LOCALIZATION IN NON-NOETHERIAN GROUP RINGS

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1. Introduction. Let k be a field and G an Abelian group of finite torsion-free rank. Brewer, Costa and Lady [**1**, Theorem A] showed that if k has characteristic 0 then each localization of the group algebra kG at a prime ideal is a regular local ring. They also showed (in the same theorem) that if k has characteristic $p>0$, then kG is locally Noetherian (i.e. each localization of kG at a prime ideal is a Noetherian ring) if and only if G is an extension of a finitely generated group by a torsion p'-group. The purpose of this note is to examine this theorem in a more general setting.

Let R be a ring (with identity) and P a semiprime ideal of R. An element c of R is *regular* if $cr \neq 0$ and $rc \neq 0$ for every non-zero element r of R. Let

$$\mathscr{C}_R(P) = \{c \in R : c+P \text{ is a regular element of the ring } R/P\}.$$

We shall write $\mathscr{C}(P)$ for $\mathscr{C}_R(P)$ when there is no ambiguity about the ring R. We shall say that P is *localizable* if R satisfies the right and left Ore conditions with respect to $\mathscr{C}(P)$; i.e. given r in R and c in $\mathscr{C}(P)$ there exist elements r_1, r_2 in R and c_1, c_2 in $\mathscr{C}(P)$ with

$$rc_1 = cr_1 \quad \text{and} \quad c_2 r = r_2 c.$$

If P is a localizable semiprime ideal of R let

$$T(P) = \{r \in R : crd = 0 \text{ for some elements } c, d \text{ in } \mathscr{C}(P)\}.$$

Then $T = T(P)$ is an ideal of R and $c+T$ is a regular element of the ring R/T for each element c in $\mathscr{C}(P)$. Moreover, we can form the partial (right and left) quotient ring of R/T with respect to $\{c+T : c \in \mathscr{C}(P)\}$ and we denote it by R_P.

Let k be a field and G a group. Let $\mathfrak{g}$ be the augmentation ideal of the group algebra kG. We first consider when $\mathfrak{g}$ is localizable. This is certainly the case if G is locally nilpotent. For, given any elements r in kG and c in $\mathscr{C}(\mathfrak{g})$ there exists a finitely generated subgroup H such that $r \in kH$ and $c \in \mathscr{C}(\mathfrak{h})$, where $\mathfrak{h}$ is the augmentation ideal of kH. But $\mathscr{C}(\mathfrak{h}) \leq \mathscr{C}(\mathfrak{g})$ and it is well known that $\mathfrak{h}$ is localizable. Hence $\mathfrak{g}$ is localizable.

Our first main result is the following one.

THEOREM A. *Let k be a field of characteristic* 0, *G a poly-(finitely generated Abelian or locally finite) group and $\mathfrak{g}$ the augmentation ideal of the group algebra kG. Then the following statements are equivalent.*

(i) *$\mathfrak{g}$ is localizable.*

(ii) *$\mathfrak{g}$ has the AR property.*

(iii) *G is an extension of a locally finite group by a nilpotent group having each upper central factor of finite torsion-free rank.*

Recall that an ideal I of a ring R has the *AR property* if for any right ideal E and left

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ideal L there exists a positive integer n such that

$$E \cap I^n \leq EI \quad \text{and} \quad L \cap I^n \leq IL.$$

For any prime p let $\mathfrak{H}_p$ denote the class of groups G having a finite chain

$$1 = H_0 \leq H_1 \leq \ldots \leq H_n = G$$

of normal subgroups H_i of G such that H_i/H_{i-1} is finitely generated Abelian or locally finite-p' for each $1 \leq i \leq n$. A result for fields of non-zero characteristic corresponding to Theorem A is the following.

THEOREM B. *Let k be a field of characteristic $p > 0$, G an $\mathfrak{H}_p$-group and $\mathfrak{g}$ the augmentation ideal of the group algebra $R = kG$. Then the following statements are equivalent.*

(i) *$\mathfrak{g}$ is localizable.*

(ii) *$\mathfrak{g}$ has the AR property.*

(iii) *G centralizes all p-chief factors.*

By a *p-chief factor* of G we mean a chief factor each of whose non-trivial elements has order a power of p.

We call a ring S with Jacobson radical J *quasi-local* provided S/J is a simple Artinian ring. Let P be a localizable prime ideal of a ring R and $T = T(P)$. Then the ideal $PR_P = \{(x+T)(c+T)^{-1} : x \in P,\ c \in \mathscr{C}(P)\}$ of R_P is contained in the Jacobson radical of R_P and the ring R_P/PR_P is isomorphic to the (classical) quotient ring of R/P. Thus, by [**2**, Theorems 4.1 and 4.4], R_P is a quasi-local ring provided the ring R/P is a (right and left) Goldie ring; on the other hand if R_P is a (right and left) Noetherian ring then so is R_P/PR_P and hence R/P is a Goldie ring. (Note that all chain conditions will be assumed to hold on both sides unless specified otherwise.) We shall call a semiprime ideal Q of R an *annihilator* semiprime ideal if R/Q satisfies the ascending chain condition on right annihilators and on left annihilators. Of course, if R is a commutative ring then all prime ideals of R are localizable annihilator prime ideals.

