Ergod. Th. & Dynam. Sys. (1990), **10**, 177–183
Printed in Great Britain

An explosion point for the set of endpoints of the Julia set of $\lambda \exp(z)$

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(*Received* 17 *May* 1988 *and revised* 11 *June* 1988)

Abstract. The Julia set J_λ of the complex exponential function $E_\lambda : z \to \lambda e^z$ for a real parameter $\lambda (0 < \lambda < 1/e)$ is known to be a *Cantor bouquet* of rays extending from the set A_λ of endpoints of J_λ to ∞. Since A_λ contains all the repelling periodic points of E_λ, it follows that $J_\lambda = \mathrm{Cl}(A_\lambda)$. We show that A_λ is a totally disconnected subspace of the complex plane $\mathbb{C}$, but if the point at ∞ is added, then $\hat{A}_\lambda = A_\lambda \cup \{\infty\}$ is a connected subspace of the Riemann sphere $\hat{\mathbb{C}}$. As a corollary, A_λ has topological dimension 1. Thus, ∞ is an *explosion point* in the topological sense for $\hat{A}_\lambda$. It is remarkable that a space with an explosion point occurs 'naturally' in this way.

1. *Introduction*

The Julia set J_λ of each member of the family of complex exponential functions $E_\lambda : z \to \lambda e^z$ for a real parameter $\lambda (0 < \lambda < 1/e)$ was shown by Devaney [**DK**] (also see [**D1, D3, DG**]) to be a *Cantor bouquet* of rays in the complex plane $\mathbb{C}$ extending from the set A_λ of endpoints of J_λ to ∞. Since A_λ contains all the repelling periodic points of E_λ, it follows that $J_\lambda = \mathrm{Cl}(A_\lambda)$. Devaney and Goldberg [**DG**] show that $\hat{A}_\lambda = A_\lambda \cup \{\infty\}$ is precisely the set of points of $\hat{J}_\lambda = J_\lambda \cup \{\infty\}$ that are accessible from the domain $\hat{\mathbb{C}} - \hat{J}_\lambda$ in the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Using this fact, we show in Theorem 3 that A_λ is totally disconnected, but that $\hat{A}_\lambda$ is connected, so that ∞ is an explosion point for A_λ. (A point x in a space X is an *explosion point* iff X is connected and $X - \{x\}$ is totally disconnected.) It also follows, from the fact that $\hat{A}_\lambda$ is connected, but nowhere dense in $\mathbb{C}$, that A_λ has topological dimension 1. We find it remarkable that a space with an explosion point occurs 'naturally' in this way, as the set of accessible points of a Julia set.

One ingredient in our proof of Theorem 3 is Theorem 2, interesting in its own right, which asserts that the set consisting of the principal points of all the prime ends of a bounded plane domain is connected, and thus, topologically one-dimensional.

Notation. Let X be a space. By $\mathrm{Cl}(A)$, we mean the closure of the set $A \subset X$. By $\mathrm{Bd}(A)$, called *boundary* A, we mean $\mathrm{Cl}(A) \cap \mathrm{Cl}(X - A)$. By E^n we denote Euclidean n-space and by S^n the unit sphere in E^{n+1}. Topologically, $\mathbb{C}$ is the plane

† Supported in part by NSF/Alabama EPSCoR grant number RII-8610669.

E^2 and $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the sphere S^2. For any $X \subset \mathbb{C}$ we denote by $\hat{X}$, the set $X \cup \{\infty\} \subset \hat{\mathbb{C}}$.

Topological dimension. An excellent exposition of topological dimension theory for separable metric spaces may be found in [**HW**], which also contains a very short section on Hausdorff dimension. A survey of Hausdorff dimension for topologists may be found in [**Ke**]. Let X be a separable metric space. We define the *topological dimension* of X, denoted $\dim(X)$, inductively as follows:

(1) $\dim(X) = -1$ iff $X = \emptyset$.

(2) $\dim(X) \le n$, where $n = 0, 1, 2, \ldots$, iff for every point $x \in X$, for every neighborhood V of x, there exists a neighborhood U of x such that $U \subset V$ and $\dim(\mathrm{Bd}(U)) \le n-1$.

(3) $\dim(X) = n$ iff $\dim(X) \le n$ and $\dim(X) \not\le n-1$.

