ON RECURRENCE FORMULAE

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(Received 20th October, 1955)

1. Introductory. The recurrence formulae for the Bessel, Legendre, hypergeometric and other such functions can all be related to each other by means of the E-functions. In this paper it will be shown how, starting from known recurrence formulae for the hypergeometric function, others can be derived. The E-function formulae are deduced in § 2, and the others in § 3.

2. E-function formulae. Many recurrence formulae are known ([1], [2]) for the hypergeometric function. From these the following four have been selected for consideration.

$$\gamma F(\alpha, \beta-1 ; \gamma ; z)-\gamma F(\alpha-1, \beta ; \gamma ; z)+(\alpha-\beta) z F(\alpha, \beta ; \gamma+1 ; z)=0. \quad \ldots\ldots(1)$$

$$\gamma(\alpha-\beta) F(\alpha, \beta ; \gamma ; z)-\alpha(\gamma-\beta) F(\alpha+1, \beta ; \gamma+1 ; z)+\beta(\gamma-\alpha) F(\alpha, \beta+1 ; \gamma+1 ; z)=0. \quad \ldots\ldots(2)$$

$$\gamma(\gamma+1) F(\alpha, \beta ; \gamma ; z)-\gamma(\gamma+1) F(\alpha, \beta ; \gamma+1 ; z)-\alpha \beta z F(\alpha+1, \beta+1 ; \gamma+2 ; z)=0. \quad \ldots\ldots(3)$$

$$(\beta-\alpha) F(\alpha, \beta ; \gamma ; z)+\alpha F(\alpha+1, \beta ; \gamma ; z)-\beta F(\alpha, \beta+1 ; \gamma ; z)=0. \quad \ldots\ldots(4)$$

Now in (1) replace α, β, γ and z by α_1, α_2, ρ_1-1 and $-1/z$, and multiply by $\Gamma(\alpha_1)$, $\Gamma(\alpha_2)$ and $1/\Gamma(\rho_1)$; then the formula can be written

$$(\alpha_1-\alpha_2) E(\alpha_1, \alpha_2 : \rho_1 : z)=(\alpha_2-1) z E(\alpha_1, \alpha_2-1 : \rho_1-1 : z)-(\alpha_1-1) z E(\alpha_1-1, \alpha_2 : \rho_1-1 : z) ;$$

and, when this is generalised, it becomes

$$(\alpha_1-\alpha_2) z^{-1} E(p ; \alpha_r : q ; \rho_s : z)=(\alpha_2-1) E(\alpha_1, \alpha_2-1, \ldots, \alpha_p-1 : q ; \rho_s-1 : z)$$
$$-(\alpha_1-1) E(\alpha_1-1, \alpha_2, \alpha_3-1, \ldots, \alpha_p-1 : q ; \rho_s-1 : z). \quad \ldots\ldots(5)$$

In the same manner, from formulae (2) to (4) the following can be derived.

$$(\alpha_1-\alpha_2) E(p ; \alpha_r : q ; \rho_s : z)=(\rho_1-\alpha_2) E(\alpha_1+1, \alpha_2, \ldots, \alpha_p : \rho_1+1, \rho_2, \ldots, \rho_q : z)$$
$$-(\rho_1-\alpha_1) E(\alpha_1, \alpha_2+1, \alpha_3, \ldots, \alpha_p : \rho_1+1, \rho_2, \ldots, \rho_q : z). \quad \ldots\ldots(6)$$

$$E(p ; \alpha_r : q ; \rho_s : z)=\rho_1 E(p ; \alpha_r : \rho_1+1, \rho_2, \ldots, \rho_q : z)$$
$$-z^{-1} E(p ; \alpha_r+1 : \rho_1+2, \rho_2+1, \ldots, \rho_q+1 : z). \quad \ldots\ldots\ldots\ldots(7)$$

$$(\alpha_1-\alpha_2) E(p ; \alpha_r : q ; \rho_s : z)=E(\alpha_1+1, \alpha_2, \ldots, \alpha_p : q ; \rho_s : z)$$
$$-E(\alpha_1, \alpha_2+1, \alpha_3, \ldots, \alpha_p : q ; \rho_s : z). \quad \ldots\ldots(8)$$