A ring R is called a *regular local ring* if R is Noetherian quasi-local with Jacobson radical M such that there exists a finite chain

$$M = M_0 > M_1 > \ldots > M_t = 0$$

of ideals M_i of R such that M_{i-1}/M_i is generated by a central regular element of R/M_i for each $1 \leq i \leq t$. In this case, Walker [**12**, Theorem 2.7] proved that R is prime and t is the global dimension of R, the Krull dimension of R, the homological dimension of the R-module R/M and the supremum of the lengths of chains of prime ideals of R, and we call t the *dimension* of R.

If G is a group and p a prime or zero then by $O_{p'}(G)$ we mean the intersection of all the normal subgroups N of G for which G/N has no non-trivial finite-p' normal subgroup. By a finite-O' group we shall mean an arbitrary finite group. Let $\mathfrak{N}_p$ denote the class of groups G such that $G/O_{p'}(G)$ is a nilpotent group each of whose upper central factors is an extension of a finitely generated group by a torsion p'-group. For such a group G let

$h(G)$ denote the sum of the torsion-free ranks of the upper central factors of $G/O_{p'}(G)$. It is not hard to prove that $h(G)$ is an invariant for G.

THEOREM C. *Let k be a field of characteristic* 0, *G an $\mathfrak{N}_0$-group and P an annihilator prime ideal of the group algebra $R = kG$. Then P is localizable and R_P is a regular local ring of dimension at most $h(G)$.*

The situation for fields of non-zero characteristic is rather different. Firstly we have:

THEOREM D1. *Let k be a field of characteristic $p > 0$, G an $\mathfrak{N}_p$-group and P an annihilator prime ideal of the group algebra $R = kG$. Then P is localizable and R_P is a Noetherian ring.*

If p is a prime let $\mathfrak{N}_p^*$ denote the class of $\mathfrak{N}_p$-groups G such that each upper central factor of $G/O_{p'}(G)$ is an extension of a free Abelian group of finite rank by a torsion p'-group. For $\mathfrak{N}_p^*$-groups we have the following result.

THEOREM D2. *Let k be a field of characteristic $p > 0$, G an $\mathfrak{N}_p^*$-group and P an annihilator prime ideal of the group algebra $R = kG$. Then R_P is a regular local ring of dimension at most $h(G)$.*

Note that Theorems C and D2 generalize not only [**1**, Theorem A] but also [**9**, Theorem B].

2. Proofs of Theorems A and B. Let R be a ring and I an ideal of R. Define a chain of ideals

$$I = I^1 \geqslant I^2 \geqslant \dots \geqslant I^\alpha \geqslant I^{\alpha+1} \geqslant \dots,$$

where, for all ordinals α,

$$I^{\alpha+1} = II^\alpha + I^\alpha I,$$

and

$$I^\alpha = \bigcap_{\beta<\alpha} I^\beta$$

if α is a limit ordinal. There exists an ordinal ρ such that $I^\rho = I^{\rho+1}$, and for the least such ordinal ρ write

$$\kappa(I) = I^\rho.$$

Now let R be a Noetherian ring and let J be the Jacobson radical of R. Then

$$\kappa(J) = \kappa(J)J + J\kappa(J)$$

and since $\kappa(J)$ is finitely generated both as a right ideal and as a left ideal it follows, by Nakayama's Lemma, that

$$\kappa(J) = 0.$$

This fact has a simple consequence for localizations of prime ideals. Let P be a localizable prime ideal of R such that R_P is a Noetherian ring. If $T = T(P)$ then $PR_P = \{(p+T)(c+T)^{-1} : p \in P,\ c \in \mathscr{C}(P)\}$ is the Jacobson radical of R_P and so

$$\kappa(PR_P) = 0.$$

This gives immediately

LEMMA 2.1. *Let P be a localizable prime ideal of a ring R such that R_P is a Noetherian ring. Then $\kappa(P) \leq T(P)$.*

We wish to push this lemma somewhat further. If P is a localizable prime ideal of R we define

$$T_r(P) = \{r \in R : rc = 0 \text{ for some } c \text{ in } \mathscr{C}(P)\},$$

and

$$T_l(P) = \{r \in R : cr = 0 \text{ for some } c \text{ in } \mathscr{C}(P)\}.$$

Recall the following well-known result.

LEMMA 2.2. *Let R be a ring which satisfies the ascending chain condition on right annihilators and let P be a localizable prime ideal of R. Then $T(P) = T_r(P)$.*

Proof. Let $r \in R$ and $c \in \mathscr{C}(P)$ with $cr = 0$. If $r(x)$ denotes the right annihilator of the element x of R then

$$r(c) \leq r(c^2) \leq \dots$$

and there exists a positive integer n such that

$$r(c^n) = r(c^{n+1}).$$

There exist elements s in R and d in $\mathscr{C}(P)$ such that $c^n s = rd$. Then $cr = 0$ implies $rd = 0$. It follows that $T(P) = T_r(P)$.