(4) $\dim(X) = \infty$ iff $\dim(X) \not\le n$, for all $n = -1, 0, 1, 2, \ldots$.

All spaces in this paper are separable metric spaces, in particular, subsets of the plane E^2 and the sphere S^2. The topological dimension of a nonempty space is always a nonnegative integer. Unlike Hausdorff dimension or fractal dimension, topological dimension is a topological invariant. That is, if two spaces X and Y are *homeomorphic* (X is the one-to-one, continuous image of Y under a map with a continuous inverse), then $\dim(X) = \dim(Y)$.

Consider the middle third Cantor set C (the remainder of the closed interval $[0, 1]$ after deleting the open interval $(\frac{1}{3}, \frac{2}{3})$, and deleting the middle third of each resulting closed interval, *ad infinitum*) which has topological dimension 0 and Hausdorff dimension $\ln 2/\ln 3$. A homeomorphic image of C, say a 'middle fifth' Cantor set, may have a different Hausdorff dimension, but will still have topological dimension 0. As further examples, each of the interval $[0, 1]$, the unit circle S^1, and the real line E^1 has topological dimension 1; while $\dim(E^2) = \dim(S^2) = 2$.

Some properties of dim we will need subsequently include:

(A) If $A \subset B$, then $\dim(A) \le \dim(B)$.

(B) If $A \subset E^n$, then $\dim(Bd(A)) \le n-1$.

(C) If a space X is connected and consists of more than one point, then $\dim(X) \ge 1$.

Let $\mathrm{Hdim}(X)$ denote the Hausdorff dimension of a space X. The fundamental relationship between topological and Hausdorff dimension is

(D) $\dim(X) \le \mathrm{Hdim}(X)$.

Since, $\mathrm{Hdim}(E^n) = n$, and $\mathrm{Hdim}(A) \le \mathrm{Hdim}(B)$ whenever $A \subset B$, we have

(E) If $X \subset E^n$, then $\mathrm{Hdim}(X) \le n$.

It is a theorem of McMullen [**Mc**] that $\mathrm{Hdim}(J_\lambda) = 2$. It is an open question as to the Hausdorff dimension of A_λ. However, as a consequence of our main result, $1 \le \mathrm{Hdim}(A_\lambda) \le 2$.

Prime ends. We begin with some preliminaries concerning prime end theory. Useful introductions to prime end theory can be found in [**Br, CL, M, P**].

A *domain* is a connected, simply connected open subset of the plane E^2 or the sphere S^2. We use D to denote the open unit disk in E^2 with center 0. The boundary $\mathrm{Bd}(U)$ of a bounded domain U is always a *continuum* (compact and connected).

However, the boundary of a domain in S^2 might be a single point. Prime end theory, as defined below, cannot be applied to the latter case. Therefore, we consider only domains in E^2 or S^2 whose boundary is a *nondegenerate* continuum (consisting of more than one point).

If $Q \subset E^2$ is the homeomorphic image of the open interval $(0, 1)$, we say that Q is an *open arc*. A *crosscut* of a domain U is an open arc $Q \subset U$ such that $\mathrm{Cl}(Q)$ is an arc whose endpoints a and b lie in $\mathrm{Bd}(U)$ and are distinct. A *chain of crosscuts* of U is a collection $\{Q_i\}_{i=1}^{\infty}$ of crosscuts of U such that

(1) Q_i separates Q_{i-1} from Q_{i+1} in U for all $i > 1$,

(2) $\mathrm{Cl}(Q_i) \cap \mathrm{Cl}(Q_j) = \varnothing$ for all $i \neq j$, and

(3) Q_i converges to a point $p \in \mathrm{Bd}(U)$.

Two chains $\{Q_i\}_{i=1}^{\infty}$ and $\{S_i\}_{i=1}^{\infty}$ of crosscuts of U are said to be *equivalent* iff their union contains a collection of crosscuts of U with infinitely many entries from each satisfying (1) and (2) above. A *prime end* E of U is an equivalence class of chains of crosscuts. A representative of E is said to *define* E.

