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ON THE STRUCTURE OF THE SET OF SOLUTIONS OF THE DARBOUX PROBLEM FOR HYPERBOLIC EQUATIONS

by F. S. DE BLASI and J. MYJAK

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1. Introduction and main result

Consider the Darboux problem

$$\begin{aligned} z_{xy} &= f(x,y,z) \\ z(x,0) = \phi(x), &\qquad z(0,y) = \psi(y), \end{aligned} \tag{1}$$

where $\phi, \psi: I \to R^d$ ($I = [0,1]$) are given absolutely continuous functions with $\phi(0) = \psi(0)$, and the mapping $f: Q \times R^d \to R^d$ ($Q = I \times I$) satisfies the following hypotheses:

(A_1) $f(.,.,z)$ is measurable for every $z \in R^d$;

(A_2) $f(x,y,.)$ is continuous for a.a. (almost all) $(x,y) \in Q$;

(A_3) there exists an integrable function $\alpha: Q \to [0, +\infty)$ such that $|f(x,y,z)| \leqq \alpha(x,y)$ for every $(x,y,z) \in Q \times R^d$.

Let $C(Q, R^d)$ denote the Banach space of all continuous functions from Q to R^d endowed with the metric of uniform convergence.

By a *solution* of problem (1) we mean a function $z \in C(Q, R^d)$ satisfying

$$z(x,y) = \phi(x) + \psi(y) - \phi(0) + \int_0^x \int_0^y f(\xi, \eta, z(\xi, \eta))\, d\xi\, d\eta,$$

for every $(x,y) \in Q$.

The purpose of this note is to prove the following

Theorem. *Let $f: Q \times R^d \to R^d$ satisfy* (A_1), (A_2), (A_3). *Let $\phi, \psi: I \to R^d$ be absolutely continuous functions with $\phi(0) = \psi(0)$. Then the set ζ_f of all solutions of the problem* (1) *is an R_δ-set in $C(Q, R^d)$.*

Recall that a subset of a metric space is called an R_δ-set if it is the intersection of a decreasing sequence of compact absolute retracts. It is known that an R_δ-set is acyclic, in particular it is nonempty compact and connected.

Hukuhara [5] and Aronszajn [1] have proved that the set of solutions of the Cauchy problem $x' = f(t,x)$, $x(0) = x_0$, where $f: I \times R^d \to R^d$ is continuous and bounded, is an R_δ-set in $C(I, R^d)$. Recently, by using topological degree arguments, Górniewicz and

Pruszko [4] have established an analogous result for the Darboux problem (1), under the main hypothesis that f be continuous with respect to all variables. In this note the set of solutions of problem (1) is shown to be an R_δ-set also when f satisfies hypotheses of Carathéodory type. Our approach is different from that used in [4].

Remark 1. The statement of the theorem remains true when the condition (A_3) is replaced by the following one: there exist integrable functions $\alpha, \beta: Q \to [0, +\infty)$ such that $|f(x,y,z)| \leqq \alpha(x,y) + \beta(x,y)|z|$ for each $(x,y,z) \in Q \times R^d$.

2. Preliminaries

The following lemma can be proved as in [2, Lemma 2].

Lemma 1. *Let $f: Q \times R^d \to R^d$ satisfy (A_1), (A_2), (A_3). Then for every $\varepsilon > 0$ there exists a locally lipschitzian function $g: Q \times R^d \to R^d$ such that*

$$\iint_Q \sup_{z \in R^d} |g(\xi,\eta,z) - f(\xi,\eta,z)| \, d\xi \, d\eta < \varepsilon.$$

Recall that a subset A of a metric space is called contractible if there exist a point $x_0 \in A$ and a continuous function $h: I \times A \to A$ such that $h(0,x) = x_0$ and $h(1,x) = x$ for each $x \in A$.

Lemma 2 [6]. *Let A be a nonempty compact subset of a metric space X. Then A is an R_δ-set in X if and only if A is the intersection of a decreasing sequence of compact contractible subsets of X.*

Let $L_1(Q, R^d)$ be the Banach space of the (equivalence classes of) Lebesgue integrable functions $v: Q \to R^d$, with the norm $\iint_Q |v(\xi,\eta)| \, d\xi \, d\eta$.

