9

Hulls and Husks

Given a coherent sheaf F over a proper scheme, the quot-scheme parametrizes all quotients $F \twoheadrightarrow Q$. In many applications, it is necessary to understand not only surjections $F \twoheadrightarrow Q$ but also "almost surjections" $F \to G$. Such objects are called *quotient husks*. Special cases appeared in Kollár (2008a); Pandharipande and Thomas (2009); Alexeev and Knutson (2010); and Kollár (2011b). In this chapter, we study quotient husks, prove that they have a fine moduli space QHusk(F), and then apply this to families of hulls.

The notion of the *hull* of a coherent sheaf F is the generalization of the concept of reflexive hull of a module over a normal domain. In Section 9.1 we discuss the absolute case, denoted usually by $F^{[**]}$, and in Section 9.2 the relative case, denoted by F^H. For many applications, the key is the following.

Question 9.1 Let $f\colon X \to S$ be a proper morphism and F a coherent sheaf on X. Do the hulls $F_s^{[**]}$ of the fibers F_s form a coherent sheaf that is flat over S?

If the answer is yes, the resulting sheaf is called the *universal hull* of F over S. Local criteria for its existence are studied in Section 9.3.

In order to get global criteria, husks and quotient husks are defined in Section 9.4. In Section 9.5, the first main result of the Chapter proves that if $X \to S$ is projective and F is a coherent sheaf on X then the functor of all quotient husks with a given Hilbert polynomial has a fine moduli space $\mathrm{QHusk}_p(X)$, which is a proper algebraic space over S. The proof closely follows the arguments given in Kollár (2008a).

This is used in a global study of hulls in Section 9.6. A third answer to our question is given in Section 9.7 in terms of a decomposition of S into locally closed subschemes. Local versions of these results are studied in Section 9.8.

Assumptions In this chapter we are mostly interested in schemes of finite type over an arbitrary base scheme.

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However, the results of Section 9.1 work for Noetherian schemes that have a dimension function dim( ) such that closed points have dimension 0, and if $W_1 \subsetneq W_2$ is a maximal (with respect to inclusion) irreducible subscheme of an irreducible $W_2 \subset X$, then $\dim W_1 = \dim W_2 - 1$. (That is, X is catenary (Stacks, 2022, tag 02I0).) This holds for schemes of finite type over a local CM scheme; see Stacks (2022, tags 00NM and 02JT).

9.1 Hulls of Coherent Sheaves

We use the results on S_2 sheaves, to be discussed in Section 10.1.

Let X be an integral, normal scheme and F a coherent sheaf on X. The reflexive hull of F is the double dual $F^{**} := \mathcal{H}om_X(\mathcal{H}om_X(F, \mathscr{O}_X), \mathscr{O}_X)$. We would like to extend this notion to arbitrary schemes and arbitrary coherent sheaves. For this, the key properties of the reflexive hull are the following:

- F^{**} is S_2, and
- F^{**} is the smallest S_2 sheaf containing $F/(\text{torsion})$.

These are the properties that we use to define the hull of a sheaf. Note, however, that for this, we need to agree what the "torsion subsheaf" of a sheaf should be. Two natural candidates, $\text{emb}(F)$ and $\text{tors}(F)$, are discussed in (10.1).

Here we work with $\text{tors}(F)$, the largest subsheaf whose support has dimension $< \dim F$. An advantage is that $\text{pure}(F) := F/\text{tors}(F)$ is pure dimensional; but one needs the dimension function to be reasonable. A theory of hulls using $\text{emb}(F)$ is developed in Kollár (2017).

A useful property of pure sheaves is the following.

Lemma 9.2 *Let $p\colon X \to Y$ be a finite morphism and F a coherent sheaf on X. Then F is pure and S_m iff p_*F is pure and S_m.*

Proof The last remark of (10.2) implies that the depth is preserved by push-forward. Thus the only question is whether (co)dimension is preserved or not; this is where our assumptions on the dimension function come in. □

Definition 9.3 (Hull of a sheaf) Let X be a scheme and F a coherent sheaf on X. Set $n = \dim F$. The *hull* of F is a coherent sheaf $F^{[**]}$ together with a map $q\colon F \to F^{[**]}$ such that

(9.3.1) $\text{Supp}(\ker q)$ has dimension $\leq n-1$,

(9.3.2) $\text{Supp}(\text{coker}\, q)$ has dimension $\leq n-2$, and

(9.3.3) $F^{[**]}$ is pure and S_2.

We sometimes say *S_2-hull* or *pure hull* if we want to emphasize these properties. We see below that a hull is unique and it exists if X is excellent.

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By definition, $F^{[**]} = (F/\operatorname{tors}(F))^{[**]}$, hence it is enough to construct hulls of pure, coherent sheaves.

The notation $F^{[**]}$ is chosen to emphasize the close connection between the hull and the reflexive hull F^{**}; see (9.4). We introduce a relative version, denoted by F^H in (9.8).

The following property is clear from the definition.

(9.3.4) Let G be a pure, coherent, S_2 sheaf and $F \subset G$ a subsheaf. Then $G = F^{[**]}$ iff $\dim(G/F) \leq \dim G - 2$.

From (9.2) and (10.10) we obtain the following base change properties of hulls.

(9.3.5) Let $p\colon X \to Y$ be a finite morphism. Then $p_*(F^{[**]}) = (p_*F)^{[**]}$.

(9.3.6) Let $g\colon Z \to X$ be flat, pure dimensional, with S_2 fibers. Then there is a natural isomorphism $g^*(F^{[**]}) = (g^*F)^{[**]}$.

Proposition 9.4 *Let X be an irreducible, normal scheme and F a torsion free coherent sheaf on X. Then $F^{[**]} = F^{**} := \mathcal{H}om_X(\mathcal{H}om_X(F, \mathcal{O}_X), \mathcal{O}_X)$.*

Proof F is locally free outside a codimension ≥ 2 subset $Z \subset X$. Thus the natural map $F \to F^{**}$ is an isomorphism over $X \setminus Z$. Since F^{**} is S_2 by (10.8), it satisfies the assumptions of (9.3). □

This can be used to construct the hull over schemes of finite type over a field. Indeed, we may assume that X is affine and $X = \operatorname{Supp} F$. By Noether normalization, there is a finite surjection $p\colon X \to \mathbb{A}^n$. Thus, by (9.3.5) and (9.4), $F^{[**]}$ can be identified with $(p_*F)^{**}$, as a $p_*\mathcal{O}_X$-module. Hulls also exist over excellent schemes; see Kollár (2017) for a more general result.

Proposition 9.5 *Let F be a pure, coherent sheaf on an excellent scheme X.*

(9.5.1) *There is a closed subset $Z \subset \operatorname{Supp} F$ of dimension $\leq \dim F - 2$ such that F is S_2 over $X \setminus Z$.*

(9.5.2) *Let $Z \subset \operatorname{Supp} F$ be any closed subset of dimension $\leq \dim F - 2$ such that F is S_2 over $U := X \setminus Z$. Then $F^{[**]} = j_*(F|_U)$, and, for every coherent sheaf G, every morphism $G|_U \to F|_U$ uniquely extends to $G \to F^{[**]}$.*

Proof The first claim follows from (10.27). To see (2), note that $j_*(F|_U)$ is coherent by (10.26), S_2 over U by assumption, and $\operatorname{depth}_Z j_*(F|_U) \geq 2$ by (10.6). Thus $j_*(F|_U)$ is a hull of F and we get $\tau : G \to j_*(G|_U) \to j_*(F|_U)$.

Let $F^{[**]}$ be any hull of F. Then $F^{[**]}|_U$ is a hull of $F|_U$; let $W \subset U$ be the support of their quotient. Then $\operatorname{codim}_X W \geq 2$ hence $F^{[**]}|_U = F|_U$ by (10.6.2). Thus we get a map $F^{[**]} \to j_*(F|_U)$. Applying (10.6) again gives that $F^{[**]} = j_*(F|_U)$. □

Corollary 9.6 *Let* $0 \to F_1 \to F_2 \to F_3$ *be an exact sequence of coherent sheaves of the same dimension. Then the hulls also form an exact sequence* $0 \to F_1^{[**]} \to F_2^{[**]} \to F_3^{[**]}$. □

9.7 (Quasi-coherent hulls) Following (9.5.2), one should define the hull of a torsion-free, quasi-coherent sheaf F as $F^{[**]} := \varinjlim (j_Z)_*(F|_{X\setminus Z})$, where Z runs through all codimension ≥ 2 closed subsets of $\operatorname{Supp} F$. It is easy to see that $F^{[**]}$ is S_2, as defined in Grothendieck (1968, exp.III).

9.2 Relative Hulls

Next we develop a relative version of the notion of hull for coherent sheaves on a scheme X over a base scheme S.

In the absolute case, the hull is an S_2 sheaf that we can associate to any coherent sheaf on X, in particular, the hull does not have embedded points.

