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BRAUER CHARACTERS AND GROTHENDIECK RINGS

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Let G be a group of finite order g, A a splitting field of G of characteristic p (which may be 0) and $R = AG$ the group algebra of G over A. In [**2**], the author studied some of the properties of the Grothendieck ring $K(R)$ of the category of all finitely generated R-modules, and derived a number of consequences. This paper continues the study carried out in [**2**]. The study is concerned with the structure and irreducible representations of $K(R)$. The ring $K(R)$ is proved to be semisimple and the primitive idempotents of $K(R)$ are explicitly constructed. If the ring $K(R)$ is identified with the 'algebra of representations', then Robinson's idempotent [**3**; **4**; **5**] follow from our description as a special case.

Through out the paper, the following notations will be kept fixed. Let $\mathscr{C}_1 = \{1\}, \mathscr{C}_2, \ldots, \mathscr{C}_n$ be the p-regular conjugate classes of G, g_i be the number of elements in $\mathscr{C}_i$, $\{M_1, M_2, \ldots, M_n\}$ be a full set of pairwise non-isomorphic irreducible R-modules, m_i the equivalence class of R-modules isomorphic to M_i, $\varphi^1, \varphi^2, \ldots, \varphi^n$ be the irreducible Brauer characters of G at the prime p, $\varphi_j{}^i$ be the value of φ^i at an element of $\mathscr{C}_j$, Z be the ring of rational integers and C be the field of complex numbers. Then $\{m_1, m_2, \ldots, m_n\}$ is a basis of $K(R)$ over Z. Therefore there are n^3 unique integers c_{ijk} in Z such that

$$(*) \qquad m_i m_j = \sum_{k=1}^{n} c_{ijk} m_k.$$

The n^3 equations (*) expressed in terms of φ^i take the forms

$$(**) \qquad \varphi^i \varphi^j = \sum_{k=1}^{n} c_{ijk} \varphi^k.$$

LEMMA 1. *The ring $K(R)$ is a torsion-free Z-module.*

Proof. Let m an element of $K(R)$ and k an element of Z such that $km = 0$. If $m = \sum_1^n z_i m_i$, then $0 = km = \sum_1^n k z_i m_i = 0$. Since $\{m_1, m_2, \ldots, m_n\}$ is a basis of $K(R)$, it follows that $kz_i = 0$ for all i. Therefore either $m = 0$ or else $k = 0$. Hence $K(R)$ is a torsion-free Z-module.

LEMMA 2. *The ring $K(G) = K(R) \otimes_Z C$ is an n dimensional vector space over C.*

Proof. By Lemma 1, $K(R)$ is a torsion-free Z-module. The ring Z is a Dedekind domain with the rational field Q as the field of fractions. Therefore it follows from the material of Section 22, pp. 144-145 of [**1**] that $K(R) \otimes_Z Q$ is a

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vector space of dimension n over Q. Since C is an extension of Q, it follows that $(K(R) \otimes_Z Q) \otimes_Q C$ is a vector space of dimension n over C. By the associativity of tensor products, we have that

$$(K(R) \otimes_Z Q) \otimes_Q C \cong K(R) \otimes_Z (Q \otimes_Q C) \cong K(R) \otimes_Z C = K(G).$$

Hence $K(G)$ is a vector space of dimension n over C.

THEOREM 3. *The ring $K(R)$ is semisimple.*

Proof. The ring $K(R)$ is a torsion-free module of rank n over Z. Since every submodule of $K(R)$ is a torsion-free module of rank at most n, it follows that $K(R)$ is an Artinian ring. Hence the (Jacobson) radical of $K(R)$ is nilpotent and it is the largest nilpotent ideal of $K(R)$. Thus the radical of $K(R)$ consists of all the nilpotent elements of $K(R)$. If m is any nilpotent element of $K(R)$, then $m \otimes k$ is a nilpotent element of $K(G)$ for any k in C. Therefore it suffices to show that $K(G)$ is semisimple.

