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Stability of Vector Bundles on Curves and Degenerations

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Abstract. We introduce a weaker notion of (semi)stability for vector bundles on reducible curves that does not depend on a choice of polarization and suffices for many applications of degeneration techniques. We explore the basic properties of this alternate notion of (semi)stability. In a complementary direction, we record a proof of the existence of semistable extensions of vector bundles in suitable degenerations.

1 Introduction

Typically, when considering (semi)stability of vector bundles on a reducible curve X, one works with respect to a polarization that assigns weights to the different irreducible components of X. The reason for this is that subsheaves of vector bundles may have different ranks on different components of X, and one needs to determine how to weigh these ranks. Put differently, the Hilbert polynomials of subsheaves will now depend nontrivially on the choice of an ample line bundle. Such an approach is very natural for working with coarse moduli spaces, but it introduces undesirable technical complications. This note is based on the observation that if one is interested in (semi)stability in the context of degeneration techniques (typically applied to higher-rank Brill–Noether theory), a weaker definition suffices, in which one only considers constant-rank subsheaves. The resulting definition is independent of polarization and both better-behaved and easier to verify than the standard one. Although it is not well suited to coarse moduli space constructions, it is still an open condition in families, and hence works well in the context of moduli stacks.

These ideas have been applied in the proofs of new existence results in higher-rank Brill–Noether theory by the author and Teixidor i Bigas [OT] and by Zhang [Zha]. The latter provides the best known results towards the existence portion of the Bertram–Feinberg–Mukai conjecture and particularly benefits from the theory introduced here, as the construction requires considering vector bundles that are unstable on some components of the curve. The traditional definition of stability is not well suited to block by block constructions, making such situations very complicated to analyze, but our theory allows for much simpler arguments. In addition, there are hints that our approach may have a role to play in specialization arguments, where *a priori* one would expect to want to make use of the usual stronger notion of semistability; see Remark 3.2.

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In a complementary direction, we record in Proposition 4.1 a natural statement that does not seem to have appeared in the literature regarding the existence of semistable extensions of vector bundles with respect to a chosen polarization.

We assume throughout that all curves are proper, geometrically reduced, and geometrically connected, but not necessarily irreducible.

We now introduce the notion of ℓ-semistability (short for "limit semistability").

Definition 1.1 Let X be a nodal (possibly reducible) curve and let $\mathscr{E}$ be a vector bundle of rank r on X. We say that $\mathscr{E}$ is ***ℓ*-*semistable*** (resp., ***ℓ*-*stable***) if for all proper subsheaves $\mathscr{F} \subseteq \mathscr{E}$ having constant rank r' on every component of X, we have

$$\frac{\chi(\mathscr{F})}{r'} \leq \frac{\chi(\mathscr{E})}{r} \quad \left(\text{resp., } \frac{\chi(\mathscr{F})}{r'} < \frac{\chi(\mathscr{E})}{r}\right).$$

Thus, if X is irreducible, we recover the usual definition of (semi)stable, while in the reducible case we have a weaker definition, which does not involve a polarization. Note that an ℓ-stable vector bundle may not be stable with respect to any polarization; see Example 3.1.

Our main results are the following observations.

Proposition 1.2 *Both ℓ-semistability and ℓ-stability are open in families.*

Proposition 1.3 *Both ℓ-semistability and ℓ-stability are closed under tensor product with line bundles.*

Consequently, we find the following corollary.

Corollary 1.4 *Let $\pi: X \to S$ be a family of curves, with S the spectrum of a DVR, such that π has smooth generic fiber X_η and nodal special fiber X_0. Let $\mathscr{E}$ be a vector bundle on X. If $\mathscr{E}$ is ℓ-semistable (resp., ℓ-stable) on X_0, then $\mathscr{E}$ is semistable (resp., stable) on X_η.*

Corollary 1.5 *Let X be a nodal curve, $\mathscr{E}$ a vector bundle on X, and $\mathscr{L}$ a line bundle on X. If $\mathscr{E} \otimes \mathscr{L}$ is semistable (resp., stable) with respect to some polarization on X, then $\mathscr{E}$ is ℓ-semistable (resp., ℓ-stable).*

Finally, we demonstrate that ℓ-semistability behaves very well with respect to gluing of subcurves.