In the relative case, assume for simplicity that $f\colon X \to S$ is smooth; then $\mathscr{O}_X$ should be its own "relative hull." Note, however, that the structure sheaf $\mathscr{O}_X$ has no embedded points if and only if the base scheme S has no embedded points. Thus if we want to say that $\mathscr{O}_X$ is its own relative hull then we have to distinguish embedded points that are caused by S (these are allowed) from other embedded points (these are forbidden).

The distinction between these two types of embedded points seems to be meaningful only if F is generically flat (3.26).

Definition 9.8 (Relative hull) Let $f\colon X \to S$ be a morphism of finite type and F a coherent sheaf on X. Let n be the relative dimension of $\operatorname{Supp} F \to S$. A *hull* (or *relative hull*) of F over S is a coherent sheaf F^H together with a morphism $q\colon F \to F^H$ such that[1]

(9.8.1) $\operatorname{Supp}(\ker q) \to S$ has fiber dimension $\leq n-1$,

(9.8.2) $\operatorname{Supp}(\operatorname{coker} q) \to S$ has fiber dimension $\leq n-2$,

(9.8.3) there is a closed subset $Z \subset X$ with complement $U := X \setminus Z$ such that $Z \to S$ has fiber dimension $\leq n-2$, $(F/\ker q) \to F^H$ is an isomorphism over U, $F^H|_U$ is flat over S with pure, S_2 fibers, and $\operatorname{depth}_Z F^H \geq 2$.

Note that $\operatorname{Supp}(\operatorname{coker} q) \subset Z$ by (3), hence in fact (3) implies (2). We state the latter separately to emphasize the parallels with (9.3).

Note that, while the hull always exists, the relative hull frequently does not; see (9.13) for a criterion. We have the following obvious comparisons.

[1] F^{h} would have been more consistent, but it is frequently used to denote the Henselization.

Claim 9.8.4 Assume that F^H exists and S is reduced. Then $(F^H)_g = (F_g)^{[**]}$ for every generic point $g \in S$. □

Claim 9.8.5 Assume that F^H exists and S is S_2. Then $F^H = F^{[**]}$. □

The converse fails. As an example, let $f\colon X := \mathbb{A}^2_{st} \to S := \mathbb{A}^1_t$ be the projection and $F \subset \mathscr{O}_X$ the ideal sheaf of the point $(0,0)$. Then $F^{[**]} = \mathscr{O}_X$, but $F \to \mathscr{O}_X$ is not a relative hull since $\operatorname{coker}(F \to \mathscr{O}_X)$ has codimension 1 on X_0.

Lemma 9.9 *Let $(0, T)$ be the spectrum of a DVR, $f\colon X \to T$ a morphism of finite type, and $q\colon F \to G$ a map between pure, coherent sheaves on X that are flat over T. Then G is a relative hull of F iff G_g is the hull of F_g, G_0 is S_1, and $q_0\colon F_0 \to G_0$ is an isomorphism outside a subset $Z_0 \subset \operatorname{Supp} G_0$ of codimension ≥ 2.*

Proof Assume that $G = F^H$ and let $Z \subset X$ be as in (9.8). By assumption, $G|_{X\setminus Z}$ has S_2 fibers thus $G|_{X\setminus Z}$ is S_2. Hence G is S_2 since $\operatorname{depth}_Z G \geq 2$ and so G_0 is S_1 and $q_0\colon F_0 \to G_0$ is an isomorphism outside $X_0 \cap Z$.

Conversely, if (1–3) hold then G is S_2 by (1–2). By (9.5) there is a closed subset $Z_1 \subset X_0$ of codimension ≥ 2 such that F_0 is S_2 over $X_0 \setminus Z_1$. Thus $q\colon F \to G$ satisfies the conditions (9.8.1–3) where Z is the union of three closed sets: Z_0, Z_1 and the closure of $\operatorname{Supp}(\operatorname{coker} q_g)$. □

Corollary 9.10 *Let $(0, T)$ be the spectrum of a DVR, $f\colon X \to T$ a morphism of finite type and F a pure, coherent sheaf on X that is flat over T. Then $F = F^H$ $\Leftrightarrow$ F is S_2 $\Leftrightarrow$ F_g is S_2 and F_0 is S_1.* □

Corollary 9.11 (Bertini theorem for relative hulls) *Let $(0, T)$ be the spectrum of a DVR, $X \subset \mathbb{P}^n_T$ a quasi-projective scheme and F a coherent sheaf on X with relative hull $q\colon F \to F^H$. Then $q|_L\colon F|_L \to F^H|_L$ is the relative hull of $F|_L$ for a general hyperplane $L \subset \mathbb{P}^n_T$.*

Proof We use (10.18) and (10.19) both for the special fiber X_0 and the generic fiber X_g. We get open subsets $U_0 \subset \check{\mathbb{P}}^n_0$ and $U_g \subset \check{\mathbb{P}}^n_g$ such that $F^H|_{L_0}$ is S_1 for $L_0 \in U_0$, $(F/\operatorname{tors}(F))|_{L_0} = (F|_{L_0})/\operatorname{tors}(F|_{L_0})$ for $L_0 \in U_0$, the natural map $(F|_{L_0})/\operatorname{tors}(F|_{L_0}) \to G_{L_0}$ is an isomorphism outside a subset of codimension ≥ 2 for $L_0 \in U_0$, and $F^H|_{L_g}$ is the hull of $F|_{L_g}$ for $L_g \in U_g$.

Let $W_T \subset \check{\mathbb{P}}^n_T$ denote the closure of $\check{\mathbb{P}}^n_g \setminus U_g$. For dimension reasons, W_T does not contain $\check{\mathbb{P}}^n_0$. Thus any hyperplane corresponding to a section through a point of $U_0 \setminus W_T$ works. □

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Definition 9.12 (Vertical purity) Let $g\colon X \to S$ be a finite type morphism and G a coherent sheaf on X. We say that G is *vertically* pure of dimenion n, if for every $W \in \operatorname{Ass}(G)$, every fiber of $g|_W\colon W \to S$ is either empty or has pure dimension n,

Let F be a coherent sheaf on X such that $\operatorname{Supp} F \to S$ has relative dimension n. Let $\{W_i\colon i \in I\} \subset \operatorname{Ass}(F)$ be those associated subschemes for which the generic fiber of $g|_{W_i}\colon W_i \to S$ has dimension $< n$. Set $Z := \cup_{i\in I} W_i$. The *vertically pure quotient* of F is $\operatorname{vpure}(F) := F/\operatorname{tors}_Z(F)$, using the notation of (10.1). Note that if $q\colon F \to F^H$ is a relative hull, then $\operatorname{vpure}(F) = \operatorname{im} q$.

Next we state the precise conditions needed for the existence of relative hulls. Then we show that a relative hull is unique, generalizing (9.5).

Lemma 9.13 *Let $f\colon X \to S$ be a morphism of finite type and F a coherent sheaf on X. Let n denote the maximum fiber dimension of* $\operatorname{Supp} F \to S$*. Then F has a relative hull iff*

(9.13.1) *F is generically flat (3.26), and*

(9.13.2) *there is an open $j : U \hookrightarrow X$ such that* $\operatorname{vpure}(F)|_U$ *is a flat family of S_2 sheaves and* $(\operatorname{Supp} F \setminus U) \to S$ *has fiber dimension $\leq n - 2$.*

If this holds, then

(9.13.3) *$F^H = j_*(\operatorname{vpure}(F)|_U)$ is the unique relative hull of F over S, and*

(9.13.4) *any $\tau_U : G|_U \to F|_U$ uniquely extends to $\tau : G \to F^H$.*

Proof If $q\colon F \to F^H$ is a relative hull, then $\operatorname{vpure}(F) = \operatorname{im} q$, so the conditions (9.13.1–2) are satisfied.

Conversely, if the conditions (9.13.1–2) are satisfied, then we can harmlessly replace F by $\operatorname{vpure}(F)$. Then $j_*(F|_U)$ is coherent by (10.26), $F \to j_*(F|_U)$ is an isomorphism over U by construction, and $\operatorname{depth}_Z j_*(F|_U) \geq 2$ by (10.6).

The last claim follows from the universal property of the push-forward and it implies that F^H is independent of the choice of U. □

Corollary 9.14 *Let $f\colon X \to S$ be a morphism of finite type and G a coherent sheaf on X that is flat over S with pure, S_2 fibers of dimension n. Let $F \subset G$ be a subsheaf. Then $G = F^H$ iff the fiber dimension of* $\operatorname{Supp}(G/F) \to S$ *is $\leq n - 2$.* □

9.3 Universal Hulls

For many applications, a key question is to understand the behavior of relative hulls under a base change.