Lemma 3. *Suppose that a sequence $\{v_n\} \subset L_1(Q, R^d)$ satisfies:*

(i) *$|v_n(x,y)| \leqq \alpha(x,y)$ for almost all $(x,y) \in Q$ $(\alpha \in L_1(Q, R^d))$;*

(ii) *for each $(x,y) \in Q$ the sequence*

$$\left\{ \int_0^x \int_0^y v_n(\xi,\eta) \, d\xi \, d\eta \right\} \tag{2}$$

is Cauchy.

Then $\{v_n\}$ is weakly Cauchy in $L_1(Q, R^d)$.

Proof. Clearly $\{v_n\}$ is norm bounded in $L_1(Q, R^d)$. Let E be a measurable subset of Q. Let $\varepsilon > 0$. Let $P \subset Q$ be an elementary set (that is a set which can be expressed as a union of a finite number of pairwise disjoint rectangles) such that $\iint_{E \Delta P} \alpha(\xi,\eta) \, d\xi \, d\eta < \varepsilon/4$ $(E \Delta P = (E \backslash P) \cup (P \backslash E))$. As P is an elementary set, by virtue of (ii) one can find an $n_0 \in \mathbb{N}$ such that $\left| \iint_P (v_m(\xi,\eta) - v_n(\xi,\eta)) \, d\xi \, d\eta \right| < \varepsilon/4$ if $m, n \geqq n_0$. Then, by an easy computation, one obtains

$$\left|\iint_E (v_m(\xi,\eta)-v_n(\xi,\eta))\,d\xi\,d\eta\right| \leqq 2\iint_{E\Delta P} \alpha(\xi,\eta)\,d\xi\,d\eta$$

$$+\left|\iint_P (v_m(\xi,\eta)-v_n(\xi,\eta))\,d\xi\,d\eta\right| < 2\frac{\varepsilon}{4}+\frac{\varepsilon}{2}=\varepsilon$$

$(m,n\geqq n_0)$, which shows that the sequence $\{\iint_E v_n(\xi,\eta)\,d\xi\,d\eta\}$ is Cauchy. By using [3, Theorem IV.8.7] one can complete the proof.

Denote by $\mathscr{K}$ the family of all nonempty compact convex subsets of R^d. Recall that a multifunction $G:Q\to\mathscr{K}$ is said to be measurable if the set $\{(x,y)\in Q \mid G(x,y)\cap U\neq\varnothing\}$ is (Lebesgue) measurable for every open subset U of R^d. A multifunction $G:R^d\to\mathscr{K}$ is said to be upper semi-continuous (u.s.c.) if the set $\{u\in R^d \mid G(u)\subset U\}$ is open for every open subset U of R^d.

Consider the (multivalued) Darboux problem

$$\begin{gathered} z_{xy}\in F(x,y,z) \\ z(x,0)=\phi(x), \qquad z(0,y)=\psi(y), \end{gathered} \tag{3}$$

where the functions $\phi,\psi:I\to R^d$ are as above, and the multifunction $F:Q\times R^d\to\mathscr{K}$ satisfies the following hypotheses:

(H_1) $F(.,.,z)$ is measurable for every $z\in R^d$;

(H_2) $F(x,y,.)$ is u.s.c. for a.a. $(x,y)\in Q$;

(H_3) there exists an integrable function $\alpha:Q\to[0,+\infty)$ such that $\sup\{|u| \mid u\in F(x,y,z)\}\leqq\alpha(x,y)$ for every $(x,y,z)\in Q\times R^d$.

By a *solution* of (3) we mean a function $z\in C(Q,R^d)$ such that there exists an integrable function $v:Q\to R^d$ satisfying

$$v(x,y)\in F(x,y,z(x,y)) \quad \text{for a.a.} \quad (x,y)\in Q,$$

and

$$z(x,y)=\phi(x)+\psi(y)-\phi(0)+\int_0^x\int_0^y v(\xi,\eta)\,d\xi\,d\eta \quad \text{for every} \quad (x,y)\in Q.$$

We denote by μ the Lebesgue measure in R^2 and by B the unit closed ball in R^d.