Proposition 1.6 *Let $X = Y \cup Z$ be a nodal curve, with the subcurves Y and Z meeting at a single node P. Given a vector bundle $\mathscr{E}$ on X of rank r, if $\mathscr{E}|_Y$ and $\mathscr{E}|_Z$ are ℓ-semistable on Y and Z respectively, then $\mathscr{E}$ is ℓ-semistable on X. If, further, there do not exist subsheaves $\mathscr{F}_Y \subseteq \mathscr{E}|_Y$ and $\mathscr{F}_Z \subseteq \mathscr{E}|_Z$ of some constant rank r' that glue to one another at P and satisfy*

$$\chi(\mathscr{F}_Y)/r' = \chi(\mathscr{E}|_Y)/r \quad \textit{and} \quad \chi(\mathscr{F}_Z)/r' = \chi(\mathscr{E}|_Z)/r,$$

then $\mathscr{E}$ is ℓ-stable.

Here, when we say that $\mathscr{F}_Y$ and $\mathscr{F}_Z$ glue to one another at P, we mean that in a neighborhood of P they are obtained as the restrictions to Y and Z respectively of a subbundle of $\mathscr{E}$.

Corollary 1.7 *Let X be a curve of compact type and let $\mathscr{E}$ be a vector bundle on X. If $\mathscr{E}|_Y$ is semistable for every component Y of X, then $\mathscr{E}$ is ℓ-semistable. If, moreover, there does not exist a vector subbundle $\mathscr{F} \subseteq \mathscr{E}$ which is weakly destabilizing on every component of X, then $\mathscr{E}$ is ℓ-stable.*

Here, a subbundle $\mathscr{F} \subseteq \mathscr{E}|_Y$ is *weakly destabilizing* if $\chi(\mathscr{F})/r' = \chi(\mathscr{E}|_Y)/r$, where r' and r are the ranks of $\mathscr{F}$ and $\mathscr{E}$, respectively. Corollary 1.7 may be deduced from Corollary 1.5 and analogous results in the literature on usual stability (see, for instance, [Tei95, Proposition 1.2]), but there are many ℓ-semistable vector bundles that are not of the form considered in Corollary 1.7, and Proposition 1.6 provides a powerful tool for building them one block at a time. We hope that systematically considering such bundles will lead to better existence results in higher-rank Brill–Noether theory (for instance, in the direction of the Bertram–Feinberg–Mukai conjecture) than those that have been obtained to date.

2 Proofs

We now give the proofs of the claimed results.

Proof of Proposition 1.2 The main observation is that if $\pi: X \to S$ is a (flat, proper) family of curves where S is connected and locally Noetherian, and $\mathscr{E}$ is a coherent sheaf on X, flat over S, if there exists $s \in S$ such that $\mathscr{E}$ has the same rank r generically on every component of the fiber X_s, then the same is true for all $s \in S$. It clearly suffices to handle the case that S is irreducible, so in this case, we first prove that the statement holds for the generic point η of S and then for all $s' \in S$. Since the hypotheses are preserved under base change and the conclusion may be tested after base change, we therefore reduce to the case that S is the spectrum of a DVR, and we wish to prove that $\mathscr{E}$ has rank r on every component of the generic fiber X_η if and only it has rank r on every component of the special fiber X_0. However, by flatness over S, the open subset of X on which $\mathscr{E}$ is locally free must meet every component of X_0; indeed, the support of any torsion of $\mathscr{E}$ cannot contain any generic point of the special fiber without creating torsion over S. Since every component of X_η must contain at least one component of X_0 in its closure, and every component of X_0 is in the closure of some component of X_η, the desired statement follows.

It thus follows that the locus in a given Quot scheme that consists of quotient sheaves having equal rank on each component is a union of connected components of the Quot scheme, and is in particular proper. The proposition then follows from the usual argument for openness of (semi)stability (see, for instance, [HL97, Prop. 2.3.1]). ∎

Proof of Proposition 1.3 Let $\mathscr{E}$ be a vector bundle of rank r on a curve X and let $\mathscr{L}$ be a line bundle. Observe that for any $r' < r$, tensoring with $\mathscr{L}$ induces a bijection between subsheaves of $\mathscr{E}$ of pure rank r' and subsheaves of $\mathscr{E} \otimes \mathscr{L}$ of pure rank r'. It

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is thus enough to observe that for any subsheaf $\mathscr{F} \subseteq \mathscr{E}$ of pure rank r', we have

$$\frac{\chi(\mathscr{E})}{r} - \frac{\chi(\mathscr{F})}{r'} = \frac{\chi(\mathscr{E} \otimes \mathscr{L})}{r} - \frac{\chi(\mathscr{F} \otimes \mathscr{L})}{r'},$$

which we prove by showing that $\chi(\mathscr{F} \otimes \mathscr{L}) = \chi(\mathscr{F}) + r' \deg \mathscr{L}$ and $\chi(\mathscr{E} \otimes \mathscr{L}) = \chi(\mathscr{E}) + r \deg \mathscr{L}$. This is presumably standard, but since the proof contains minor subtleties in the case of a reducible curve, we include it for the sake of completeness.