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Notation 9.15 Let $f\colon X \to S$ be a morphism of finite type and F a coherent sheaf satisfying (9.13.1–2). As in (3.18.1), for any $g\colon T \to S$ we get

$$X \stackrel{g_X}{\longleftarrow} X_T := X \times_S T \stackrel{f_T}{\longrightarrow} T.$$

Set $U_T := g_X^{-1}(U)$ and $F_T := g_X^* F$. The relative hulls F^H and $(F_T)^H$ exists, and, as in (3.27.2), we have *restriction maps*

$$r_T^S\colon (F^H)_T \to (F_T)^H. \tag{9.15.1}$$

Definition 9.16 Let $f\colon X \to S$ be a morphism of finite type and F a coherent sheaf on X satisfying (9.13.1–2).

We say that F^H is a *universal hull* of F at $x \in X$ if the restriction map r_T^S (9.15.2) is an isomorphism along $g_X^{-1}(x)$ for every $g\colon T \to S$. F^H is a *universal hull* of F if this holds at every $x \in X$. Equivalently, iff the functor $F \mapsto F^H$ *commutes with base change.*

We say that $F \mapsto F^H$ is *universally flat* if $(F_T)^H$ is flat over T for every $g\colon T \to S$.

The following theorem gives several characterizations of universal hulls.

Theorem 9.17 *Let $f\colon X \to S$ be a morphism of finite type and F a coherent sheaf on X that has a relative hull F^H over S. The following are equivalent.*

(9.17.1) *F^H is a universal hull of F.*

(9.17.2) *$F \mapsto F^H$ is universally flat.*

(9.17.3) *F^H is flat over S with pure, S_2 fibers.*

(9.17.4) *F^H is flat over S with pure, S_2 fibers over closed points of S.*

(9.17.5) *$r_s^S\colon F^H \to (F_s)^H$ is surjective for every closed point $s \in S$.*

(9.17.6) *$(F_A)^H$ is a universal hull of F_A for every Artinian scheme $A \to S$.*

Proof The only obvious implications are (3) ⇒ (4) and (1) ⇒ (5), but (4) ⇒ (3) directly follows from the openness of the S_2-condition (10.11).

Note that the properties in (3) are preserved by base change, thus $(F^H)_T$ is flat over T and $\big((F^H)_T\big)_t$ is S_2 for every point $t \in T$. By (9.14) this implies that $(F^H)_T$ is the relative hull of F_T. Therefore $(F^H)_T = (F_T)^H$, so $F \mapsto F^H$ is universally flat and commutes with base change. That is, (3) ⇒ (2) and (3) ⇒ (1) both hold.

If (4) holds, then $\big(F^H\big)_s = (F_s)^H$ by (9.3.4), thus (4) ⇒ (5). Applying (10.71) to every localization of S at closed points shows that (5) ⇒ (4).

Next we show that (2) ⇒ (6). We may assume that $S = \operatorname{Spec} A$, where (A, m) is a local, Artinian ring. Choose the smallest $r \geq 0$ such that $m^{r+1} = 0$;

so $m^r \simeq \oplus_i A/m$, the sum of a certain number of copies of A/m. This gives an injection $j_r\colon \oplus_i F_s \hookrightarrow F$ which then extends to $j_r^H\colon \oplus_i (F_s)^H \hookrightarrow F^H$.

Since F^H is flat over A, the image $j_r^H(\oplus_i (F_s)^H)$ is also isomorphic to $(m^r)\otimes_A F^H$ which is the same as $\oplus_i (F^H)_s$. Thus $(F_s)^H = (F^H)_s$ and, by the above arguments, (2) implies the properties (1–5) for local, Artinian base schemes.

In order to see (6) ⇒ (5), we may replace S by its completion at s. For $r \in \mathbb{N}$ set $A_r := \operatorname{Spec}_S \mathscr{O}_S/m_s^r$. By base change we get $f_r\colon X_r \to A_r$ and $F_r := F|_{X_r}$. By assumption, $(F_r)^H$ is flat over A_r and we have proved that $F \mapsto F^H$ commutes with base change over Artinian schemes. Set $\tilde{F} := \varprojlim (F_r)^H$. Then $\tilde{F}$ is flat over S, coherent by Hartshorne (1977, II.9.3.A), agrees with F over U, and $\tilde{F} \to F_s^H$ is surjective. Thus $\tilde{F} = F^H$ by (9.14), giving (5). □

We can restate the characterization (9.17.3) as follows.

Corollary 9.18 *Let $f\colon X \to S$ be a morphism of finite type, $q\colon F \to G$ a map of coherent sheaves on X. Let n denote the maximum fiber dimension of* $\operatorname{Supp}(F) \to S$. *Then G is the universal hull of F over S iff the following hold.*

(9.18.1) *$q_s\colon F_s \to G_s$ is an isomorphism at all n-dimensional points of X_s for every $s \in S$.*

(9.18.2) *G is flat with purely n-dimensional, S_2 fibers over S, and*

(9.18.3) $\operatorname{Supp}(\operatorname{coker}(q)) \to S$ *has fiber dimension $\leq n-2$.* □

Combining (9.18) and (10.12) shows that a relative hull is a universal hull over a dense open subset of the base. Thus Noetherian induction gives the following. A much more precise form will be proved in (9.40).

Corollary 9.19 *Let $f\colon X \to S$ be a proper morphism and F a coherent sheaf on X. Then there is a locally closed decomposition $j\colon S' \to S$ such that $j_X^* F$ has a universal hull.* □

The following example illustrates several aspects of (9.17).

Example 9.20 Let $g\colon X \to S$ be a flat family of projective varieties, S reduced and connected, with g-ample line bundle L. As in (2.35), we get the relative affine cone $C_S(X) := \operatorname{Spec}_S \oplus_{m\in\mathbb{N}} g_*\mathscr{O}_X(m)$, with vertex $V \simeq S$. Note that $C_S(X) \setminus V$ is a $\mathbb{G}_m$-bundle over X, so flat over S. By contrast, $C_S(X)$ is flat over S iff $h^0(X_s, L_s^m)$ is independent of $s \in S$ for all $m \in \mathbb{N}$.

The simplest examples where h^0 jumps are given by taking $X = C \times \operatorname{Jac}(C)$ for some smooth curve C of genus ≥ 2 and L a universal line bundle of relative degree $0 < d < 2g-2$.

In these cases, the structure sheaf of $C_S(X)$ is its own relative hull, but it is not a universal hull.

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9.4 Husks of Coherent Sheaves

Definition 9.21 Let X be a scheme and F a coherent sheaf on X. An n-dimensional *quotient husk* of F is a quasi-coherent sheaf G together with a homomorphism $q\colon F \to G$ such that

(9.21.1) G is pure of dimension n and

(9.21.2) $q\colon F \to G$ is surjective at all generic points of $\operatorname{Supp} G$.

A quotient husk is called a *husk,* if $n = \dim F$ and

(9.21.3) $q\colon F \to G$ is an isomorphism at all n-dimensional points of X.

Note 9.21.4 If $h \in \operatorname{Ann}(F)$, then $hG \subset G$ is supported in dimension $< n$, thus it is 0. Therefore G is also an $\mathscr{O}_X/\operatorname{Ann}(F)$ sheaf, so we get the same husks if we replace X with any subscheme containing $\operatorname{Spec}_X(\mathscr{O}_X/\operatorname{Ann}(F))$.

Any coherent sheaf F has a maximal husk $M(F) := \varinjlim (j_Z)_*(F|_{X\setminus Z})$, where Z runs through all closed subsets of $\operatorname{Supp} F$ such that $\dim Z < \dim F$. If $\dim F \geq 1$ then $M(F)$ is never coherent, but it is the union of coherent husks.

Lemma 9.22 *Let F be a coherent sheaf on X and $q\colon F \to G$ an n-dimensional (quotient) husk of F.*

(9.22.1) *Let $g\colon X \to Z$ be a finite morphism. Then g_*G is an n-dimensional (quotient) husk of g_*F.*

(9.22.2) *Let $h\colon Y \to X$ be a flat morphism of pure relative dimension r with S_1 fibers. Then h^*G is an $(n+r)$-dimensional (quotient) husk of h^*F.*

Proof If g is a finite morphism and M is a sheaf then the associated primes of g_*M are the images of the associated primes of M. This implies (1). Similarly, if h is flat then the associated primes of h^*M are the preimages of the associated primes of M. Since h^*G is S_1 by (10.10), we get (2). □

9.23 (Bertini theorem for (quotient) husks) Let F be a coherent sheaf on a quasi-projective variety $X \subset \mathbb{P}^n$ and $q\colon F \to G$ a coherent (quotient) husk. Let $H \subset \mathbb{P}^n$ be a general hyperplane. Then $G|_H$ is pure by (10.18). If, in addition, H does not contain any of the associated primes of $\operatorname{coker} q$ then $q|_H\colon F|_H \to G|_H$ is also a (quotient) husk.

Definition 9.24 Let X be a scheme and F a coherent sheaf on X. Set $n := \dim F$. A husk $q\colon F \to G$ is called *tight* if $q\colon F/\operatorname{tors}(F) \hookrightarrow G$ is an isomorphism at all $(n-1)$-dimensional points of X.

