir}}$ coming from the perfect obstruction theory (3.1). We show in this section that $[C^{[\mathbf{n}\lambda]}]_{\text{PT}}^{\text{vir}}$ agrees with the virtual fundamental class constructed in Corollary 2.9 and compute the class in K -theory of the virtual normal bundle N^{vir} .

We start by describing the class in K -theory of (the restriction of) the perfect obstruction theory $\mathbb{E} \in K_0^{\mathbf{T}}(C^{[\mathbf{n}\lambda]})$. To ease readability we omit various pullbacks whenever they are clear from the context. Recall the following identities in K -theory:

$$\begin{aligned} \mathbb{E} &= -\mathbf{R}\pi_* \mathbf{R}\mathcal{H}om(\mathbb{I}, \mathbb{I})_0^{\vee} \in K_0^{\mathbf{T}}(P_n(X, \beta)^{\mathbf{T}}), \\ \mathbb{F} &= \sum_{(i,j) \in \lambda} i_* \mathcal{O}_{C \times C^{[\mathbf{n}\lambda]}}(\mathcal{Z}_{ij}) \otimes L_1^{-i} \otimes L_2^{-j} \otimes \mathfrak{t}_1^{-i} \mathfrak{t}_2^{-j} \in K_0^{\mathbf{T}}(X \times C^{[\mathbf{n}\lambda]}), \\ \mathbb{I} &= \mathcal{O}_{X \times P_n(X, \beta)^{\mathbf{T}}} - \mathbb{F} \in K_0^{\mathbf{T}}(X \times P_n(X, \beta)^{\mathbf{T}}), \end{aligned}$$

where the various maps fit in the following diagram.

$$\begin{array}{ccccc} C & \xrightarrow{i} & X & \longleftarrow & X \times C^{[\mathbf{n}\lambda]} & \xrightarrow{\pi} & C^{[\mathbf{n}\lambda]} \\ & \searrow p & & & \downarrow i & \swarrow \pi & \\ & & & & C \times C^{[\mathbf{n}\lambda]} & & \end{array}$$

Again, to ease notation, we keep denoting $i \times \text{id}_{C^{[n,\lambda]}}$, $p \times \text{id}_{C^{[n,\lambda]}}$ by i, p and by π the composition $\pi \circ i$. We compute

$$\begin{aligned} \mathbb{E}^\vee &= \mathbf{R}\pi_* \mathbf{R}\mathcal{H}om(\mathcal{O}, \mathbb{F}) + \mathbf{R}\pi_* \mathbf{R}\mathcal{H}om(\mathbb{F}, \mathcal{O}) - \mathbf{R}\pi_* \mathbf{R}\mathcal{H}om(\mathbb{F}, \mathbb{F}) \\ &= \mathbf{R}\pi_* \mathbb{F} - (\mathbf{R}\pi_*(\mathbb{F} \otimes K_X))^\vee \otimes \mathfrak{t}_1 \mathfrak{t}_2 - \mathbf{R}\pi_* \mathbf{R}\mathcal{H}om(\mathbb{F}, \mathbb{F}), \end{aligned}$$

where in the second equality we applied Grothendieck duality⁴ and the projection formula and $K_X = K_C \otimes L_1^{-1} \otimes L_2^{-1}$. Define $\Lambda^\bullet(V) = \sum_{i=0}^{\text{rk} V} (-1)^i \Lambda^i V$ for a locally free sheaf V and extend it by linearity to any class in $K_0^{\mathbf{T}}(C)$. By [CG10, Lemma 5.4.9], for every \mathbf{T} -equivariant coherent sheaf $\mathcal{F} \in K_0^{\mathbf{T}}(C)$, we have

$$\mathbf{L}i^* i_* \mathcal{F} = \Lambda^\bullet N_{C/X}^* \otimes \mathcal{F} \in K_0^{\mathbf{T}}(C),$$

where $N_{C/X} = L_1 \otimes \mathfrak{t}_1 \oplus L_2 \otimes \mathfrak{t}_2$ is the \mathbf{T} -equivariant normal bundle of $i : C \rightarrow X$; an analogous statement holds for $i : C \times C^{[n,\lambda]} \rightarrow X \times C^{[n,\lambda]}$. Therefore,

$$\begin{aligned} &\mathbf{R}\pi_* \mathbf{R}\mathcal{H}om(\mathbb{F}, \mathbb{F}) \\ &= \sum_{(i,j),(l,k) \in \lambda} \mathbf{R}\pi_* \mathbf{R}\mathcal{H}om_X(i_* \mathcal{O}_{C \times C^{[n,\lambda]}}(\mathcal{Z}_{ij}), i_* \mathcal{O}_{C \times C^{[n,\lambda]}}(\mathcal{Z}_{lk}) \otimes L_1^{i-l} L_2^{j-k}) \mathfrak{t}_1^{i-l} \mathfrak{t}_2^{j-k} \\ &= \sum_{(i,j),(l,k) \in \lambda} \mathbf{R}\pi_* \mathbf{R}\mathcal{H}om_C(\mathbf{L}i^* i_* \mathcal{O}_{C \times C^{[n,\lambda]}}(\mathcal{Z}_{ij}), \mathcal{O}_{C \times C^{[n,\lambda]}}(\mathcal{Z}_{lk}) \otimes L_1^{i-l} L_2^{j-k}) \mathfrak{t}_1^{i-l} \mathfrak{t}_2^{j-k} \\ &= \sum_{(i,j),(l,k) \in \lambda} \mathbf{R}\pi_*((\mathcal{O} - L_1 \mathfrak{t}_1 - L_2 \mathfrak{t}_2 + L_1 L_2 \mathfrak{t}_1 \mathfrak{t}_2)(\mathcal{Z}_{lk} - \mathcal{Z}_{ij}) \otimes L_1^{i-l} L_2^{j-k}) \mathfrak{t}_1^{i-l} \mathfrak{t}_2^{j-k}, \end{aligned}$$

where in the second line we used adjunction in the derived category. To simplify the notation, for any $(i, j), (l, k) \in \lambda$ set $\Delta_{ij;lk} = \mathcal{Z}_{lk} - \mathcal{Z}_{ij}$, which is an effective divisor if $(i, j) \leq (l, k)$. Putting this all together we have the following identity in $K_0^{\mathbf{T}}(C^{[n,\lambda]})$:

$$\begin{aligned} \mathbb{E}^\vee &= \sum_{(i,j) \in \lambda} \mathbf{R}\pi_*(\mathcal{O}_{C \times C^{[n,\lambda]}}(\mathcal{Z}_{ij}) \otimes L_1^{-i} L_2^{-j}) \mathfrak{t}_1^{-i} \mathfrak{t}_2^{-j} \\ &\quad - \sum_{(i,j) \in \lambda} (\mathbf{R}\pi_*(\mathcal{O}_{C \times C^{[n,\lambda]}}(\mathcal{Z}_{ij}) \otimes K_X \otimes L_1^{-i} L_2^{-j} \mathfrak{t}_1^{-i} \mathfrak{t}_2^{-j}))^\vee \otimes \mathfrak{t}_1 \mathfrak{t}_2 \\ &\quad - \sum_{(i,j),(l,k) \in \lambda} \mathbf{R}\pi_*((\mathcal{O} - L_1 \mathfrak{t}_1 - L_2 \mathfrak{t}_2 + L_1 L_2 \mathfrak{t}_1 \mathfrak{t}_2)(\Delta_{ij;lk}) \otimes L_1^{i-l} L_2^{j-k}) \mathfrak{t}_1^{i-l} \mathfrak{t}_2^{j-k}. \end{aligned} \tag{4.1}$$

THEOREM 4.1. *There is an identity of virtual fundamental classes*

$$[C^{[n,\lambda]}]_{\text{PT}}^{\text{vir}} = [C^{[n,\lambda]}]^{\text{vir}} \in A_*(C^{[n,\lambda]}),$$

where the class on the left-hand-side is induced by (3.1) by Graber–Pandharipande localization and that on the right-hand side is constructed in Corollary 2.9.

Proof. By a result of Siebert [Sie04, Theorem 4.6] any two virtual fundamental classes coincide if the classes in K -theory of their perfect obstruction theory agree. The class in K -theory of the dual of the induced PT perfect obstruction theory is the \mathbf{T} -fixed part of \mathbb{E}^\vee by

⁴ Even though X is not proper, we can pass to a compactification $X \hookrightarrow \overline{X}$ and use Grothendieck duality exploiting the fact that the sheaves involved have proper support, see [CK18, footnote 4].

[GP99, Proposition 1]

$$\begin{aligned}
 (\mathbb{E}^\vee)^{\text{fix}} &= \mathbf{R}\pi_* \mathcal{O}_{C \times C^{[n\lambda]}}(\mathcal{Z}_{00}) - \sum_{(i,j) \in \lambda} \mathbf{R}\pi_* \mathcal{O}_{C \times C^{[n\lambda]}} + \sum_{\substack{(i,j) \in \lambda \\ i \geq 1}} \mathbf{R}\pi_* \mathcal{O}_{C \times C^{[n\lambda]}}(\Delta_{i-1,j;ij}) \\
 &+ \sum_{\substack{(i,j) \in \lambda \\ j \geq 1}} \mathbf{R}\pi_* \mathcal{O}_{C \times C^{[n\lambda]}}(\Delta_{i,j-1;ij}) - \sum_{\substack{(i,j) \in \lambda \\ i,j \geq 1}} \mathbf{R}\pi_* \mathcal{O}_{C \times C^{[n\lambda]}}(\Delta_{i-1,j-1;ij}). \tag{4.2}
 \end{aligned}$$

We explained in § 2.1 how to associate a graph with any Young diagram λ . Note that boxes $(i, j) \in \lambda$ are in bijection with the vertices V , boxes $(i, j) \in \lambda$, such that $i \geq 1$ (respectively, $j \geq 1$) are in bijection with vertical (respectively, horizontal) edges E and boxes $(i, j) \in \lambda$, such that $i, j \geq 1$ are in bijection with squares Q of the associated graph. By Lemma 2.1

$$V - E + Q - 1 = 0. \tag{4.3}$$

From the universal exact sequences

$$\begin{aligned}
 0 \rightarrow \mathcal{O} \rightarrow \mathcal{O}(\mathcal{Z}_{00}) \rightarrow \mathcal{O}_{\mathcal{Z}_{00}}(\mathcal{Z}_{00}) \rightarrow 0, \\
 0 \rightarrow \mathcal{O} \rightarrow \mathcal{O}(\Delta_{ij;lk}) \rightarrow \mathcal{O}_{\Delta_{ij;lk}}(\Delta_{ij;lk}) \rightarrow 0,
 \end{aligned} \tag{4.4}$$

whenever $\Delta_{ij;lk}$ is an effective divisor, we deduce the identity in K -theory

$$\mathbf{R}\pi_* \mathcal{O}(\mathcal{Z}_{00}) = \mathbf{R}\pi_* \mathcal{O} + \mathbf{R}\pi_* \mathcal{O}_{\mathcal{Z}_{00}}(\mathcal{Z}_{00}), \tag{4.5}$$

and similarly for \mathcal{Z}_{00} replaced by $\Delta_{ij;lk}$. Substituting (4.5) in (4.2) and noting that

$$(1 - V + E - Q)\mathbf{R}\pi_* \mathcal{O} = 0,$$

we obtain

$$\begin{aligned}
 (\mathbb{E}^\vee)^{\text{fix}} &= \mathbf{R}\pi_* \mathcal{O}_{\mathcal{Z}_{00}}(\mathcal{Z}_{00}) + \sum_{\substack{(i,j) \in \lambda \\ i \geq 1}} \mathbf{R}\pi_* \mathcal{O}_{\Delta_{i-1,j;ij}}(\Delta_{i-1,j;ij}) \\
 &+ \sum_{\substack{(i,j) \in \lambda \\ j \geq 1}} \mathbf{R}\pi_* \mathcal{O}_{\Delta_{i,j-1;ij}}(\Delta_{i,j-1;ij}) - \sum_{\substack{(i,j) \in \lambda \\ i,j \geq 1}} \mathbf{R}\pi_* \mathcal{O}_{\Delta_{i-1,j-1;ij}}(\Delta_{i-1,j-1;ij}).
 \end{aligned}$$

Moreover, in the expression above, all higher direct images $\mathbf{R}^k \pi_*$ vanish for $k > 0$ by cohomology and base change, therefore

$$\begin{aligned}
 (\mathbb{E}^\vee)^{\text{fix}} &= \pi_* \mathcal{O}_{\mathcal{Z}_{00}}(\mathcal{Z}_{00}) + \sum_{\substack{(i,j) \in \lambda \\ i \geq 1}} \pi_* \mathcal{O}_{\Delta_{i-1,j;ij}}(\Delta_{i-1,j;ij}) \\
 &+ \sum_{\substack{(i,j) \in \lambda \\ j \geq 1}} \pi_* \mathcal{O}_{\Delta_{i,j-1;ij}}(\Delta_{i,j-1;ij}) - \sum_{\substack{(i,j) \in \lambda \\ i,j \geq 1}} \pi_* \mathcal{O}_{\Delta_{i-1,j-1;ij}}(\Delta_{i-1,j-1;ij}).
 \end{aligned}$$

We finally show that this is precisely the same class in K -theory as

$$T_{A_{C, \mathbf{n}\lambda}}|_{C^{[n\lambda]}} - \mathcal{E}|_{C^{[n\lambda]}} \in K_0(C^{[n\lambda]}),$$

where $\mathcal{E} \rightarrow A_{C, \mathbf{n}\lambda}$ is the vector bundle constructed in Theorem 2.8. In fact, in the notation of § 2.4, we have

$$\begin{aligned}
 \mathcal{X}_{ij}|_{C^{[n\lambda]} \times C} &= \Delta_{i-1,j;ij}, \quad i \geq 1, \\
 \mathcal{Y}_{ij}|_{C^{[n\lambda]} \times C} &= \Delta_{i,j-1;ij}, \quad j \geq 1, \\
 \Gamma_{ij}^1|_{C^{[n\lambda]} \times C} &= \Gamma_{ij}^2|_{C^{[n\lambda]} \times C} = \Delta_{i-1,j-1;ij}, \quad i, j \geq 1.
 \end{aligned}$$

Moreover, the explicit description of the tangent bundle of the Hilbert scheme of points on a smooth curve in terms of its universal subscheme [ACGH85, Lemma IV.2.3] yields

$$T_{A_{C,n\lambda}} = \pi_* \mathcal{O}_{\mathcal{Z}_{00}}(\mathcal{Z}_{00}) \oplus \bigoplus_{\substack{(i,j) \in \lambda \\ i \geq 1}} \pi_* \mathcal{O}_{\mathcal{X}_{ij}}(\mathcal{X}_{ij}) \oplus \bigoplus_{\substack{(i,j) \in \lambda \\ j \geq 1}} \pi_* \mathcal{O}_{\mathcal{Y}_{ij}}(\mathcal{Y}_{ij}),$$

by which we conclude that

$$(\mathbb{E}^\vee)^{\text{fix}} = T_{A_{C,n\lambda}}|_{C^{[n\lambda]}} - \mathcal{E}|_{C^{[n\lambda]}} \in K_0(C^{[n\lambda]}). \quad \square$$

In virtue of Theorem 4.1, we now denote by $T_{C^{[n\lambda]}}^{\text{vir}}$ the class in K -theory of the dual of the perfect obstruction theory (2.2) and that induced by the fixed part of (3.1), which we showed to agree.

In order to compute stable pair invariants (3.3) one needs to express the virtual normal bundle (3.4) in terms of K -theoretic classes which are easier to handle. For instance, we could express N^{vir} in terms of pullbacks of the line bundles L_1, L_2, K_C and the universal divisors $\Delta_{ij;lk}$, but that would lead to cumbersome expressions difficult to manipulate.

Example 4.2. As a concrete example, we compute the weight space of \mathbb{E}^\vee relative to the character $\mathfrak{t}_1 \mathfrak{t}_2$, which we denote by $\mathbb{E}_{\mathfrak{t}_1 \mathfrak{t}_2}^\vee$.

An application of Grothendieck duality and projection formula on $\pi : C \times C^{[n\lambda]} \rightarrow C^{[n\lambda]}$ yields

$$\mathbf{R}\pi_* L(\Delta) = -(\mathbf{R}\pi_*(K_C \otimes L^{-1}(-\Delta)))^\vee \in K_0(C^{[n\lambda]}), \quad (4.6)$$

for any divisor $\Delta \subset C \times C^{[n\lambda]}$ and any line bundle L on C . As in the proof of Theorem 4.1, combining (4.3), (4.4), (4.6), the identity $K_X = K_C \otimes L_1^{-1} \otimes L_2^{-1}$ and some vanishing of higher direct images yields

$$\begin{aligned} \mathbb{E}_{\mathfrak{t}_1 \mathfrak{t}_2}^\vee &= -(\pi_*(K_X \otimes \mathcal{O}_{\mathcal{Z}_{00}}(\mathcal{Z}_{00})))^\vee - \sum_{\substack{(i,j) \in \lambda \\ i \geq 1}} (\pi_*(K_X \otimes \mathcal{O}_{\Delta_{i-1,j;ij}}(\Delta_{i-1,j;ij})))^\vee \\ &\quad - \sum_{\substack{(i,j) \in \lambda \\ j \geq 1}} (\pi_*(K_X \otimes \mathcal{O}_{\Delta_{i,j-1;ij}}(\Delta_{i,j-1;ij})))^\vee + \sum_{\substack{(i,j) \in \lambda \\ i,j \geq 1}} (\pi_*(K_X \otimes \mathcal{O}_{\Delta_{i-1,j-1;ij}}(\Delta_{i-1,j-1;ij})))^\vee. \end{aligned}$$

The situation notably simplifies if we impose X to be Calabi–Yau.