A non-empty subset S of a ring R will be called an *Ore set* if

(i) S is multiplicatively closed,

(ii) for all elements r of R and t of S there exist elements r_1, r_2 of R and t_1, t_2 of S such that $rt_1 = tr_1$ and $t_2 r = r_2 t$, and

(iii) $\{r \in R : rt = 0 \text{ for some } t \text{ in } S\} = \{r \in R : tr = 0 \text{ for some } t \text{ in } S\}$.

In this case let $T(S) = \{r \in R : rt = 0 \text{ for some } t \text{ in } S\}$. The partial quotient ring of R with respect to S will be denoted by R_S.

LEMMA 2.3. *Let P be a localizable prime ideal of a ring R such that there exists an Ore set S with $S \leq \mathscr{C}(P)$ and R_S Noetherian. Then $\kappa(P) \leq T_r(P)$.*

Proof. By Lemmas 2.1 and 2.2,

$$\kappa(PR_S) \leq T_r(PR_S),$$

where $PR_S = \{(p + T(S))(t + T(S))^{-1} : p \in P,\ t \in S\}$. If $r \in \kappa(P)$ then there exist c in $\mathscr{C}(P)$ and t in S such that $rct = 0$. Since $t \in \mathscr{C}(P)$ it follows that $r \in T_r(P)$. Hence $\kappa(P) \leq T_r(P)$.

Let k be a field and G a group. Then the augmentation ideal of the group algebra kG will be denoted by $\mathfrak{g}_k$ or simply $\mathfrak{g}$ when there is no ambiguity about k.

LEMMA 2.4. *Let k be a field and G a group such that $\mathfrak{g}_k$ is localizable. If H is any subgroup of G then $\mathfrak{h}_k$ is localizable.*

Proof. Let $r \in kH$ and $c \in \mathscr{C}(\mathfrak{h})$. Then $c \in \mathscr{C}(\mathfrak{g})$ and there exist elements s, d in kG with d in $\mathscr{C}(\mathfrak{g})$ such that $rd = cs$. Let T be a transversal to the right cosets of H in G. Then $kG = \bigoplus_{t \in T} (kH)t$. It follows that there exist elements s', d' in kH with d' in $\mathscr{C}(\mathfrak{h})$ such that $rd' = cs'$. It follows that $\mathfrak{h}$ is localizable.

Proof of Theorem A. The equivalence of (ii) and (iii) is proved in [**11**, Theorem D]. Also (ii) implies (i) by [**10**, Lemma 2.2]. Thus it is sufficient to prove that (i) implies (iii). Let

$$1 = H_0 \leqslant H_1 \leqslant \ldots \leqslant H_n = G \tag{1}$$

be a finite chain of subgroups H_i of G such that H_{i-1} is normal in H_i and H_i/H_{i-1} is finitely generated Abelian or locally finite for each $1 \leqslant i \leqslant n$. We prove the result by induction on n. If $n = 1$ then $\mathfrak{g}$ has the AR property by [**10**, Theorem C]. So suppose $n > 1$ and let $H = H_{n-1}$. By Lemma 2.4 we can suppose that $\mathfrak{h}$ has the AR property and G/H is either finitely generated Abelian or locally finite.

Suppose that G/H is finitely generated Abelian. Since $\mathfrak{h}$ has the AR property it follows that $S = \{1 - a : a \in \mathfrak{h}\}$ is an Ore set in kH and the ring $(kH)_S$ is Noetherian (see [**10**, Lemma 2.2 and Corollary C1]). Because S is G-invariant, S is an Ore set in R, where $R = kG$ (see the proof of [**6**, Lemma 13.3.5 (ii)]), and by [**6**, Theorem 10.2.6] R_S is a Noetherian ring. Since $S \leqslant \mathscr{C}(\mathfrak{g})$ we can apply Lemma 2.3 to obtain

$$\kappa(\mathfrak{g}) \leqslant T_r(\mathfrak{g}).$$

On the other hand, suppose that G/H is locally finite. By [**11**, proof of Theorem E], for every finitely generated right ideal E of R there exists a positive integer m such that

$$E \cap \mathfrak{g}^m \leqslant E\mathfrak{g},$$

and by [**10**, Lemma 2.1] we conclude

$$\mathfrak{g}^\omega = \bigcap_{m=1}^{\infty} \mathfrak{g}^m = T_r(\mathfrak{g}).$$

