P 57. Let m, n be relatively prime positive integers: $(m, n) = 1$. Write

$$f(x) = \frac{(1-x^{mn})(1-x)}{(1-x^m)(1-x^n)} ;$$

and show

(i) $f(x)$ is a polynomial of degree $(m-1)(n-1)$ whose non-zero coefficients are alternately $+1$ and -1.

(ii) the number of non-zero coefficients is

$$Mm + Nn - 2MN$$

where M, N are integers defined by $Mm - Nn = 1$, $0 < M < n$.

J. D. Dixon, California Institute of Technology

Solution by L. Carlitz, Duke University.

It follows from

$$Mm - Nn = 1$$

that

$$x^{Nn} f(x) = \frac{(1-x^{mn})(x^{Nn} - x^{Mm})}{(1-x^m)(1-x^n)}$$

$$= \frac{(1-x^{mn})(1-x^{Mm})}{(1-x^m)(1-x^n)} - \frac{(1-x^{mn})(1-x^{Nn})}{(1-x^m)(1-x^n)}$$

$$= (1 + x^n + \ldots + x^{n(m-1)})(1 + x^m + \ldots + x^{m(M-1)})$$

$$- (1 + x^m + \ldots + x^{m(n-1)})(1 + x^n + \ldots + x^{n(N-1)}) .$$

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Clearly $f(x)$ is a polynomial.

Now if

$$rn + sm = pm + qn$$

where $0 \leq r < m$, $0 \leq s < M$, $0 \leq p < n$, $0 \leq q < N$, it follows at once that

$$(r-q)n = (p-s)m ,$$

so that $n \mid p-s$, which implies $p = s$, $q = r$. Consequently we get

$$(*) \qquad x^{Nn} f(x) = \sum_{\substack{N \leq r < m \\ 0 \leq s < M}} x^{rn+sm} - \sum_{\substack{M \leq p < n \\ 0 \leq q < N}} x^{pm+qn} .$$

Moreover there is no further cancellation. Hence the number of non-vanishing terms on the right of (*) (which is also the number of terms in $f(x)$) is equal to

$$M(m-N) + N(n-M) = Mm + Nn - 2MN ,$$

as asserted.

Finally to show that the nonvanishing coefficients in $f(x)$ are alternately positive and negative, we write

$$f(x) = (1-x)(1 + x^{m} + \ldots + x^{m(n-1)}) \sum_{r=0}^{\infty} x^{rn}$$

$$= (1-x) \sum_{s=0}^{n-1} \sum_{r=0}^{\infty} x^{rn+sm}$$

Since $\deg f(x) = (m-1)(n-1)$ we need only consider those terms in which $r<m$. Thus we have

$$f(x) = (1-x)(1 + x^{k_1} + x^{k_2} + x^{k_3} + \ldots) + \ldots$$

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where $0 < k_1 < k_2 < k_3 < \ldots.$ It follows that

$$f(x) = 1 - x + x^{k_1} - x^{k_1+1} + x^{k_2} - x^{k_2+1} + \ldots.$$

The assertion about the alternating signs is now evident. For example if $k_1 = 1$, $k_2 = 3$, we get

$$f(x) = 1 - x^2 + x^3 - \ldots.$$

Also solved by N. Kimura and the proposer.

Editor's comment: This problem has some connection with the following:

If $e = mn - m - n$, $(m, n) = 1$, show that every integer $k > e$ is represented by

$$sm + yn, \quad x \geq 0, \quad y \geq 0$$

x and y integers

and of the integers in the interval $[0, e]$ exactly half are represented. (This can be proved by observing that k is represented if and only if $e - k$ is not; since $k < 0$ is not represented, every $k > e$ is; and the statement about $[0, e]$ also follows immediately.)

If $\nu(k)$ denotes the number of representations, clearly

$$\sum_{k=0}^{\infty} \nu(k)x^k = g(x) = \frac{1}{(1-x^m)(1-x^n)}$$

$$= \frac{a_1}{(1-x)^2} + \frac{a_2}{1-x} + \sum_{r=1}^{m-1} \frac{b_r}{(1-\rho^r x)}$$

$$+ \sum_{s=1}^{n-1} \frac{c_s}{(1-\sigma^s x)}$$

where

$$\rho = \exp(2\pi i/m) ,$$

$$\sigma = \exp(2\pi i/n)$$

The numerators of the partial fractions are easily evaluated:

$$a_1 = {}^1/mn , \quad a_2 = \frac{m+n-2}{2mn} \cdot$$

$$b_r = \frac{1}{m(1-\rho^{-nr})} , \quad c_s = \frac{1}{n(1-\sigma^{-ms})} .$$

Comparing coefficients we obtain

$$\nu(k) = \frac{k}{mn} + \frac{m+n}{2mn}$$

$$+ \frac{1}{m} \sum_{r=1}^{m-1} \frac{\rho^{kr}}{1-\rho^{-nr}} + \frac{1}{n} \sum_{s=1}^{n-1} \frac{\rho^{ks}}{1-\rho^{-ms}}$$

$$= \frac{k}{mn} + O(1) .$$

For example, the number of solutions of $k = 2x + 3y$ is

$$\nu(k) = \frac{k}{6} + \frac{5}{12} + \frac{1}{4}(-1)^k$$

$$+ \frac{1}{3} \cos \frac{2\pi k}{3} - \frac{1}{3\sqrt{3}} \sin \frac{2\pi k}{3} .$$

Returning to the original problem, if

$$f(x) = d_0 + d_1 x + \ldots + d_{e+1} x^{e+1} ,$$

then

$$(1-x)\, g(x) = d_0 + \ldots + d_{e+1} x^{e+1}$$

$$+ \text{higher terms};$$

thus

$$d_k = \nu(k) - \nu(k-1) \quad (k=0,1,\ldots,e+1),$$ (and therefore $\nu(k) = d_0 + d_1 + \ldots + d_k$, $k = 0,\ldots,e+1$). For the above example this is

$$d_k = \frac{1}{6} + \frac{1}{2}(-1)^k + \frac{1}{3}\cos\frac{2\pi k}{3} - \frac{1}{\sqrt{3}}\sin\frac{2\pi k}{3},$$

and in general

$$d_k = \frac{1}{mn} + \frac{1}{m}\sum_{r=1}^{m-1}\frac{\rho^{rk}(1-\rho^{-r})}{1-\rho^{-rn}} + \frac{1}{n}\sum_{s=1}^{n-1}\frac{\sigma^{sk}(1-\sigma^{-s})}{1-\sigma^{-sm}}.$$

P 60. Let $[x]$ denote the largest integer not exceeding x. Prove that

$$\sum_{n=1}^{n}[\sqrt{i}] = [\sqrt{n}]\,(6n - 2[\sqrt{n}]^2 - 3[\sqrt{n}] + 5)/6 .$$

Can one give a simpler closed expression for the left hand side?

Leo Moser, University of Alberta

See the note by H. W. Gould on page 275.

Also solved by L. Artiaga, M. M. Brisebois, L. Carlitz, T. M. K. Davison, N. Kimura, and the proposer.

P 61. Find all solutions of $\phi(n) = \tau(n)$. Here $\phi(n)$ is Euler's totient function and $\tau(n)$ is the number of divisors of n .

Leo Moser, University of Alberta

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Solution by W. J. Blundon, Memorial University of Newfoundland.

It is required to solve in integers the equation $Q(n) = 1$, where $Q(n) = \varphi(n)/\tau(n)$. Since $\varphi(n)$ and $\tau(n)$ are multiplicative, so is $Q(n)$.

If $p > q$ (p, q primes), then $Q(p) = \frac{p-1}{2} > \frac{q-1}{2} = Q(q)$, so that, for p prime, $Q(p)$ increases strictly with p. Also, for $k \geq 1$, $Q(p^{k+1})/Q(p^k) = p(1+k)/(2+k) \geq 2(1+k)/(2+k) > 1$, so that, for fixed p, $Q(p^k)$ increases strictly with k. Clearly $\min Q(p^k) = Q(2) = 1/2$.

Enumerating all $Q(p^k) \leq 2$, we have

$Q(2) = 1/2$	$Q(3) = 1$	$Q(5) = 2$
$Q(4) = 2/3$	$Q(9) = 2$	
$Q(8) = 1$		
$Q(16) = 8/5$		

Using the fact that $Q(n)$ is multiplicative, we easily enumerate the complete set of solutions to $Q(n) = 1$ as follows:

$1 = Q(3)$

$1 = Q(8)$

$1 = Q(2) \cdot Q(5) = Q(10)$

$1 = Q(2) \cdot Q(9) = Q(18)$

$1 = Q(3) \cdot Q(8) = Q(24)$

$1 = Q(2) \cdot Q(3) \cdot Q(5) = Q(30)$.

Thus all solutions of $\varphi(n) = \tau(n)$ are given by

$n = 1, 3, 8, 10, 18, 24, 30$.

Also solved by L. Carlitz, N. Kimura, M. V. Subba Rao, and the proposer.