Thus the hull $q\colon F \to F^{[**]}$ defined in (9.3) is a tight husk of F. We see below that the hull is the maximal tight husk.

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Lemma 9.25 *Let X be a scheme and F a coherent sheaf on X with hull $q\colon F \to F^{[**]}$. Let $r\colon F \to G$ be any tight husk. Then q extends uniquely to an injection $q_G\colon G \hookrightarrow F^{[**]}$. Therefore $F^{[**]}$ is the unique tight husk that is S_2.*

Proof After replacing F with $F/\operatorname{tors}(F)$ we may assume that F is pure. Set $Z := \operatorname{Supp}(\operatorname{coker} r) \cup \operatorname{Supp}(F^{[**]}/F)$. Then Z has codimension ≥ 2 and F is S_2 on $X \setminus Z$. Using (9.5.2) we get that $G \subset j_*(G|_{X\setminus Z}) = j_*(F|_{X\setminus Z}) = F^{[**]}$. If G is also S_2, then, (9.5.2) gives that $G = F^{[**]}$. □

Lemma 9.26 *Let X be a projective scheme, F a coherent sheaf of pure dimension n and $F \to G$ a quotient husk. The following are equivalent.*
(9.26.1) $G = F^{[**]}$.
(9.26.2) *G is S_2 and $\chi(X, F(t)) - \chi(X, G(t))$ has degree $\leq n-2$.*
(9.26.3) $\chi(X, F^{[**]}(t)) \equiv \chi(X, G(t))$ *(identical as polynomials).*

Proof The exact sequence $0 \to K \to F \to G \to Q \to 0$ defines K, Q and

$$\chi(X, F(t)) - \chi(X, G(t)) \equiv \chi(X, K(t)) - \chi(X, Q(t)).$$

Note that K has pure dimension n and $\dim Q \leq n-1$. If $G = F^{[**]}$ then $K = 0$ and $\dim Q \leq n-2$ which implies (2) and (1) ⇒ (3) is obvious.

Conversely, assume that $\chi(X, F(t)) - \chi(X, G(t))$ has degree $\leq n-2$. Since $\deg \chi(X, Q(t)) \leq n-1$, we see that $\deg \chi(X, K(t)) \leq n-1$. However, K has pure dimension n, thus in fact $K = 0$ and so G is a tight husk of F. If G is S_2 then (9.25) implies that $G = F^{[**]}$, hence (2) ⇒ (1).

Finally, if (3) holds, then $\chi(X, F(t)) - \chi(X, G(t))$ has degree $\leq n-2$, hence, as we proved, G is a tight husk of F. By (9.25.1) G is a subsheaf of $F^{[**]}$. Thus $G = F^{[**]}$ since they have the same Hilbert polynomials. □

Definition 9.27 (Husks over a base scheme) Let $f\colon X \to S$ be a morphism and F a coherent sheaf on X. A *quotient husk* of F over S is a quasi-coherent sheaf G on X, together with a homomorphism $q\colon F \to G$ such that
(9.27.1) G is flat and pure over S, and
(9.27.2) $q_s\colon F_s \to G_s$ is a quotient husk for every $s \in S$.
A quotient husk is called a *husk* if
(9.27.3) $q_s\colon F_s \to G_s$ is a husk for every $s \in S$.
We sometimes omit "over S" if our choice of S is clear from the context. The following properties are useful.
(9.27.4) Husks are preserved by base change. That is, let $q\colon F \to G$ be a (quotient) husk over S and $g\colon T \to S$ a morphism. Set $X_T := X \times_S T$ and let

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$g_X\colon X_T \to X$ be the first projection. Then $g_X^*q\colon g_X^*F \to g_X^*G$ is a (quotient) husk over T.

(9.27.5) Assume that f is proper and we have $q\colon F \to G$ where G is flat and pure over S. By (10.54.1) there is a largest open S° such that $q^\circ\colon F^\circ \to G^\circ$ is a quotient husk over $S^\circ \subset S$.

9.5 Moduli Space of Quotient Husks

Definition 9.28 Let $f\colon X \to S$ be a proper morphism and F a coherent sheaf on X. Let $\mathcal{QH}usk(F/S)(*)$ (resp. $\mathcal{H}usk(F/S)(*)$) be the functor that to an S-scheme $g\colon T \to S$ associates the set of all coherent quotient husks (resp. husks) of g_X^*F, where $g_X\colon T \times_S X \to X$ is the projection.

We write $\mathcal{QH}usk(F)$ and $\mathcal{H}usk(F)$ if the choice of S is clear.

By (10.54.1) $\mathcal{H}usk(F/S)(*)$ is an open subfunctor of $\mathcal{QH}usk(F/S)(*)$.

If f is projective, H is an f-ample divisor class and $p(t)$ is a polynomial, then $\mathcal{QH}usk_p(F/S)(*)$ (resp. $\mathcal{H}usk_p(F/S)(*)$) denote the subfunctors of all coherent quotient husks (resp. husks) of g_X^*F with Hilbert polynomial $p(t)$.

The main existence theorem of this section is the following.

Theorem 9.29 *Let $f\colon X \to S$ be a projective morphism and F a coherent sheaf on X. Let H be an f-ample divisor class and $p(t)$ a polynomial. Then $\mathcal{QH}usk_p(F/S)$ has a fine moduli space* $\mathrm{QHusk}_p(F/S) \to S$*, which is a proper algebraic space over S.*

When S is a point, the projectivity of $\mathrm{QHusk}_p(F)$ is proved in Lin (2015), see also Wandel (2015).

As we noted, $\mathcal{H}usk_p(F/S)$ is represented by an open subspace $\mathrm{Husk}_p(F/S) \subset \mathrm{QHusk}_p(F/S)$, which is usually not closed. There are, however, many important cases when $\mathrm{Husk}_p(F/S)$ is also proper over S.

Corollary 9.30 *Let $f\colon X \to S$ be a projective morphism and F a coherent sheaf that is generically flat over S (3.26). Let H be an f-ample divisor class and $p(t)$ a polynomial. Then $\mathcal{H}usk_p(F/S)$ has a fine moduli space* $\mathrm{Husk}_p(F/S) \to S$ *which is a proper algebraic space over S.*

The implication (9.29) ⇒ (9.30) is proved in (9.31), where we also establish the valuative criteria of properness and separatedness for $\mathcal{QH}usk(F/S)$.

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As a preliminary step, note that the problem is local on S, thus we may assume that S is affine. Then f, X, F are defined over a finitely generated subalgebra of $\mathscr{O}_S$, hence we may assume in the sequel that S is of finite type.

9.31 (The valuative criteria of separatedness and properness) More generally, we show that $\mathcal{QH}usk(F/S)$ satisfies the valuative criteria of separatedness and properness whenever f is proper.

Let T be the spectrum of an excellent DVR with closed point $0 \in T$ and generic point $t \in T$. Given $g\colon T \to S$, let $g_X\colon T \times_S X \to X$ denote the projection. We have the coherent sheaf $g_X^* F$ and, over the generic point, a quotient husk $q_t\colon g_X^* F_t \to g_X^* G_t$. We aim to extend it to a quotient husk $\tilde{q}\colon g_X^* F \to \tilde{G}$.

Let $K \subset g_X^* F$ be the largest subsheaf that agrees with $\ker q_t$ over the generic fiber. Then $g_X^* F/K$ is a coherent sheaf on X_T and none of its associated primes is contained in X_0. Thus $g_X^* F/K$ is flat over T. Let $Z_0 \subset X_0$ be the union of the embedded primes of $(g_X^* F/K)_0$.

By construction q_t descends to a morphism $q_t'\colon (g_X^* F/K)_t \hookrightarrow g_X^* G_t$. Let $Z_t \subset \operatorname{Supp}(g_X^* F/K)_t$ be the closed subset where q_t' is not an isomorphism and $Z_T \subset X_T$ its closure. Finally set $Z = Z_0 \cup (Z_T \cap X_0)$.

The restriction of the sheaf $g_X^* F/K$ to $X_T \setminus (Z_0 \cup Z_T)$ is flat and pure over T and $g_X^* G_t$ is pure on $X_t = X_T \setminus X_0$. Furthermore, when restricted to $X_T \setminus (X_0 \cup Z_T)$, both of these sheaves are naturally isomorphic to $g_X^* F/K$. Thus we can glue them to get a single sheaf G' defined on $X_T \setminus Z$ that is is flat and pure over T.

Let $j\colon X_T \setminus Z \hookrightarrow X_T$ be the injection. By (10.6.6), $\tilde{G} := j_* G'$ is the unique extension that is flat and pure over T, hence $\tilde{q}\colon g_X^* F \to g_X^* F/K \to \tilde{G}$ is the unique quotient husk extending $q_t\colon F_t \to G_t$. Thus $\mathcal{QH}usk(F/S)$ satisfies the valuative criteria of separatedness and properness.

If f is projective then $\tilde{G}_0$ has the same Hilbert polynomial as G_t.