Thus, in any case,

$$\kappa(\mathfrak{g}) \leqslant T_r(\mathfrak{g}).$$

Returning to the chain (1) we note that G has a finite series

$$1 = K_0 \leqslant K_1 \leqslant \ldots \leqslant K_q = G$$

of subgroups K_i such that K_{i-1} is normal in K_i and K_i/K_{i-1} is infinite cyclic or locally finite for $1 \leqslant i \leqslant q$. If $G = G_1 \geqslant G_2 \geqslant \ldots$ is the lower central series of G then, arguing as in the proof of [**11**, Theorem D], G_{q+1}/G_{q+2} is a torsion group. It follows that if $U = G_{q+1}$ then

$$\mathfrak{u} \leqslant \mathfrak{g}^\omega.$$

Suppose that $\mathfrak{u} \leqslant \mathfrak{g}^\alpha$ for some ordinal α. If $u \in U$ and $x \in G$ then

$$1 - [u, x] = u^{-1}x^{-1}\{(1-x)(1-u) - (1-u)(1-x)\} \in \mathfrak{g}^{\alpha+1}.$$

Since G_{q+1}/G_{q+2} is a torsion group it follows that $\mathfrak{u} \leq \mathfrak{g}^{\alpha+1}$. Thus $\mathfrak{u} \leq \kappa(\mathfrak{g}) \leq T_r(\mathfrak{g})$. Now it is easy to prove that U is a locally finite group (see the proof of [**11**, Theorem D]). This proves (iii).

Theorem A has the following consequence.

COROLLARY A. *Let k be a field of characteristic 0 and G a hyper-(Abelian or locally finite) group and let $\mathfrak{g}$ be the augmentation ideal of the group algebra $R = kG$. Then the following statements are equivalent.*

(i) *$\mathfrak{g}$ is localizable, $T_1(\mathfrak{g}) = T_r(\mathfrak{g})$ and $R_{\mathfrak{g}}$ is a Noetherian ring.*

(ii) *$\mathfrak{g}$ has the AR property.*

(iii) *G is an extension of a locally finite group by a nilpotent group with each upper central factor of finite torsion-free rank.*

To prove Corollary A, by the theorem we need show only that (i) and (ii) are equivalent. By [**10**, Lemmas 2.1 and 2.2 and Corollary C1], (ii) implies (i). In order to prove that (i) implies (ii) we require some notation.

Let P be a localizable prime ideal of a ring R. If E is a right ideal of R then the *P-closure* of E is

$$\mathrm{cl}_P E = \{r \in R : rc \in E \text{ for some } c \text{ in } \mathscr{C}(P)\}.$$

Then $\mathrm{cl}_P E$ is a right ideal containing E. We call E *P-closed* provided $E = \mathrm{cl}_P E$. There are similar definitions for left ideals. The next lemma is elementary.

LEMMA 2.5. *Let P be a localizable prime ideal of a ring R such that $T(P) = T_r(P)$. Then the ring R_P is right Noetherian if and only if R satisfies the ascending chain condition on P-closed right ideals.*

To complete the proof of Corollary A, suppose that (i) holds. By Lemma 2.5, $R = kG$ satisfies the ascending chain condition on $\mathfrak{g}$-closed right ideals. By [**11**, Lemma B and the proof of Lemma A], G is poly-(locally finite or finitely generated Abelian) and so (iii) follows by Theorem A. This completes the proof of Corollary A.

We now turn our attention to Theorem B.

Proof of Theorem B. (ii) and (iii) are equivalent by [**11**, Theorem E]. Moreover, (ii) implies (i) by [**10**, Lemma 2.2]. It remains to prove that (i) implies (iii).

Suppose that (i) holds. In order to prove (iii) it is sufficient to prove that if A is a minimal normal subgroup of G and a p-group then A is central. There exists a chain

$$A = H_0 \leq H_1 \leq \ldots \leq H_n = G$$

of normal subgroups H_i of G such that H_i/H_{i-1} is finitely generated Abelian or locally finite-p' for each $1 \leq i \leq n$. The result is proved by induction on n. The case $n = 0$ is clear since A is finite. So suppose $n > 0$ and let $H = H_{n-1}$. By induction we can suppose that $\mathfrak{h}$ has the AR property in kH. Then following the argument used in the proof of Theorem A we obtain

$$\kappa(\mathfrak{g}) \leq T_r(\mathfrak{g}).$$

If A is not central then $A = [A, G]$ and it follows that

$$\mathfrak{o} \leq \kappa(\mathfrak{g})$$

so that A is a p'-group, a contradiction. (The argument is very like that in the proof of Theorem A and so the details are left to the reader.)

In the same way that Theorem A gives Corollary A, Theorem B gives the following result. The proof is virtually identical to that of Corollary A and so is omitted.