PROPOSITION 4.3. *If X is Calabi–Yau, we have an identity*

$$N^{\text{vir}} = -T_{C^{[n\lambda]}}^{\text{vir},\vee} \otimes \mathfrak{t}_1 \mathfrak{t}_2 + \Omega - \Omega^\vee \otimes \mathfrak{t}_1 \mathfrak{t}_2 \in K_{\mathbf{T}}^0(C^{[n\lambda]}),$$

where $\Omega, \Omega^\vee \in K_{\mathbf{T}}^0(C^{[n\lambda]})$ have no weight spaces corresponding to the characters $(\mathfrak{t}_1 \mathfrak{t}_2)^0, \mathfrak{t}_1 \mathfrak{t}_2$.

Proof. If X is Calabi–Yau, the perfect obstruction theory (3.1) satisfies

$$\mathbb{E}^\vee = -\mathbb{E} \otimes \mathfrak{t}_1 \mathfrak{t}_2 \in K_{\mathbf{T}}^0(C^{[n\lambda]}) \quad (4.7)$$

by \mathbf{T} -equivariant Serre duality. Setting $\mathbb{E}^\vee = W_+ - W_-$, where $W_+, W_- \in K_{\mathbf{T}}^0(C^{[n\lambda]})$ are classes of \mathbf{T} -equivariant vector bundles, (4.7) implies that

$$W_- = W_+^\vee \otimes \mathfrak{t}_1 \mathfrak{t}_2.$$

We have that $(\mathbb{E}^\vee)^{\text{fix}} = T_{C^{[n\lambda]}}^{\text{vir}}$, therefore

$$\mathbb{E}^\vee = T_{C^{[n\lambda]}}^{\text{vir}} - T_{C^{[n\lambda]}}^{\text{vir},\vee} \otimes \mathfrak{t}_1 \mathfrak{t}_2 + \Omega - \Omega^\vee \otimes \mathfrak{t}_1 \mathfrak{t}_2,$$

which concludes the argument. □

Remark 4.4. A simple computation shows that we could take Ω to be of the form

$$\begin{aligned} \Omega = & \sum_{\substack{(i,j) \in \lambda \\ (i,j) \neq (0,0)}} \mathbf{R}\pi_*(\mathcal{O}_{C \times C^{[\mathbf{n}_\lambda]}}(\mathcal{Z}_{ij}) \otimes L_1^{-i} L_2^{-j}) \mathbf{t}_1^{-i} \mathbf{t}_2^{-j} \\ & - \sum_{\substack{(i,j),(l,k) \in \lambda \\ (i,j) \neq (l,k) \\ (i,j) \neq (l+1,k+1)}} \mathbf{R}\pi_*(\mathcal{O}(\Delta_{ij;lk}) \otimes L_1^{i-l} L_2^{j-k}) \mathbf{t}_1^{i-l} \mathbf{t}_2^{j-k} \\ & + \sum_{\substack{(i,j),(l,k) \in \lambda \\ (i,j) \neq (l-1,k) \\ (i,j) \neq (l,k+1)}} \mathbf{R}\pi_*(\mathcal{O}(\Delta_{ij;lk}) \otimes L_1^{i-l+1} L_2^{j-k}) \mathbf{t}_1^{i-l+1} \mathbf{t}_2^{j-k}. \end{aligned}$$

All other choices $\tilde{\Omega}$ must be of the form

$$\tilde{\Omega} = \Omega + A + A^\vee \otimes \mathbf{t}_1 \mathbf{t}_2,$$

for any $A \in K_{\mathbf{T}}^0(C^{[\mathbf{n}_\lambda]})$ having no weight spaces corresponding to the characters $(\mathbf{t}_1 \mathbf{t}_2)^0, \mathbf{t}_1 \mathbf{t}_2$. In particular, this implies that the parity of $\text{rk } \Omega$ is independent by the choice of Ω .

5. Universality

5.1 Universal expression

In the previous sections, given a triple (C, L_1, L_2) with C an irreducible smooth projective curve and L_1, L_2 line bundles on C , we reduced stable pair invariants (with no insertions) of $\text{Tot}_C(L_1 \oplus L_2)$ to the computation of

$$\int_{[C^{[\mathbf{n}_\lambda]}]_{\text{vir}}} e^{\mathbf{T}(-N_{C,L_1,L_2}^{\text{vir}})} \in \mathbb{Q}(s_1, s_2), \tag{5.1}$$

where \mathbf{n}_λ is a reversed plane partition and the virtual normal bundle $N_{C,L_1,L_2}^{\text{vir}}$ is the \mathbf{T} -moving part of the class in K -theory (4.1).

We state our main results, describing the generating series of (5.1) in terms of three universal functions exploiting the universality techniques used in [EGL01, Theorem 4.2] for surfaces. Furthermore, we find explicit expressions for these universal series under the anti-diagonal restriction $s_1 + s_2 = 0$.

THEOREM 5.1. *Let C be a genus g smooth irreducible projective curve and L_1, L_2 line bundles over C . We have an identity*

$$\sum_{\mathbf{n}_\lambda} q^{|\mathbf{n}_\lambda|} \int_{[C^{[\mathbf{n}_\lambda]}]_{\text{vir}}} e^{\mathbf{T}(-N_{C,L_1,L_2}^{\text{vir}})} = A_\lambda^{g-1} \cdot B_\lambda^{\text{deg } L_1} \cdot C_\lambda^{\text{deg } L_2} \in \mathbb{Q}(s_1, s_2)[[q]],$$

where $A_\lambda, B_\lambda, C_\lambda \in \mathbb{Q}(s_1, s_2)[[q]]$ are fixed universal series for $i = 1, 2, 3$ which only depend on λ . Moreover,

$$A_\lambda(s_1, s_2) = A_{\bar{\lambda}}(s_2, s_1), \quad B_\lambda(s_1, s_2) = C_{\bar{\lambda}}(s_2, s_1).$$

Proof. The proof is similar to [EGL01, Theorem 4.2]. Consider the map

$$Z : \mathcal{K} := \{(C, L_1, L_2) : C \text{ curve, } L_1, L_2 \text{ line bundles}\} \rightarrow \mathbb{Q}(s_1, s_2)[[q]]$$

given by

$$Z(C, L_1, L_2) = C_{g, \deg L_1 \deg L_2}^{-1} \cdot \sum_{\mathbf{n}_\lambda} q^{|\mathbf{n}_\lambda|} \int_{[C^{[\mathbf{n}_\lambda]}]^{vir}} e^{\mathbf{T}}(-N_{C, L_1, L_2}^{vir}),$$

where $C_{g, \deg L_1 \deg L_2}$ is the leading term of the generating series of the integrals (5.1).

By Proposition 5.2 the integral (5.1) is *multiplicative* and by Corollary 5.4 it is a polynomial on $g, \deg L_1, \deg L_2$. This implies that Z factors through

$$\mathcal{K} \xrightarrow{\gamma} \mathbb{Q}^3 \xrightarrow{Z'} \mathbb{Q}(s_1, s_2)[[q]],$$

where $\gamma(C, L_1, L_2) = (g - 1, \deg L_1, \deg L_2)$ and Z' is a linear map.

A basis of \mathbb{Q}^3 is given by the images

$$e_1 = \gamma(\mathbb{P}^1, \mathcal{O}, \mathcal{O}), \quad e_2 = \gamma(\mathbb{P}^1, \mathcal{O}(1), \mathcal{O}), \quad e_3 = \gamma(\mathbb{P}^1, \mathcal{O}, \mathcal{O}(1)),$$

and the image of a generic triple (C, L_1, L_2) can be written as

$$\gamma(C, L_1, L_2) = (1 - g - \deg L_1 - \deg L_2) \cdot e_1 + \deg L_1 \cdot e_2 + \deg L_2 \cdot e_3.$$

We conclude that

$$Z'(C, L_1, L_2) = Z'(e_1)^{1-g} \cdot (Z'(e_1)^{-1} Z'(e_2))^{\deg L_1} \cdot (Z'(e_1)^{-1} Z'(e_3))^{\deg L_2},$$

which gives the universal series we were looking for. The second claim just follows by interchanging the role of L_1 and L_2 . □

We devote the remainder of § 5 to prove the multiplicativity and polynomiality of (5.1). In § 6 we compute the leading term of the generating series of (5.1), whereas in § 7 we explicitly compute the integral (5.1) in the toric case under the anti-diagonal restriction. These computations will lead to the proof of the second part of the main Theorem 1.3 (see Theorem 8.1).

5.2 Multiplicativity

We now show that the integral (5.1) is multiplicative. First, note that if $C = C' \sqcup C''$ is a smooth projective curve with two connected components, the construction of Theorem 2.8 does not directly work and we need to adjust it to define a virtual fundamental class.

For any reversed plane partition \mathbf{n}_λ there is an induced stratification

$$C^{[\mathbf{n}_\lambda]} = \bigsqcup_{\mathbf{n}'_\lambda + \mathbf{n}''_\lambda = \mathbf{n}_\lambda} C'^{[\mathbf{n}'_\lambda]} \times C''^{[\mathbf{n}''_\lambda]}.$$

We set

$$A_{C, \mathbf{n}_\lambda} := \bigsqcup_{\mathbf{n}'_\lambda + \mathbf{n}''_\lambda = \mathbf{n}_\lambda} A_{C', \mathbf{n}'_\lambda} \times A_{C'', \mathbf{n}''_\lambda},$$

which is a smooth projective variety. Let $\mathcal{E}_{C', \mathbf{n}'_\lambda}, \mathcal{E}_{C'', \mathbf{n}''_\lambda}$ denote the vector bundles over $A_{C', \mathbf{n}'_\lambda}, A_{C'', \mathbf{n}''_\lambda}$ of Theorem 2.8. We define a vector bundle $\mathcal{E}_{C, \mathbf{n}_\lambda}$ over $A_{C, \mathbf{n}_\lambda}$ by declaring its restriction to any connected component $A_{C', \mathbf{n}'_\lambda} \times A_{C'', \mathbf{n}''_\lambda}$ to be

$$\mathcal{E}_{C, \mathbf{n}_\lambda}|_{A_{C', \mathbf{n}'_\lambda} \times A_{C'', \mathbf{n}''_\lambda}} = \mathcal{E}_{C', \mathbf{n}'_\lambda} \boxplus \mathcal{E}_{C'', \mathbf{n}''_\lambda}.$$

By Theorem 2.8, there exists a section s of $\mathcal{E}_{C, \mathbf{n}_\lambda}$ such that

$$C^{[\mathbf{n}_\lambda]} \cong Z(s) \hookrightarrow A_{C, \mathbf{n}_\lambda},$$

and, therefore, an induced virtual fundamental class $[C^{[\mathbf{n}_\lambda]}]^{vir}$ satisfying

$$[C^{[\mathbf{n}_\lambda]}]^{vir}_{C'^{[\mathbf{n}'_\lambda]} \times C''^{[\mathbf{n}''_\lambda]}} = [C'^{[\mathbf{n}'_\lambda]}]^{vir} \boxtimes [C''^{[\mathbf{n}''_\lambda]}]^{vir}. \tag{5.2}$$

By iterating this construction, there exists a natural virtual fundamental class on $C^{[n_\lambda]}$ for any smooth projective curve C (with any number of connected components).

PROPOSITION 5.2. *Let (C, L_1, L_2) be a triple where $C = C' \sqcup C''$ and $L_i = L'_i \oplus L''_i$ for $i = 1, 2$, where L'_i are line bundles on C' and L''_i are line bundles on C'' . Then*

$$\begin{aligned} & \sum_{\mathbf{n}_\lambda} q^{|\mathbf{n}_\lambda|} \int_{[C^{[n_\lambda]}]^{vir}} e^{\mathbf{T}}(-N_{C, L_1, L_2}^{vir}) \\ &= \sum_{\mathbf{n}_\lambda} q^{|\mathbf{n}_\lambda|} \int_{[C'^{[n'_\lambda]}]^{vir}} e^{\mathbf{T}}(-N_{C', L'_1, L'_2}^{vir}) \cdot \sum_{\mathbf{n}_\lambda} q^{|\mathbf{n}_\lambda|} \int_{[C''^{[n''_\lambda]}]^{vir}} e^{\mathbf{T}}(-N_{C'', L''_1, L''_2}^{vir}). \end{aligned}$$

Proof. Let \mathbf{n}_λ be a fixed reversed plane partition. We claim that the restriction of the virtual normal bundle to the connected component $C'^{[n'_\lambda]} \times C''^{[n''_\lambda]} \subset C^{[n_\lambda]}$ decomposes as

$$N_{C, L_1, L_2}^{vir}|_{C'^{[n'_\lambda]} \times C''^{[n''_\lambda]}} = N_{C', L'_1, L'_2}^{vir} \boxplus N_{C'', L''_1, L''_2}^{vir}. \tag{5.3}$$

In fact, N_{C, L_1, L_2}^{vir} is a linear combination of K -theoretic classes of the form

$$\mathbf{R}\pi_*(\mathcal{O}_{C \times C^{[n_\lambda]}}(\Delta) \otimes L_1^a L_2^b) \otimes \mathfrak{t}^\mu \in K_{\mathbf{T}}^0(C^{[n_\lambda]}),$$

where Δ is a \mathbb{Z} -linear combination of the universal divisors \mathcal{Z}_{ij} on $C \times C^{[n_\lambda]}$, $a, b \in \mathbb{Z}$ and \mathfrak{t}^μ is a \mathbf{T} -character and note that

$$L_1^a L_2^b = L_1'^a L_2'^b \oplus L_1''^a L_2''^b \in \text{Pic}(C' \sqcup C'').$$

Consider the induced stratification

$$C \times C^{[n_\lambda]} = \bigsqcup_{\mathbf{n}'_\lambda + \mathbf{n}''_\lambda = \mathbf{n}_\lambda} (C' \sqcup C'') \times C'^{[n'_\lambda]} \times C''^{[n''_\lambda]}.$$

Denote by Δ', Δ'' the corresponding universal divisor on $C' \times C'^{[n'_\lambda]}, C'' \times C''^{[n''_\lambda]}$ and consider the projection maps

$$\begin{aligned} q_1 &: C' \times C'^{[n'_\lambda]} \times C''^{[n''_\lambda]} \rightarrow C' \times C'^{[n'_\lambda]}, \\ q_2 &: C'' \times C'^{[n'_\lambda]} \times C''^{[n''_\lambda]} \rightarrow C'' \times C''^{[n''_\lambda]}. \end{aligned}$$

On every component $(C' \sqcup C'') \times C'^{[n'_\lambda]} \times C''^{[n''_\lambda]}$ we have

$$\mathcal{O}_{C \times C^{[n_\lambda]}}(\Delta)|_{(C' \sqcup C'') \times C'^{[n'_\lambda]} \times C''^{[n''_\lambda]}} = q_1^* \mathcal{O}_{C' \times C'^{[n'_\lambda]}}(\Delta') \oplus q_2^* \mathcal{O}_{C'' \times C''^{[n''_\lambda]}}(\Delta''),$$

and, similarly,

$$\begin{aligned} & \mathcal{O}_{C \times C^{[n_\lambda]}}(\Delta) \otimes L_1^a L_2^b|_{(C' \sqcup C'') \times C'^{[n'_\lambda]} \times C''^{[n''_\lambda]}} \\ &= q_1^*(\mathcal{O}_{C' \times C'^{[n'_\lambda]}}(\Delta') \otimes L_1'^a L_2'^b) \oplus q_2^*(\mathcal{O}_{C'' \times C''^{[n''_\lambda]}}(\Delta'') \otimes L_1''^a L_2''^b). \end{aligned}$$

Consider the following cartesian diagram given by the natural projections.

$$\begin{array}{ccc} C' \times C'^{[n'_\lambda]} \times C''^{[n''_\lambda]} & \xrightarrow{q_1} & C' \times C'^{[n'_\lambda]} \\ \downarrow \pi & & \downarrow \pi_1 \\ C'^{[n'_\lambda]} \times C''^{[n''_\lambda]} & \xrightarrow{\tilde{q}_1} & C'^{[n'_\lambda]} \end{array}$$

Flat base change yields

$$\mathbf{R}\pi_* q_1^*(\mathcal{O}_{C' \times C'^{[n'_\lambda]}}(\Delta') \otimes L_1'^a L_2'^b) = \tilde{q}_1^* \mathbf{R}\pi_{1*}(\mathcal{O}_{C' \times C'^{[n'_\lambda]}}(\Delta') \otimes L_1'^a L_2'^b),$$

and analogously for C'' , which implies

$$\begin{aligned} & \mathbf{R}\pi_* (\mathcal{O}_{C \times C^{[n_\lambda]}}(\Delta) \otimes L_1^a L_2^b) \otimes \mathfrak{t}^\mu |_{C'^{[n'_\lambda]} \times C''^{[n''_\lambda]}} \\ &= \mathbf{R}\pi_{1*} (\mathcal{O}_{C' \times C'^{[n'_\lambda]}}(\Delta') \otimes L_1'^a L_2'^b) \otimes \mathfrak{t}^\mu \boxplus \mathbf{R}\pi_{2*} (\mathcal{O}_{C'' \times C''^{[n''_\lambda]}}(\Delta'') \otimes L_1''^a L_2''^b) \otimes \mathfrak{t}^\mu, \end{aligned}$$

and proves the claim (5.3). Combining (5.2) and (5.3) yields

$$\int_{[C^{[n_\lambda]}]_{\text{vir}}} e^{\mathbf{T}(-N_{C,L_1,L_2}^{\text{vir}})} = \sum_{\mathbf{n}'_\lambda + \mathbf{n}''_\lambda = \mathbf{n}_\lambda} \int_{[C'^{[n'_\lambda]}]_{\text{vir}}} e^{\mathbf{T}(-N_{C',L'_1,L'_2}^{\text{vir}})} \cdot \int_{[C''^{[n''_\lambda]}]_{\text{vir}}} e^{\mathbf{T}(-N_{C'',L''_1,L''_2}^{\text{vir}})},$$

which concludes the proof. □

5.3 Chern numbers dependence

We now show that the integral 5.1 is a polynomial in the Chern numbers of the triple (C, L_1, L_2) . Our strategy is to express the integral on a product of Picard varieties $\text{Pic}^n(C)$, via the Abel–Jacobi map, where the integrand is a polynomial expression on tautological classes. Through this section, we follow the notation as in [ACGH85, § VIII.2] and [KT17, §§ 9, 10.1].