If Z_n is the ring of all n by n matrices over Z, then the mapping T from $K(R)$ to Z_n given by $mT = (a_{ij}^m)$, where $mm_i = \sum_j a_{ij}^m m_j$, is a regular representation of $K(R)$ over Z [**2**]. Hence $K(R)T \otimes_Z C$ is a regular representation of $K(G)$. The n by n matrix $\Phi = (\varphi_j^i)$, where $\varphi^i = (\varphi_1^i, \varphi_2^i, \ldots, \varphi_n^i)$, is a non-singular matrix. It is proved in [**2**] that

$$\Phi^{-1} m_k T \Phi = \begin{bmatrix} \varphi_1^k, 0, \ldots, 0 \\ 0, \varphi_2^k, \ldots, 0 \\ \ldots\ldots\ldots\ldots \\ 0, 0, \ldots, \varphi_n^k \end{bmatrix}$$

for $k = 1, 2, \ldots, n$. Hence Φ diagonalizes the matrices $m_1T, m_2T, \ldots, m_nT$ simultaneously. Since $\{m_1T, m_2T, \ldots, m_nT\}$ is a basis of $K(R)T$, it follows that $K(G)$ is a completely reducible algebra over C. Hence $K(G)$ is semisimple, so that $K(G)$ has no non-zero nilpotent elements. Therefore $K(R)$ has no non-zero nilpotent elements and hence it is semisimple.

Irreducible representations of $K(G)$. The algebra $K(G)$ is a commutative algebra over an algebraically closed field C and hence every irreducible representation of $K(G)$ is one-dimensional. Therefore an irreducible representation of $K(G)$ can be considered as an algebra homomorphism from $K(G)$ to C.

THEOREM 4. *The mapping θ^k from $K(G)$ to C defined by*

$$\left(\sum_{j=1}^{n} z_j m_j\right)\theta^k = \sum_{j=1}^{n} z_j \varphi^j{}_k$$

is an irreducible representation of $K(G)$. Moreover $\{\theta^1, \theta^2, \ldots, \theta^n\}$ is a full set of pairwise inequivalent irreducible representations of $K(G)$ over C.

Proof. By definition of θ^k, it is a vector space homomorphism. Let $\sum_{i=1}^n x_i m_i$

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and $\sum_1^n y_i m_i$ be two arbitrary elements of $K(G)$. Then we have

$$\begin{aligned}\Big(\sum_i x_i m_i \cdot \sum_j y_j m_j\Big)\theta^k &= \Big(\sum_i \sum_j x_i y_i m_i m_j\Big)\theta^k \\ &= \sum_i \sum_j \sum_l x_i y_j c_{ijl} m_l \theta^k \text{ by } (*) \\ &= \sum_i \sum_j x_i y_j \sum_l c_{ijl} \varphi_k{}^l \\ &= \sum_i \sum_j x_i y_j \varphi_k{}^i \varphi_k{}^j \text{ by } (**) \\ &= \Big(\sum_i x_i \varphi_k{}^i\Big)\Big(\sum_j y_j \varphi_k{}^j\Big) \\ &= \Big(\sum_i x_i m_i\Big)\theta^k \Big(\sum_j y_j m_j\Big)\theta^k.\end{aligned}$$

Hence θ^k is a representation of $K(G)$. Since the degree of θ^k is 1, it is obviously irreducible.

Suppose π is an isomorphism of the algebra $K(G)\theta^i$ to $K(G)\theta^j$. Then $\theta^i\pi = \theta^j$ Since $m_k\theta^i\pi = m_k\theta^j$, we get $\varphi_i{}^k\pi = \varphi_j{}^k$ for all $k = 1, 2, \ldots, n$. But $\varphi_i{}^k\pi = (1\varphi_i{}^k)\pi = 1\pi\varphi_i{}^k = \varphi_i{}^k = \varphi_j{}^k$. Hence $i = j$. Therefore it follows that $\theta^1, \theta^2, \ldots, \theta^n$ are pairwise inequivalent irreducible representations of $K(G)$. Since the degree of $K(G)$ over C is n, $\{\theta^1, \theta^2, \ldots, \theta^n\}$ is a full set of pairwise inequivalent irreducible representations of $K(G)$.