Let D be a sufficiently ample effective divisor supported on the smooth locus of X such that $\mathscr{L}(D)$ has a section s that is nonvanishing at the nodes of X. Then using the short exact sequences induced by s gives us

$$\chi\big(\mathscr{F} \otimes \mathscr{L}(D)\big) = \chi(\mathscr{F}) + r'(\deg \mathscr{L} + \deg D),$$
$$\chi\big(\mathscr{E} \otimes \mathscr{L}(D)\big) = \chi(\mathscr{E}) + r(\deg \mathscr{L} + \deg D).$$

Then the exact sequences induced by the canonical inclusion $\mathscr{L} \hookrightarrow \mathscr{L}(D)$ yield the desired identities. ■

Note that Corollary 1.4 is an immediate consequence of Proposition 1.2, and Corollary 1.5 is an immediate consequence of Proposition 1.3.

Proof of Proposition 1.6 Let $\mathscr{F}$ be a subsheaf of $\mathscr{E}$ of constant rank r'. Let $\mathscr{F}_Y$ (resp., $\mathscr{F}_Z$) denote the quotient of $\mathscr{F}|_Y$ (resp., $\mathscr{F}|_Z$) by its torsion subsheaf, or equivalently, the image of $\mathscr{F}|_Y$ inside $\mathscr{E}|_Y$ (resp., of $\mathscr{F}|_Z$ inside $\mathscr{E}|_Z$). Then there is an integer r_P between 0 and r' described as follows. If $\mathscr{Q}$ is the cokernel of

$$\mathscr{F} \hookrightarrow \mathscr{F}_Y \oplus \mathscr{F}_Z,$$

then one checks that injectivity is preserved after restriction to P, so we have an induced exact sequence

$$0 \longrightarrow \mathscr{F}|_P \longrightarrow \mathscr{F}_Y|_P \oplus \mathscr{F}_Z|_P \longrightarrow \mathscr{Q}|_P \longrightarrow 0,$$

and we let r_P be the dimension of $\mathscr{Q}|_P$. We then have that $r_P = r'$ if and only if $\mathscr{F}$ is locally free at P, and furthermore, if $\widetilde{\mathscr{F}}_Y$ denotes the saturation of $\mathscr{F}_Y$ at P in $\mathscr{E}|_Y$, and similarly for $\widetilde{\mathscr{F}}_Z$, then r_P has the property that the quotients $\widetilde{\mathscr{F}}_Y/\mathscr{F}_Y$ and $\widetilde{\mathscr{F}}_Z/\mathscr{F}_Z$ each have dimension at least $r' - r_P$ at P. Now, we carry out the following calculation:

$$\begin{aligned}
\frac{\chi(\mathscr{F})}{r'} &\leq \frac{1}{r'}\big(\chi(\mathscr{F}_Y) + \chi(\mathscr{F}_Z) - r_P\big)\\
&\leq \frac{1}{r'}\big(\chi(\widetilde{\mathscr{F}}_Y) + \chi(\widetilde{\mathscr{F}}_Z) - 2(r' - r_P) - r_P\big)\\
&\leq \frac{1}{r}\big(\chi(\mathscr{E}|_Y) + \chi(\mathscr{E}|_Z)\big) - \frac{2r' - r_P}{r'}\\
&= \frac{\chi(\mathscr{E})}{r} + 1 - \frac{2r' - r_P}{r'} = \frac{\chi(\mathscr{E})}{r} - \frac{r' - r_P}{r'} \leq \frac{\chi(\mathscr{E})}{r}.
\end{aligned}$$

Thus, we get that $\mathscr{E}$ is ℓ-semistable, and is in fact ℓ-stable unless there exists some $\mathscr{F}$ with $\chi(\widetilde{\mathscr{F}}_Y)/r' = \chi(\mathscr{E}|_Y)/r$, $\chi(\widetilde{\mathscr{F}}_Z)/r' = \chi(\mathscr{E}|_Z)/r$, and $r_P = r'$. The condition $r_P = r'$ implies that $\mathscr{F}_Y = \widetilde{\mathscr{F}}_Y$ and $\mathscr{F}_Z = \widetilde{\mathscr{F}}_Z$ and that $\mathscr{F}_Y$ must glue to $\mathscr{F}_Z$ at P, as desired. ■

Corollary 1.7 follows immediately from Proposition 1.6 by induction on the number of components of X.