5.3.1 *Tautological integrals on $\text{Pic}^n(C)$.* Let C be a smooth curve of genus g . If $n > 2g - 2$, the Abel–Jacobi map

$$\begin{aligned} \text{AJ} : C^{(n)} &\rightarrow \text{Pic}^n(C) \\ Z &\rightarrow [\mathcal{O}_C(Z)] \end{aligned}$$

is a projective bundle. In fact, consider the diagram

$$\begin{array}{ccc} C^{(n)} \times C & \xrightarrow{\text{AJ} \times \text{id}} & \text{Pic}^n(C) \times C \\ \downarrow \pi & & \downarrow \bar{\pi} \\ C^{(n)} & \xrightarrow{\text{AJ}} & \text{Pic}^n(C) \end{array}$$

and a Poincaré line bundle \mathcal{P} on $\text{Pic}^n(C) \times C$, normalized by fixing

$$\mathcal{P}|_{\text{Pic}^n(C) \times \{c\}} \cong \mathcal{O}_{\text{Pic}^n(C)},$$

for a certain $c \in C$. Then

$$C^{(n)} \cong \mathbb{P}(\bar{\pi}_* \mathcal{P}).$$

The universal divisor $\mathcal{Z} \subset C^{(n)} \times C$ satisfies [KT17, (67)]

$$\mathcal{O}(\mathcal{Z}) \cong (\text{AJ} \times \text{id})^* \mathcal{P} \otimes \pi^* \mathcal{O}(1) \quad \text{and} \quad \mathcal{O}(1) \cong \mathcal{O}(\mathcal{Z})|_{C^{(n)} \times \{c\}},$$

and we denote the first Chern class of the latter by

$$\omega := c_1(\mathcal{O}(1)) \in H^2(C^{(n)}, \mathbb{Z}).$$

Now consider the product of Abel–Jacobi maps

$$\text{AJ} : C^{(n_1)} \times \dots \times C^{(n_s)} \rightarrow \text{Pic}^{n_1}(C) \times \dots \times \text{Pic}^{n_s}(C),$$

where each $n_i > 2g - 2$. We denote by \mathcal{P}_i (the pullback of) the Poincaré line bundle on $\text{Pic}^{n_i}(C) \times C$, each normalized at a point $c_i \in C$, and by ω_i the first Chern classes of the tautological bundles on $C^{(n_i)}$. Finally, we denote by $\mathcal{Z}_i \subset C^{(n_i)} \times C$ the universal divisors and by \mathcal{I}_i (the pullback of) their ideal sheaves, which in this case are line bundles.

We are interested in studying integrals of the form

$$\int_{C^{(n_1)} \times \dots \times C^{(n_s)}} f, \tag{5.4}$$

where f is a polynomial in the Chern classes of the K -theoretic classes

$$\mathbf{R}\pi_* \mathbf{R}\mathcal{H}om\left(\bigotimes_{i \in I} \mathcal{I}_i, \bigotimes_{j \in J} \mathcal{I}_j \otimes L_k\right),$$

L_k are line bundles on C and I, J are families of indices (possibly with repetitions).

We assume now that $n_i \gg 0$ for all $i = 1, \dots, s$. Applying the projection formula and flat base change yields

$$\begin{aligned} & \mathbf{R}\pi_* \mathbf{R}\mathcal{H}om\left(\bigotimes_{i \in I} \mathcal{I}_i, \bigotimes_{j \in J} \mathcal{I}_j \otimes L_k\right) \\ &= \mathbf{R}\pi_* \mathbf{R}\mathcal{H}om\left(\bigotimes_{i \in I} (\text{AJ} \times \text{id})^* \mathcal{P}_i^* \otimes \pi^* \mathcal{O}_{C^{(n_i)}}(-1), \bigotimes_{j \in J} (\text{AJ} \times \text{id})^* \mathcal{P}_j^* \otimes \pi^* \mathcal{O}_{C^{(n_j)}}(-1) \otimes L_k\right) \\ &= \mathcal{F} \otimes \mathbf{R}\pi_* \left(\bigotimes_{i \in I} (\text{AJ} \times \text{id})^* \mathcal{P}_i \otimes \bigotimes_{j \in J} (\text{AJ} \times \text{id})^* \mathcal{P}_j^* \otimes L_k\right) \\ &= \mathcal{F} \otimes \text{AJ}^* \mathbf{R}\bar{\pi}_* \left(\bigotimes_{i \in I} \mathcal{P}_i \otimes \bigotimes_{j \in J} \mathcal{P}_j^* \otimes L_k\right), \end{aligned}$$

where $\mathcal{F} = \bigotimes_{i \in I} \mathcal{O}_{C^{(n_i)}}(1) \otimes \bigotimes_{j \in J} \mathcal{O}_{C^{(n_j)}}(-1)$. The Chern classes of the last expression are a linear combination of

$$\prod_{i=1}^s \omega_i^{m_i} \cdot \text{AJ}^* \text{ch}_l\left(\mathbf{R}\bar{\pi}_* \left(\bigotimes_{i \in I} \mathcal{P}_i \otimes \bigotimes_{j \in J} \mathcal{P}_j^* \otimes L_k\right)\right),$$

where ch denotes the Chern character for certain $m_i \in \mathbb{Z}_{\geq 0}$. Integrating this class yields

$$\begin{aligned} & \int_{C^{(n_1)} \times \dots \times C^{(n_s)}} \prod_{i=1}^s \omega_i^{m_i} \cdot \text{AJ}^* \text{ch}_l\left(\mathbf{R}\bar{\pi}_* \left(\bigotimes_{i \in I} \mathcal{P}_i \otimes \bigotimes_{j \in J} \mathcal{P}_j^* \otimes L_k\right)\right) \\ &= \int_{\text{Pic}^{n_1}(C) \times \dots \times \text{Pic}^{n_s}(C)} \text{AJ}_* \prod_{i=1}^s \omega_i^{m_i} \cdot \text{ch}_l\left(\mathbf{R}\bar{\pi}_* \left(\bigotimes_{i \in I} \mathcal{P}_i \otimes \bigotimes_{j \in J} \mathcal{P}_j^* \otimes L_k\right)\right). \end{aligned}$$

Using a standard identity [Ful98, §3.1] we can express the pushforward $\text{AJ}_* \omega_i^{m_i}$ in terms of Segre classes (and, therefore, Chern characters) of $\bar{\pi}_* \mathcal{P}_i$. These Chern characters appearing in the integral are computed by Grothendieck–Riemann–Roch:

$$\text{ch}\left(\mathbf{R}\bar{\pi}_* \left(\bigotimes_{i \in I} \mathcal{P}_i \otimes \bigotimes_{j \in J} \mathcal{P}_j^* \otimes L_k\right)\right) = \bar{\pi}_* \left(\prod_{i \in I} \text{ch}(\mathcal{P}_i) \prod_{j \in J} \text{ch}(\mathcal{P}_j^*) \cdot e^{c_1(L_k)} \cdot \text{td}(TC)\right).$$

The Chern character of the Poincaré line bundle is (cf. [ACGH85, p. 335])

$$\text{ch}(\mathcal{P}_i) = 1 + n_i[c_i] + \gamma_i - \theta_i[c_i] \in H^*(\text{Pic}^{n_i}(C) \times C, \mathbb{Z}).$$

Here, in the decomposition

$$H^2(\text{Pic}^{n_i}(C) \times C) \cong H^2(\text{Pic}^{n_i}(C)) \oplus (H^1(\text{Pic}^{n_i}(C)) \otimes H^1(C)) \oplus H^2(C),$$

we have

$$\begin{aligned}
 [c_i] &\in H^2(C), \\
 \theta_i &\in H^2(\text{Pic}^{n_i}(C)), \\
 \gamma_i &= [\Delta]^{1,1} \in H^1(C) \otimes H^1(C) \cong H^1(\text{Pic}^{n_i}(C)) \otimes H^1(C),
 \end{aligned}$$

where $\Delta \subset C \times C$ is the diagonal and θ_i is the theta divisor. All of this results in

$$\int_{C^{(n_1)} \times \dots \times C^{(n_s)}} f = \int_{\text{Pic}^{n_1}(C) \times \dots \times \text{Pic}^{n_s}(C) \times C} \tilde{f},$$

where \tilde{f} is a polynomial in the classes

$$\theta_i, [c_i], \gamma_i, c_1(T_C), c_1(L_k).$$

5.3.2 *Extension to all n.* In the previous section we assumed that all $n_i > 2g - 2$; we now explain how to remove this assumption, following closely [KT17, § 10.1].

Let $n \in \mathbb{Z}_{\geq 0}$ and $N > 2g - 2$. Then $C^{(N)} \cong \mathbb{P}(\pi_* \mathcal{P})$, where \mathcal{P} is the Poincaré line bundle on $\text{Pic}^N(C) \times C$ normalized at $c \in C$. We can embed $C^{(n)}$ in $C^{(N)}$ as the zero section of a vector bundle

$$\begin{array}{ccc}
 & \pi_* \mathcal{O}(\mathcal{W})|_{C^{(N)} \times (N-n)c} & \\
 & \downarrow \uparrow^s & \\
 C^{(n)} \cong Z(s) & \xleftarrow{\iota} & C^{(N)},
 \end{array}$$

by sending $Z \mapsto Z + (N - n)c$, where $\mathcal{W} \subset C^{(N)} \times C$ is the universal divisor and $(N - n)c \subset C$ is an Artinian thickened point. Moreover, the section is regular, therefore

$$\iota_*[C^{(n)}] = e(\mathcal{G}) \cap [C^{(N)}] \in H_*(C^{(N)}),$$

where $\mathcal{G} = \pi_* \mathcal{O}(\mathcal{W})|_{C^{(N)} \times (N-n)c}$. Finally, if we denote by $\mathcal{Z} \subset C^{(n)} \times C$ the universal divisor, we have that $(\iota \times \text{id})^* \mathcal{W} = \mathcal{Z}((N - n)c)$.

Recall that we are interested in the integrals (5.4). Choose $N_i > 2g - 2$, denote by \mathcal{I}'_i the universal ideal sheaves of the universal divisors \mathcal{W}_i on $C^{(N_1)} \times \dots \times C^{(N_s)} \times C$ and by π' the projection map. By base change we can write

$$\begin{aligned}
 &\mathbf{R}\pi_* \mathbf{R}\mathcal{H}om\left(\bigotimes_{i \in I} \mathcal{I}_i, \bigotimes_{j \in J} \mathcal{I}_j \otimes L_k\right) \\
 &= \mathbf{L}\iota^* \mathbf{R}\pi'_* \mathbf{R}\mathcal{H}om\left(\bigotimes_{i \in I} \mathcal{I}'_i(-(N_i - n_i)c_i), \bigotimes_{j \in J} \mathcal{I}'_j(-(N_j - n_j)c_j) \otimes L_k\right),
 \end{aligned}$$

therefore

$$\int_{C^{(n_1)} \times \dots \times C^{(n_s)}} f = \int_{C^{(N_1)} \times \dots \times C^{(N_s)}} f' \cdot \prod_{i=1}^s e(\mathcal{G}_i),$$

where each $\mathcal{G}_i = \pi_* \mathcal{O}(\mathcal{W}_i)|_{C^{(N_i)} \times (N_i - n_i)c_i}$ and f' is a polynomial in the Chern classes of

$$\mathbf{R}\pi'_* \mathbf{R}\mathcal{H}om\left(\bigotimes_{i \in I} \mathcal{I}'_i, \bigotimes_{j \in J} \mathcal{I}'_j \otimes \bigotimes_{i \in I} \mathcal{O}((N_i - n_i)c_i) \otimes \bigotimes_{j \in J} \mathcal{O}(-(N_j - n_j)c_j) \otimes L_k\right).$$

The exact sequence

$$0 \rightarrow \mathcal{O}(\mathcal{W}_i - (N_i - n_i)c_i) \rightarrow \mathcal{O}(\mathcal{W}_i) \rightarrow \mathcal{O}(\mathcal{W}_i)|_{C^{(N_i)} \times (N_i - n_i)c_i} \rightarrow 0$$

yields the identity in K -theory

$$\mathcal{G}_i = \mathbf{R}\pi'_* \mathcal{I}_i'^* - \mathbf{R}\pi'_* (\mathcal{I}_i'^* \otimes \mathcal{O}((N_i - n_i)c_i)),$$

by which we conclude that we can apply the construction in §5.3.1 to express

$$\int_{C^{(n_1)} \times \dots \times C^{(n_s)}} f = \int_{\text{Pic}^{N_1}(C) \times \dots \times \text{Pic}^{N_s}(C) \times C} \tilde{f},$$

where \tilde{f} is a polynomial in the classes

$$\theta_i, [c_i], \gamma_i, c_1(T_C), c_1(L_k), c_1(\mathcal{O}((N_i - n_i)c_i)). \tag{5.5}$$

Smooth projective curves of the same genus are diffeomorphic to each other, therefore $\text{Pic}^n(C)$ is diffeomorphic to a g -dimensional complex torus. By the intersection theory on $\text{Pic}^n(C)$ developed in [ACGH85, § VIII.2] we immediately obtain the following result.

PROPOSITION 5.3. *Let f be a polynomial in the Chern classes of the K -theory classes*

$$\mathbf{R}\pi_* \mathbf{R}\mathcal{H}om\left(\bigotimes_{i \in I} \mathcal{I}_i, \bigotimes_{j \in J} \mathcal{I}_j \otimes L_k\right),$$

where L_k are line bundles on C and I, J are families of indices (possibly with repetitions). Then

$$\int_{C^{(n_1)} \times \dots \times C^{(n_s)}} f = \int_{\text{Pic}^{n_1}(C) \times \dots \times \text{Pic}^{n_s}(C) \times C} \tilde{f},$$

where \tilde{f} is a polynomial in the classes (5.5). In particular, the integral is a polynomial in the genus $g = g(C)$ and the degrees of the line bundles $\{L_k\}_k$.

As a corollary, we obtain that the localized contributions on $C^{[n_\lambda]}$ only depend on the Chern numbers of (C, L_1, L_2) .

COROLLARY 5.4. *Let C be a genus g irreducible smooth projective curve and L_1, L_2 line bundles on C . Then the intersection numbers (5.1) are polynomials in $g, \deg L_1, \deg L_2$.*

Proof. Let $i : C^{[n_\lambda]} \hookrightarrow A_{C, \mathbf{n}_\lambda}$ be the embedding of Theorem 2.8 and to ease the notation set $A_{C, \mathbf{n}_\lambda} = C^{(n_1)} \times \dots \times C^{(n_s)}$. We claim that

$$N_{C, L_1, L_2}^{\text{vir}} = i^* \tilde{N}_{C, L_1, L_2}^{\text{vir}},$$

for a certain class $\tilde{N}_{C, L_1, L_2}^{\text{vir}} \in K_{\mathbf{T}}^0(A_{C, \mathbf{n}_\lambda})$. In fact, $N_{C, L_1, L_2}^{\text{vir}}$ is a linear combination of classes in K -theory of the form

$$\mathbf{R}\pi_*(\mathcal{O}_{C \times C^{[n_\lambda]}}(\Delta) \otimes L_1^a L_2^b) \otimes \mathfrak{t}^\mu \in K_{\mathbf{T}}^0(C^{[n_\lambda]}),$$

where Δ is a \mathbb{Z} -linear combination of the universal divisors \mathcal{Z}_{ij} on $C \times C^{[n_\lambda]}$, $a, b \in \mathbb{Z}$ and \mathfrak{t}^μ is a \mathbf{T} -character. Each of such universal divisors Δ can be expressed, in the Picard group of $C^{[n_\lambda]}$, as a linear combination of the divisors $i^* \mathcal{Z}_{00}, i^* \mathcal{X}_{ij}, i^* \mathcal{Y}_{ij}$, with notation as in §2.4; applying base change proves the claim. Therefore, the integral (5.1) can be expressed as an intersection number on the product of symmetric powers of curves $A_{C, \mathbf{n}_\lambda}$

$$\int_{[C^{[n_\lambda]}]_{\text{vir}}} e^{\mathbf{T}}(-N_{C, L_1, L_2}^{\text{vir}}) = \int_{A_{C, \mathbf{n}_\lambda}} e^{\mathbf{T}}(\mathcal{E} - \tilde{N}_{C, L_1, L_2}^{\text{vir}}),$$

where \mathcal{E} is the vector bundle of Theorem 2.8. In particular, the K -theory class of $\mathcal{E} - \tilde{N}_{C,L_1,L_2}^{\text{vir}}$ is a linear combination of classes of the form

$$\mathbf{R}\pi_* \mathbf{R}\mathcal{H}om\left(\bigotimes_{i \in I} \mathcal{I}_i, \bigotimes_{j \in J} \mathcal{I}_j \otimes L_k\right) \otimes \mathfrak{t}^\mu,$$

where L_k are line bundles on C and I, J are families of indices (possibly with repetitions). By Proposition 5.3 this integral is a polynomial in g and the degrees of L_k . We conclude the proof by noticing that all line bundles L_k possibly occurring are a linear combination of L_1, L_2, K_C . \square

6. The leading term

We compute the leading term of the generating series of the integrals (5.1), which is essential for the computation of the full generating series in Theorem 8.1.