Corollary B. *Let k be a field of characteristic $p > 0$, G a hyper-(finitely generated Abelian or locally finite-p') group and $\mathfrak{g}$ the augmentation ideal of the group algebra $R = kG$. Then the following statements are equivalent.*

(i) *$\mathfrak{g}$ is localizable, $T_1(\mathfrak{g}) = T_r(\mathfrak{g})$ and $R_{\mathfrak{g}}$ is a Noetherian ring.*
(ii) *$\mathfrak{g}$ has the AR property.*
(iii) *G is an $\mathfrak{H}_p$-group and G centralizes all p-chief factors.*

Corollaries A and B should be compared with [**11**, Theorem C], where it is proved that if k is any field, G a locally nilpotent group and $\mathfrak{g}$ the augmentation ideal of $R = kG$ then statements (i) and (ii) of Corollary B are equivalent. In fact, for any group G, (ii) implies (i) (see [**10**, Lemmas 2.1 and 2.2 and Corollary C1]). This leaves the question of whether (i) always implies (ii).

3. Proofs of Theorems C, D1 and D2. The key result required is an old result of D. G. Higman (see [**6**, Lemma 7.2.2]). We call a ring R a *Higman extension* of a ring S if S is a subring of R with the same identity and there exists a finite collection of units u_i $(1 \leq i \leq n)$ in R such that

(i) n is a unit in R,
(ii) $u_i S = S u_i$ $(1 \leq i \leq n)$, $u_i S \neq u_j S$ $(1 \leq i \neq j \leq n)$,
(iii) $\{S u_i u_j : 1 \leq j \leq n\} = \{S u_j : 1 \leq j \leq n\}$ $(1 \leq i \leq n)$, and
(iv) $R = u_1 S + \ldots + u_n S$.

Higman's Lemma can be expressed in the following form.

Lemma 3.1. *Any Higman extension of a semiprime Artinian ring is semiprime Artinian.*

Corollary 3.2. *Let R be a Higman extension of a semiprime Goldie ring S. Let I be an ideal of R such that $\mathscr{C}_S(0) \leq \mathscr{C}_R(I)$. Then $Ic = 0$ for some element c of $\mathscr{C}_R(I)$.*

Proof. By [**2**, Theorems 4.1 and 4.4], S has a semiprime Artinian quotient ring Q. By [**6**, Lemma 13.3.5], $\mathscr{C}_S(0)$ is an Ore set in the ring R and we denote the partial quotient ring of R with respect to $\mathscr{C}_S(0)$ by Q_1. Clearly Q_1 is a Higman extension of Q and so, by the lemma, Q_1 is semiprime Artinian. Because $\mathscr{C}_S(0) \leq \mathscr{C}_R(I)$, it follows that $IQ_1 = \{ac^{-1} : a \in I,\ c \in \mathscr{C}_S(0)\}$ is an ideal of Q_1 and so is generated by a central idempotent element bd^{-1} (say) with b in I and d in $\mathscr{C}_S(0)$. Then $I(d-b) = 0$ and $d - b \in \mathscr{C}_R(I)$.

An ideal I of a ring R has a *weak centralizing set of generators* if there exists a finite chain of ideals

$$0 = I_0 \leq I_1 \leq \ldots \leq I_n = I$$

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such that, for each $1 \leqslant j \leqslant n$, I_j/I_{j-1} is generated by a finite collection of central elements of R/I_{j-1} or is $\mathscr{C}(I)$-torsion (i.e. for all a in I_j there exist c_1 and c_2 in $\mathscr{C}(I)$ such that $ac_1 \in I_{j-1}$ and $c_2 a \in I_{j-1}$). If each of the factors I_j/I_{j-1} $(1 \leqslant j \leqslant n)$ is generated by a finite collection of central elements of R/I_{j-1} then we say that I has a *centralizing set of generators*.

We extend these definitions in the following way. Let R be a ring and G a group of automorphisms of R. If $r \in R$ and $g \in G$ then

$$r^g$$

will denote the action of g on r. An element c of R will be called *G-central* if c is central in R and

$$c^g = c$$

for all g in G. Then G-invariant ideals having a *weak G-centralizing set of generators* or a *G-centralizing set of generators* will have the obvious meaning.

We say that an ideal I of a ring R has the *right fAR property* if for every finitely generated right ideal E there exists a positive integer n such that $E \cap I^n \leqslant EI$. The ideal I will be said to have the *right fAR property locally* if for every finitely generated right ideal E there exists a positive integer n such that

$$E \cap I^n \leqslant \mathrm{cl}_I\,(EI),$$

i.e. for each element r in $E \cap I^n$ there exists c in $\mathscr{C}(I)$ such that $rc \in EI$.

Suppose that I is an ideal of R such that I has the right fAR property locally. Let E be a finitely generated right ideal of R and suppose

$$x \in \bigcap_{n=1}^{\infty} \mathrm{cl}_I\,(E+I^n).$$

If $F = E + xR$ then there exists a positive integer m such that

$$F \cap I^m \leqslant \mathrm{cl}_I\,(FI).$$

There exist c in $\mathscr{C}(I)$ and e in E such that $xc - e \in F \cap I^m$ and so $(xc-e)d \in FI \leqslant E + xI$ for some element d of $\mathscr{C}(I)$. It follows that $x \in \mathrm{cl}_I E$. Hence

$$\bigcap_{n=1}^{\infty} \mathrm{cl}_I\,(E+I^n) = \mathrm{cl}_I E \tag{2}$$

for all finitely generated right ideals E of R. We require this fact in the proof of the next result.