Finally note that if F is generically flat over S and $q_t\colon g_X^* F_t \to g_X^* G_t$ is a husk then $K \subset g_X^* F$ is zero at the generic points of $X_0 \cap \operatorname{Supp} g_X^* F$, thus $\tilde{q}\colon g_X^* F \to g_X^* F/K \to \tilde{G}$ is a husk.

This shows that if F is generically flat over S then $\operatorname{Husk}(F/S)$ is closed in $\operatorname{QHusk}(F/S)$ hence (9.30) follows from (9.29).

9.32 (Construction of $\operatorname{QHusk}_p(F/S)$) We may assume that $X = \mathbb{P}^N_S$ for some N; the only consequence we actually need is that $f_* \mathscr{O}_X = \mathscr{O}_S$, and this holds after any base change.

We use the existence and basic properties of quot-schemes (9.33) and hom-schemes (9.34). Also, as we discuss in (9.35), there is fixed m such that $G_s(m)$ is generated by global sections and its higher cohomologies vanish for all

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quotient husks of $F_s \to G_s$ with Hilbert polynomial $p(t)$. Thus each $G_s(m)$ can be written as a quotient of $\mathscr{O}_{X_s}^{p(m)}$. Let

$$Q_{p(t)} := \mathrm{Quot}^{\circ}_{p(t)}(\mathscr{O}_X^{p(m)}) \subset \mathrm{Quot}(\mathscr{O}_X^{p(m)})$$

be the universal family of quotients $q_s\colon \mathscr{O}_{X_s}^{p(m)} \twoheadrightarrow M_s$ that have Hilbert polynomial $p(t)$, are pure, have no higher cohomologies and the induced map

$$q_s\colon H^0(X_s, \mathscr{O}_{X_s}^{p(m)}) \to H^0(X_s, M_s)$$

is an isomorphism. Openness of purity is the $m = 1$ case of (10.12), the other two properties are discussed in (9.35).

Let $\pi\colon Q_{p(t)} \to S$ be the structure map, $\pi_X\colon Q_{p(t)} \times_S X \to X$ the second projection and M the universal sheaf on $Q_{p(t)} \times_S X$.

By (10.54.1) the hom-scheme $\mathbf{Hom}(\pi_X^* F, M)$ (9.34) has an open subscheme $W_{p(t)}$ parametrizing maps from F to M that are surjective outside a subset of dimension $\le n-1$. Let $\sigma\colon W_{p(t)} \to Q_{p(t)}$ be the structure map and $\sigma_X\colon W_{p(t)} \times_S X \to Q_{p(t)} \times_S X$ the fiber product.

Note that $W_{p(t)}$ parametrizes triples

$$w := [F_w \stackrel{r_w}{\to} G_w \stackrel{q_w}{\twoheadleftarrow} \mathscr{O}_{X_w}^{p(m)}(-m)],$$

where $r_w\colon F_w \to G_w$ is a quotient husk with Hilbert polynomial $p(t)$ and $q_w(m)\colon \mathscr{O}_{X_w}^{p(m)} \to G_w(m)$ is a surjection that induces an isomorphism on the spaces of global sections.

Let $w' \in W_{p(t)}$ be another point corresponding to the triple

$$[F_{w'} \stackrel{r_{w'}}{\to} G_{w'} \stackrel{q_{w'}}{\twoheadleftarrow} \mathscr{O}_{X_{w'}}^{p(m)}(-m)]. \quad \text{such that} \quad [F_w \stackrel{r_w}{\to} G_w] \simeq [F_{w'} \stackrel{r_{w'}}{\to} G_{w'}].$$

The difference between w and w' comes from the different ways that we can write $G_w \simeq G_{w'}$ as quotients of $\mathscr{O}_{X_w}(-m)^{\bigoplus p(t)}$. Since we assume that the induced maps

$$q_w(m), q_{w'}(m)\colon H^0(X_w, \mathscr{O}_{X_w}^{p(m)}) \rightrightarrows H^0(X_w, G_w(m)) = H^0(X_w, G_{w'}(m))$$

are isomorphisms, the different choices of q_w and $q_{w'}$ correspond to different bases in $H^0(X_w, G_w(mH))$. Thus the fiber of $\mathrm{Mor}(*, W_{p(t)}) \to \mathcal{QH}usk_p(F/S)(*)$ over $\pi \circ \sigma(w) = \pi \circ \sigma(w') =: s \in S$ is a principal homogeneous space under the algebraic group $\mathrm{GL}(p(m), k(s)) = \mathrm{Aut}(H^0(X_s, G_s(m)))$.

Thus the group scheme $\mathrm{GL}(p(m), S)$ acts on $W_{p(t)}$ and, arguing as in (8.56),

$$\mathrm{QHusk}_p(F/S) = W_{p(t)}/\mathrm{GL}(p(m), S).$$

9.33 (Quot-schemes) Let $f\colon X \to S$ be a morphism and F a coherent sheaf on X. $\mathcal{Q}uot(F/S)(*)$ denotes the functor that to a scheme $g\colon T \to S$ associates

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the set of all quotients of g_X^*F that are flat over T with proper support, where $g_X\colon T\times_S X\to X$ is the projection.

If $F=\mathscr{O}_X$, then a quotient can be identified with a subscheme of X, thus $\mathcal{Q}uot(\mathscr{O}_X/S)=\mathcal{H}ilb(X/S)$, the Hilbert functor.

If H is an f-ample divisor class and $p(t)$ a polynomial, then $\mathcal{Q}uot_p(F/S)(*)$ denotes those flat quotients that have Hilbert polynomial $p(t)$.

By Grothendieck (1962, lect.IV), $\mathcal{Q}uot_p(F/S)$ is bounded, proper, separated and it has a fine moduli space $\mathrm{Quot}_p(F/S)$. See Sernesi (2006, sec.4.4) for a detailed proof.

Note that one can write F as a quotient of $\mathscr{O}_{\mathbb{P}^n}(-m)^r$ for some m,r, thus $\mathcal{Q}uot_p(F/S)$ can be viewed as a subfunctor of $\mathcal{Q}uot(\mathscr{O}_{\mathbb{P}^n}^r/S)$. The theory of the latter is essentially the same as the study of the Hilbert functor.

9.34 (Hom-schemes) Let $f\colon X\to S$ be a morphism and F,G quasi-coherent sheaves on X. Let $\mathrm{Hom}_S(F,G)$ be the set of $\mathscr{O}_X$-linear maps of F to G.

For $q\colon T\to S$, we have $\mathrm{Hom}_S(F_T,G_T)$, where $g_X\colon T\times_S X\to X$ is the projection and $F_T=g_X^*F$, $G_T=g_X^*G$.

As a special case of Grothendieck (1960, III.7.7.8–9), if f is proper, F,G are coherent and G is flat over S, then this functor is represented by an S-scheme $\mathbf{Hom}_S(F,G)$. That is, for any $g\colon T\to S$, there is a natural isomorphism

$$\mathrm{Hom}_T(F_T,G_T)\simeq \mathrm{Mor}_S(T,\mathbf{Hom}_S(F,G)).$$

To see this, note first that there is a natural identification between

(9.34.1) homomorphisms $\phi\colon F\to G$, and

(9.34.2) quotients $\Phi\colon (F+G)\twoheadrightarrow Q$ that induce an isomorphism $\Phi|_G\colon G\simeq Q$.

Next let $\pi\colon \mathrm{Quot}_S(F+G)\to S$ denote the quot-scheme parametrizing quotients of $F+G$ with universal quotient $u\colon \pi_X^*(F+G)\to Q$, where π_X denotes the induced map $\pi_X\colon \mathrm{Quot}_S(F+G)\times_S X\to X$.

Consider now the restriction of u to $u_G\colon \pi_X^*G\to Q$. By (10.54) there is an open subset

$$\mathrm{Quot}_S^\circ(F+G)\subset \mathrm{Quot}_S(F+G)$$

that parametrizes those quotients $v\colon F+G\to Q$ that induce an isomorphism $v_G\colon G\simeq Q$. Thus $\mathbf{Hom}_S(F,G)=\mathrm{Quot}_S^\circ(F+G)$. □

9.35 (Boundedness of quotient husks) Let us say that a set of sheaves $\{F_\lambda\colon \lambda\in\Lambda\}$ is *bounded* if there is fixed m such that, $F_\lambda(m)$ is generated by global sections and its higher cohomologies vanish for all $\lambda\in\Lambda$.

By an argument going back to Mumford (1966, lec.14), a set of pure sheaves $\{F_\lambda\colon \lambda\in\Lambda\}$ on $\mathbb{P}^N$ with given Hilbert polynomial is bounded iff their

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restrictions to general linear subspaces of codimension $d-1$ are bounded; see Huybrechts and Lehn (1997, 3.3.7) for a stronger result.

Since being a quotient husk commutes with restriction to general linear subspaces (9.23), after replacing S by the Grassmannian $\mathrm{Gr}_S(\mathbb{P}^{N-d+1}, \mathbb{P}^N)$, it is sufficient to prove boundedness in relative dimension 1.