From these formulae others can be deduced. For instance, on applying (5) to the terms on the right of (5), it is found that

$$(\alpha_1-\alpha_2-1)(\alpha_1-\alpha_2+1)(\alpha_1-\alpha_2) E(p ; \alpha_r : q ; \rho_s : z)$$
$$=(\alpha_1-\alpha_2-1)(\alpha_2-1)(\alpha_2-2) z^2 E(\alpha_1, \alpha_2-2, \ldots, \alpha_p-2 : q ; \rho_s-2 : z)$$
$$-2(\alpha_1-\alpha_2)(\alpha_1-1)(\alpha_2-1) z^2 E(\alpha_1-1, \alpha_2-1, \alpha_3-2, \ldots, \alpha_p-2 : q ; \rho_s-2 : z)$$
$$+(\alpha_1-\alpha_2+1)(\alpha_1-1)(\alpha_1-2) z^2 E(\alpha_1-2, \alpha_2, \alpha_3-2, \ldots, \alpha_p-2 : q ; \rho_s-2 : z). \quad \ldots(9)$$

Again, on applying (5) to the terms on the right of (8), we get

$$(\alpha_1-\alpha_2-1)(\alpha_1-\alpha_2+1)(\alpha_1-\alpha_2) E(p ; \alpha_r : q ; \rho_s : z)$$
$$=(\alpha_1-\alpha_2-1)(\alpha_2-1) z E(\alpha_1+1, \alpha_2-1, \ldots, \alpha_p-1 : q ; \rho_s-1 : z)$$
$$-(\alpha_1+\alpha_2-1)(\alpha_1-\alpha_2) z E(\alpha_1, \alpha_2, \alpha_3-1, \ldots, \alpha_p-1 : q ; \rho_s-1 : z)$$
$$+(\alpha_1-\alpha_2+1)(\alpha_1-1) z E(\alpha_1-1, \alpha_2+1, \alpha_3-1, \ldots, \alpha_p-1 : q ; \rho_s-1 : z). \quad \ldots\ldots(10)$$

https://doi.org/10.1017/S2040618500033402 Published online by Cambridge University Press

Finally, apply (5) to the terms on the right of (6), and so obtain

$$(\alpha_1-\alpha_2-1)(\alpha_1-\alpha_2+1)(\alpha_1-\alpha_2)E(p\,;\,\alpha_r:q\,;\,\rho_s:z)$$
$$=(\alpha_1-\alpha_2-1)(\rho_1-\alpha_2)(\alpha_2-1)zE(\alpha_1+1,\alpha_2-1,\ldots,\alpha_p-1:\rho_1,\rho_2-1,\ldots,\rho_q-1:z)$$
$$-\{\rho_1(\alpha_1+\alpha_2-1)-2\alpha_1\alpha_2\}zE(\alpha_1,\alpha_2,\alpha_3-1,\ldots,\alpha_p-1:\rho_1,\rho_2-1,\ldots,\rho_q-1:z)$$
$$+(\alpha_1-\alpha_2+1)(\rho_1-\alpha_1)(\alpha_1-1)zE(\alpha_1-1,\alpha_2+1,\alpha_3-1,\ldots,\alpha_p-1:\rho_1,\rho_2-1,\ldots,\rho_q-1:z)\ldots(11)$$

The formula for the derivative is

$$\frac{d}{dz}E(p\,;\,\alpha_r:q\,;\,\rho_s:z)=\frac{1}{z^2}E(p\,;\,\alpha_r+1:q\,;\,\rho_s+1:z).\quad\ldots\ldots\ldots\ldots(12)$$

3. Special Cases. The recurrence formulae for the Bessel and Legendre functions are special cases of the formulae of § 2.