THEOREM 5. *If $\eta^1, \eta^2, \ldots, \eta^n$ are the projective indecomposable Brauer characters of G, then*

$$f_i = \frac{g_i}{g} \sum_{j=1}^n \overline{\eta_i{}^j} m_j$$

is a primitive idempotent of $K(G)$ for $i = 1, 2, \ldots, n$.

Proof. By Theorem 3, $K(G)$ is a semisimple algebra of dimension n over C. Since $K(G)$ is commutative, $K(G)$ has n primitive idempotents, say $f_1, f_2, \ldots, f_n$. There are unique complex numbers a_{ij} such that $f_i = \sum_{j=1}^n a_{ij} m_j$. If θ^i is the irreducible representation of $K(G)$ corresponding to the simple ideal $f_iK(G)$, then $f_i\theta^i = 1$ and $f_j\theta^i = 0$ for $i \neq j$. Hence

$$f_i\theta^k = \delta_{ik} = \sum_{j=1}^n a_{ij} m_j \theta^k = \sum_{j=1}^n a_{ij} \varphi_k{}^j.$$

Multiplying both sides of

$$\delta_{ik} = \sum_{j=1}^n a_{ij}\varphi_k{}^j \text{ by } g_k\overline{\eta_k{}^l}$$

and summing over k, we obtain that

$$\sum_k \delta_{ik} g_k \overline{\eta_k{}^l} = \sum_{j=1}^n a_{ij} \sum_{k=1}^n g_k \varphi_k{}^j \overline{\eta_k{}^l}.$$

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By orthogonality relations for Brauer characters [**1**, p. 600], we have $\sum_{k=1}^{n} g_k \varphi_k{}^i \overline{\eta_k{}^l} = g\, \delta_{il}$, so that $a_{ij} = g_i \overline{\eta_i{}^j}/g$. Hence

$$f_i = \frac{g_i}{g} \sum_{j=1}^{n} \overline{\eta_i{}^j} m_j$$

is a primitive idempotent of $K(G)$ for $i = 1, 2, \ldots, n$.

Example. We illustrate the construction of the primitive idempotents of $K(G)$ by means of an example. Let $L_2(5)$ be the simple group of order 60 and A a splitting field of characteristic 5. The group G has three 5-regular conjugate classes containing 1, 15 and 20 elements. Hence $K(G)$ has dimension 3 over C. Let $\{m_1, m_2, m_3\}$ be a basis of $K(G)$ determined by the equivalence classes of irreducible $AL_2(5)$-modules. The irreducible Brauer characters φ^1, φ^2 and φ^3 are given in [**2**] by

$$\varphi^1 = (1, 1, 1), \quad \varphi^2 = (3, -1, 0) \quad \text{and} \quad \varphi^3 = (5, 1, -1).$$

The Cartan matrix of $AL_2(5)$ is

$$\begin{bmatrix} 2, & 1, & 0 \\ 1, & 3, & 0 \\ 0, & 0, & 1 \end{bmatrix}.$$

Therefore the projective indecomposable Brauer characters η^1, η^2, η^3 at prime 5 are given by

$$\eta^1 = (5, 1, 2), \quad \eta^2 = (10, -2, 1) \quad \text{and} \quad \eta^3 = (5, 1, -1).$$

Hence the primitive idempotents are:

$$f_1 = \frac{1}{60}[5m_1 + 10m_2 + 5m_3],$$

$$f_2 = \frac{15}{60}[m_1 - 2m_2 + m_3],$$

$$f_3 = \frac{20}{60}[2m_1 + m_2 - m_3].$$

REFERRNCES

1. C. W. Curtis and I. Reiner, *Representation theory of finite groups and associative algebras* (Interscience, New York, 1962), pp. 144–145 and pp. 598–600.
2. B. M. Puttaswamaiah, *Determination of Brauer characters*, Can. J. Math. *26* (1974) 746–752.
3. G. deB. Robinson, *The algebras of representations and classes of finite groups*, J. Mathematical Phys. *12* (1971), 2212–2215.
4. ——— *Tensor product representations*, J. of Algebra *20* (1972), 118–123.
5. ——— *The dual of Frobenius' reciprocity theorem*, Can. J. Math. *25* (1973), 1051–1059.

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