PROPOSITION 6.1. *Let C be a smooth projective curve of genus g and L_1, L_2 line bundles on C . Then under the anti-diagonal restriction $s_1 + s_2 = 0$ we have*

$$\int_{C^{[0_\lambda]}} e^{\mathbf{T}}(-N_{C,L_1,L_2}^{\text{vir}}) = (-1)^{|\lambda|(g-1+\text{deg } L_2)+n(\lambda) \text{ deg } L_1+n(\bar{\lambda}) \text{ deg } L_2} \left(s_1^{|\lambda|} \cdot \prod_{\square \in \lambda} h(\square)\right)^{2g-2-\text{deg } L_1-\text{deg } L_2},$$

where 0_λ is the unique reversed plane partition of size 0 and underlying Young diagram λ .

Proof. We have that $C^{[0_\lambda]} \cong \text{pt}$ and $[C^{[0_\lambda]}]^{\text{vir}} = [\text{pt}] \in A_*(\text{pt})$, therefore \mathbb{E}^\vee is completely \mathbf{T} -movable and

$$\int_{C^{[0_\lambda]}} e^{\mathbf{T}}(-N_{C,L_1,L_2}^{\text{vir}}) = e^{\mathbf{T}}(-N_{C,L_1,L_2}^{\text{vir}}) \in \mathbb{Q}(s_1, s_2),$$

where by (4.1), (4.6) we express the class in K -theory of the virtual normal bundle as

$$\begin{aligned} N_{C,L_1,L_2}^{\text{vir}} &= \sum_{(i,j) \in \lambda} (\mathbf{R}\Gamma(L_1^{-i} L_2^{-j}) \mathfrak{t}_1^{-i} \mathfrak{t}_2^{-j} - (\mathbf{R}\Gamma(K_C L_1^{-i-1} L_2^{-j-1}) \mathfrak{t}_1^{-i} \mathfrak{t}_2^{-j})^\vee \otimes \mathfrak{t}_1 \mathfrak{t}_2) \\ &\quad - \sum_{(i,j),(l,k) \in \lambda} (\mathbf{R}\Gamma((\mathcal{O}_C - L_1 \mathfrak{t}_1) L_1^{i-l} L_2^{j-k}) \mathfrak{t}_1^{i-l} \mathfrak{t}_2^{j-k} \\ &\quad - (\mathbf{R}\Gamma((\mathcal{O}_C - L_1 \mathfrak{t}_1) K_C L_1^{i-l-1} L_2^{j-k-1}) \mathfrak{t}_1^{i-l} \mathfrak{t}_2^{j-k})^\vee \otimes \mathfrak{t}_1 \mathfrak{t}_2). \end{aligned}$$

Applying Riemann–Roch, every line bundle L on C satisfies

$$\mathbf{R}\Gamma(L) = \mathbb{C}^{\text{deg } L+1-g} \in K^0(\text{pt}) \cong \mathbb{Z},$$

therefore we can write the virtual normal bundle as

$$\begin{aligned} N_{C,L_1,L_2}^{\text{vir}} &= \sum_{\mu} (\mathbb{C}^{m_\mu+1-g} \mathfrak{t}^\mu - \mathbb{C}^{m_\mu+g-1-\text{deg } L_1-\text{deg } L_2} \mathfrak{t}^{-\mu} \mathfrak{t}_1 \mathfrak{t}_2) \\ &\quad - \sum_{\nu} (\mathbb{C}^{m_\nu+1-g} \mathfrak{t}^\nu - \mathbb{C}^{m_\nu+g-1-\text{deg } L_1-\text{deg } L_2} \mathfrak{t}^{-\nu} \mathfrak{t}_1 \mathfrak{t}_2), \end{aligned} \tag{6.1}$$

where the weights μ range among

$$\begin{aligned} &(-i, -j) \in \mathbb{Z}^2 \text{ such that } (i, j) \in \lambda, (i, j) \neq (0, 0), \\ &(i - l + 1, j - k) \in \mathbb{Z}^2 \text{ such that } ((i - j), (l - k)) \in \lambda, (i, j) \neq ((l - 1, k), (l, k + 1)), \end{aligned} \tag{6.2}$$

the weights ν range among

$$(i - l, j - k) \in \mathbb{Z}^2 \text{ such that } ((i - j), (l - k)) \in \lambda, (i, j) \neq ((l, k), (l + 1, k + 1)), \tag{6.3}$$

and, for a weight $\mu = (\mu_1, \mu_2)$, we set $m_\mu = \mu_1 \deg L_1 + \mu_2 \deg L_2$. In particular, the weights $1, \mathbf{t}_1 \mathbf{t}_2$ do not appear in (6.1), as the virtual tangent bundle of the \mathbf{T} -fixed locus has rank zero and by the explicit description of the weight space of $\mathbf{t}_1 \mathbf{t}_2$ in Example 4.2. For every weight μ , we compute

$$e^{\mathbf{T}}(\mathbb{C}^{m_\mu+1-g} \mathbf{t}^\mu - \mathbb{C}^{m_\mu+g-1-\deg L_1-\deg L_2} \mathbf{t}^{-\mu} \mathbf{t}_1 \mathbf{t}_2) = \frac{(\mu \cdot s)^{m_\mu+1-g}}{(-\mu \cdot s + s_1 + s_2)^{m_\mu+g-1-\deg L_1-\deg L_2}},$$

where $s = (s_1, s_2)$ and $\mu \cdot s$ is the standard inner product. With a simple manipulation we end up with

$$\begin{aligned} e^{\mathbf{T}}(-N_{C, L_1, L_2}^{\text{vir}}) &= (-1)^{\sum_{\nu \neq (a,a)} m_\nu + \sum_{\mu \neq (b,b)} m_\mu + (1-g+\deg L_1+\deg L_2)(\#\{\nu \neq (a,a)\} - \#\{\mu \neq (b,b)\})} \\ &\cdot \frac{\prod_{\nu \neq (a,a)} (\nu \cdot s)^{m_\nu+1-g} (\nu \cdot s - s_1 - s_2)^{-m_\nu+1-g+\deg L_1+\deg L_2}}{\prod_{\mu \neq (b,b)} (\mu \cdot s)^{m_\mu+1-g} (\mu \cdot s - s_1 - s_2)^{-m_\mu+1-g+\deg L_1+\deg L_2}} \\ &\cdot \frac{\prod_{\nu=(a,a)} a^{a(\deg L_1+\deg L_2)+1-g} (1-a)^{(1-a)(\deg L_1+\deg L_2)+1-g} (s_1+s_2)^{2-2g+\deg L_1+\deg L_2}}{\prod_{\mu=(b,b)} b^{b(\deg L_1+\deg L_2)+1-g} (1-b)^{(1-b)(\deg L_1+\deg L_2)+1-g} (s_1+s_2)^{2-2g+\deg L_1+\deg L_2}}. \end{aligned}$$

Following the proof of Lemma A.1, we see that for any weight $\nu = (a, a)$ (with $a \in \mathbb{Z}$) there is either a weight $\mu = (a, a)$ or $\mu = (1 - a, 1 - a)$ (and vice versa). This implies that

$$\#\{\nu = (a, a)\} = \#\{\mu = (b, b)\}$$

and

$$\frac{\prod_{\nu=(a,a)} a^{a(\deg L_1+\deg L_2)+1-g} (1-a)^{(1-a)(\deg L_1+\deg L_2)+1-g} (s_1+s_2)^{2-2g+\deg L_1+\deg L_2}}{\prod_{\mu=(b,b)} b^{b(\deg L_1+\deg L_2)+1-g} (1-b)^{(1-b)(\deg L_1+\deg L_2)+1-g} (s_1+s_2)^{2-2g+\deg L_1+\deg L_2}} = 1.$$

In particular, the anti-diagonal restriction $s_1 + s_2 = 0$ is well-defined; we obtain

$$\begin{aligned} &\frac{\prod_{\nu \neq (a,a)} (\nu \cdot s)^{m_\nu+1-g} (\nu \cdot s - s_1 - s_2)^{-m_\nu+1-g+\deg L_1+\deg L_2}}{\prod_{\mu \neq (b,b)} (\mu \cdot s)^{m_\mu+1-g} (\mu \cdot s - s_1 - s_2)^{-m_\mu+1-g+\deg L_1+\deg L_2}} \Big|_{s_1+s_2=0} \\ &= \left(s_1^{\#\nu - \#\mu} \cdot \frac{\prod_{\nu=(\nu_1, \nu_2)} (\nu_1 - \nu_2)^{2-2g+\deg L_1+\deg L_2}}{\prod_{\mu=(\mu_1, \mu_2)} (\mu_1 - \mu_2)^{2-2g+\deg L_1+\deg L_2}} \right), \end{aligned}$$

where the product is over all $\mu, \nu \neq (a, a)$. Moreover, it is immediate to check that $\#\mu - \#\nu = |\lambda|$. By Lemma A.2, we conclude that the last expression is equal to

$$\sigma(\lambda)^{\deg L_1+\deg L_2} \cdot \left(s_1^{-|\lambda|} \cdot \prod_{\square \in \lambda} h(\square)^{-1} \right)^{2-2g+\deg L_1+\deg L_2},$$

where $\sigma(\lambda)$ is defined in Lemma A.2. We are only left with a sign computation; we conclude by Lemma A.3

$$\begin{aligned} &(-1)^{\sum_{\nu \neq (a,a)} m_\nu + \sum_{\mu \neq (b,b)} m_\mu + (1-g+\deg L_1+\deg L_2)|\lambda|} \sigma(\lambda)^{\deg L_1+\deg L_2} \\ &= (-1)^{\rho(\lambda)+|\lambda|(k_1+k_2+g-1)} \sigma(\lambda)^{k_1+k_2} \\ &= (-1)^{|\lambda|(k_2+g-1)+n(\lambda)k_1+n(\bar{\lambda})k_2}, \end{aligned}$$

where $\rho(\lambda)$ was defined in Lemma A.3 and we set $k_i = \deg L_i$. □

7. Toric computations

7.1 Torus action

Let U_0, U_∞ be the standard open cover of \mathbb{P}^1 . The torus \mathbb{C}^* acts on the coordinate functions of \mathbb{P}^1 as $t \cdot x = tx$ (respectively, $t \cdot x = t^{-1}x$) in the chart U_0 (respectively, U_∞). The \mathbb{C}^* -representation of the tangent space at the \mathbb{C}^* -fixed points of \mathbb{P}^1 is

$$\begin{aligned} T_{\mathbb{P}^1,0} &= \mathbb{C} \otimes \mathfrak{t}^{-1}, \\ T_{\mathbb{P}^1,\infty} &= \mathbb{C} \otimes \mathfrak{t}. \end{aligned}$$

We prove some identities of \mathbb{C}^* -representations that are useful later in this section.

LEMMA 7.1. *Let $Z = Z_0 \sqcup Z_\infty \subset \mathbb{P}^1$ be a closed subscheme, where Z_0 (respectively, Z_∞) is a closed subscheme of length n_0 (respectively, n_∞) supported on 0 (respectively, ∞). For any $a \in \mathbb{Z}$, we have the following identities in $K_{\mathbb{C}^*}^0(\text{pt})$:*

$$\begin{aligned} \mathbf{R}\Gamma(K_{\mathbb{P}^1}^a) &= \begin{cases} \sum_{i=a}^{-a} \mathfrak{t}^i & a \leq 0, \\ - \sum_{i=-a+1}^{a-1} \mathfrak{t}^i & a \geq 1, \end{cases} \\ \mathbf{R}\Gamma(\mathcal{O}_Z(Z)) &= \sum_{i=1}^{n_0} \mathfrak{t}^{-i} + \sum_{i=1}^{n_\infty} \mathfrak{t}^i, \\ \mathbf{R}\Gamma(\mathcal{O}_Z \otimes K_{\mathbb{P}^1}^a) &= \sum_{i=0}^{n_0-1} \mathfrak{t}^{a+i} + \sum_{i=0}^{n_\infty-1} \mathfrak{t}^{-a-i}, \\ \mathbf{R}\text{Hom}(\mathcal{O}_Z, K_{\mathbb{P}^1}^a) &= - \sum_{i=0}^{n_0-1} \mathfrak{t}^{a-i-1} - \sum_{i=0}^{n_\infty-1} \mathfrak{t}^{-a+i+1}. \end{aligned}$$

Proof. We have that

$$\begin{aligned} \mathbf{R}\Gamma(K_{\mathbb{P}^1}^a) &= \chi(\mathbb{P}^1, K_{\mathbb{P}^1}^a) \\ &= \frac{\mathfrak{t}^a}{1-\mathfrak{t}} - \frac{\mathfrak{t}^{-a}}{1-\mathfrak{t}^{-1}} \\ &= \begin{cases} \sum_{i=a}^{-a} \mathfrak{t}^i & a \leq 0, \\ - \sum_{i=-a+1}^{a-1} \mathfrak{t}^i & a \geq 1, \end{cases} \end{aligned}$$

where in the second line we applied the classical K -theoretic localization [Tho92] on \mathbb{P}^1 .

Second, we have

$$\begin{aligned} \mathbf{R}\Gamma(\mathcal{O}_Z(Z)) &= H^0(\mathbb{P}^1, \mathcal{O}_Z(Z)) \\ &= H^0(U_0, \mathcal{O}_{Z_0}(Z_0)|_{U_0}) + H^0(U_\infty, \mathcal{O}_{Z_\infty}(Z_\infty)|_{U_\infty}) \\ &= H^0(U_0, \mathcal{O}_{Z_0} \otimes K_{\mathbb{P}^1}|_{U_0})^* + H^0(U_\infty, \mathcal{O}_{Z_\infty} \otimes K_{\mathbb{P}^1}|_{U_\infty})^* \\ &= \sum_{i=1}^{n_0} \mathfrak{t}^{-i} + \sum_{i=1}^{n_\infty} \mathfrak{t}^i, \end{aligned}$$

where in the second line we used Čech cohomology and in the third line we used [Oko17, Exercise 3.4.5].

Third, by applying Čech cohomology as before we have

$$\begin{aligned} \mathbf{R}\Gamma(\mathcal{O}_Z \otimes K_{\mathbb{P}^1}^a) &= H^0(U_0, \mathcal{O}_{Z_0} \otimes K_{\mathbb{P}^1}^a|_{U_0}) + H^0(U_\infty, \mathcal{O}_{Z_\infty} \otimes K_{\mathbb{P}^1}^a|_{U_\infty}) \\ &= \sum_{i=0}^{n_0-1} t^{a+i} + \sum_{i=0}^{n_\infty-1} t^{-a-i}. \end{aligned}$$

Finally, combining Serre duality and the previous result yields

$$\begin{aligned} \mathbf{R}\mathrm{Hom}(\mathcal{O}_Z, K_{\mathbb{P}^1}^a) &= -\mathbf{R}\Gamma(\mathcal{O}_Z \otimes K_{\mathbb{P}^1}^{1-a})^* \\ &= -\sum_{i=0}^{n_0-1} t^{a-i-1} - \sum_{i=0}^{n_\infty-1} t^{-a+i+1}. \quad \square \end{aligned}$$

The \mathbb{C}^* -action on \mathbb{P}^1 naturally lifts to a \mathbb{C}^* -action on the Hilbert scheme of point $\mathbb{P}^{1[n]}$, whose \mathbb{C}^* -fixed locus consists of length n closed subschemes $Z \subset \mathbb{P}^1$ supported on $0, \infty$. Therefore, there is an induced \mathbb{C}^* -action on $A_{\mathbb{P}^1, \mathbf{n}_\lambda}$, whose \mathbb{C}^* -fixed locus is zero-dimensional and reduced. This \mathbb{C}^* -action restricts to a \mathbb{C}^* -action on $\mathbb{P}^{1[\mathbf{n}_\lambda]}$, whose \mathbb{C}^* -fixed locus is necessarily zero-dimensional and reduced. Moreover, the perfect obstruction theory (2.2) is naturally \mathbb{C}^* -equivariant, as all the ingredients of Theorem 2.8 are.

PROPOSITION 7.2. *Let $\underline{Z} \in \mathbb{P}^{1[\mathbf{n}_\lambda], \mathbb{C}^*}$ be a \mathbb{C}^* -fixed point. Then $T_{\mathbb{P}^{1[\mathbf{n}_\lambda], \underline{Z}}}^{\mathrm{vir}}$ is completely \mathbb{C}^* -movable. In particular, the induced perfect obstruction theory on $\mathbb{P}^{1[\mathbf{n}_\lambda], \mathbb{C}^*}$ is trivial.*

Proof. We need to show that the class in K -theory

$$T_{A_{C, \mathbf{n}_\lambda}}|_{\underline{Z}} - \mathcal{E}|_{\underline{Z}} \in K_{\mathbb{C}^*}^0(\mathrm{pt})$$

does not have \mathbb{C}^* -fixed part. Recall that we have an identity in $K_{\mathbb{C}^*}^0(\mathrm{pt})$

$$T_{A_{C, \mathbf{n}_\lambda}}|_{\underline{Z}} = \mathbf{R}\Gamma(\mathbb{P}^1, \mathcal{O}_{Z_{00}}(Z_{00})) + \sum_{\substack{(i,j) \in \lambda \\ i \geq 1}} \mathbf{R}\Gamma(\mathbb{P}^1, \mathcal{O}_{X_{ij}}(X_{ij})) + \sum_{\substack{(i,j) \in \lambda \\ j \geq 1}} \mathbf{R}\Gamma(\mathbb{P}^1, \mathcal{O}_{Y_{ij}}(Y_{ij})),$$

where, for simplicity, we denoted by $X_{ij} = Z_{ij} - Z_{i-1,j}$ and by $Y_{ij} = Z_{ij} - Z_{i,j-1}$. Moreover, we have

$$\mathcal{E}|_{\underline{Z}} = \sum_{\substack{(i,j) \in \lambda \\ i, j \geq 1}} \mathbf{R}\Gamma(\mathbb{P}^1, \mathcal{O}_{W_{ij}}(W_{ij})) \in K_{\mathbb{C}^*}^0(\mathrm{pt}),$$

where for simplicity we used the notation $W_{ij} = Z_{ij} - Z_{i-1,j-1}$. Therefore, the virtual tangent space is a sum of classes of the form $\mathbf{R}\Gamma(\mathbb{P}^1, \mathcal{O}_Z(Z))$, with $Z \subset \mathbb{P}^1$ a closed subscheme, which is entirely \mathbb{C}^* -movable by the description in Lemma 7.1. □

7.2 Case I: Calabi–Yau

We compute the integral (5.1) for $C = \mathbb{P}^1$ in the case of $L_1 \otimes L_2 = K_{\mathbb{P}^1}$, showing that it coincides (up to a sign) with the topological Euler characteristic $e(\mathbb{P}^{1[\mathbf{n}_\lambda]})$.