Bessel functions. In (7) take $p=0$, $q=1$, $\rho_1=n$ and replace z by $4/z^2$; then, since

$$J_n(z)=(\tfrac{1}{2}z)^nE(\;:n+1:4/z^2),\quad\ldots\ldots\ldots\ldots(13)$$
$$J_{n-1}(z)-2nz^{-1}J_n(z)+J_{n+1}(z)=0.\quad\ldots\ldots\ldots\ldots(14)$$

Next, in (10) put $p=2$, $q=0$, $\alpha_1=\frac{1}{2}+n$, $\alpha_2=\frac{1}{2}-n$, and replace z by $2z$; then, since

$$E(\tfrac{1}{2}+n,\tfrac{1}{2}-n::2z)=\sec n\pi\,\sqrt{(2\pi z)}e^zK_n(z),\quad\ldots\ldots\ldots\ldots(15)$$
$$2nz^{-1}K_n(z)=K_{n+1}(z)-K_{n-1}(z).\quad\ldots\ldots\ldots\ldots(16)$$

Legendre functions. In (10) put $p=2$, $q=1$, $\alpha_1=-n$, $\alpha_2=n+1$, $\rho_1=m+2$, and replace z by $2/(z-1)$; then, since

$$T_n^{-m}(z)=-\pi^{-1}\sin n\pi\left(\frac{1-z}{1+z}\right)^{\frac{1}{2}m}E\left(-n,n+1:m+1:\frac{2}{z-1}\right),\ldots\ldots\ldots\ldots(17)$$
$$(2n+1)\sqrt{(1-z^2)}T_n^{-m-1}(z)=T_{n-1}^{-m}(z)-T_{n+1}^{-m}(z).\quad\ldots\ldots\ldots\ldots(18)$$

Again, in (11) put $p=2$, $q=1$, $\alpha_1=-n$, $\alpha_2=n+1$, $\rho_1=m+1$ and replace z by $2/(z-1)$; then

$$(n+m+1)T_{n+1}^{-m}(z)-(2n+1)zT_n^{-m}(z)+(n-m)T_{n-1}^{-m}(z)=0.\quad\ldots\ldots\ldots\ldots(19)$$

Next, in (7) put $p=2$, $q=1$, $\alpha_1=\frac{1}{2}(n+m)$, $\alpha_2=\frac{1}{2}(n+m+1)$, $\rho_1=n+\frac{1}{2}$, and replace z by $-z^2$; then, since

$$Q_n^m(z)=2^{m-1}(z^2-1)^{\frac{1}{2}m}z^{-n-m-1}E(\tfrac{1}{2}n+\tfrac{1}{2}m+\tfrac{1}{2},\tfrac{1}{2}n+\tfrac{1}{2}m+1:n+\tfrac{3}{2}:-z^2),\quad\ldots\ldots\ldots(20)$$
$$Q_{n-1}^m(z)-(2n+1)\sqrt{(z^2-1)}Q_n^{m-1}(z)-Q_{n+1}^m(z)=0.\quad\ldots\ldots\ldots\ldots(21)$$

From (12),
$$\frac{d}{dz}E(\;:n+1:4/z^2)=-\tfrac{1}{2}z\,E(\;:n+2:4/z^2),$$

and therefore, from (13),

$$J_n'(z)-nz^{-1}J_n(z)=-J_{n+1}(z).\quad\ldots\ldots\ldots\ldots(22)$$

Hence, from (14),
$$2J_n'(z)=J_{n-1}(z)-J_{n+1}(z).\quad\ldots\ldots\ldots\ldots(23)$$

The other derivatives can be derived in much the same way. For instance, for $K_n(z)$, apply (9) to the expression on the right of (12), and then use formula (15).

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