Lemma 4. *Let $F:Q\times R^d\to\mathscr{K}$ satisfy* (H_1), (H_2), (H_3) *and let $\phi,\psi:I\to R^d$ be absolutely continuous functions with $\phi(0)=\psi(0)$. In addition, suppose that there exists a locally lipschitzian function $g:Q\times R^d\to R^d$ such that $g(x,y,z)\in F(x,y,z)$ for each $(x,y,z)\in(Q\backslash Q_0)\times R^d$, where $\mu(Q_0)=0$. Then the set ζ_F of all solutions of problem* (3) *is a* (*nonempty*) *compact contractible subset of $C(Q,R^d)$.*

Proof. Since the solution of problem (3) (with g in the place of f) belongs to ζ_F, one has that $\zeta_F \neq \emptyset$.

Let us show that ζ_F is compact. To this end consider any sequence $\{z_n\} \subset \zeta_F$. Taking into account the uniform continuity of ϕ, ψ and assumption (H_3) one can easily show that the functions z_n are equicontinuous and equibounded. By Ascoli–Arzelà's Theorem, passing to a subsequence (without change of notation), we can assume that $\{z_n\}$ converges uniformly on Q, to a function $z_0 \in C(Q, R^d)$. For each $n \in \mathbb{N}$ we have

$$z_n(x,y) = \phi(x) + \psi(y) - \phi(0) + \int_0^x \int_0^y v_n(\xi,\eta)\, d\xi\, d\eta,$$

where $v_n(\xi,\eta) \in F(\xi,\eta,z_n(\xi,\eta))$ for a.a. $(\xi,\eta) \in Q$. From (4) it follows that the sequence (2) converges for each $(x,y) \in Q$. Since $L_1(Q,R^d)$ is weakly complete, by Lemma 3 there exists a $v_0 \in L_1(Q,R^d)$ such that $\{v_n\}$ converges weakly to v_0. By Mazur's Theorem there exists a sequence $\{w_n\}$ of finite convex combinations of v_n's

$$w_n = \sum_{i=0}^{k(n)} \alpha_i^n v_{n+i}, \quad \left(\alpha_i^n \geqq 0, \sum_{i=0}^{k(n)} \alpha_i^n = 1\right),$$

such that

$$\iint_Q |w_n(\xi,\eta) - v_0(\xi,\eta)|\, d\xi\, d\eta \to 0 \quad \text{as} \quad n \to +\infty.$$

Thus, passing to a subsequence (without change of notation), we can assume that $w_n(x,y) \to v_0(x,y)$ for each $(x,y) \in Q \backslash Q_0$, where $\mu(Q_0) = 0$. Let $\tilde{Q} \supset Q_0$, $\mu(\tilde{Q}) = 0$, be such that for every $(x,y) \in Q \backslash \tilde{Q}$, the multifunction $F(x,y,\cdot)$ is u.s.c. and, moreover, $v_n(x,y) \in F(x,y,z_n(x,y))$ for $n = 1,2,\ldots$.

Let $(x,y) \in Q \backslash \tilde{Q}$ and let $\varepsilon > 0$. Since $F(x,y,.)$ is u.s.c. at $z_0(x,y)$, there exists $n_0 = n_0(x,y,\varepsilon) \in \mathbb{N}$ such that $v_n(x,y) \in F(x,y,z_0(x,y)) + \varepsilon B$ for every $n \geqq n_0$. This implies $w_n(x,y) \in F(x,y,z_0(x,y)) + \varepsilon B$ for $n \geqq n_0$. From this we deduce that $v_0(x,y) \in F(x,y,z_0(x,y))$. Since (x,y) is arbitrary in $Q \backslash \tilde{Q}$ it is proved that $v_0(x,y) \in F(x,y,z_0(x,y))$ for a.a. $(x,y) \in Q$. Moreover, from (4) for each $(x,y) \in Q$ we have

$$\sum_{i=0}^{k(n)} \alpha_i^n z_{n+i}(x,y) = \phi(x) + \psi(y) - \phi(0) + \int_0^x \int_0^y w_n(\xi,\eta)\, d\xi\, d\eta$$

and so, letting $n \to +\infty$, we get

$$z_0(x,y) = \phi(x) + \psi(y) - \phi(0) + \int_0^x \int_0^y v_0(\xi,\eta)\, d\xi\, d\eta.$$

Hence $z_0 \in \zeta_F$ and the compactness of ζ_F is established.