By the Riemann Mapping Theorem, there is a conformal homeomorphism $\varphi : U \to D$ with the properties that crosscuts of U are carried by φ to crosscuts of D, and that the collection of endpoints of images of crosscuts of U is dense in $\mathrm{Bd}(D)$. The content of Caratheodory's **[C]** main theorem may be expressed by saying that the conformal homeomorphism φ induces a one-to-one correspondence between the prime ends of U and the points of $\mathrm{Bd}(D)$. Indeed, if $\{Q_i\}_{i=1}^{\infty}$ is any chain of crosscuts of U defining a fixed prime end E, then there is a unique $e \in \mathrm{Bd}(D)$ (independent of the representative $\{Q_i\}_{i=1}^{\infty}$ of E) such that $\varphi(Q_i) \to e$.

Define the set

$$P(E) = \{p \in \mathrm{Bd}(U) \mid \text{for some } \{Q_i\}_{i=1}^{\infty} \text{ defining } E,\ Q_i \to p\},$$

called the set of *principal points* of E. It can be shown that $P(E)$ is a continuum. Thus, we usually call $P(E)$ the *principal continuum* of E.

The principal set of U. Define the *principal set* of the domain U to be the collection of all the principal points of all the prime ends of U; that is,

$$P(U) = \bigcup \{P(E) \mid E \text{ is a prime end of } U\}.$$

In Theorem 2, we prove that $P(U)$ is connected. However, it is not generally the case that $P(U)$ is closed.

Accessible points. If $R \subset E^2$ is the continuous, one-to-one image of the half-line $[0, \infty)$, we say that R is a *ray*. An *endcut* of the domain U is a ray $R \subset U$ such that $\mathrm{Cl}(R)$ is an arc and one endpoint of R is in $\mathrm{Bd}(U)$. A point $p \in \mathrm{Bd}(U)$ is *accessible* (from U) iff there is an endcut R in U whose endpoint in $\mathrm{Bd}(U)$ is p. It can be shown **[Br, UY]** that each accessible point of $\mathrm{Bd}(U)$ is the principal continuum of some prime end. Conversely, if E is a prime end of U and $P(E) = \{p\}$, then p is accessible. Therefore, the union of all degenerate principal continua of U coincides with the set of accessible points of $\mathrm{Bd}(U)$.

Radial limits. Let $\psi : D \to U$ be a conformal homeomorphism, and let $R_\theta = \{r e^{i\theta} \mid 0 \leq r < 1\} = [0, e^{i\theta})$ be a *radial ray* in D from the center 0 of D to the point

https://doi.org/10.1017/S0143385700005460 Published online by Cambridge University Press

$e^{i\theta} \in \mathrm{Bd}(D)$. We refer to $\mathrm{Cl}(\psi(R_\theta)) - \psi(R_\theta)$ as the *radial cluster set* of ψ at $e^{i\theta}$. It can be shown **[CL, P]** that there is a prime end E_θ (corresponding to $e^{i\theta}$ under the correspondence induced by ψ^{-1}) such that

$$P(E_\theta) = \mathrm{Cl}(\psi(R_\theta)) - \psi(R_\theta).$$

The correspondence between radial rays of D and prime ends of U is one-to-one.

For a *radial limit* of ψ at $e^{i\theta}$ to exist, we mean that

$$P(E_\theta) = \mathrm{Cl}(\psi(R_\theta)) - \psi(R_\theta) = \{p\}.$$

The set of radial limit points of U coincides with the set of accessible points of $\mathrm{Bd}(U)$.

THEOREM 1. *Let U be a bounded plane domain, $X \subset U$ a connected set such that* $\mathrm{Cl}(X) \cap \mathrm{Bd}(U) \neq \emptyset$, and $\varphi : U \to D$ *a conformal homeomorphism. Then for at least one prime end E of U,* $\mathrm{Cl}(X) \cap P(E) \neq \emptyset$. *Indeed, for all* $e \in \mathrm{Cl}(\varphi(X)) \cap \mathrm{Bd}(D)$ *with E the prime end of U corresponding to e,* $\mathrm{Cl}(X) \supset P(E)$.