THEOREM 7.3. *Let L_1, L_2 be line bundles on \mathbb{P}^1 such that $L_1 \otimes L_2 = K_{\mathbb{P}^1}$. For any reversed plane partition \mathbf{n}_λ , we have*

$$\left(\int_{[\mathbb{P}^1[\mathbf{n}_\lambda]_{\text{vir}}]} e^{\mathbf{T}(-N_{\mathbb{P}^1, L_1, L_2}^{\text{vir}})} \right) \Big|_{s_1+s_2=0} = (-1)^{\deg L_1(c_\lambda+|\lambda|)+|\lambda|+|\mathbf{n}_\lambda|} \cdot e_{(\mathbb{P}^1[\mathbf{n}_\lambda])}.$$

Proof. By Graber and Pandharipande [GP99], there is an induced perfect obstruction theory and virtual fundamental class on the \mathbb{C}^* -fixed locus $\mathbb{P}^1[\mathbf{n}_\lambda]^{\mathbb{C}^*}$, both trivial by Proposition 7.2. By Proposition 4.3

$$N_{\mathbb{P}^1, L_1, L_2}^{\text{vir}} = -T_{\mathbb{P}^1[\mathbf{n}_\lambda]}^{\text{vir}, \vee} \otimes \mathbf{t}_1 \mathbf{t}_2 + \Omega - \Omega^\vee \otimes \mathbf{t}_1 \mathbf{t}_2 \in K_{\mathbf{T}}^0(\mathbb{P}^1[\mathbf{n}_\lambda]),$$

and applying the virtual localization formula with respect to the \mathbb{C}^* -action yields

$$\begin{aligned} \int_{[\mathbb{P}^1[\mathbf{n}_\lambda]_{\text{vir}}]} e^{\mathbf{T}(-N_{\mathbb{P}^1, L_1, L_2}^{\text{vir}})} &= \left(\sum_{\underline{Z} \in \mathbb{P}^1[\mathbf{n}_\lambda]^{\mathbb{C}^*}} \frac{e^{\mathbf{T} \times \mathbb{C}^*}(T_{\mathbb{P}^1[\mathbf{n}_\lambda], \underline{Z}}^{\text{vir}, \vee} \otimes \mathbf{t}_1 \mathbf{t}_2)}{e^{\mathbf{T} \times \mathbb{C}^*}(T_{\mathbb{P}^1[\mathbf{n}_\lambda], \underline{Z}}^{\text{vir}})} \cdot \frac{e^{\mathbf{T} \times \mathbb{C}^*}(\Omega|_{\underline{Z}}^\vee \otimes \mathbf{t}_1 \mathbf{t}_2)}{e^{\mathbf{T} \times \mathbb{C}^*}(\Omega|_{\underline{Z}})} \right) \Big|_{s_3=0} \\ &= \left(\sum_{\underline{Z} \in \mathbb{P}^1[\mathbf{n}_\lambda]^{\mathbb{C}^*}} \frac{e^{\mathbf{T} \times \mathbb{C}^*}(\mathbf{V}_{\underline{Z}}^\vee \otimes \mathbf{t}_1 \mathbf{t}_2)}{e^{\mathbf{T} \times \mathbb{C}^*}(\mathbf{V}_{\underline{Z}})} \right) \Big|_{s_3=0}, \end{aligned}$$

where s_3 is the generator of the equivariant cohomology $H_{\mathbb{C}^*}^*(\text{pt})$ and we denoted by

$$\mathbf{V}_{\underline{Z}} = T_{\mathbb{P}^1[\mathbf{n}_\lambda], \underline{Z}}^{\text{vir}} + \Omega|_{\underline{Z}} \in K_{\mathbf{T} \times \mathbb{C}^*}^0(\text{pt})$$

the \mathbb{C}^* -equivariant lift of the K -theoretic class $T_{\mathbb{P}^1[\mathbf{n}_\lambda] + \underline{Z}}^{\text{vir}}, \Omega|_{\underline{Z}} \in K_{\mathbf{T}}^0(\text{pt})$. Under the Calabi–Yau restriction $s_1 + s_2 = 0$, we have by Lemma 7.6

$$\left(\frac{e^{\mathbf{T} \times \mathbb{C}^*}(\mathbf{V}_{\underline{Z}}^\vee \otimes \mathbf{t}_1 \mathbf{t}_2)}{e^{\mathbf{T} \times \mathbb{C}^*}(\mathbf{V}_{\underline{Z}})} \right) \Big|_{s_1+s_2=0} = (-1)^{\text{rk } \mathbf{V}_{\underline{Z}}}.$$

Moreover, by Lemma 7.5

$$\text{rk } \mathbf{V}_{\underline{Z}} = \deg L_1(c_\lambda + |\lambda|) + |\lambda| + |\mathbf{n}_\lambda| \pmod{2}.$$

Therefore, we conclude that

$$\begin{aligned} \left(\sum_{\underline{Z} \in \mathbb{P}^1[\mathbf{n}_\lambda]^{\mathbb{C}^*}} \frac{e^{\mathbf{T} \times \mathbb{C}^*}(\mathbf{V}_{\underline{Z}}^\vee \otimes \mathbf{t}_1 \mathbf{t}_2)}{e^{\mathbf{T} \times \mathbb{C}^*}(\mathbf{V}_{\underline{Z}})} \right) \Big|_{s_1+s_2=0} &= \sum_{\underline{Z} \in \mathbb{P}^1[\mathbf{n}_\lambda]^{\mathbb{C}^*}} (-1)^{\text{rk } \mathbf{V}_{\underline{Z}}} \\ &= (-1)^{\deg L_1(c_\lambda+|\lambda|)+|\lambda|+|\mathbf{n}_\lambda|} \cdot e_{(\mathbb{P}^1[\mathbf{n}_\lambda])}, \end{aligned}$$

as the Euler characteristic of a \mathbb{C}^* -scheme coincides with the Euler characteristic of its \mathbb{C}^* -fixed locus (in our case the number of fixed points). □

Exploiting the close formula for the generating series of the topological Euler characteristic proved in Theorem 2.11, we derive the following close expression in the Calabi–Yau case.

COROLLARY 7.4. *Let L_1, L_2 be line bundles on \mathbb{P}^1 such that $L_1 \otimes L_2 = K_{\mathbb{P}^1}$ and λ be a Young diagram. We have*

$$\sum_{\mathbf{n}_\lambda} q^{|\mathbf{n}_\lambda|} \cdot \left(\int_{[\mathbb{P}^1[\mathbf{n}_\lambda]_{\text{vir}}]} e^{\mathbf{T}(-N_{\mathbb{C}, L_1, L_2}^{\text{vir}})} \right) \Big|_{s_1+s_2=0} = (-1)^{\deg L_1(c_\lambda+|\lambda|)+|\lambda|} \cdot \prod_{\square \in \lambda} (1 - (-q)^{h(\square)})^{-2},$$

where the sum is over all reversed plane partition \mathbf{n}_λ .

We devote the remainder of this section to prove the technical lemmas we used in Theorem 7.3.

LEMMA 7.5. *Let C be a smooth projective curve of genus g , L_1, L_2 be line bundles on C such that $L_1 \otimes L_2 = K_C$ and let $\Omega \in K_{\mathbf{T}}^0(C^{[\mathbf{n}\lambda]})$ be the K -theory class of Remark 4.4. Then*

$$\text{rk}(T_{C^{[\mathbf{n}\lambda]}}^{\text{vir}} + \Omega) = \text{deg } L_1(c_\lambda + |\lambda|) + (1 - g)|\lambda| + |\mathbf{n}\lambda| \pmod 2.$$

Proof. Let $\underline{Z} \in C^{[\mathbf{n}\lambda]}$. Then

$$\begin{aligned} \text{rk } T_{C^{[\mathbf{n}\lambda]}}^{\text{vir}} &= \text{rk } T_{C^{[\mathbf{n}\lambda]}}^{\text{vir}}|_{\underline{Z}} \\ &= n_{00} + \sum_{\substack{(i,j) \in \lambda \\ i \geq 1}} (n_{ij} - n_{i-1,j}) + \sum_{\substack{(i,j) \in \lambda \\ j \geq 1}} (n_{ij} - n_{i,j-1}) - \sum_{\substack{(i,j) \in \lambda \\ i,j \geq 1}} (n_{ij} - n_{i-1,j-1}). \end{aligned}$$

For any line bundle L on C and Δ a \mathbb{Z} -linear combination of the universal divisors \mathcal{Z}_{ij} on $C \times C^{[\mathbf{n}\lambda]}$, Riemann–Roch yields

$$\begin{aligned} \text{rk } \mathbf{R}\pi_*(\mathcal{O}(\Delta) \otimes L) &= \chi(C, L \otimes \mathcal{O}(\Delta)|_{\underline{Z}}) \\ &= \text{deg } L + \text{deg } \Delta|_{\underline{Z}} + 1 - g. \end{aligned}$$

As Ω is a sum of K -theoretic classes of the form $\mathbf{R}\pi_*(\mathcal{O}(\Delta) \otimes L)$, for suitable L, Δ , we have

$$\begin{aligned} \text{rk } \Omega &= \sum_{\substack{(i,j) \in \lambda \\ (i,j) \neq (0,0)}} (n_{ij} - i \cdot \text{deg } L_1 - j \cdot (2g - 2 - \text{deg } L_1) + 1 - g) \\ &\quad - \sum_{\substack{(i,j),(l,k) \in \lambda \\ (i,j) \neq (l,k) \\ (i,j) \neq (l+1,k+1)}} (n_{lk} - n_{ij} - (i - l) \text{deg } L_1 - (j - k)(2g - 2 - \text{deg } L_1) + 1 - g) \\ &\quad + \sum_{\substack{(i,j),(l,k) \in \lambda \\ (i,j) \neq (l-1,k) \\ (i,j) \neq (l,k+1)}} (n_{lk} - n_{ij} - (i - l + 1) \text{deg } L_1 - (j - k)(2g - 2 - \text{deg } L_1) + 1 - g). \end{aligned}$$

Denote by \equiv the congruence modulo two. We have

$$n_{00} - \sum_{\substack{(i,j) \in \lambda \\ (i,j) \neq (0,0)}} n_{ij} \equiv |\mathbf{n}\lambda|.$$

Denote by V, E and Q the number of vertices, edges and squares of the graph, respectively, associated with λ as in Lemma 2.1. We have

$$\begin{aligned} \sum_{\substack{(i,j) \in \lambda \\ (i,j) \neq (0,0)}} 1 - \sum_{\substack{(i,j),(l,k) \in \lambda \\ (i,j) \neq (l,k) \\ (i,j) \neq (l+1,k+1)}} 1 + \sum_{\substack{(i,j),(l,k) \in \lambda \\ (i,j) \neq (l-1,k) \\ (i,j) \neq (l,k+1)}} 1 &= V - 1 - (|\lambda|^2 - V - Q) + (|\lambda|^2 - E) \\ &= |\lambda|, \end{aligned}$$

where in the last line we used Lemma 2.1. The coefficient of $\text{deg } L_1$ in $\text{rk } \Omega$ is

$$\sum_{\substack{(i,j) \in \lambda \\ (i,j) \neq (0,0)}} (i - j) - \sum_{\substack{(i,j),(l,k) \in \lambda \\ (i,j) \neq (l,k) \\ (i,j) \neq (l+1,k+1)}} (i - j - l + k) + \sum_{\substack{(i,j),(l,k) \in \lambda \\ (i,j) \neq (l-1,k) \\ (i,j) \neq (l,k+1)}} (i - j + l - k + 1)$$

$$\begin{aligned} &\equiv \sum_{(i,j) \in \lambda} (i+j) + \sum_{\substack{(i,j),(l,k) \in \lambda \\ (i,j)=(l,k) \\ (i,j)=(l+1,k+1)}} (i+j+l+k) + \sum_{\substack{(i,j),(l,k) \in \lambda \\ (i,j)=(l-1,k) \\ (i,j)=(l,k+1)}} (i+j+l+k) + \sum_{\substack{(i,j),(l,k) \in \lambda \\ (i,j) \neq (l-1,k) \\ (i,j) \neq (l,k+1)}} 1 \\ &\equiv c_\lambda + 2E + |\lambda| \\ &\equiv c_\lambda + |\lambda|. \end{aligned}$$

Finally,

$$\begin{aligned} &\sum_{\substack{(i,j),(l,k) \in \lambda \\ (i,j) \neq (l,k) \\ (i,j) \neq (l+1,k+1)}} (n_{lk} - n_{ij}) + \sum_{\substack{(i,j),(l,k) \in \lambda \\ (i,j) \neq (l-1,k) \\ (i,j) \neq (l,k+1)}} (n_{lk} - n_{ij}) \\ &+ \sum_{\substack{(i,j) \in \lambda \\ i \geq 1}} (n_{ij} - n_{i-1,j}) + \sum_{\substack{(i,j) \in \lambda \\ j \geq 1}} (n_{ij} - n_{i,j-1}) - \sum_{\substack{(i,j) \in \lambda \\ i,j \geq 1}} (n_{ij} - n_{i-1,j-1}) \equiv 0. \end{aligned}$$

Combining all these identities together, we conclude that

$$\text{rk}(T_{\mathbb{C}^{\mathbf{n}_\lambda}}^{\text{vir}} + \Omega) = \text{deg } L_1(c_\lambda + |\lambda|) + |\lambda|(1-g) + |\mathbf{n}_\lambda| \pmod{2}. \quad \square$$

LEMMA 7.6. Let $\underline{Z} \in \mathbb{P}^{1[\mathbf{n}_\lambda], \mathbb{C}^*}$ be a \mathbb{C}^* -fixed point and set

$$\mathbf{V}_{\underline{Z}} = T_{\mathbb{P}^{1[\mathbf{n}_\lambda]}}^{\text{vir}}|_{\underline{Z}} + \Omega|_{\underline{Z}} \in K_{\mathbf{T} \times \mathbb{C}^*}^0(\text{pt}),$$

where $\Omega \in K_{\mathbf{T}}^0(\mathbb{P}^{1[\mathbf{n}_\lambda]})$ is as in Remark 4.4. Then we have

$$\left(\frac{e^{\mathbf{T} \times \mathbb{C}^*}(\mathbf{V}_{\underline{Z}}^\vee \otimes \mathbf{t}_1 \mathbf{t}_2)}{e^{\mathbf{T} \times \mathbb{C}^*}(\mathbf{V}_{\underline{Z}})} \right) \Big|_{s_1+s_2=0} = (-1)^{\text{rk } \mathbf{V}_{\underline{Z}}}.$$

Proof. Denote by $\mathbf{V}_{\underline{Z}}^{CY}$ the sub-representation of $\mathbf{V}_{\underline{Z}}$ consisting of weight spaces corresponding to the characters $(\mathbf{t}_1 \mathbf{t}_2)^a$, for all $a \in \mathbb{Z}$, with respect to the $\mathbf{T} \times \mathbb{C}^*$ -action. We claim that $\mathbf{V}_{\underline{Z}}^{CY}$ is of the form

$$\mathbf{V}_{\underline{Z}}^{CY} = A_{\underline{Z}} + A_{\underline{Z}}^\vee \otimes \mathbf{t}_1 \mathbf{t}_2,$$

for a suitable $A_{\underline{Z}} \in K_{\mathbf{T} \times \mathbb{C}^*}^0(\text{pt})$.

Step I. Assuming the claim, set

$$\tilde{\mathbf{V}}_{\underline{Z}} = \mathbf{V}_{\underline{Z}} - \mathbf{V}_{\underline{Z}}^{CY}$$

and $\tilde{\mathbf{V}}_{\underline{Z}} = \sum_{\mu} \mathbf{t}^\mu - \sum_{\nu} \mathbf{t}^\nu \in K_{\mathbf{T} \times \mathbb{C}^*}^0(\text{pt})$, where none of the characters $\mathbf{t}^\mu, \mathbf{t}^\nu$ is a power of $\mathbf{t}_1 \mathbf{t}_2$. Write $e^{\mathbf{T} \times \mathbb{C}^*}(\mathbf{t}^\mu) = \mu \cdot s$, where $s = (s_1, s_2, s_3)$. Then we conclude that

$$\begin{aligned} &\left(\frac{e^{\mathbf{T} \times \mathbb{C}^*}(\mathbf{V}_{\underline{Z}}^\vee \otimes \mathbf{t}_1 \mathbf{t}_2)}{e^{\mathbf{T} \times \mathbb{C}^*}(\mathbf{V}_{\underline{Z}})} \right) \Big|_{s_1+s_2=0} = \left(\frac{e^{\mathbf{T} \times \mathbb{C}^*}(\tilde{\mathbf{V}}_{\underline{Z}}^\vee \otimes \mathbf{t}_1 \mathbf{t}_2)}{e^{\mathbf{T} \times \mathbb{C}^*}(\tilde{\mathbf{V}}_{\underline{Z}})} \right) \Big|_{s_1+s_2=0} \\ &= \left(\prod_{\mu} \frac{-\mu \cdot s + s_1 + s_2}{\mu \cdot s} \cdot \prod_{\nu} \frac{\nu \cdot s}{-\nu \cdot s + s_1 + s_2} \right) \Big|_{s_1+s_2=0} \\ &= (-1)^{\text{rk } \mathbf{V}_{\underline{Z}}}, \end{aligned}$$

where we used that no $\mu \cdot s, \nu \cdot s$ is a multiple of $s_1 + s_2$ and that $\text{rk } \mathbf{V}_{\underline{Z}} = \text{rk } \tilde{\mathbf{V}}_{\underline{Z}} \pmod{2}$.