Lemma 3.3. *Let Q be a localizable annihilator semiprime ideal of a ring R such that Q has a weak centralizing set of generators and Q has the right fAR property locally. Then R_Q is a right Noetherian ring.*

Proof. Let Y be a right ideal of R_Q and $Y_1 = \{r \in R : r + T \in Y\}$, where $T = T(Q)$. Then Y_1 is a Q-closed right ideal of R. Moreover, Y is a finitely generated right ideal of R_Q if and only if there exists a finitely generated right ideal Y_2 of R such that $Y_1 = \mathrm{cl}_Q Y_2$.

Suppose there exists a Q-closed right ideal of R which is not the Q-closure of a finitely generated right ideal. By Zorn's Lemma there exists a Q-closed right ideal E maximal with respect to not being the Q-closure of a finitely generated right ideal. Suppose that $Q \leqslant E$. Since Q has a weak centralizing set of generators it follows that QR_Q is a finitely generated right ideal of the ring R_Q. But the ring R_Q/QR_Q is isomorphic to the classical right quotient ring B of the ring R/Q and, by [**4**, Theorem], B is semiprime Artinian. It follows that $(E/T)R_Q$ is a finitely generated right ideal of R_Q. This implies that E is the Q-closure of a finitely generated right ideal of R, a contradiction.

Thus $Q \not\leqslant E$. Because Q has a weak centralizing set of generators there exists a finitely generated right ideal X_1, an ideal X and an element c of Q such that $\mathrm{cl}_Q X_1 = X \leqslant E$, c is central modulo X and $c \notin E$. Let $F = \{r \in R : cr \in E\}$. Then F is a Q-closed right ideal of R and $E \leqslant F$. Let $G = E + cR$. The choice of E entails that there exist a positive integer n and elements g_i $(1 \leqslant i \leqslant n)$ of G such that

$$G \leqslant \mathrm{cl}_Q(g_1R + \ldots + g_nR).$$

For each $1 \leqslant i \leqslant n$ let

$$g_i = e_i + cr_i$$

with e_i in E and r_i in R. Let $H = e_1R + \ldots + e_nR$.

Suppose $E \neq F$. Then by the choice of E there exists a finitely generated right ideal M such that $F = \mathrm{cl}_Q M$. Let $e \in E$. Then $e \in G$ and hence there exists an element d in $\mathscr{C}(Q)$ such that

$$ed = \sum_{i=1}^{n} e_i s_i + cu$$

for some elements $s_i (1 \leqslant i \leqslant n)$ and u in R. It follows that $u \in F$ and hence $e \in \mathrm{cl}_Q(H + cM)$. But this implies that $E = \mathrm{cl}_Q(H + cM)$ and, because $H + cM$ is a finitely generated right ideal, we have a contradiction. Thus $E = F$. In this case $E \leqslant \mathrm{cl}_Q(H + cE)$. Using the fact that c is central modulo $X = \mathrm{cl}_Q X_1$, it follows that

$$E \leqslant \bigcap_{s=1}^{\infty} \mathrm{cl}_Q(H + X_1 + c^sE) \leqslant \bigcap_{s=1}^{\infty} \mathrm{cl}_Q(H + X_1 + Q^s) = \mathrm{cl}_Q(H + X_1),$$

by (2). Hence $E = \mathrm{cl}_Q(H + X_1)$, another contradiction. The result follows.

LEMMA 3.4. *Let k be a field of characteristic $p \geqslant 0$ and G an $\mathfrak{N}_p$-group. Let P be an annihilator prime ideal of the group algebra $R = kG$. Then P is localizable, P has a weak centralizing set of generators and R_P is a Noetherian ring.*

Proof. There exists an infinite chain

$$1 = H_0 \leqslant H_1 \leqslant \ldots \leqslant H_\alpha \leqslant H_{\alpha+1} \leqslant \ldots \leqslant H_\rho = G$$

of normal subgroups H_α of G such that for all ordinals α,

(i) $H_{\alpha+1}/H_\alpha$ is an infinite cyclic group or a finite-p group and $[H_{\alpha+1}, G] \leqslant H_\alpha$, or

(ii) $H_{\alpha+1}/H_\alpha$ is a finite-p' group,

and

$$H_\alpha = \bigcup_{\beta < \alpha} H_\beta$$

if α is a limit ordinal. Moreover, all but a *finite* number of the factors $H_{\alpha+1}/H_\alpha$ are finite-p' groups. For each ordinal α with $0 \leqslant \alpha \leqslant \rho$ let $R^{(\alpha)} = kH_\alpha$ and $P^{(\alpha)} = P \cap kH_\alpha$. Then $P^{(\alpha)}$ is a G-invariant annihilator semiprime ideal of $R^{(\alpha)}$ for each ordinal α with $0 \leqslant \alpha \leqslant \rho$. To see that $R^{(\alpha)}/P^{(\alpha)}$ satisfies the ascending chain condition on right annihilators one need merely note that for any non-empty subset X of $R^{(\alpha)}$,

$$R^{(\alpha)} \cap \{r \in R : Xr \leqslant P\} = \{r \in R^{(\alpha)} : Xr \leqslant P^{(\alpha)}\}.$$