If $\dim F_s = 1$, then we can choose m such that $F_s(m)$ is generated by global sections and its H^1 vanishes for all $s \in S$. Since $\mathrm{coker}(F_s \to G_s)$ has dimension 0, we get that $G_s(m)$ is also generated by global sections and its H^1 vanishes.

9.6 Hulls and Hilbert Polynomials

Recall that we use $\preceq$ (resp. $\equiv$) to denote the lexicographic ordering (resp. identity) of polynomials, see (5.14).

Let $f: X \to S$ be a projective morphism with relatively ample line bundle $\mathscr{O}_X(1)$. For a coherent sheaf F on X we aim to understand flatness of F and of its hull F^H in terms of the Hilbert polynomials $\chi(X_s, F_s(t))$ of the fibers F_s. Note that the $\chi(X_s, F_s(t))$ carry no information about the nilpotents in $\mathscr{O}_S$, so we assume that S is reduced.

As we noted in (3.20), $s \mapsto \chi(X_s, F_s(*))$ is an upper semi-continuous function on S and F is flat over S iff this function is locally constant.

The next result says that the same holds for $s \mapsto \chi(X_s, F_s^{[**]}(*))$. This does not follow directly from (3.20), since in general there is no sheaf on X whose fibers are $F_s^{[**]}$.

Theorem 9.36 *Let $f: X \to S$ be a projective morphism with relatively ample line bundle $\mathscr{O}_X(1)$ and F a mostly flat family of coherent, S_2 sheaves (3.26). Assume that S is reduced. Then $s \mapsto \chi(X_s, F_s^{[**]}(*))$ is an upper semi-continuous function and the following are equivalent.*

(9.36.1) *$s \mapsto \chi(X_s, F_s^{[**]}(*))$ is locally constant on S.*

(9.36.2) *$r_s^S : F_s \to F_s^{[**]}$ is an isomorphism for $s \in S$.*

(9.36.3) *F is flat over S with S_2 fibers (9.17).*

Proof We follow the method of (5.30). By generic flatness (Eisenbud, 1995, 14.4), there is a dense open subset $S^\circ \subset S$ such that F^H is flat with S_2 fibers $(F^H)_s = F_s^{[**]}$ over S°. Thus the function $s \mapsto \chi(X_s, F_s^{[**]}(t))$ is locally constant on S°, hence constructible on S by Noetherian induction. Thus it is enough to prove upper semicontinuity when $(0 \in S)$ is the spectrum of a DVR with generic point g.

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Then F is S_2 and flat over S. Thus $\chi(X_0, F_0(t)) \equiv \chi(X_g, F_g(t))$ and F_0 is S_1, hence the restriction map (9.15) $r_0^S: F_0 \to F_0^H$ is an injection. The exact sequence

$$0 \to F_0 \to F_0^H \to Q_0 \to 0$$

defines Q_0 and $\chi(X_0, F_0^H(t)) \equiv \chi(X_0, F_0(t)) + \chi(X_0, Q_0(t))$. This gives that

$$\chi(X_0, F_0^H(t)) \geq \chi(X_0, F_0(t)) \equiv \chi(X_g, F_g(t)).$$

Equality holds iff $r_0^S: F_0 \to F_0^H$ is an isomorphism, that is, when F_0 is S_2.

We have thus proved that if $s \mapsto \chi(X_s, F_s^{[**]}(t))$ is locally constant and S is regular, one-dimensional, then F^H is flat over S with S_2 fibers. We show in (9.41) that this implies the general case. □

Complement 9.36.4 If $\dim Q_0 = 0$, then we get that $\chi(X_0, F_0^H) \geq \chi(X_g, F_g)$ and equality holds iff r_0^S is an isomorphism.

Proposition 9.37 *Let $f: X \to S$ be a projective morphism with relatively ample line bundle $\mathcal{O}_X(1)$ and F a mostly flat family of coherent, S_2 sheaves. Then F^H is a universal hull iff for every local, Artinian ring (A, m_A) with residue field $k = A/m_A$ and every morphism* $\operatorname{Spec} A \to S$, *we have*

$$\chi(X_A, (F_A)^H(t)) \equiv \chi(X_k, (F_k)^H(t)) \cdot \operatorname{length} A.$$

Proof We show that the condition holds iff $(F_A)^H$ is flat over A and then conclude using (9.17.6).

Let $U \subset X$ be the largest open set where F is flat with S_2 fibers. Pick a maximum length filtration of A and lift it to a filtration

$$0 = G_0^U \subset G_1^U \subset \cdots \subset G_r^U = F_A|_{U_A}$$

such that $G_{i+1}^U/G_i^U \simeq F_k|_{U_k}$ and $r = \operatorname{length} A$. By pushing it forward to X_A we get a filtration

$$0 = G_0 \subset G_1 \subset \cdots \subset G_r = (F_A)^H$$

such that $G_{i+1}/G_i \subset (F_k)^H$. Therefore

$$\chi(X_A, (F_A)^H(t)) \leq \chi(X_k, (F_k)^H(t)) \cdot \operatorname{length} A.$$

Equality holds iff $G_{i+1}/G_i = (F_k)^H$ for every i, that is, iff F_A^H is flat over A. □

The next result roughly says that local constancy of H^0 implies flatness for globally generated shaves. It is similar to Grauert's theorem on direct images; the key difference is that we do not have a flat sheaf to start with.

Proposition 9.38 *Let $f\colon X \to S$ a proper morphism to a reduced scheme and F a mostly flat family of coherent, S_2 sheaves on X. Assume that*
(9.38.1) *$s \mapsto h^0(X_s, F_s^H)$ is a locally constant function on S, and*
(9.38.2) *F_s^H is generated by its global sections for every $s \in S$.*
Then F^H is a universal hull and $f_*(F^H)$ is locally free.

Proof Assume first that S is the spectrum of a DVR. We may replace F by $F^{[**]}$, hence assume that F is flat over S. Then $F_s \hookrightarrow F_s^H$ is an injection and we have inequalities

$$h^0(X_g, F_g) \le h^0(X_s, F_s) \le h^0(X_s, F_s^H). \tag{9.38.3}$$

By (1) these are equalities. Since F_s^H is generated by its global sections, this implies that $F_s = F_s^H$. As we explain in (9.41), this implies that F^H is a universal hull for every S. The last claim then follows from Grauert's theorem. □

9.7 Moduli Space of Universal Hulls

Definition 9.39 Let $f\colon X \to S$ be a morphism and F a coherent sheaf on X. As in (3.16.1)), for a scheme $g\colon T \to S$ set $\mathcal{H}ull(F/S)(T) = \{\emptyset\}$ if g_X^*F has a universal hull, and $\mathcal{H}ull(F/S)(T) = \emptyset$ otherwise, where $g_X\colon T \times_S X \to X$ is the projection.

If f is projective and p is a polynomial we set $\mathcal{H}ull_p(F/S)(T) = 1$ if g_X^*F has a universal hull with Hilbert polynomial p.

The following result is the key to many applications of the theory.

Theorem 9.40 (Flattening decomposition for universal hulls) *Let $f\colon X \to S$ be a projective morphism and F a coherent sheaf on X. Then*
(9.40.1) *$\mathcal{H}ull_p(F/S)$ has a fine moduli space* $\mathrm{Hull}_p(F/S)$.
(9.40.2) $\mathrm{Hull}_p(F/S) \to S$ *is a locally closed embedding (10.83).*
(9.40.3) *The structure map* $\mathrm{Hull}(F/S) = \amalg_p \mathrm{Hull}_p(F/S) \to S$ *is a locally closed decomposition (10.83).*

Proof Let n be the relative dimension of $\operatorname{Supp} F/S$ and $S_n \subset S$ the closed subscheme parametrizing n-dimensional fibers. We construct $\mathrm{Hull}_n(F/S)$, the fine moduli space of n-dimensional universal hulls. Then repeat the argument for $S \setminus S_n$.

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Let $\pi\colon \operatorname{Husk}(F/S) \to S$ be the structure map, $\pi_X\colon \operatorname{Husk}(F/S) \times_S X \to X$ the second projection and $q_{\text{univ}}\colon \pi_X^* F \to G_{\text{univ}}$ the universal husk. The set of points $y \in \operatorname{Husk}(F/S)$ such that $(G_{\text{univ}})_y$ is S_2 and has pure dimension n is open by (10.12). The fiber dimension of

$$\operatorname{Supp}\operatorname{coker}[\pi_X^* F \to G_{\text{univ}}] \to \operatorname{Husk}(F/S)$$

is upper semi-continuous. Thus there is a largest open set $W_n \subset \operatorname{Husk}(F/S)$ parametrizing husks $F_s \to G_s$ such that G_s is S_2, has pure dimension n and $\dim \operatorname{Supp} G_s/F_s \le n-2$. By (9.18), $\operatorname{Hull}_n(F/S) = W_n$.