It remains to prove that ζ_F is contractible. Let u_0 be the (unique) solution of the Darboux problem $z_{xy} = g(x,y,z)$, $z(x,0) = \phi(x)$, $z(0,y) = \psi(y)$. Let $u \in \zeta_F$ be arbitrary. For

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$t\in[0,1)$ consider the Darboux problem

$$\begin{aligned} &z_{xy}=g(x,y,z)\\ &z(x,t)=u(x,t), \qquad z(t,y)=u(t,y), \qquad (x,y)\in Q_t \end{aligned} \tag{5}$$

where $Q_t=[t,1]\times[t,1]$. Denote by $z^{(t)}:Q_t\to R^d$ the (unique) solution of problem (5).

For $t\in[0,1)$ define $u^{(t)}:Q\to R^d$ by

$$u^{(t)}(x,y)=\begin{cases} z^{(t)}(x,y), & \text{if } (x,y)\in Q_t\\ u(x,y), & \text{if } (x,y)\in Q\backslash Q_t.\end{cases}$$

Moreover, set $u^{(1)}=u$. Observe that $u^{(0)}=u_0$.

We claim that for every $t\in[0,1]$, $u^{(t)}$ is a solution of problem (3). Indeed, let $t\in[0,1]$. For every $(x,y)\in Q\backslash Q_t$ we have

$$u^{(t)}(x,y)=u(x,y)=\phi(x)+\psi(y)-\phi(0)+\int_0^x\int_0^y v(\xi,\eta)\,d\xi\,d\eta, \tag{6}$$

where $v(\xi,\eta)\in F(\xi,\eta,u^{(t)}(\xi,\eta))$ for a.a. $(\xi,\eta)\in Q\backslash Q_t$.

For $(x,y)\in Q_t$ we have

$$u^{(t)}(x,y)=z^{(t)}(x,y)=u(x,t)+u(t,y)-u(t,t)+\int_t^x\int_t^y v^{(t)}(\xi,\eta)\,d\xi\,d\eta, \tag{7}$$

where $v^{(t)}(\xi,\eta)=g(\xi,\eta,z^{(t)}(\xi,\eta))\in F(\xi,\eta,z^{(t)}(\xi,\eta))$ and so $v^{(t)}(\xi,\eta)\in F(\xi,\eta,u^{(t)}(\xi,\eta))$ for a.a. $(\xi,\eta)\in Q_t$. From (7), by virtue of (6), we have

$$\begin{aligned} u^{(t)}(x,y)=\;&\phi(x)+\psi(t)-\phi(0)+\int_0^x\int_0^t v(\xi,\eta)\,d\xi\,d\eta\\ &+\phi(t)+\psi(y)-\phi(0)+\int_0^t\int_0^y v(\xi,\eta)\,d\xi\,d\eta\\ &-\phi(t)-\psi(t)+\phi(0)-\int_0^t\int_0^t v(\xi,\eta)\,d\xi\,d\eta+\int_t^x\int_t^y v^{(t)}(\xi,\eta)\,d\xi\,d\eta\\ =\;&\phi(x)+\psi(y)-\phi(0)+\int_0^x\int_0^y [\chi_{Q\backslash Q_t}(\xi,\eta)v(\xi,\eta)+\chi_{Q_t}(\xi,\eta)v^{(t)}(\xi,\eta)]\,d\xi\,d\eta, \end{aligned}$$

where χ_A denotes the characteristic function of A. It follows that $u^{(t)}$ is a solution of (3). Thus $u^{(t)}\in\zeta_F$ for each $u\in\zeta_F$ and $t\in I$.

Now define the function $h:I\times\zeta_F\to\zeta_F$ by $h(t,u)=u^{(t)}$. Suppose that $I\times\zeta_F$ is given the metric $\max\{|t_1-t_2|, \|u_1-u_2\|\}$, $(t_1,u_1),(t_2,u_2)\in I\times\zeta_F$ ($\|u_1-u_2\|=\max_{(x,y)\in Q}|u_1(x,y)-u_2(x,y)|$). We are going to prove that h is continuous.

Under our assumptions ζ_F is a bounded subset of $C(Q,R^d)$, thus there is a constant $m>0$ such that for every $u\in\zeta_F$ one has $u(x,y)\in mB$, $(x,y)\in Q$. Since the set $Q\times mB$ is compact and convex, the restriction of the function g to $Q\times mB$ is lipschitzian with some constant $L>0$.