If N is the ideal of $R^{(\alpha)}$ containing $P^{(\alpha)}$ such that $N/P^{(\alpha)}$ is the sum of all nilpotent ideals of $R^{(\alpha)}/P^{(\alpha)}$ then N is G-invariant and, by [**3**, Theorem 1], $N/P^{(\alpha)}$ is nilpotent. It follows that NR is an ideal of R and $(NR)^s \leqslant P$ for some positive integer s. Hence $NR \leqslant P$ and it follows that $P^{(\alpha)}$ is semiprime.

Next we claim that, for each ordinal α with $0 \leqslant \alpha \leqslant \rho$,

$$P^{(\alpha)} \text{ is a localizable ideal of } R^{(\alpha)} \text{ such that } P^{(\alpha)} \text{ has a weak } G\text{-centralizing set of generators and } R^{(\alpha)}_{P^{(\alpha)}} \text{ is a Noetherian ring.} \tag{3}$$

The action of G on the ring $R^{(\alpha)}$ is by conjugation.

Suppose that (3) is false and let α be the least ordinal for which it fails to be true. Clearly $\alpha > 0$. Suppose first that α is not a limit ordinal. Let $A = H_{\alpha-1}$, $B = H_\alpha$, $P_1 = P^{(\alpha-1)}$, $P_2 = P^{(\alpha)}$, $S = R^{(\alpha-1)}$ and $T = R^{(\alpha)}$. Then $P_1 = P_2 \cap S$. By hypothesis, P_1 is localizable in S. Hence T satisfies the right and left Ore conditions with respect to $\mathscr{C}_S(P_1)$ (see [**6**, Lemma 13.3.5]). Let $U = \mathscr{C}_S(P_1)$ and

$$K = \{t \in T : tu \in P_2 \text{ for some } u \text{ in } U\}.$$

Then K is a G-invariant ideal of T and $P_2 \leqslant K$. By [**7**, Lemma 7], $P_2 < K$ implies the existence of an element t of K which is central in kG modulo P. But $tu \in P_2$ for some u in U and hence $u \in P_2 \cap S = P_1$, a contradiction. Thus $K = P_2$ and it follows that

$$U \leqslant C_T(P_2).$$

A similar argument shows that $T_l(P_1) = T_r(P_1)$.

Now suppose that B/A is a finite-p' group. By [**4**, Theorem], S/P_1 has a semiprime Artinian quotient ring and, by [**2**, Theorem 4.4], S/P_1 is a Goldie ring. Thus we can apply Corollary 3.2 to obtain that P_2/P_1T is $\mathscr{C}_T(P_2)$-torsion. It follows that P_2 has a weak G-centralizing set of generators. By hypothesis S_U is a Noetherian ring and hence T_U, being a finitely generated S_U-module, is a Noetherian ring. Hence, by [**8**, Theorem 2.2 Corollary 1], P_2T_U is localizable and it follows that P_2 is localizable and T_{P_2} is Noetherian.

Next suppose that B/A is infinite cyclic. By [**9**, Lemma 2.1], either $P_2 = P_1T$ or there exists an element c of P_2 which is G-central and regular modulo P_1T (and hence regular modulo P_1R) such that $P_2/(P_1T + cT)$ is $\mathscr{C}_T(P_2)$-torsion. As before, P_2 has a weak G-centralizing set of generators, P_2 is localizable and T_{P_2} is a Noetherian ring.

The other possibility is that B/A is a finite-p group. Then P_2/P_1T has a G-centralizing set of generators (see [**7**, Lemma 7]) and again P_2 has the desired properties. Thus α is a limit ordinal.

Let $\beta<\alpha$. Since $P^{(\beta)}$ is localizable it follows that $R^{(\alpha)}$ satisfies the right and left Ore conditions with respect to

$$\mathscr{C}_{R^{(\beta)}}(P^{(\beta)})$$

(see [**6**, Lemma 13.3.5]) and as above

$$\mathscr{C}_{R^{(\beta)}}(P^{(\beta)}) \leqslant \mathscr{C}_{R^{(\alpha)}}(P^{(\alpha)}).$$

It follows that

$$\mathscr{C}_{R^{(\alpha)}}(P^{(\alpha)}) = \bigcup_{0\leqslant\beta<\alpha} \mathscr{C}_{R^{(\beta)}}(P^{(\beta)}).$$

Consequently, $P^{(\alpha)}$ is localizable.