Proof. Since φ is a homeomorphism, it follows that $\mathrm{Cl}(\varphi(X)) \cap \mathrm{Bd}(D) \neq \emptyset$. Let $e \in \mathrm{Cl}(\varphi(X)) \cap \mathrm{Bd}(D)$, and let E be the prime end of U corresponding to e. Let $p \in P(E)$ and let $\{Q_i\}_{i=1}^{\infty}$ be a chain of crosscuts of U defining E such that $Q_i \to p$. Then $\{\varphi(Q_i)\}_{i=1}^{\infty}$ is a chain of crosscuts of D and $\varphi(Q_i) \to e$. For $i > 1$, $\mathrm{Cl}(\varphi(Q_i))$ separates $\mathrm{Cl}(D)$ between e and $\mathrm{Cl}(\varphi(Q_{i-1}))$. Since $\varphi(X)$ is connected and $\mathrm{Cl}(\varphi(X))$ contains e, $\varphi(X) \cap \varphi(Q_i) \neq \emptyset$ for almost all i. Hence, $X \cap Q_i \neq \emptyset$ for almost all i. Thus, $p \in \mathrm{Cl}(X)$. Therefore, $P(E) \subset \mathrm{Cl}(X)$. □

THEOREM 2. *Let U be a bounded plane domain. Then $P(U)$, the principal set of U, is connected. Consequently,* $\dim(P(U)) = 1$.

Proof. By way of contradiction, suppose that $J \cup L = P(U)$ is a separation of $P(U)$ between points $p \in J$ and $q \in L$, where J and L are disjoint, relatively open sets. There are disjoint, bounded open sets G and H in E^2 such that $G \cap P(U) = J$ and $H \cap P(U) = L$ **[Ku, II**, p. 128]. Thus, $(E^2 - (G \cup H)) \cap P(U) = \emptyset$. Since $\mathrm{Bd}(G) \subset E^2 - (G \cup H)$, we have $\mathrm{Bd}(G) \cap P(U) = \emptyset$.

Now $\mathrm{Bd}(G)$ contains a continuum K which separates p from q. That is, p and q are in different components of $E^2 - K$. Since $\mathrm{Bd}(U)$ is a continuum containing p and q, $K \cap \mathrm{Bd}(U) \neq \emptyset$. Since $U \cup \{p, q\}$ is connected, it also follows that $K \cap U \neq \emptyset$.

Let Z be a component of $K \cap U$. By the Boundary Bumping Theorem, we have $\mathrm{Cl}(Z) \cap \mathrm{Bd}(U) \neq \emptyset$. It follows from Theorem 1 that $\mathrm{Cl}(Z) \cap P(E) \neq \emptyset$ for some prime end E of U. Since $\mathrm{Cl}(Z) \subset K \subset \mathrm{Bd}(G)$ and $\mathrm{Bd}(G) \cap P(U) = \emptyset$, we have a contradiction. Therefore, $P(U)$ is connected. By property (C) of topological dimension, $\dim(P(U)) \geq 1$. Since $P(U) \subset \mathrm{Bd}(U)$, properties (A) and (B) imply that $\dim(P(U)) \leq 1$. □

Remark. The above exposited prime end theory, including Theorems 1 and 2, applies to the case that U is a domain in S^2 with nondegenerate boundary.

The Julia set of $z \to \lambda e^z$. The *Julia set* $J(F)$ of a complex analytic map $F : \mathbb{C} \to \mathbb{C}$ (for rational maps, $F:\hat{\mathbb{C}}\to\hat{\mathbb{C}}$) is defined by

$$J(F)=\{z\in\mathbb{C}\,|\,\{F^n\}_{n=1}^{\infty} \text{ is not normal at } z\}.$$

For entire maps (as well as rational maps) it is a theorem of Baker [**B**], generalizing classical theorems of Julia [**J**] and Fatou [**F**], that

$$J(F)=\mathrm{Cl}\,(\{z\in\mathbb{C}\,|\,z \text{ is a repelling periodic point for } F\}).$$

An introduction to Julia sets and complex analytic dynamics can be found in [**Bl, D3**]. A summary of results concerning the exponential function and certain other entire functions can be found in [**D1**].

Let $J_\lambda=J(E_\lambda)$ be the Julia set of the entire transcendental function $E_\lambda:z\to\lambda e^z$ for a positive real parameter λ. Answering a question of Fatou [**F**], Misiurewicz [**Mi**] has shown that for $\lambda=1$, $J_1=\mathbb{C}$. Misiurewicz's proof also appears in [**D3**], where Devaney points out that Misiurewicz's argument easily extends to show that $J_\lambda=\mathbb{C}$ for all $\lambda>1/e$.