Since hulls are unique (9.13), $\operatorname{Hull}(F/S) \to S$ is a monomorphism (10.82). In order to prove that each $\operatorname{Hull}_p(F/S) \to S$ is a locally closed embedding, we check the valuative criterion (10.84).

Let $(0, T)$ be the spectrum of a DVR with generic point g and $p\colon T \to S$ a morphism such that the hulls of F_g and of F_0 have the same Hilbert polynomials. Let G_g denote the hull of F_g and extend G_g to a husk $F_T \to G_T$. By assumption and by flatness

$$\chi(X_0, (G_T)_0(t)) \equiv \chi(X_g, (G_T)_g(t)) \equiv \chi(X_g, (F_g)^H(t)) \equiv \chi(X_0, (F_0)^H(t)).$$

Hence $(G_T)_0 = (F_0)^H$ by (9.26) and so G_T is the relative hull of F_T. Thus G_T defines the lifting $T \to \operatorname{Hull}_p(F/S)$. □

9.41 (End of the proof of 9.36 and 9.38) By definition, F has a universal hull over $\operatorname{Hull}(F/S)$, thus we need to show that $\tau\colon \operatorname{Hull}(F/S) \to S$ is an isomorphism.

By (9.40), τ is a locally closed decomposition, and, by (10.83.2), a proper, locally closed decomposition is an isomorphism if S is reduced.

To check properness, let T be the spectrum of a DVR and $p\colon T \to S$ a morphism. We already proved for both (9.36) and (9.38) that $(p^*F)^H$ is a universal hull. Thus $p\colon T \to S$ lifts to $\tilde{p}\colon T \to \operatorname{Hull}(F/S)$, so $\operatorname{Hull}(F/S) \to S$ is proper. □

Let $f\colon X \to S$ be a morphism. Two coherent sheaves F, G on X are called *relatively isomorphic* or *f-isomorphic* if there is a line bundle L_S on S such that $F \simeq G \otimes f^* L_S$. We are interested in understanding all morphisms $q\colon T \to S$ such that the hulls of $q_X^* F$ and $q_X^* G$ are relatively isomorphic, that is, there is a line bundle L_T on T such that $(q_X^* F)^H \simeq (q_X^* G)^H \otimes f_T^* L_T$.

Proposition 9.42 *Let $f\colon X \to S$ be a flat, projective morphism with S_2 fibers such that $H^0(X_s, \mathscr{O}_{X_s}) \simeq k(s)$ for every $s \in S$. Let M_1, M_2 be mostly flat families of divisorial sheaves on X. Then there is a locally closed subscheme $i\colon S^{riso} \hookrightarrow$*

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S such that, for any $q\colon T \to S$, the pull-backs $q_X^*M_1$ and $q_X^*M_2$ have relatively isomorphic hulls iff q factors as $q\colon T \to S^{riso} \hookrightarrow S$.

Proof Set $L := \mathcal{H}om_X(M_1, M_2)$. Then $q_X^*M_1$ and $q_X^*M_2$ have relatively isomorphic hulls iff L is relatively isomorphic to $\mathscr{O}_X$.

We may assume that S is connected. Then $p(*) := \chi(X_s, \mathscr{O}_{X_s}(*))$ is independent of $s \in S$. Thus $i\colon S^{\mathrm{riso}} \hookrightarrow S$ factors through $\mathrm{Hull}_p(L/S) \to S$. After replacing S by $\mathrm{Hull}_p(L/S)$ it remains to prove the special case when L is flat over S. The latter follows from (3.21). □

9.43 (Pure quotients) We get a similar flattening decomposition for pure quotients. The proofs are essentially the same as for hulls, so we just state the results.

Let $f\colon X \to S$ be a morphism of finite type and F a coherent sheaf on X. We say that F is *f-pure* or *relatively pure,* if F is flat over S and has pure fibers (10.1). We say that $q\colon F \to G$ is an *f-pure quotient* or *relatively pure quotient* of F if G is f-pure and $G_s = \mathrm{pure}(F_s)$ for every $s \in S$. Note that $\ker q$ is then the largest subsheaf $K \subset F$ such that $\dim(\mathrm{Supp}\, K_s) < \dim(\mathrm{Supp}\, F_s)$ for every $s \in S$. In particular, a relatively pure quotient is unique.

This gives the functor of relatively pure quotients $\mathcal{P}ureq(F/S)$. If f is projective, it can be decomposed $\mathcal{P}ureq(F/S) = \amalg_p \mathcal{P}ureq_p(F/S)$ using Hilbert polynomials. As in (9.40), we get the following.

Claim 9.43.1 Let $f\colon X \to S$ be a projective morphism and F a coherent sheaf on X. The functor of pure quotients is represented by a locally closed decomposition $\mathrm{Pureq}(F/S) \to S$. □

Arguing as in (9.36) gives the following.

Corollary 9.43.2 Let S be a reduced scheme, $g\colon X \to S$ a projective morphism and F a coherent sheaf on X. Then F has a g-pure quotient $F \twoheadrightarrow G$ iff $s \mapsto \chi(\mathrm{pure}(F_s)(*))$ is locally constant on S. □

9.8 Non-projective Versions

The proofs in Section 9.7 used in an essential way the projectivity of $X \to S$. Here we consider similar questions for non-projective morphisms in two cases. If $X \to S$ is affine then a good theory seems possible only if S is local and complete. Then we study the case when $X \to S$ is proper.

For affine morphisms we have the following variant of (9.40).

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Theorem 9.44 *Let (S, m_S) be a complete local ring, R a finite type S-algebra and F a finite R-module that is mostly flat with S_2 fibers over S (3.28). Then there is a quotient $S \twoheadrightarrow S^u$ that represents Hull(F/S) for local, Artinian S-algebras.*

Since universal hulls commute with completion (9.17.6), (9.44) implies the same statement for complete, local S-algebras. That is, for every local morphism $h\colon (S, m_S) \to (T, m_T)$, the hull $(F_T)^H$ is universal iff there is a factorization $h\colon S \twoheadrightarrow S^u \to T$.

Note that, compared with (9.40), we only identify the stratum containing the closed point of Spec S.

Proof We follow the usual method of deformation theory; Artin (1976); Seshadri (1975); Hartshorne (2010). As a first step we construct S^u.

For an ideal $I \subset S$ set $F_I := F \otimes (R/IR)$. First, we claim that if $(F_I)^H$ and $(F_J)^H$ are universal hulls then so is $(F_{I\cap J})^H$. Start with the exact sequence

$$0 \to S/(I \cap J) \to S/I + S/J \to S/(I + J) \to 0. \tag{9.44.1}$$

F is mostly flat over S, thus (9.44.1) stays left exact after tensoring by F and taking the hull. Thus we obtain the exact sequence

$$0 \to (F_{I\cap J})^H \to (F_I)^H + (F_J)^H \to (F_{I+J})^H. \tag{9.44.2}$$

$(F_J)^H \to (F_{I+J})^H$ is surjective since $(F_J)^H$ is a universal hull, hence (9.44.2) is also right exact.

Set $k := S/m_S$. Since $(F_I)^H$ is a universal hull, $(F_I)^H \otimes k \simeq (F_m)^H$, and the same holds for J and $I + J$. Thus tensoring (9.44.2) with k yields

$$(F_{I\cap J})^H \otimes k \to (F_m)^H + (F_m)^H \xrightarrow{p} (F_m)^H \to 0. \tag{9.44.3}$$

Since $\ker p \simeq (F_m)^H$ we see that $(F_{I\cap J})^H \otimes k \to (F_m)^H$ is surjective. By (9.17) this implies that $(F_{I\cap J})^H$ is a universal hull.

Let $I^u \subset S$ be the intersection of all those ideals I such that $(F_I)^H$ is a universal hull and $S^u := S/I^u$. By (9.17.6) $(F_{S^u})^H$ is a universal hull.

By construction, if $h\colon S \twoheadrightarrow W := S/I_W$ is a quotient such that $(F_W)^H$ is a universal hull then $I^u \subset I_W$. We still need to prove that if (A, m_A) is a local Artinian S-algebra such that $(F_A)^H$ is a universal hull then $h\colon S \to A$ factors through S^u.

Let $K := A/m_A$ denote the residue field. $F/m_S F$ has a hull by (9.5), so $I^u \subset m_S$. Thus $S \to A \to K$ factors through S^u. Working inductively we may assume that there is an ideal $J_A \subset A$ such that $J_A \simeq K$ and $h'\colon S \to A' := A/J$ factors through S^u. Therefore $h\colon S \to A$ factors through $S \to S/m_S I^u$. Note that $I^u/m_S I^u$ is a finite dimensional k-vector space, call it V_k, and we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V_k & \longrightarrow & S/m_S I^u & \longrightarrow & S^u & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \lambda} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h'} & & \\ 0 & \longrightarrow & K & \longrightarrow & A & \longrightarrow & A & \longrightarrow & 0 \end{array} \tag{9.44.4}$$

for some k-linear map $\lambda\colon V_k \to K$. If $\lambda = 0$ then h factors through S^u, this is what we want. If $\lambda \neq 0$ then we show that there is an ideal $J^u \subsetneq I^u$ such that F has a universal hull over S/J^u. This contradicts our choice of I^u and proves the theorem.