Let $(\tilde{t},\tilde{u})\in I\times\zeta_F$. Let $\varepsilon>0$ and choose $0<\delta<\varepsilon/(7e^L)$. Let $\tau>0$ be so that $\iint_\Delta \alpha(\xi,\eta)\,d\xi\,d\eta<\delta$, where $\Delta=\{(x,y)\in Q\,|\,x,y\in[\tilde{t}-\tau,\tilde{t}+\tau]\}$. Let $(t,u)\in I\times\zeta_F$ be such that $|t-\tilde{t}|<\tau$, $\|u-\tilde{u}\|<\delta$. Let $t>\tilde{t}$ (the proof is similar when $t<\tilde{t}$).

Suppose $(x,y)\in Q_t$. As

$$h(t,u)(x,y)=u(x,t)+u(t,y)-u(t,t)+\int_t^x\int_t^y g(\xi,\eta,h(t,u)(\xi,\eta))\,d\xi\,d\eta$$

$$h(\tilde{t},\tilde{u})(x,y)=\tilde{u}(x,\tilde{t})+\tilde{u}(\tilde{t},y)-\tilde{u}(\tilde{t},\tilde{t})+\int_{\tilde{t}}^x\int_{\tilde{t}}^y g(\xi,\eta,h(\tilde{t},\tilde{u})(\xi,\eta))\,d\xi\,d\eta \tag{8}$$

we have

$$\begin{aligned}|h(t,u)(x,y)-h(\tilde{t},\tilde{u})(x,y)| &\leqq |u(x,t)-\tilde{u}(x,\tilde{t})|+|u(t,y)-\tilde{u}(\tilde{t},y)|\\ &+|u(t,t)-\tilde{u}(\tilde{t},\tilde{t})|+\iint_\Delta \alpha(\xi,\eta)\,d\xi\,d\eta\\ &+\int_t^x\int_t^y |g(\xi,\eta,h(t,u)(\xi,\eta))-g(\xi,\eta,h(\tilde{t},\tilde{u})(\xi,\eta))|\,d\xi\,d\eta\\ &<2\delta+2\delta+2\delta+\delta+L\int_t^x\int_t^y |h(t,u)(\xi,\eta)-h(\tilde{t},\tilde{u})(\xi,\eta))|\,d\xi\,d\eta.\end{aligned}$$

From this, using Gronwall's inequality, we obtain $|h(t,u)(x,y)-h(\tilde{t},\tilde{u})(x,y)|\leqq 7\delta e^L<\varepsilon$.

Suppose $(x,y)\in Q_{\tilde{t}}\backslash Q_t$. As $h(\tilde{t},\tilde{u})(x,y)$ is still given by (8) while

$$h(t,u)(x,y)=u(x,y)=u(x,\tilde{t})+u(\tilde{t},y)-u(\tilde{t},\tilde{t})+\int_{\tilde{t}}^x\int_{\tilde{t}}^y g(\xi,\eta,u(\xi,\eta))\,d\xi\,d\eta$$

we have

$$\begin{aligned}|h(t,u)(x,y)-h(\tilde{t},\tilde{u})(x,y)| &\leqq |u(x,\tilde{t})-\tilde{u}(x,\tilde{t})|+|u(\tilde{t},y)-\tilde{u}(\tilde{t},y)|\\ &+|u(\tilde{t},\tilde{t})-\tilde{u}(\tilde{t},\tilde{t})|+2\iint_\Delta \alpha(\xi,\eta)\,d\xi\,d\eta<\delta+\delta+\delta+2\delta=5\delta<\varepsilon.\end{aligned}$$

Finally, if $(x,y)\in Q\backslash Q_{\tilde{t}}$ we have $|h(t,u)(x,y)-h(\tilde{t},\tilde{u})(x,y)|=|u(x,y)-\tilde{u}(x,y)|<\delta<\varepsilon$. Hence $|h(t,u)(x,y)-h(\tilde{t},\tilde{u})(x,y)|<\varepsilon$ for every $(x,y)\in Q$, which implies $\|h(t,u)-h(\tilde{t},\tilde{u})\|\leqq\varepsilon$. This shows that h is continuous at $(\tilde{t},\tilde{u})$. As $(\tilde{t},\tilde{u})$ is arbitrary, h is continuous on $I\times\zeta_F$. Moreover, $h(0,u)=u_0$ and $h(1,u)=u$, for every $u\in\zeta_F$. Hence ζ_F is contractible and the proof of Lemma 4 is complete.