Step II. We now prove our claim on $\mathbf{V}_{\underline{Z}}^{CY}$. First, by Proposition 7.2 $T_{\mathbb{P}^{1[\mathbf{n}_\lambda], \underline{Z}}}^{\text{vir}}$ is \mathbb{C}^* -movable, which implies that there are no weight spaces corresponding to a power of $\mathbf{t}_1 \mathbf{t}_2$.

It is clear that the $\mathbf{T} \times \mathbb{C}^*$ -weight spaces of $\Omega|_{\underline{Z}}$ relative to the characters $(t_1 t_2)^a$ are given by $\Omega'|_{\underline{Z}}$, where

$$\begin{aligned} \Omega' = & \sum_{\substack{(a,a) \in \lambda \\ a \neq 0}} \mathbf{R}\pi_*(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1[\mathbf{n}_\lambda]}(\mathcal{Z}_{aa}) \otimes K_{\mathbb{P}^1}^{-a})(t_1 t_2)^{-a} - \sum_{\substack{(i,j),(l,k) \in \lambda \\ (i,j)=(l+a,k+a) \\ a \neq 0,1}} \mathbf{R}\pi_*(\mathcal{O}(\Delta_{ij;lk}) \otimes K_{\mathbb{P}^1}^a)(t_1 t_2)^a \\ & + \sum_{\substack{(i,j),(l,k) \in \lambda \\ (i,j)=(l+a-1,k+a) \\ a \neq 0,1}} \mathbf{R}\pi_*(\mathcal{O}(\Delta_{ij;lk}) \otimes K_{\mathbb{P}^1}^a)(t_1 t_2)^a, \end{aligned}$$

as we just considered the weight spaces of $(t_1 t_2)^a$ in Ω of Remark 4.4. We note that $\Omega'|_{\underline{Z}}$ is a sum of $\mathbf{T} \times \mathbb{C}^*$ -representations of the form

$$\mathbf{R}\Gamma(\mathcal{O}_{\mathbb{P}^1}(Z_a) \otimes K_{\mathbb{P}^1}^{-a}) \otimes (t_1 t_2)^{-a}, \quad \text{if } a \geq 1,$$

or

$$\mathbf{R}\Gamma(\mathcal{O}_{\mathbb{P}^1}(-Z_a) \otimes K_{\mathbb{P}^1}^a) \otimes (t_1 t_2)^a, \quad \text{if } a \geq 2,$$

where $Z_a \subset \mathbb{P}^1$ are effective divisors. We have the following identities of \mathbb{C}^* -representations:

$$\begin{aligned} \mathbf{R}\Gamma(\mathcal{O}_{\mathbb{P}^1}(Z_a) \otimes K_{\mathbb{P}^1}^{-a}) &= \mathbf{R}\Gamma(K_{\mathbb{P}^1}^{-a}) - \mathbf{R}\text{Hom}(\mathcal{O}_{Z_a}, K_{\mathbb{P}^1}^{-a}), \quad a \geq 1, \\ \mathbf{R}\Gamma(\mathcal{O}_{\mathbb{P}^1}(-Z_a) \otimes K_{\mathbb{P}^1}^a) &= \mathbf{R}\Gamma(K_{\mathbb{P}^1}^a) - \mathbf{R}\Gamma(\mathcal{O}_{Z_a} \otimes K_{\mathbb{P}^1}^a), \quad a \geq 2. \end{aligned}$$

By Lemma 7.1, their \mathbb{C}^* -fixed part is

$$\begin{aligned} (\mathbf{R}\Gamma(K_{\mathbb{P}^1}^a))^{\text{fix}} &= \begin{cases} 1 & a \leq -1, \\ -1 & a \geq 2, \end{cases} \\ (\mathbf{R}\text{Hom}(\mathcal{O}_{Z_a}, K_{\mathbb{P}^1}^{-a}))^{\text{fix}} &= 0, \quad a \geq 1, \\ (\mathbf{R}\Gamma(\mathcal{O}_{Z_a} \otimes K_{\mathbb{P}^1}^a))^{\text{fix}} &= 0, \quad a \geq 2. \end{aligned}$$

Combining everything together, we conclude that

$$V_{\underline{Z}}^{CY} = \sum_{\substack{(a,a) \in \lambda \\ a \neq 0}} (t_1 t_2)^{-a} + \sum_{\substack{(i,j),(l,k) \in \lambda \\ (i,j)=(l+a,k+a) \\ a \neq 0,1}} \text{sgn}(a)(t_1 t_2)^a - \sum_{\substack{(i,j),(l,k) \in \lambda \\ (i,j)=(l+a-1,k+a) \\ a \neq 0,1}} \text{sgn}(a)(t_1 t_2)^a,$$

where sgn is the usual sign function. Our claim follows by Lemma A.1. □

7.3 Case II: trivial vector bundle

If $L_1 = L_2 = \mathcal{O}_{\mathbb{P}^1}$, the integral (5.1) amounts to the leading term computation of §6 and a vanishing result, which relies on a combinatorial fact about the topological vertex formalism for stable pairs developed in [PT09b].

PROPOSITION 7.7. *Under the anti-diagonal restriction $s_1 + s_2 = 0$ we have an identity*

$$\sum_{\mathbf{n}_\lambda} q^{|\mathbf{n}_\lambda|} \int_{[\mathbb{P}^1[\mathbf{n}_\lambda]]^{\text{vir}}} e^{\mathbf{T}(-N_{\mathbb{P}^1, \mathcal{O}, \mathcal{O}}^{\text{vir}})} \Big|_{s_1+s_2=0} = (-s_1^2)^{-|\lambda|} \cdot \prod_{\square \in \lambda} h(\square)^{-2}.$$

Proof. The leading term is computed in Proposition 6.1, therefore we just need to prove that the integral vanishes for $|\mathbf{n}_\lambda| > 0$. We apply Graber–Pandharipande virtual localization [GP99]

with respect to the \mathbb{C}^* -action on $\mathbb{P}^{1[\mathbf{n}_\lambda]}$

$$\int_{[\mathbb{P}^{1[\mathbf{n}_\lambda]}]_{\text{vir}}} e^{\mathbf{T}}(-N_{\mathbb{P}^1, \mathcal{O}, \mathcal{O}}^{\text{vir}}) = \left(\sum_{\underline{Z} \in \mathbb{P}^{1[\mathbf{n}_\lambda], \mathbb{C}^*}} e^{\mathbf{T} \times \mathbb{C}^*}(-T_{\mathbb{P}^{1[\mathbf{n}_\lambda], \underline{Z}}}^{\text{vir}} - N_{\mathbb{P}^1, \mathcal{O}, \mathcal{O}, \underline{Z}}^{\text{vir}}) \right) \Big|_{s_3=0},$$

where s_3 is the generator of $H_{\mathbb{C}^*}^*(\text{pt})$. The \mathbb{C}^* -action on $\mathbb{P}^{1[\mathbf{n}_\lambda]}$ is just the restriction of the natural $\mathbf{T} \times \mathbb{C}^*$ -action on $P_n(\mathbb{P}^1 \times \mathbb{C}^2, d[\mathbb{P}^1])$, where $d = |\lambda|$ and $n = |\lambda| + |\mathbf{n}_\lambda|$. This means that, as $\mathbf{T} \times \mathbb{C}^*$ -representation,

$$\mathbb{E}^\vee|_{\underline{Z}} = T_{\mathbb{P}^{1[\mathbf{n}_\lambda], \underline{Z}}}^{\text{vir}} + N_{\mathbb{P}^1, \mathcal{O}, \mathcal{O}, \underline{Z}}^{\text{vir}} \in K_{\mathbf{T} \times \mathbb{C}^*}^0(\text{pt})$$

can be described via the topological vertex formalism of Pandharipande and Thomas [PT09b, Theorem 3], which states

$$\mathbb{E}^\vee|_{\underline{Z}} = V_{0, \underline{Z}} + V_{\infty, \underline{Z}} + E_{\underline{Z}} \in K_{\mathbf{T} \times \mathbb{C}^*}^0(\text{pt}),$$

where $V_{0, \underline{Z}}, V_{\infty, \underline{Z}}$ are the vertex terms corresponding to the two toric charts of $\mathbb{P}^1 \times \mathbb{C}^2$ and $E_{\underline{Z}}$ is the edge term. By [MPT10, Lemma 22] we have that $e^{\mathbf{T} \times \mathbb{C}^*}(-V_{0, \underline{Z}} - V_{\infty, \underline{Z}})$ is divisible by $(s_1 + s_2)$ if $|\mathbf{n}_\lambda| > 0$, whereas $e^{\mathbf{T} \times \mathbb{C}^*}(-E_{\underline{Z}})$ is easily seen to be coprime with $(s_1 + s_2)$.⁵ This implies that the anti-diagonal restriction $s_1 + s_2 = 0$ is well-defined on every localized term and satisfies

$$e^{\mathbf{T} \times \mathbb{C}^*}(\mathbb{E}^\vee|_{\underline{Z}})|_{s_1+s_2=0} = 0,$$

by which we conclude the required vanishing. □

Remark 7.8. We could prove Proposition 7.7 without relying on the vertex formalism for stable pairs, by simply carrying out a detailed (but probably longer) analysis of the weight space of $\mathfrak{t}_1 \mathfrak{t}_2$ as in Lemma 7.6.

8. Summing up the theory

8.1 Anti-diagonal restriction

We combine the computations in §§6 and 7 to prove the second part of Theorem 1.3 from the introduction.

THEOREM 8.1. *Under the anti-diagonal restriction $s_1 + s_2 = 0$ the three universal series are*

$$\begin{aligned} A_\lambda(q, s_1, -s_1) &= (-s_1^2)^{|\lambda|} \cdot \prod_{\square \in \lambda} h(\square)^2, \\ B_\lambda(-q, s_1, -s_1) &= (-1)^{n(\lambda)} \cdot s_1^{-|\lambda|} \cdot \prod_{\square \in \lambda} h(\square)^{-1} \cdot \prod_{\square \in \lambda} (1 - q^{h(\square)}), \\ C_\lambda(-q, s_1, -s_1) &= (-1)^{n(\tilde{\lambda})} \cdot (-s_1)^{-|\lambda|} \cdot \prod_{\square \in \lambda} h(\square)^{-1} \cdot \prod_{\square \in \lambda} (1 - q^{h(\square)}). \end{aligned}$$

Proof. Let $C_{g, \text{deg } L_1 \text{ deg } L_2} \in \mathbb{Q}(s_1, s_2)$ be the leading term of the generating series of Theorem 5.1. We can write

$$\begin{aligned} C_{g, \text{deg } L_1 \text{ deg } L_2}^{-1} \cdot \sum_{\mathbf{n}_\lambda} q^{|\mathbf{n}_\lambda|} \int_{[C^{[\mathbf{n}_\lambda]}]_{\text{vir}}} e^{\mathbf{T}}(-N_{C, L_1, L_2}^{\text{vir}}) &= \tilde{A}_{\lambda, 1}^{g-1} \cdot \tilde{A}_{\lambda, 2}^{\text{deg } L_1} \cdot \tilde{A}_{\lambda, 3}^{\text{deg } L_2} \\ &= \exp((g-1) \cdot \log \tilde{A}_{\lambda, 1} + \text{deg } L_1 \cdot \log \tilde{A}_{\lambda, 2} + \text{deg } L_2 \cdot \log \tilde{A}_{\lambda, 3}), \end{aligned}$$

⁵ In fact, $E_{\underline{Z}}$ is the \mathbf{T} -representation of the tangent space at a \mathbf{T} -fixed point of $\text{Hilb}^{|\lambda|}(\mathbb{C}^2)$. The claim follows by noting that the \mathbf{T} -fixed locus coincides with the fixed locus of the subtorus $\{t_1 t_2 = 1\} \subset \mathbf{T}$ preserving the Calabi–Yau form of \mathbb{C}^2 , which is zero-dimensional and reduced.

for suitable $\tilde{A}_{\lambda,i} \in 1 + \mathbb{Q}(s_1, s_2)[[q]]$, where $\log \tilde{A}_i$ are well-defined as \tilde{A}_i are power series starting with 1. The claim is therefore reduced to the computation of the leading term (under the anti-diagonal restriction $s_1 + s_2 = 0$) and to the solution of a linear system. The leading term is computed in Proposition 6.1, which also shows that it is invertible in $\mathbb{Q}(s_1, s_2)$. The linear system is solved by computing the generating series of the integrals (5.1), under the anti-diagonal restriction, on a basis of \mathbb{Q}^3 . The classes

$$\gamma(\mathbb{P}^1, \mathcal{O}, \mathcal{O}), \quad \gamma(\mathbb{P}^1, \mathcal{O}(-1), \mathcal{O}(-1)), \quad \gamma(\mathbb{P}^1, \mathcal{O}, \mathcal{O}(-2))$$

are linearly independent and we computed their generating series in Proposition 7.7 and Corollary 7.4. □

8.2 degree 1

In degree 1, we consider the Young diagram consisting of a single box and the corresponding double nested Hilbert scheme is $C^{[n]} \cong C^{(n)}$, the symmetric power of a smooth projective curve C , with universal subscheme $\mathcal{Z} \subset C \times C^{(n)}$. Given line bundles L_1, L_2 on C , the class (4.1) in K -theory of the virtual normal bundle is

$$\begin{aligned} N_{C,L_1,L_2}^{\text{vir}} &= -\mathbf{R}\pi_*(L_1L_2 \otimes \mathcal{O}_{\mathcal{Z}}) \otimes \mathbf{t}_1 \mathbf{t}_2 + \mathbf{R}\pi_*L_1 \otimes \mathbf{t}_1 + \mathbf{R}\pi_*L_2 \otimes \mathbf{t}_2 \\ &= -(L_1L_2)^{[n]} \otimes \mathbf{t}_1 \mathbf{t}_2 + \mathcal{O}_{C^{(n)}}^{\text{deg } L_1+1-g} \otimes \mathbf{t}_1 + \mathcal{O}_{C^{(n)}}^{\text{deg } L_2+1-g} \otimes \mathbf{t}_2, \end{aligned}$$

where $(L_1L_2)^{[n]}$ is the tautological bundle with fibers $(L_1L_2)^{[n]}|_{\mathcal{Z}} = H^0(C, L_1L_2 \otimes \mathcal{O}_{\mathcal{Z}})$. This yields

$$\int_{[C^{[n]}]_{\text{vir}}} e^{\mathbf{T}}(-N_{C,L_1,L_2}^{\text{vir}}) = s_1^{g-1-\text{deg } L_1} s_2^{g-1-\text{deg } L_2} \int_{C^{(n)}} e((L_1L_2)^{[n]}).$$

By the universal structure of Theorem 5.1, we just need to compute the (generating series of the) last integral for $L_1 = L_2 = \mathcal{O}_C$ and $L_1 \otimes L_2 \cong K_C$, which yields the explicit universal series

$$\begin{aligned} A_{\square}(q, s_1, s_2) &= s_1 s_2, \\ B_{\square}(q, s_1, s_2) &= s_1^{-1}(1 + q), \\ C_{\square}(q, s_1, s_2) &= s_2^{-1}(1 + q). \end{aligned}$$

8.3 GW/PT correspondence

Let $X = \text{Tot}_C(L_1 \oplus L_2)$ be a local curve. Combining Theorem 5.1, Theorem 8.1 and the description of the \mathbf{T} -fixed locus of $P_n(X, d[C])$ in Proposition 3.1, we obtain our main result.

THEOREM 8.2. *The generating series of stable pair invariants satisfies*

$$\text{PT}_d(X; q) = \sum_{\lambda \vdash d} (q^{-|\lambda|} A_{\lambda}(q))^{g-1} \cdot (q^{-n(\lambda)} B_{\lambda}(q))^{\text{deg } L_1} \cdot (q^{-n(\bar{\lambda})} C_{\lambda}(q))^{\text{deg } L_2},$$

where $A_{\lambda,i}$ are the universal series of Theorem 5.1. Moreover, under the anti-diagonal restriction $s_1 + s_2 = 0$

$$\begin{aligned} \text{PT}_d(X; -q)|_{s_1+s_2=0} &= (-1)^{d \text{deg } L_2} \sum_{\lambda \vdash d} q^{d(1-g) - \text{deg } L_1 n(\lambda) - \text{deg } L_2 n(\bar{\lambda})} \\ &\quad \cdot \prod_{\square \in \lambda} (s_1 h(\square))^{2g-2 - \text{deg } L_1 - \text{deg } L_2} (1 - q^{h(\square)})^{\text{deg } L_1 + \text{deg } L_2}. \end{aligned}$$

Comparing with Bryan and Pandharipande’s results (cf. Theorem 1.5 and [BP08, § 8] for the fully equivariant result in degree 1) we obtain a proof of the GW/stable pairs correspondence for local curves.