However, for $0<\lambda\le 1/e$ Devaney [**D1, D3, DG, DK**] has shown that J_λ is nowhere dense in $\mathbb{C}$. For $0<\lambda<1/e$, it is the boundary of the basin of attraction of the unique attracting fixed point of E_λ, and this basin is a domain. Since J_λ changes abruptly at the parameter value $\lambda=1/e$ from being nowhere dense in $\mathbb{C}$ to being all of $\mathbb{C}$. Devaney has called J_λ an 'exploding' Julia set.

By a *Cantor bouquet* we mean a set homeomorphic to $C\times[0,\infty)$, where C is the Cantor set. Devaney shows that J_λ contains an increasing union of Cantor bouquets C_n of rays extending from a set A_n of endpoints ($C\times\{0\}$) to ∞ in $\mathbb{C}$. On the endpoint set A_n of each C_n, the dynamics of E_λ is that of the shift map on $2n+1$ symbols. The 'stems' $\{a\}\times(0,\infty)$ of the bouquet C_n are permuted according to this shift. Under iteration of E_λ, the endpoints of C_n have bounded orbits, but all points of the stems tend to ∞.

Devaney shows that $J_\lambda=\mathrm{Cl}\,(\bigcup_{n=1}^{\infty} C_n)$. Each component R of J_λ can be coordinatized by $[0,\infty)$. The endpoint set (those points having coordinate 0) of J_λ is denoted by A_λ, and $A_\lambda\supset\bigcup_{n=1}^{\infty} A_n$. As all the repelling periodic points of E_λ are in A_λ, we have $J_\lambda=\mathrm{Cl}\,(A_\lambda)$. We also call J_λ a *Cantor bouquet.* However, it turns out that in $\hat{\mathbb{C}}$, even though for each n, $\hat{C}_n=C_n\cup\{\infty\}$ is homeomorphic to the cone over the Cantor set, $\hat{J}_\lambda=J_\lambda\cup\{\infty\}$ is not homeomorphic to the cone over the Cantor set.

Let U be the domain $\hat{\mathbb{C}}-\hat{J}_\lambda$. Devaney and Goldberg [**DG**] show that for a conformal homeomorphism $\psi:D\to U$, all radial limits exist. Moreover, they show that the set of radial limits, and thus the set of accessible points, of U is $\hat{A}_\lambda$. Therefore, $P(U)=\hat{A}_\lambda$. As they observe, this means that the rays comprising J_λ are *inaccessible* except for their endpoints! In particular, they show that the point ∞ in $\hat{J}_\lambda$ is the radial limit of ψ at a dense set of points in $\mathrm{Bd}\,(D)$, and that each point $a\in A_\lambda$ is the radial limit of a unique point in $\mathrm{Bd}\,(D)$.

Explosion points. A space Z is *totally disconnected* iff each pair of distinct points p and q in Z can be separated; that is, there are disjoint open sets U and V in Z such that $p\in U$, $q\in V$, and $U\cup V=Z$. We say that $x\in X$ is an *explosion point* (or *dispersion point*) for a space X iff X is connected and $X-\{x\}$ is totally disconnected.

Such weakly connected spaces have been constructed in the plane, for example, by Knaster and Kuratowski, Sierpinski [**Ku**], and Lelek [**L**]. Bula and Oversteegen [**BO**] have recently characterized smooth Cantor bouquets with a dense set of endpoints. If the Cantor bouquets in Julia sets are smooth, an open question, then all are homeomorphic! Lelek's example would thus be the topological prototype for Cantor bouquet Julia sets.

THEOREM 3. *The set A_λ of endpoints of the Julia set J_λ of $E_\lambda : z \to \lambda e^z (0 < \lambda < 1/e)$ is totally disconnected, but $\hat{A}_\lambda = A_\lambda \cup \{\infty\}$ is connected. Thus, ∞ is an explosion point for $\hat{A}_\lambda$, and* $\dim(A_\lambda) = 1$.