It is easier to write down the obstruction map in scheme-theoretic language. To simplify notation, we may assume that $m_S I^u = 0$. Thus set $X := \operatorname{Spec}_S R$ and let $i\colon U \hookrightarrow X$ be the largest open set over which $\tilde{F}$ (the sheaf associated to F) is flat over S. For any $S \to T$ by base change we get $i\colon U_T \hookrightarrow X_T$. Let $\mathcal{F}_T$ denote the restriction of $\tilde{F}_T$ to U_T. Then $i_*\mathcal{F}_T$ is the sheaf associated to $(F_T)^H$ and we have a commutative diagram

$$\begin{array}{ccccccc} V_k \otimes_k i_*\mathcal{F}_k & \longrightarrow & i_*\mathcal{F}_S & \longrightarrow & i_*\mathcal{F}_{S^u} & \xrightarrow{\delta} & V_k \otimes_k R^1 i_*\mathcal{F}_k \\ \downarrow{\scriptstyle \lambda} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h'} & & \downarrow{\scriptstyle \lambda} \\ i_*\mathcal{F}_K & \longrightarrow & i_*\mathcal{F}_A & \longrightarrow & i_*\mathcal{F}_{A'} & \xrightarrow{\Delta} & R^1 i_*\mathcal{F}_K. \end{array} \tag{9.44.5}$$

Here $\Delta = 0$ since $i_*\mathcal{F}_A$ is a universal hull. The right-hand square factors as

$$\begin{array}{ccccc} \delta\colon i_*\mathcal{F}_{S^u} & \longrightarrow & i_*\mathcal{F}_k & \xrightarrow{\delta_k} & V_k \otimes_k R^1 i_*\mathcal{F}_k \\ \downarrow{\scriptstyle h'} & & \downarrow{\scriptstyle h_k} & & \downarrow{\scriptstyle \lambda\otimes\mathbf{1}} \\ \Delta\colon i_*\mathcal{F}_{A'} & \longrightarrow & i_*\mathcal{F}_K & \xrightarrow{\Delta_K} & K \otimes_k R^1 i_*\mathcal{F}_k. \end{array} \tag{9.44.6}$$

By assumption $\Delta_K = 0$. Choosing a basis $\{v_j\}$ of V_k, this means that the components $\delta_{k,j} : i_*\mathcal{F}_k \to R^1 i_*\mathcal{F}_k$ are linearly dependent over K. So they are linearly dependent over k, that is, there is a nonzero $\mu\colon V_k \to k$ such that $\mu \circ \delta_k = 0$. Set $J^u := m_S I^u + \ker \mu$ and $S' := S/J^u$. Note that $I^u/J^u \simeq k$. The extension $I^u/J^u \to S' \to S^u$ gives the exact sequence

$$(I^u/J^u) \otimes_k i_*\mathcal{F}_k \hookrightarrow i_*\mathcal{F}_{S'} \to i_*\mathcal{F}_{S^u} \xrightarrow{\mu\circ\delta} (I^u/J^u) \otimes_k R^1 i_*\mathcal{F}_k. \tag{9.44.7}$$

Since $\mu \circ \delta = 0$ the map $i_*\mathcal{F}_{S'} \to i_*\mathcal{F}_{S^u}$ is surjective and so is the composite $i_*\mathcal{F}_{S'} \to i_*\mathcal{F}_{S^u} \to i_*\mathcal{F}_k$. Thus $i_*\mathcal{F}_{S'}$ is a universal hull by (9.17). This contradicts the choice of S^u. □

One can see that (9.44) does not hold for arbitrary local schemes S, but the following consequence was pointed out by E. Szabó.

Corollary 9.45 *The conclusion of (9.44) remains true if S is a Henselian local ring, that is the localization of an algebra of finite type over a field or over an excellent DVR.*

Proof There is a general theorem (Artin, 1969, 1.6) about representing functors over Henselian local rings; we check that its conditions are satisfied.

Let $\hat{S}$ denote the completion of S. As in (3.16.1), define a functor on local S-algebras by setting $\mathcal{F}(T) = \{\emptyset\}$ if $(F_T)^H$ is a universal hull and $\mathcal{F}(T) = \emptyset$ otherwise.

It is easy to see that if $\mathcal{F}(T) = \{\emptyset\}$, then there is a factorization $S \to T' \to T$ such that T' is of finite type over S and $\mathcal{F}(T') = \{\emptyset\}$. So $\mathcal{F}$ is locally of finite presentation over S, as in Artin (1969, 1.5). The universal family over $(\hat{S})^u$ gives an effective versal deformation of the fiber over m_S. The existence of S^u now follows from Artin (1969, 1.6). □

Next we present an alternative approach to hulls and husks that does not use projectivity, works for proper algebraic spaces, but leaves properness of Husk(F/S) unresolved. The proofs were worked out jointly with M. Lieblich.

Theorem 9.46 *Let S be a Noetherian algebraic space and $p\colon X \to S$ a proper morphism of algebraic spaces. Let F be a coherent sheaf on X. Then $\mathcal{QH}usk(F/S)$ is separated and it has a fine moduli space* QHusk(F/S).

Proof Let $f\colon X \to S$ be a proper morphism. The functor of flat families of coherent sheaves $\mathcal{F}lat(X/S)$ is represented by an algebraic stack Flat(X/S) which is locally of finite type, but very non-separated; see Laumon and Moret-Bailly (2000, 4.6.2.1).

Let $\sigma\colon$ Flat$(X/S) \to S$ be the structure morphism and $U_{X/S}$ the universal family. By (10.12), there is an open substack Flat$^n(X/S) \subset$ Flat(X/S) parametrizing pure sheaves of dimension n. Let $U^n_{X/S}$ be the corresponding universal family. Consider $X \times_S$ Flat$^n(X/S)$ with coordinate projections π_1, π_2. The stack $\mathbf{Hom}(\pi_1^* F, \pi_2^* U^n_{X/S})$ parametrizes all maps from the sheaves F_s to pure, n-dimensional sheaves N_s.

We claim that QHusk(F/S) is an open substack of $\mathbf{Hom}(\pi_1^* F, \pi_2^* U^n_{X/S})$. Indeed, by (10.54), for a map of sheaves $M \to N$ with N flat over S, it is an open condition to be an isomorphism at the generic points of the support.

As we discussed in (9.31), QHusk(F/S) satisfies the valuative criteria of separatedness and properness, so the diagonal of QHusk(F/S) is a monomorphism. Every algebraic stack with this property is an algebraic space; see Laumon and Moret-Bailly (2000, sec.8). □

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The connected components of QHusk(F/S) are not proper over S, this fails even for the quot-scheme, but the following should be true.

Conjecture 9.47 *Every irreducible component of* QHusk(F/S) *is proper.*

The construction of Hull(F/S) given in (9.40) applies to algebraic spaces as well, but it does not give boundedness. Nonetheless, we claim that Hull(F/S) is of finite type. First, it is locally of finite type since QHusk(F/S) is. Second, we claim that red Hull(F/S) is dominated by an algebraic space of finite type. In order to see this, consider the (reduced) structure map red Hull$(F/S) \to$ red S. It is an isomorphism at the generic points, hence there is an open dense $S^\circ \subset$ red S such that S° is isomorphic to an open subspace of red Hull(F/S). Repeating this for red $S \setminus S^\circ$, by Noetherian induction we eventually write red Hull(F/S) as a disjoint union of finitely many locally closed subspaces of red S. These imply that Hull(F/S) is of finite type. Using (9.13.4), we get the following.

Theorem 9.48 (Flattening decomposition for hulls) *Let $f\colon X \to S$ be a proper morphism of algebraic spaces and F a coherent sheaf on X. Then*

(9.48.1) *$\mathcal{H}ull(F/S)$ is separated and it has a fine moduli space* Hull(F/S)*,*

(9.48.2) Hull(F/S) *is an algebraic space of finite type over S, and*

(9.48.3) *the structure map* Hull$(F/S) \to S$ *is a surjective monomorphism.* □

Example 9.49 Let C, D be two smooth projective curves. Pick points $p, q \in C$ and $r \in D$. Let X be the surface obtained from the blow-up $B_{(p,r)}(C \times D)$ by identifying $\{q\} \times D$ with the birational transform of $\{p\} \times D$. Note that X is a proper, but non-projective scheme and there is a natural proper morphism $\pi\colon X \to C'$ where C' is the nodal curve obtained from C by identifying the points p, q. Then Hull$(\mathcal{O}_X/S) = C \setminus \{q\}$. The natural map $C \setminus \{q\} \to C'$ is a surjective monomorphism, but not a locally closed embedding.

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