BULLETIN **13** (1), 1970

THE MAXIMAL NUMBER OF TRIANGLES OF MAXIMAL PERIMETER LENGTH DETERMINED BY A FINITE SET

BY

BÉLA BOLLOBÁS

Suppose there are N not necessarily distinct points on a plane in such a position that any triangle (degenerate or non-degenerate) determined by these points has perimeter length at most 1. Denote by m the number of triangles with maximal perimeter length, (briefly, the number of *maximal* triangles), and put $f(N) = \max m$ where the maximum is taken over all permissible configurations. At the Colloquim on Graph Theory in Calgary, 1969, P. Erdös proposed the problem of determining $f(N)$. He conjectured that the following construction gives the maximal number: place approximately half of the points at a point A and the others at B where $AB = 1/2$. The aim of this note is to prove this conjecture.

THEOREM. $f(2n) = n^2(n-1)$, $f(2n+1) = (2n-1)n(n+1)/2$, *and if* $N > 4$ *the only extremal configuration is the one described above.*

Proof. For $N = 3$ and 4 the theorem is evident.

Suppose $N > 4$ and choose any permissible set S consisting of N points which contains $f(N)$ maximal triangles. If one of the maximal triangles is degenerate then all of them must be degenerate and the theorem follows easily.

On the other hand we shall show that if the maximal triangles are not degenerate then there are at most $N(N-1)^2/9$ maximal triangles. As the construction described above gives more maximal triangles, $f(N)$ certainly must be greater than this number, so this will imply the theorem.

Let X be an arbitrary point of S. Construct a graph G on the remaining points of S. Connect two points A and B if and only if the triangle XAB is maximal.

If ABC is a triangle in the graph G then putting $XA = a$, $XB = b$, $XC = c$, $AB = c'$, $BC = a'$ and $CA = b'$, the equalities $a' + b + c = a + b' + c = a + b + c'$ imply immediately that $a - b = a' - b'$, etc. (see Figure 1).

We claim that the graph does not contain a complete quadrangle. For if $ABCD$ is a complete quadrangle then it can be assumed that the line AB does not separate the points C and D. By the previous paragraph $CA - CB = DA - DB$, so C and D are on a hyperbola with foci A and B. Consequently it can be assumed that the triangle ABC contains the point D (see Figure 1). But then none of the triangles XDA, XDB and XDC can be maximal, contradicting the hypothesis.

According to a well-known theorem of Turán [1], if a graph with v vertices does not contain a complete k-gon then it has at most $v^2(k-2)/2(k-1)$ edges. So our

Received by the editors August 23, 1969.

graph G can have at most $(N-1)^2/3$ edges, i.e., there are at most $(N-1)^2/3$ maximal triangles with a given vertex. Consequently the set S determines at most $N(N-1)^2/9$ maximal triangles, and the proof is complete.

It is interesting to note that though the points are allowed to move on a *plane*, the extremal configuration is on a *line*. Moreover, it seems likely the *same* extremal configuration will be obtained if sufficiently many points are distributed in E^3.

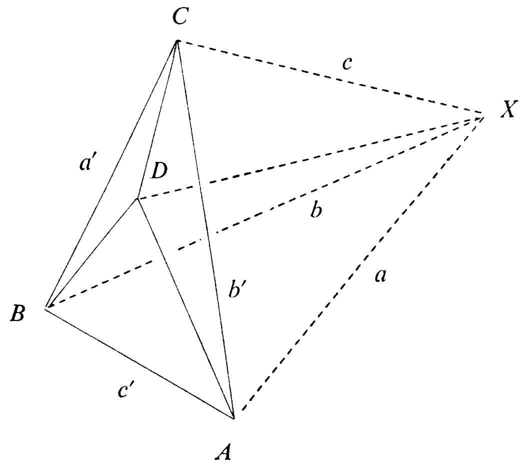

Figure 1.

Actually, the lowest dimensional space in which the extremal distribution is known to be different for any number of points is E^{11}, where a better configuration is obtained if roughly a twelfth of the points are put into each vertex of a simplex.

REFERENCE

1. P. Turán, *An extremal problem of graph theory*, Mat és Fiz. Lapok **48** (1941), 436–452 (in Hungarian). See also P. Turán, *On the theory of graphs*, Coll. Math. **3** (1954), 19–30.

MATHEMATICAL INSTITUTE,
OXFORD AND MATHEMATICAL INSTITUTE,
BUDAPEST

https://doi.org/10.4153/CMB-1970-009-9 Published online by Cambridge University Press