3 Further Discussion

It is instructive to compare ℓ-semistability to usual semistability in the case of rank 2 and degree $2g-2$. This case is the subject of the Bertram–Feinberg–Mukai conjecture, and has consequently received a great deal of attention. A vector bundle $\mathscr{E}$ of rank 2 and degree $2g-2$ has $\chi(\mathscr{E}) = 0$, so we see that although the usual definition of (semi)stability calls for a polarization, the resulting definition in fact does not depend on the polarization. This, therefore, presents a natural context in which to compare the definitions.

Example 3.1 Let X be a chain of smooth curves $Y_1, \dots, Y_n$, glued together at nodes. Let $\mathscr{E}$ be a vector bundle of rank 2 and degree $2g-2$, satisfying the condition for ℓ-stability of Corollary 1.7. Suppose further that $d_1 := \deg \mathscr{E}|_{Y_1}$ is even, and $\mathscr{E}|_{Y_1}$ is strictly semistable. Then we see that even though $\mathscr{E}$ is ℓ-stable, it is not stable on X. Indeed, let g_1 be the genus of Y_1; then if $d_1 \geq 2g_1$, the condition for stability is violated by the subsheaf of $\mathscr{E}$ consisting of sections that vanish on the complement of Y_1, while if $d_1 \leq 2g_1 - 2$, the condition for stability is violated by the subsheaf of $\mathscr{E}$ consisting of sections that vanish on Y_1.

The peculiarities of the case $\chi(\mathscr{E}) = 0$ lead to subtleties in certain aspects of Teixidor i Bigas' [Tei04, Tei08]. These subtleties are addressed by twisting arguments, and we take the opportunity to discuss how they fit into the context of ℓ-stability.

Remark 3.2 In [Tei08], a key point is to place suitable conditions on the vector bundle $\mathscr{E}_0$ obtained as a specialization from a semistable vector bundle on the smooth generic fiber; this is carried out in Claim 2.3. However, Claim 2.3 does not apply directly to the case of interest, because when $\chi = 0$ it is not possible to choose a polarization with the required non-integrality property. Instead, as described at the beginning of [Tei08, §3], one uses twisting to complete the argument, as follows. Let $\pi: X \to S$ be the family of curves used for the degeneration and let $\mathscr{E}_\eta$ be a vector bundle on the generic fiber X_η, with canonical determinant. Choose D any divisor on X of nonzero relative degree; rather than extending $\mathscr{E}_\eta$ right away, we instead twist by D and then extend $\mathscr{E}_\eta(D|_{X_\eta})$ to a bundle $\mathscr{E}'$ semistable with respect to a polarization satisfying the stated condition.[1] Then Claim 2.3 of *loc. cit.* shows that $\mathscr{E}'$ has the desired properties on each irreducible component and in a neighborhood of each node, and it follows that if we set $\mathscr{E}_0 = \mathscr{E}'(-D)|_{X_0}$, we obtain an extension of the original $\mathscr{E}_\eta$ which has the desired properties.

For us, the relevant point is that, because semistability is not preserved by twisting, there is no reason to think that $\mathscr{E}_0$ is semistable, but at least it follows from Corollary 1.5 that it is ℓ-semistable. This hints that even though for specialization arguments

[1]In fact, the argument in [Tei08] is slightly more complicated, but can be simplified to the above using Proposition 4.1.

it is natural to try to take advantage of the stronger properties afforded by semistability (with respect to a polarization), there may also be a role for ℓ-semistable vector bundles.

Remark 3.3 In [Tei04], in order to stay within the framework of usual stability, one needs to make an argument similar to that of Remark 3.2, because the situation is precisely as in Example 3.1, so the natural underlying vector bundles are in fact not stable under any choice of polarization. But if $\pi\colon X \to S$ and D are as in the previous remark, and $\mathscr{E}$ is a vector bundle on X underlying the relevant limit linear series and its smoothing, we show that $\mathscr{E}$ is stable on the generic fiber X_η as follows. The twisted bundle $\mathscr{E}(D)$ has nonzero Euler characteristic, so after a possible further twist to redistribute degrees, it is stable on X_0 for a suitable choice of polarization. It thus follows that $\mathscr{E}(D)|_{X_\eta}$ is stable, and hence, since X_η is smooth, that $\mathscr{E}|_{X(\eta)}$ is also stable, as desired.