3. Proof of the Theorem

By Lemma 1, for every $k\in\mathbb{N}$ there is a locally lipschitzian function $g_k:Q\times R^d\to R^d$ such that

$$\iint_Q \sup_{z\in R^d}|g_k(\xi,\eta,z)-f(\xi,\eta,z)|\,d\xi\,d\eta \leqq \frac{1}{2^k}.$$

For $n\in\mathbb{N}$ define $\tilde{\lambda}_n:Q\to[0,+\infty]$ by

$$\tilde{\lambda}_n(x,y)=\sum_{k\geqq n}\sup_{z\in R^d}|g_k(x,y,z)-f(x,y,z)|.$$

Note that each $\tilde{\lambda}_n$ is integrable on Q. Consequently there is a null set $Q_0\subset Q$ such that $\tilde{\lambda}_n(x,y)$ is finite for every $(x,y)\in Q\backslash Q_0$, and every $n\in\mathbb{N}$. Define $\lambda_n:Q\to\mathbb{R}$ by

$$\lambda_n(x,y)=\begin{cases}\tilde{\lambda}_n(x,y), & \text{if } (x,y)\in Q\backslash Q_0\\ 0, & \text{if } (x,y)\in Q_0.\end{cases}$$

For $n\in\mathbb{N}$ define the multifunction $G_n:Q\times R^d\to\mathscr{K}$ by

$$G_n(x,y,z)=f(x,y,z)+\lambda_n(x,y)B.$$

Clearly G_n satisfies hypotheses (H_1), (H_2), (H_3) (the latter with $\alpha(x,y)+\lambda_n(x,y)$ in the place of $\alpha(x,y)$). Moreover $g_n(x,y,z)\in G_n(x,y,z)$ for each $(x,y,z)\in(Q\backslash Q_0)\times R^d$.

Consider the problem

$$\begin{gathered} z_{xy}\in G_n(x,y,z)\\ z(x,0)=\phi(x),\qquad z(0,y)=\psi(y). \end{gathered}\tag{9}$$

Let ζ_{G_n} denote the set of all solutions $z:Q\to R^d$ of problem (9). By virtue of Lemma 4 (with $g=g_n$, $F=G_n$) the set ζ_{G_n} is nonempty, compact and contractible. Clearly $\zeta_{G_1}\supset\zeta_{G_2}\supset\ldots$, for $G_1(x,y,z)\supset G_2(x,y,z)\supset\ldots$ for each $(x,y,z)\in Q\times R^d$. By Lemma 2, $\tilde{\zeta}=\bigcap_{n=1}^{\infty}\zeta_{G_n}$ is an R_δ-set in $C(Q,R^d)$. To finish the proof it suffices to show that $\zeta_f=\tilde{\zeta}$.

It is obvious that $\zeta_f\subset\tilde{\zeta}$. To see the reverse inclusion suppose that $z\in\tilde{\zeta}$. Let $n\in\mathbb{N}$. For each $(x,y)\in Q$ we have

$$z(x,y)=\phi(x)+\psi(y)-\phi(0)+\int_0^x\int_0^y v_n(\xi,\eta)\,d\xi\,d\eta,$$

where $v_n(\xi,\eta)\in G_n(\xi,\eta,z(\xi,\eta))$ for a.a. $(\xi,\eta)\in Q$. Hence $v_n(\xi,\eta)=f(\xi,\eta,z(\xi,\eta))+w_n(\xi,\eta)$, where w_n is a measurable function satisfying $w_n(\xi,\eta)\in\lambda_n(\xi,\eta)B$ for a.a. $(\xi,\eta)\in Q$.

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Consequently we have

$$\left|z(x,y)-\phi(x)-\psi(y)+\phi(0)-\int_0^x\int_0^y f(\xi,\eta,z(\xi,\eta))\,d\xi\,d\eta\right|$$

$$\leqq \int_0^x\int_0^y \lambda_n(\xi,\eta)\,d\xi\,d\eta \leqq \frac{1}{2^{n-1}}.$$

Since $n\in\mathbb{N}$ is arbitrary, we conclude that $z\in\zeta_f$. This completes the proof.

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Dipartimento di Matematica
Università di Roma II
Via O. Raimondo
00173 Roma
Italy

Instytut Matematyki AGH
Al. Mickiewicza 30
30-059 Kraków
Poland