Since only a finite number of the factors $H_{\beta+1}/H_\beta$ are not finite-p' groups, there exists an ordinal γ with $0\leqslant\gamma<\alpha$ such that $H_{\beta+1}/H_\beta$ is a finite-p' group for each ordinal β with $\gamma\leqslant\beta<\alpha$. Thus H_α/H_γ is a locally finite-p' group and by the argument used earlier in the proof, $P^{(\alpha)}/P^{(\gamma)}R^{(\alpha)}$ is $\mathscr{C}(P^{(\alpha)})$-torsion. It follows that $P^{(\alpha)}$ has a weak G-centralizing set of generators.

Let $X=R^{(\alpha)}$, $Y=R^{(\gamma)}$ and $V=\mathscr{C}(P^{(\gamma)})$. Since $P^{(\gamma)}Y_V$ has a centralizing set of generators it follows that $P^{(\gamma)}Y_V$ has the AR property in Y_V (see [**5**, 2.7]). By adapting the proof of [**11**, Theorem E], we conclude that for each finitely generated right ideal E of X there exists a positive integer m such that for each element r of $E\cap X(P^{(\gamma)})^m$ there exists an element c of V such that $rc\in EP^{(\gamma)}$. Since $P^{(\alpha)}/XP^{(\gamma)}$ is a right $\mathscr{C}(P^{(\alpha)})$-torsion module it follows that $P^{(\alpha)}$ has the right fAR property locally in X. Hence by Lemma 3.3, $X_{P^{(\alpha)}}$ is a right Noetherian ring. Similarly it is a left Noetherian ring as well. This contradicts the choice of α and completes the proof of Lemma 3.5.

Theorem D1 follows at once from Lemma 3.4. Now let k be a field of characteristic $p\geqslant 0$ and G an $\mathfrak{N}_0$-group (if $p=0$) or an $\mathfrak{N}_p^*$-group (if $p\neq 0$). If P is an annihilator prime ideal of the ring $R=kG$ then by the proof of Lemma 3.5 we see that there exists a finite chain

$$0=P_0\leqslant P_1\leqslant\ldots\leqslant P_n=P$$

of ideals P_i of R such that P_i/P_{i-1} is generated by a central regular element of R/P_{i-1} or P_i/P_{i-1} is $\mathscr{C}(P)$-torsion for all $1\leqslant i\leqslant n$. Moreover, P is localizable and it follows that R_P is a regular local ring. By examining the proof of Lemma 3.5 we see that the dimension of R_P is at most $h(G)$. This completes the proof of Theorems C and D2.

Finally we mention an analogous result for integral group rings. Let $\mathfrak{X}$ denote the class of Abelian groups G which contain a free Abelian subgroup F of finite rank such that G/F is a torsion group with finite p-primary component for each prime p.

THEOREM 3.5. *Let G be a nilpotent group each of whose upper central factors is an $\mathfrak{X}$-group and let R be the integral group ring ZG. If P is an annihilator prime ideal of R then P is localizable and R_P is a Noetherian ring.*

Proof. If $P \cap Z = 0$ then the non-zero elements of Z belong to $\mathscr{C}_R(P)$. If Q is the rational field then $P' = PQ$ is an annihilator prime ideal of $S = QG$ and, by Theorem C, $R_P \cong S_{P'}$ is a regular local ring. If $P \cap Z \neq 0$ then there exists a prime p such that $p \in P$. By Theorem D1, $\bar{P} = P/pR$ is localizable and, if $\bar{R} = R/pR$, $\bar{R}_{\bar{P}}$ is Noetherian. It can easily be checked that pR has the right fAR property and P/p^nR is localizable for all integers $n \geq 1$. Let $r \in R$, $c \in \mathscr{C}(P)$. There exists a positive integer m such that

$$(rR + cR) \cap p^m R \leqslant (rR + cR)p.$$

There exist elements s in R and d in $\mathscr{C}(P)$ such that $rd - cs \in p^m R$ and, hence,

$$rd - cs = (ra + cb)p$$

for some a, b in R. Thus

$$r(d - ap) = c(s + bp)$$

and $d - ap \in \mathscr{C}(P)$. It follows that P is localizable. Also R_P/pR_P is a right Noetherian ring. By adapting the proof of Lemma 3.3, R_P is a right Noetherian ring. Similarly R_P is a left Noetherian ring.

Finally we can combine Corollaries A and B and Theorems C and D1 to characterize, for hypercentral groups G, those group algebras $R = kG$ such that every annihilator prime ideal P is localizable with R_P a Noetherian ring. Note that for such a prime ideal P, we have, by [**7**, Theorem A],

$$T_1(P) = T_r(P).$$

THEOREM 3.6. *Let k be a field of characteristic $p \geqslant 0$ and G a hypercentral group. Then a necessary and sufficient condition for every annihilator prime ideal P of the group algebra $R = kG$ to be localizable with R_P a Noetherian ring is that G be an $\mathfrak{N}_p$-group.*

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