Proof. Let $U = \hat{\mathbb{C}} - \hat{J}_\lambda$. By Devaney's and Goldberg's result, $P(U) = \hat{A}_\lambda$. Thus, by Theorem 2, $\hat{A}_\lambda$ is connected and $\dim(\hat{A}_\lambda) = 1$. Since removing a single point cannot reduce the dimension of a positive-dimensional space [**HW**], $\dim(A_\lambda) = 1$.

Let $p \neq q \in A_\lambda$. Let $\psi : D \to U$ be a conformal homeomorphism. There are points $p' \neq q' \in \mathrm{Bd}(D)$ corresponding, respectively, to the prime ends E_p and E_q of U for which p and q are the sole principal points. Since points in $\mathrm{Bd}(D)$ corresponding to prime ends of U whose principal point is ∞ are dense in the circle $\mathrm{Bd}(D)$, there are points ∞' and ∞'' in $\mathrm{Bd}(D)$, corresponding to distinct prime ends F' and F'', each of which have ∞ as their sole principal point, separating p' and q' on $\mathrm{Bd}(D)$. Let $[0, \infty')$ and $[0, \infty'')$ be radial rays in D. Thus, $Q = [0, \infty') \cup [0, \infty'')$ is a crosscut from ∞' to ∞''. See figure 1.

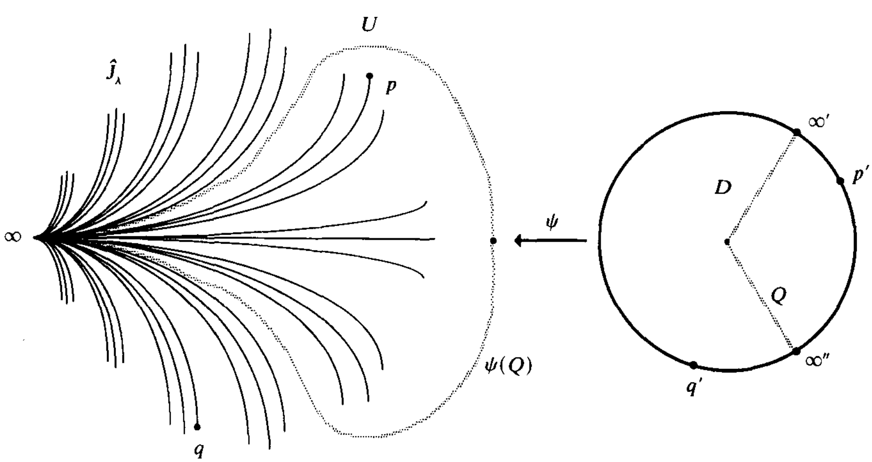

FIGURE 1

It follows that $\mathrm{Cl}(\psi(Q))$ is a simple closed curve in $\hat{\mathbb{C}}$ meeting $\hat{J}_\lambda$ only at ∞, with p and q in the two different components G and H, respectively, of $\hat{\mathbb{C}} - \mathrm{Cl}(\psi(Q))$. Then $G \cap A_\lambda$ and $H \cap A_\lambda$ separate A_λ between p and q. Since any two points in A_λ can be separated, A_λ is totally disconnected. □

Application to other entire functions. We have considered the family $E_\lambda : z \to \lambda e^z$ for a real parameter λ. The above analysis applies to the family E_λ with a complex parameter λ. For an open set of complex λ values, E_λ has a unique attracting fixed point ω_λ [**DG, DGH, D1**]. The boundary of the basin of attraction of ω_λ is a Cantor bouquet [**DG**]. All such maps are dynamically equivalent [**DG**].

There are other functions whose Julia sets have properties similar to those of the exponential function. A class of these, *critically finite* entire functions (those with only finitely many critical and asymptotic values; also called entire functions of *finite type*), is studied in [**GK, DT**]. A sufficient growth condition must be met to ensure the appearance of Cantor bouquets. The prime end analysis of [**DG**], together with our analysis above, can be applied to the Cantor bouquets that appear in the Julia sets of maps of this class to obtain results similar to Theorem 3; for example $S_\lambda : z \to \lambda \sin(z)$ and $C_\lambda : z \to \lambda \cos(z)$, for appropriate values of the parameter $\lambda \in \mathbb{C}$.

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