Although the above argument works, it seems much simpler to argue that for the original $\mathscr{E}$, although $\mathscr{E}|_{X_0}$ is strictly semistable, it is ℓ-stable by Proposition 1.6, and therefore $\mathscr{E}|_{X_\eta}$ is stable, as needed.

4 Semistable Extensions

Although the following result on specialization of vector bundles under degeneration is a straightforward application of standard techniques, it does not appear to be stated anywhere in the literature. Because it complements the main subject of this note, we take the opportunity to record its proof.

Proposition 4.1 *Let $\pi\colon X \to B$ be a flat proper morphism with B the spectrum of a DVR, generic fiber X_η a smooth curve, and special fiber X_0 a nodal curve. Suppose that X is regular. Then for any polarization w on X_0, and any semistable vector bundle $\mathscr{E}_\eta$ on X_η, there exists a vector bundle $\mathscr{E}$ on X such that $\mathscr{E}|_{X_\eta} \cong \mathscr{E}_\eta$ and $\mathscr{E}|_{X_0}$ is semistable with respect to w.*

Recall that a polarization w is a positive rational weighting of the components of X_0 adding to 1. Semistability with respect to such a polarization is equivalent to semistability with respect to an ample divisor supported on the smooth locus of X_0; clearing denominators in w describes the distribution of degrees of the divisor in question.

Proof First, recall that a reflexive sheaf on a regular 2-dimensional scheme is necessarily locally free. Thus, by extending $\mathscr{E}_\eta$ to any coherent sheaf on X and then taking the reflexive hull, we obtain a vector bundle $\mathscr{E}'$ on X extending $\mathscr{E}_\eta$. It remains to show that the desired $\mathscr{E}$ can be realized as a subsheaf of $\mathscr{E}'$. This follows the standard argument of Langton (see [HL97, Theorem 2.1.B]); all that needs to be checked is that the subsheaves considered inductively in the argument in question remain locally free at each step. But these subsheaves are obtained by considering the kernel $\mathscr{K}$ of composed maps of the form

$$\mathscr{E} \twoheadrightarrow \mathscr{E}|_{X_0} \twoheadrightarrow \mathscr{F},$$

where $\mathscr{E}$ is a vector bundle and $\mathscr{F}$ is the quotient sheaf corresponding to a maximally destabilizing subbundle of $\mathscr{E}|_{X_0}$. In particular, the kernel of $\mathscr{E}|_{X_0} \twoheadrightarrow \mathscr{F}$ is saturated, so

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$\mathscr{F}$ is pure of dimension 1. Now, purity implies that for any closed point $x \in \mathrm{Supp}(\mathscr{F})$, the stalk $\mathscr{F}_x$ has depth 1, so it follows from the Auslander–Buchsbaum formula that the projective dimension of $\mathscr{F}_x$ is also 1. Using the Tor exact sequence, we conclude that $\mathscr{K}_x$ is flat over $\mathscr{O}_{X,x}$, and hence $\mathscr{K}$ is locally free, as desired. ∎

If we drop the regularity hypothesis on X, then it is always possible to blow up X at nodes of X_0 to resolve any singularities; this only introduces chains of rational components at the nodes of X_0, and then it follows from Proposition 4.1 that we can extend any vector bundle while preserving semistability.

References

[HL97] D. Huybrechts and M. Lehn, *The geometry of moduli spaces of sheaves*. Aspects of Mathematics, E31, Friedr. Vieweg & Sohn, Braunschweig, 1997.

[OT] B. Osserman and M. Teixidor i Bigas, *Linked symplectic forms and limit linear series in rank* 2 *with special determinant*. Adv. Math. **288**(2016), 576–630. http://dx.doi.org/10.1016/j.aim.2015.09.030

[Tei95] M. Teixidor i Bigas, *Moduli spaces of vector bundles on reducible curves*. Amer. J. Math. **117**(1995), no. 1, 125–139. http://dx.doi.org/10.2307/2375038

[Tei04] ———, *Rank two vector bundles with canonical determinant*. Math. Nachr. **265**(2004), 100–106. http://dx.doi.org/10.1002/mana.200310138

[Tei08] ———, *Petri map for rank two bundles with canonical determinant*. Compos. Math. **144**(2008), no. 3, 705–720. http://dx.doi.org/10.1112/S0010437X07003442

[Zha] N. Zhang, *Towards the Bertram-Feinberg-Mukai conjecture*. To appear, J. Pure and Appl. Algebra. http://dx.doi.org/10.1016/j.jpaa.2015.09.020

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