COROLLARY 8.3. *Let X be a local curve. Under the anti-diagonal restriction $s_1 + s_2 = 0$ the GW/stable pair correspondence holds*

$$(-i)^{d(2-2g+\deg L_1+\deg L_2)} \cdot \text{GW}_d(g|\deg L_1, \deg L_2; u) = (-q)^{-(1/2) \cdot d(2-2g+\deg L_1+\deg L_2)} \text{PT}_d(X, q),$$

after the change of variable $q = -e^{iu}$. Moreover, it holds fully equivariantly in degree 1.

8.4 Resolved conifold

In some cases, the moduli space $P_n(X, d[C])$ happens to be proper, for example whenever $H^0(C, L_i) = 0$ for $i = 1, 2$; see [CKM22b, Proposition 3.1] for a similar setting for local surfaces. Under this assumption, the invariants are computed as

$$\text{PT}_d(X; q) = \text{PT}_d(X; q)|_{s_1=s_2=0}$$

and can be deduced from the anti-diagonal restriction. The resulting invariants are interesting, that is, non-zero, only in the Calabi–Yau case; in this case, they coincide with the virtual Euler characteristic and Behrend’s weighted Euler characteristic of $P_n(X, d[C])$ [Beh09]. Applying Riemann–Roch, this situation may appear only when $H^l(C, L_i) = 0$ for $l = 0, 1$ and $\deg L_i = g - 1$ for $i = 1, 2$.

An interesting example is the resolved conifold $X = \text{Tot}_{\mathbb{P}^1}(\mathcal{O}(-1) \oplus \mathcal{O}(-1))$; in this case, the invariants can be further packaged into a generating series

$$\begin{aligned} 1 + \sum_{d \geq 1} Q^d \cdot \text{PT}_d(X; -q) &= \sum_{d \geq 0} (-Qq)^d \sum_{\lambda \vdash d} q^{n(\lambda)+n(\bar{\lambda})} \prod_{\square \in \lambda} (1 - q^{h(\square)})^{-2} \\ &= \sum_{\lambda} (-Qq)^{|\lambda|} \frac{q^{n(\lambda)}}{\prod_{\square \in \lambda} (1 - q^{h(\square)})} \frac{q^{n(\bar{\lambda})}}{\prod_{\square \in \bar{\lambda}} (1 - q^{h(\square)})} \\ &= \sum_{\lambda} (-Qq)^{|\lambda|} s_{\lambda}(q) s_{\bar{\lambda}}(q) \\ &= \prod_{n \geq 1} (1 - Qq^n)^n, \end{aligned}$$

where we used some classical identities involving the Schur function s_{λ} (see, e.g., [Mac95, p. 45] and [Tak14, §2.3]). This last generating series is well-known as it agrees with the expression of the unrefined limit of the topological vertex of Iqbal, Kozçaz and Vafa [IKV09] and can be seen as a specialization both of the motivic invariants of Morrison *et al.* [MMNS12] and of the K -theoretic invariants of Kononov, Okounkov and Osinenko [KOO21].

9. K -theoretic refinement

9.1 K -theoretic invariants

Let $X = \text{Tot}_C(L_1 \oplus L_2)$ be a local curve. The perfect obstruction theory on $P_X := P_n(X, d[C])$ induces a (\mathbf{T} -equivariant) virtual structure sheaf $\mathcal{O}_{P_X}^{\text{vir}} \in K_0^{\mathbf{T}}(P_X)$ [FG10] which depends only on the K -theory class of the perfect obstruction theory [Tho22, Corollary 4.5]. The K -theoretic PT invariants are defined by virtual K -theoretic localization [FG10] for any $V \in K_0^{\mathbf{T}}(P_X)$

$$\begin{aligned} \text{PT}_{d,n}^K(X, V) &:= \chi(P_n(X, d[C]), \mathcal{O}_{P_X}^{\text{vir}} \otimes V) \\ &:= \chi\left(P_n(X, d[C])^{\mathbf{T}}, \frac{\mathcal{O}_{P_X}^{\text{vir}} \otimes V|_{P_X^{\mathbf{T}}}}{\Lambda^{\bullet} N^{\text{vir},*}}\right) \in \mathbb{Q}(\mathbf{t}_1, \mathbf{t}_2), \end{aligned}$$

where $\Lambda^\bullet(V) := \sum_{i=0}^{\text{rk } V} (-1)^i \Lambda^i V$ is defined for every locally free sheaf V and then extended by linearity to any class in K -theory.

Remark 9.1. Differently than the case of equivariant cohomology, $\chi(M, \mathcal{F}) \in K_0^{\mathbf{T}}(\text{pt})_{\text{loc}}$ is well-defined for any \mathbf{T} -equivariant coherent sheaf \mathcal{F} on a non-proper scheme M , as long as the weight spaces of \mathcal{F} are finite-dimensional; in this case, the virtual localization formula is an actual theorem, rather than an *ad hoc* definition of the invariants.

Using the description of the \mathbf{T} -fixed locus $(P_n(X, d[C]))^{\mathbf{T}}$ of Proposition 3.1, K -theoretic stable pair invariants on X (with no insertions) are reduced to intersection numbers on $C^{[n\lambda]}$. The same techniques of § 5 can be applied in this setting, yielding the following result.

PROPOSITION 9.2. *Let C be a genus g smooth irreducible projective curve and L_1, L_2 line bundles over C . We have an identity*

$$\sum_{n\lambda} q^{|\mathbf{n}\lambda|} \chi\left(C^{[n\lambda]}, \frac{\mathcal{O}_{C^{[n\lambda]}}^{\text{vir}}}{\Lambda^\bullet N_{C, L_1, L_2}^{\text{vir}, *}}\right) = A_{K, \lambda}^{g-1} \cdot B_{K, \lambda}^{\deg L_1} \cdot C_{K, \lambda}^{\deg L_2} \in \mathbb{Q}(\mathbf{t}_1, \mathbf{t}_2)[[q]],$$

where $A_{K, \lambda}, B_{K, \lambda}, C_{K, \lambda} \in \mathbb{Q}(\mathbf{t}_1, \mathbf{t}_2)[[q]]$ are fixed universal series for $i = 1, 2, 3$, only depending on λ . Moreover,

$$\begin{aligned} A_{K, \lambda}(\mathbf{t}_1, \mathbf{t}_2) &= A_{K, \bar{\lambda}}(\mathbf{t}_2, \mathbf{t}_1), \\ B_{K, \lambda}(\mathbf{t}_1, \mathbf{t}_2) &= C_{K, \bar{\lambda}}(\mathbf{t}_2, \mathbf{t}_1). \end{aligned}$$

Proof. The proof follows the same strategy as Theorem 5.1. We just need to notice that $\Lambda^\bullet(\cdot)$ is multiplicative and that, via virtual Hirzebruch–Riemann–Roch [FG10], we can express

$$\begin{aligned} \chi\left(C^{[n\lambda]}, \frac{\mathcal{O}_{C^{[n\lambda]}}^{\text{vir}}}{\Lambda^\bullet N_{C, L_1, L_2}^{\text{vir}, *}}\right) &= \int_{[C^{[n\lambda]}]^{\text{vir}}} \text{ch}(-\Lambda^\bullet N_{C, L_1, L_2}^{\text{vir}, *}) \cdot \text{td}(T_{C^{[n\lambda]}}^{\text{vir}}) \\ &= \int_{A_{C, n\lambda}} f, \end{aligned}$$

where f is a polynomial expression of classes of the same form as in Proposition 5.3. □

Denote by $\text{PT}_d^K(X; q) = \sum_{n \in \mathbb{Z}} q^n \text{PT}_{d, n}^K(X)$ the generating series of K -theoretic stable pair invariants.

COROLLARY 9.3. *Let $X = \text{Tot}_C(L_1 \oplus L_2)$ be a local curve. We have*

$$\text{PT}_d^K(X; q) = \sum_{\lambda \vdash d} (q^{-|\lambda|} A_{K, \lambda}(q))^{g-1} \cdot (q^{-n(\lambda)} B_{K, \lambda}(q))^{\deg L_1} \cdot (q^{-n(\bar{\lambda})} C_{K, \lambda}(q))^{\deg L_2}.$$

9.2 Nekrasov–Okounkov

Let M be a scheme with a perfect obstruction theory \mathbb{E} . Define the virtual canonical bundle to be $K_{\text{vir}} = \det \mathbb{E} \in \text{Pic}(M)$. Assume that K_{vir} admits a square root $K_{\text{vir}}^{1/2}$, that is a line bundle such that $(K_{\text{vir}}^{1/2})^{\otimes 2} \cong K_{\text{vir}}$. Nekrasov and Okounkov [NO16] teach us that it is much more natural to consider the twisted virtual structure sheaf

$$\widehat{\mathcal{O}}_M^{\text{vir}} = \mathcal{O}_M^{\text{vir}} \otimes K_{\text{vir}}^{1/2}.$$

In [NO16], Nekrasov and Okounkov showed the existence of square roots for P_X (and uniqueness, up to 2-torsion). Nevertheless, even if square roots could not exist on P_X as line bundles, they

exist as a class $K_{\text{vir}, P_X}^{1/2} \in K^0(P_X, \mathbb{Z}[\frac{1}{2}])$ and are unique [OT20, Lemma 5.1]. In our setting, we define *Nekrasov–Okounkov K -theoretic stable pair invariants* as⁶

$$\begin{aligned} \text{PT}_{d,n}^{\hat{K}}(X) &:= \text{PT}_{d,n}^K(X, K_{\text{vir}, P_X}^{1/2}) \\ &= \chi(P_n(X, d[C]))^{\mathbf{T}}, \frac{\mathcal{O}_{P_X^{\mathbf{T}}}^{\text{vir}} \otimes K_{\text{vir}, P_X}^{1/2}|_{P_X^{\mathbf{T}}}}{\Lambda \bullet N^{\text{vir},*}} \in \mathbb{Q}(\mathfrak{t}_1^{1/2}, \mathfrak{t}_2^{1/2}), \end{aligned}$$

which are an algebro-geometric analogue of the \widehat{A} -genus of a spin manifold. On the \mathbf{T} -fixed locus, we have an identity in $K_{\mathbf{T}}^0(P_X^{\mathbf{T}}, \mathbb{Z}[\frac{1}{2}])_{\text{loc}}$

$$\begin{aligned} \frac{\mathcal{O}_{P_X^{\mathbf{T}}}^{\text{vir}} \otimes K_{\text{vir}, P_X}^{1/2}|_{P_X^{\mathbf{T}}}}{\Lambda \bullet N^{\text{vir},*}} &= \frac{\mathcal{O}_{P_X^{\mathbf{T}}}^{\text{vir}} \otimes K_{\text{vir}, P_X^{\mathbf{T}}}^{1/2} \otimes (\det N^{\text{vir},*})^{1/2}}{\Lambda \bullet N^{\text{vir},*}} \\ &= \frac{\widehat{\mathcal{O}}_{P_X^{\mathbf{T}}}^{\text{vir}}}{\widehat{\Lambda} \bullet N^{\text{vir},*}}, \end{aligned}$$

where we define $\widehat{\Lambda} \bullet (\cdot) = \Lambda \bullet (\cdot) \otimes \det(\cdot)^{-1/2}$. Again, the same techniques of § 5 and Proposition 9.2 and can be applied in this setting, yielding the following result.

THEOREM 9.4. *Let C be a genus g smooth irreducible projective curve and L_1, L_2 line bundles over C . We have an identity*

$$\sum_{\mathbf{n}_\lambda} q^{|\mathbf{n}_\lambda|} \chi\left(C^{[\mathbf{n}_\lambda]}, \frac{\widehat{\mathcal{O}}_{P_X^{\mathbf{T}}}^{\text{vir}}}{\widehat{\Lambda} \bullet N_{C, L_1, L_2}^{\text{vir},*}}\right) = A_{\hat{K}, \lambda}^{g-1} \cdot B_{\hat{K}, \lambda}^{\deg L_1} \cdot C_{\hat{K}, \lambda}^{\deg L_2} \in \mathbb{Q}(\mathfrak{t}_1^{1/2}, \mathfrak{t}_2^{1/2})[[q]],$$

where $A_{\hat{K}, \lambda}, B_{\hat{K}, \lambda}, C_{\hat{K}, \lambda} \in \mathbb{Q}(\mathfrak{t}_1^{1/2}, \mathfrak{t}_2^{1/2})[[q]]$ are fixed universal series for $i = 1, 2, 3$. Moreover

$$\begin{aligned} A_{\hat{K}, \lambda}(\mathfrak{t}_1, \mathfrak{t}_2) &= A_{\hat{K}, \bar{\lambda}}(\mathfrak{t}_2, \mathfrak{t}_1), \\ B_{\hat{K}, \lambda}(\mathfrak{t}_1, \mathfrak{t}_2) &= C_{\hat{K}, \bar{\lambda}}(\mathfrak{t}_2, \mathfrak{t}_1). \end{aligned}$$

Denote by $\text{PT}_d^{\hat{K}}(X; q) = \sum_{n \in \mathbb{Z}} q^n \text{PT}_{d,n}^{\hat{K}}(X)$ the generating series of Nekrasov–Okounkov K -theoretic stable pair invariants.

COROLLARY 9.5. *Let $X = \text{Tot}_C(L_1 \oplus L_2)$ be a local curve. We have*

$$\text{PT}_d^{\hat{K}}(X; q) = \sum_{\lambda \vdash d} (q^{-|\lambda|} A_{\hat{K}, \lambda}(q))^{g-1} \cdot (q^{-n(\lambda)} B_{\hat{K}, \lambda}(q))^{\deg L_1} \cdot (q^{-n(\bar{\lambda})} C_{\hat{K}, \lambda}(q))^{\deg L_2}.$$

The techniques of § 6, 7 can be adapted to compute the generating series of Nekrasov–Okounkov K -theoretic invariants under the anti-diagonal restriction $\mathfrak{t}_1 \mathfrak{t}_2 = 1$. In fact, as in the proof of Theorem 8.1, we just need to compute the leading term of the generating series and the cases $g = 0, L_1 \otimes L_2 = K_{\mathbb{P}^1}$ and $L_1 = L_2 = \mathcal{O}_{\mathbb{P}^1}$. Similarly to the proof of Theorem 7.3, as we work with $C \cong \mathbb{P}^1$, applying the K -theoretic virtual localization formula [FG10] on $(\mathbb{P}^1)^{[\mathbf{n}_\lambda]}$ yields

$$\chi\left(\mathbb{P}^{1[\mathbf{n}_\lambda]}, \frac{\widehat{\mathcal{O}}_{P_X^{\mathbf{T}}}^{\text{vir}}}{\widehat{\Lambda} \bullet N_{C, L_1, L_2}^{\text{vir},*}}\right) = \left(\sum_{\underline{Z} \in \mathbb{P}^{1[\mathbf{n}_\lambda], \mathbb{C}^*}} \chi\left(\underline{Z}, \frac{1}{\widehat{\Lambda} \bullet (T_{\mathbb{P}^1[\mathbf{n}_\lambda], \underline{Z}}^{\text{vir},*} + N_{\mathbb{P}^1, L_1, L_2, \underline{Z}}^{\text{vir},*})}\right) \right) \Big|_{\mathfrak{t}_3=1},$$

⁶ To take square roots *equivariantly*, we need to replace the torus \mathbf{T} with the minimal cover where the characters $\mathfrak{t}_1^{1/2}, \mathfrak{t}_2^{1/2}$ are well-defined, see [NO16, § 7.2.1].

where t_3 is the equivariant parameter of the \mathbb{C}^* -action. For a character $t^\mu \in K_{\mathbb{T} \times \mathbb{C}^*}^0(\text{pt})$, define the operator $[t^\mu] = t^{\mu/2} - t^{-\mu/2}$ and extend it by linearity to any $V \in K_{\mathbb{T} \times \mathbb{C}^*}^0(\text{pt})$. It is proven in [FMR21, § 6.1] that

$$\chi(\text{pt}, \widehat{\Lambda}^\bullet(V^*)) = [V],$$

which satisfies $[V^*] = (-1)^{\text{rk } V} [V]$, therefore

$$\chi\left(\mathbb{P}^1[\mathbf{n}\lambda], \frac{\widehat{\mathcal{O}}_{P_X^{\text{vir}}}^{\text{vir}}}{\widehat{\Lambda}^\bullet N_{C, L_1, L_2}^{\text{vir}, *}}\right) = \left(\sum_{Z \in \mathbb{P}^1[\mathbf{n}\lambda], \mathbb{C}^*} [-T_{\mathbb{P}^1[\mathbf{n}\lambda], Z}^{\text{vir}} - N_{\mathbb{P}^1, L_1, L_2, Z}^{\text{vir}}] \right) \Big|_{t_3=1}.$$

Explicit computations yields

$$\begin{aligned} A_{\widehat{K}, \lambda}(q, t_1, t_1^{-1}) &= (-1)^{|\lambda|} \cdot F_\lambda^{-2}, \\ B_{\widehat{K}, \lambda}(-q, t_1, t_1^{-1}) &= (-1)^{n(\lambda)} \cdot F_\lambda \cdot \prod_{\square \in \lambda} (1 - q^{h(\square)}), \\ C_{\widehat{K}, \lambda}(-q, t_1, t_1^{-1}) &= (-1)^{|\lambda| + n(\bar{\lambda})} F_\lambda \cdot \prod_{\square \in \lambda} (1 - q^{h(\square)}), \end{aligned}$$

where

$$F_\lambda = \prod_{\substack{(i,j) \in \lambda \\ (i,j) \neq (a,a)}} \frac{1}{t_1^{j-i|/2} - t_1^{-|j-i|/2}} \cdot \frac{\prod_{\substack{(i,j),(l,k) \in \lambda \\ (i,j) \neq (l+a, k+a)}} (t_1^{|i-j+k-l|/2} - t_1^{-|i-j+k-l|/2})}{\prod_{\substack{(i,j),(l,k) \in \lambda \\ (i,j) \neq (l+a-1, k+a)}} (t_1^{|1+i-j+k-l|/2} - t_1^{-|1+i-j+k-l|/2})}.$$

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Appendix A. The combinatorial identities

In this appendix, we collect the proofs of some technical results on the combinatorics of Young diagrams we have used.

LEMMA A.1. *Let λ be a Young diagram and consider the Laurent polynomials in $\mathbb{Z}[t^{\pm 1}]$*

$$\begin{aligned} g_\lambda(t) &= \sum_{\substack{(a,a) \in \lambda \\ a \neq 0, a \in \mathbb{Z}}} t^{-a} + \sum_{\substack{(i,j),(l,k) \in \lambda \\ (i,j) = (l+a, k+a) \\ a \neq 0, 1, a \in \mathbb{Z}}} \text{sgn}(a) t^a - \sum_{\substack{(i,j),(l,k) \in \lambda \\ (i,j) = (l+a-1, k+a) \\ a \neq 0, 1, a \in \mathbb{Z}}} \text{sgn}(a) t^a, \\ h_\lambda(t) &= \sum_{\substack{(a,a) \in \lambda \\ a \neq 0, a \in \mathbb{Z}}} t^{-a} - \sum_{\substack{(i,j),(l,k) \in \lambda \\ (i,j) = (l+a, k+a) \\ a \neq 0, 1, a \in \mathbb{Z}}} t^a + \sum_{\substack{(i,j),(l,k) \in \lambda \\ (i,j) = (l+a-1, k+a) \\ a \neq 0, 1, a \in \mathbb{Z}}} t^a. \end{aligned}$$

Then we have

$$\begin{aligned} g_\lambda(t) &= A_\lambda(t) + A_\lambda(t^{-1}) t, \\ h_\lambda(t) &= B_\lambda(t) - B_\lambda(t^{-1}) t, \end{aligned}$$

where $A_\lambda(t), B_\lambda(t) \in \mathbb{Z}[t^{\pm 1}]$.

Proof. We prove the first claim by induction on the size of λ . If $|\lambda| = 1$ this is clear. Now suppose that the claim holds for all Young diagrams of size $|\lambda| = n$ and consider a Young diagram of size $|\tilde{\lambda}| = n + 1$ obtained by adding to a Young diagram λ a box whose lattice coordinates are $(i, j) \in \mathbb{Z}^2$:

- $(i, j) = (i, 0)$ or $(i, j) = (0, j)$, with $i, j \neq 0$, we have

$$g_{\tilde{\lambda}} = g_{\lambda};$$

- $(i, j) = (i, 1)$, $i \geq 1$, we have

$$\begin{aligned} g_{\tilde{\lambda}} &= g_{\lambda} + t^{-1} - t^{-1} \\ &= g_{\lambda}; \end{aligned}$$

- $(i, j) = (1, j)$, $j \geq 2$, we have

$$\begin{aligned} g_{\tilde{\lambda}} &= g_{\lambda} - t^{-1} - t^2 \\ &= g_{\lambda} - t^{-1} - (t^{-1})^{-1} t; \end{aligned}$$

- $(i, j) = (i, i)$, $i \geq 2$, we have

$$\begin{aligned} g_{\tilde{\lambda}} &= g_{\lambda} + t^{-i} - \sum_{l=1}^i t^{-l} + \sum_{l=2}^i t^l - \left(\sum_{l=2}^i t^l - \sum_{l=1}^{i-1} t^{-l} \right) \\ &= g_{\lambda}; \end{aligned}$$

- (i, j) , $i > j \geq 2$, we have

$$\begin{aligned} g_{\tilde{\lambda}} &= g_{\lambda} - \sum_{l=1}^j t^{-l} + \sum_{l=2}^j t^l - \left(\sum_{l=2}^j t^l - \sum_{l=1}^j t^{-l} \right) \\ &= g_{\lambda}; \end{aligned}$$

- (i, j) , $j > i \geq 2$, we have

$$\begin{aligned} g_{\tilde{\lambda}} &= g_{\lambda} - \sum_{l=1}^i t^{-l} + \sum_{l=2}^i t^l - \left(\sum_{l=2}^{i+1} t^l - \sum_{l=1}^{i-1} t^{-l} \right) \\ &= g_{\lambda} - t^{-i} - t^{i+1}; \end{aligned}$$

Therefore, the induction step is proven in all possible cases and we conclude the proof.

With an analogous analysis one proves the second claim as well. □

LEMMA A.2. *Let λ be a Young diagram. Then the following identity holds:*

$$\prod_{\substack{(i,j) \in \lambda \\ (i,j) \neq (a,a)}} (j-i) \cdot \frac{\prod_{\substack{(i,j),(l,k) \in \lambda \\ (i,j) \neq (l+a-1,k+a)}} (1+i-j+k-l)}{\prod_{\substack{(i,j),(l,k) \in \lambda \\ (i,j) \neq (l+a,k+a)}} (i-j+k-l)} = \sigma(\lambda) \cdot \prod_{\square \in \lambda} h(\square),$$

where

$$\sigma(\lambda) = \prod_{\substack{(i,j) \in \lambda \\ (i,j) \neq (a,a)}} \text{sgn}(j-i) \cdot \prod_{\substack{(i,j),(l,k) \in \lambda \\ (i,j) \neq (l+a-1,k+a)}} \text{sgn}(1+i-j+k-l) \cdot \prod_{\substack{(i,j),(l,k) \in \lambda \\ (i,j) \neq (l+a,k+a)}} \text{sgn}(i-j+k-l).$$

Proof. The sign $\sigma(\lambda)$ is easily determined, so we just need to compute the absolute value. To ease the notation, we adopt the following convention for the remainder of the proof: we set $|0| = 1$,

which is merely a formal shortcut to include in the productory trivial factors we would have otherwise excluded. The claim therefore becomes

$$\prod_{(i,j) \in \lambda} |i - j| \cdot \prod_{(i,j),(l,k) \in \lambda} \frac{|1 + i - j + k - l|}{|i - j + k - l|} = \prod_{\square \in \lambda} h(\square). \tag{A.1}$$

Denote the left-hand side of (A.1) by H_λ . We prove this claim on the induction on the size of λ . If $|\lambda| = 1$, the claim is trivially satisfied. Assume it holds for all Young diagrams of size n and consider a Young diagram λ' of size $n + 1$ obtained by a Young diagram of size λ by adding a box with lattice coordinates $(i, j) \in \mathbb{Z}^2$. We have

$$H_{\lambda'} = H_\lambda \cdot |i - j| \cdot \prod_{(l,k) \in \lambda} \frac{|1 + i - j + k - l|}{|i - j + k - l|} \cdot \frac{|-1 + i - j + k - l|}{|i - j + k - l|}.$$

To avoid confusion, we denote by $h(\square)$ (respectively, $h'(\square)$) the hooklength of $\square \in \lambda$ (respectively, $\square \in \lambda'$). The strategy now is to divide the boxes of λ' in sub-collections and compute separately each contribution of the product on the right-hand side.

Step I. Fix a box $(\tilde{i}, j) \in \lambda$, with $\tilde{i} < i$. The contribution of all boxes on the right (on the same row) of (\tilde{i}, j) is

$$\begin{aligned} & \prod_{k=j}^{\lambda_{\tilde{i}}-1} \frac{|i - j + k - \tilde{i} + 1| \cdot |i - j + k - \tilde{i} - 1|}{|i - j + k - \tilde{i}| \cdot |i - j + k - \tilde{i}|} \\ &= \frac{|i - \tilde{i} + 1| \cdot |i - \tilde{i} - 1|}{|i - \tilde{i}| \cdot |i - \tilde{i}|} \cdot \frac{|i - \tilde{i} + 2| \cdot |i - \tilde{i}|}{|i - \tilde{i} + 1| \cdot |i - \tilde{i} + 1|} \cdots \frac{|i - \tilde{i} + \lambda_{\tilde{i}} - j| \cdot |i - \tilde{i} + \lambda_{\tilde{i}} - j - 2|}{|i - \tilde{i} + \lambda_{\tilde{i}} - j - 1| \cdot |i - \tilde{i} + \lambda_{\tilde{i}} - j - 1|} \\ &= \frac{|i - \tilde{i} - 1| \cdot |i - \tilde{i} + \lambda_{\tilde{i}} - j|}{|i - \tilde{i}| \cdot |i - \tilde{i} + \lambda_{\tilde{i}} - j - 1|} \\ &= \frac{|i - \tilde{i} - 1|}{|i - \tilde{i}|} \cdot \frac{h'(\tilde{i}, j)}{h(\tilde{i}, j)}. \end{aligned}$$

We now multiply the last expression for all boxes (\tilde{i}, j) with $\tilde{i} = 0, \dots, i - 1$

$$\begin{aligned} \prod_{\tilde{i}=0}^{i-1} \frac{|i - \tilde{i} - 1|}{|i - \tilde{i}|} \cdot \frac{h'(\tilde{i}, j)}{h(\tilde{i}, j)} &= \frac{|i - 1|}{|i|} \cdot \frac{|i - 2|}{|i - 1|} \cdots \frac{|0|}{|1|} \cdot \prod_{\tilde{i}=0}^{i-1} \frac{h'(\tilde{i}, j)}{h(\tilde{i}, j)} \\ &= \frac{1}{|i|} \cdot \prod_{\tilde{i}=0}^{i-1} \frac{h'(\tilde{i}, j)}{h(\tilde{i}, j)}. \end{aligned}$$

This is the contribution of all boxes $(l, k) \in \lambda$ such that $k \geq j$. By symmetry, we obtain that the contribution of all boxes $(l, k) \in \lambda$ such that $l \geq i$ is given by

$$\frac{1}{|j|} \cdot \prod_{\tilde{j}=0}^{j-1} \frac{h'(\tilde{i}, \tilde{j})}{h(\tilde{i}, \tilde{j})}.$$

Step II. The contribution of the remaining boxes is given by

$$|i - j| \cdot \prod_{\substack{(l,k) \in \lambda \\ l < i \\ k < j}} \frac{|1 + i - j + k - l|}{|i - j + k - l|} \cdot \frac{|-1 + i - j + k - l|}{|i - j + k - l|}$$

$$\begin{aligned}
 &= |i - j| \cdot \prod_{l=0}^{i-1} \prod_{k=0}^{j-1} \frac{|1 + i - j + k - l|}{|i - j + k - l|} \cdot \frac{|-1 + i - j + k - l|}{|i - j + k - l|} \\
 &= |i - j| \cdot \prod_{l=0}^{i-1} \frac{|i - l - j - 1| \cdot |i - l|}{|i - l - j| \cdot |i - l - 1|} = |i - j| \cdot \frac{|i| \cdot |j|}{|i - j|} = |i| \cdot |j|.
 \end{aligned}$$

Step III. Using steps I and II and the induction step we have

$$\begin{aligned}
 H_{\lambda'} &= \prod_{\square \in \lambda} h(\square) \cdot \prod_{\tilde{i}=0}^{i-1} \frac{h'(\tilde{i}, j)}{h(\tilde{i}, j)} \cdot \prod_{\tilde{j}=0}^{j-1} \frac{h'(i, \tilde{j})}{h(i, \tilde{j})} \\
 &= \prod_{\square \in \lambda'} h'(\square),
 \end{aligned}$$

which concludes the proof. □

LEMMA A.3. Let λ be a Young diagram and $k_1, k_2 \in \mathbb{Z}$. Set

$$\begin{aligned}
 \rho(\lambda) &= \sum_{\substack{(i,j) \in \lambda \\ (i,j) \neq (a,a)}} (ik_1 + jk_2) + \sum_{\substack{(i,j),(l,k) \in \lambda \\ (i,j) \neq (l+a-1,k+a)}} ((i - l + 1)k_1 + (j - k)k_2) \\
 &+ \sum_{\substack{(i,j),(l,k) \in \lambda \\ (i,j) \neq (l+a,k+a)}} ((i - l)k_1 + (j - k)k_2).
 \end{aligned}$$

We have

$$(-1)^{\rho(\lambda) + |\lambda|(k_1 + k_2)} \sigma(\lambda)^{k_1 + k_2} = (-1)^{|\lambda|k_2 + n(\lambda)k_1 + n(\bar{\lambda})k_2},$$

where $\sigma(\lambda)$ was defined in Lemma A.2.

Proof. Let \equiv denote congruence modulo two. We have

$$\begin{aligned}
 &\rho(\lambda) + |\lambda|(k_1 + k_2) \\
 &\equiv \sum_{(i,j) \in \lambda} (ik_1 + jk_2) + (k_1 + k_2) \left(\sum_{\substack{(i,j) \in \lambda \\ (i,j) = (a,a)}} a + \sum_{\substack{(i,j),(l,k) \in \lambda \\ (i,j) = (l+a-1,k+a)}} a + \sum_{\substack{(i,j),(l,k) \in \lambda \\ (i,j) = (l+a,k+a)}} a \right) \\
 &+ k_1 \sum_{(i,j),(l,k) \in \lambda} 1 + |\lambda|(k_1 + k_2) \\
 &\equiv n(\lambda)k_1 + n(\bar{\lambda})k_2 + |\lambda|k_2 + (k_1 + k_2) \left(\sum_{\substack{(i,j) \in \lambda \\ (i,j) = (a,a)}} a + \sum_{\substack{(i,j),(l,k) \in \lambda \\ (i,j) = (l+a-1,k+a)}} a + \sum_{\substack{(i,j),(l,k) \in \lambda \\ (i,j) = (l+a,k+a)}} a \right),
 \end{aligned}$$

therefore the statement of the lemma reduces to the following claim:

$$(-1)^{\sum_{\substack{(i,j) \in \lambda \\ (i,j) = (a,a)}} a + \sum_{\substack{(i,j),(l,k) \in \lambda \\ (i,j) = (l+a-1,k+a)}} a + \sum_{\substack{(i,j),(l,k) \in \lambda \\ (i,j) = (l+a,k+a)}} a} \cdot \sigma(\lambda) = 1. \tag{A.2}$$

Given a lattice point $\mu = (\mu_1, \mu_2) \in \mathbb{Z}^2$ define

$$\tau(\mu) = \begin{cases} (-1)^{\mu_1} & \mu_1 = \mu_2, \\ \text{sgn}(\mu_1 - \mu_2) & \mu_1 \neq \mu_2. \end{cases}$$

1	1	1	1	1
1	-1	1	-1	
1	1	1	ij	
1	1			
1				

FIGURE A.1. The label ± 1 in every box represents the contribution to the final sign, with $(i, j) = (2, 3)$.

Note that the left-hand side of (A.2) can be rewritten as

$$\prod_{(i,j) \in \lambda} \tau(-i, -j) \prod_{(i,j),(l,k) \in \lambda} \tau(i-l, j-k)\tau(1+i-l, j-k),$$

and denote this last expression by F_λ . We prove the claim (A.2) on induction on the size of λ . If $|\lambda| = 1$, the result is clear. Assume it holds for all Young diagrams of size n and consider a Young diagram λ' of size $n + 1$ obtained from a Young diagram of size λ by adding a box with lattice coordinates $(i, j) \in \mathbb{Z}^2$; we have

$$F_{\lambda'} = F_\lambda \cdot \tau(-i, -j) \prod_{(l,k) \in \lambda} \tau(i-l, j-k)\tau(l-i, k-j)\tau(1+i-l, j-k)\tau(1+l-i, k-j).$$

We analyze the contribution of every box $(l, k) \in \lambda$ in the product above. We say that a box (l, k) is in the same diagonal as (i, j) if it is of the form

$$(l, k) = (i + a, j + a), \quad a \in \mathbb{Z}.$$

The contribution of the boxes on the same diagonal of (i, j) is

$$\tau(-a, -a)\tau(a, a)\tau(1-a, -a)\tau(1+a, a) = 1.$$

We say that a box (l, k) is in the *lower diagonal* of (i, j) if it is of the form

$$(l, k) = (i + a, j + a - 1), \quad a \in \mathbb{Z}.$$

The contribution of the boxes in the lower diagonal of (i, j) is

$$\tau(-a, 1-a)\tau(a, a-1)\tau(1-a, 1-a)\tau(1+a, a-1) = (-1)^a.$$

We say that a box (l, k) is in the *upper diagonal* of (i, j) if it is of the form

$$(l, k) = (i + a - 1, j + a), \quad a \in \mathbb{Z}.$$

The contribution of the boxes in the upper diagonal of (i, j) is

$$\tau(1-a, -a)\tau(a-1, a)\tau(2-a, -a)\tau(a, a) = (-1)^{a+1}.$$

A completely analogous analysis shows that all other boxes $(i, j) \in \lambda$ do not contribute to the product. Therefore, the contribution to the sign in the product is just given by the boxes in the upper or lower diagonal of (i, j) , as displayed in Figure A.1.

If we denote by δ_+ and δ_- the number of boxes in the upper diagonal and lower diagonal, respectively, we conclude that

$$\prod_{(l,k) \in \lambda} \tau(i-l, j-k)\tau(l-i, k-j)\tau(1+i-l, j-k)\tau(1+l-i, k-j) = (-1)^{\lceil \delta_+/2 \rceil + \lfloor \delta_-/2 \rfloor},$$

where $\lceil \cdot \rceil, \lfloor \cdot \rfloor$ denote the usual *ceiling* and *floor* functions. One readily proves that

$$(\delta_+, \delta_-) = \begin{cases} (j+1, j) & i > j, \\ (i, i) & i = j, \\ (i, i+1) & i < j. \end{cases}$$

With a direct analysis we can show that

$$(-1)^{\lceil \delta_+/2 \rceil + \lfloor \delta_-/2 \rfloor} \cdot \tau(-i, -j) = 1$$

in all the three cases, by which we conclude the inductive step. \square

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