ON THE HOLONOMY GROUP OF THE CONFORMALLY FLAT RIEMANNIAN MANIFOLD

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The main purpose of the present paper is to show that the local homogeneous holonomy group of the conformally flat Riemannian manifold is the full rotation group with some exceptions.

1. Let M be an n-dimensional conformally flat Riemannian manifold ($n \geqq 3$), the metric being given by $ds^2 = a^2 \sum dx_i^2$ in a coordinates neighborhood U with a function $a = a(x_1, \ldots, x_n)$ of class 2. We take rectangular frames in the tangent spaces at each point of U and put according to the frames

$$ds^2 = \sum \omega_i^2. \tag{1}$$

When we put

$$\omega_i = a\pi_i, \tag{2}$$

$\sum \pi_i^2$ is a flat metric, and if we take π_{ij} such that

$$d\pi_i = \pi_j \wedge \pi_{ji}, \qquad \pi_{ij} = -\pi_{ji}, \tag{3}$$

then we have by the flatness of $\sum \pi_i^2$

$$d\pi_{ij} = \pi_{ik} \wedge \pi_{kj}. \tag{4}$$

Next we put

$$da/a = b_i \pi_i \tag{5}$$

$$\omega_{ij} = \pi_{ij} + b_i \pi_j - b_j \pi_i \tag{6}$$

and we get

$$d\omega_i = \omega_j \wedge \omega_{ji}, \qquad \omega_{ij} = -\omega_{ji}. \tag{7}$$

Thus ω_{ij} are the parameters of the Riemannian connection of M in U. Now we calculate the curvature forms of M. Putting

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$$\sum b_i^2 = b^2 \tag{8}$$

$$db_i + b_k \omega_{ki} - \frac{1}{2} b^2 \pi_i = a^2 \lambda_{ik} \pi_k \tag{9}$$

we get for the curvature forms

$$\Omega_{ij} = d\omega_{ij} - \omega_{ik} \wedge \omega_{kj} = \lambda_{ik} \omega_k \wedge \omega_j + \omega_i \wedge \lambda_{jk} \omega_k, \tag{10}$$

using the relations (3), (4), (6). Taking an exterior differential of (5) we get

$$\lambda_{ij} = \lambda_{ji}. \tag{11}$$

We put $\Omega_{ij} = \frac{1}{2} R_{ijkh} \omega_k \wedge \omega_h$ $(R_{ijkh} = -R_{ijhk})$ and we have by (10)

$$R_{ijkh} = \lambda_{ik} \delta_{jh} - \lambda_{ih} \delta_{jk} - \lambda_{jk} \delta_{ih} + \lambda_{jh} \delta_{ik}.$$

Contracting with respect to j and h we get Ricci's tensor

$$R_{ik} = (n-2) \lambda_{ik} + \lambda_{jj}, \qquad R = R_{ii} = 2(n-1) \lambda_{jj}. \tag{12}$$

Hence we can represent λ_{ij} as follows,

$$\lambda_{ij} = \frac{1}{n-2} \left(R_{ij} - \frac{1}{2(n-1)} R \delta_{ij} \right). \tag{13}$$

We take a geodesic of the manifold M. It satisfies the differental equations

$$\frac{d}{ds}\left(\frac{\omega_i}{ds}\right) + \frac{\omega_j}{ds} \frac{\omega_{ji}}{ds} = 0$$

and along it we have by (5), (9)

$$\frac{\lambda_{ij} \omega_i \omega_j}{ds^2} = \frac{d}{ds}\left(\frac{1}{a} \frac{da}{ds}\right) + \left(\frac{1}{a} \frac{da}{ds}\right)^2 - \frac{1}{2} \frac{b^2}{a^2}. \tag{14}$$

This gives an interpretation of the tensor (λ_{ij}) in terms of a and b.

2. We assume that our metric is not flat in the coordinate neighborhood U and the local homogeneous holonomy group H_p (cf. [1]) at any point of U is not the full rotation group $SO(n)$. Then we take a point p at which the tensor (λ_{ij}) is not a zero tensor. When we take a suitable rectangular frame $\mathbf{e}_1, \ldots, \mathbf{e}_n$ at p, we can reduce the tensor (λ_{ij}) into the diagonal form with the diagonal elements $\lambda_1, \ldots, \lambda_n$. Then we get for (10)

$$\Omega_{ij} = (\lambda_i + \lambda_j) \omega_i \wedge \omega_j \quad \text{(not summed for } i, j) \tag{15}$$

at p. Conversely if the curvature forms are represented as (15) by a suitable choice of rectangular frames, the manifold is conformally flat under the assumption $n \geqq 4$. This can be verified by the vanishing of the conformal curvature tensor.

We denote by H_p the local homogeneous holonomy group at p and by $\mathfrak{H}_p$ the Lie algebra of H_p. Let S_{ij} be a skew symmetric matrix (s_{hl}) such that $s_{ij} = -s_{ji} = 1$ and all the other s_{hl}'s are zero. Then by virtue of (15) $\mathfrak{H}_p$ contains a subalgebra generated by S_{ij} if $\lambda_i + \lambda_j \neq 0$. Let $O(i_1, \ldots, i_k)$ be a subgroup of a full rotation group $SO(n)$ which induces the full rotation in the linear subspaces generated by $\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_k}$ and fixes the remaining fundamental vectors. As at least one of $\lambda_i + \lambda_j$ is not zero at p, $\mathfrak{H}_p$ contains S_{ij} for such i and j. Hence H_p contains $O(i, j)$. If H_p is not the full rotation group, there exist k and $i_1, \ldots, i_k$ such that H_p contains $O(i_1, \ldots, i_k)$ but does not contain $O(i_1, \ldots, i_k, j)$ for any j. We change the indices suitably and assume that H_p contains $O(1, \ldots, k)$ but does not contain $O(1, \ldots, k, j)$ for any j. Then $\mathfrak{H}_p$ does not contain S_{ar} $(a = 1, \ldots, k\,;\ r = k+1, \ldots, n)$ and we get

$$\lambda_a + \lambda_r = 0 \quad (a = 1, \ldots, k\,;\ r = k+1, \ldots, n).$$

By putting $\lambda = \lambda_1$ we have

$$\lambda_1 = \ldots = \lambda_k = \lambda, \qquad \lambda_{k+1} = \ldots = \lambda_n = -\lambda \qquad (\lambda \neq 0). \tag{16}$$

In this section and the next we assume that the indices run as follows,

$$a, b = 1, \ldots, k \qquad r, s = k+1, \ldots, n \qquad i, j = 1, \ldots, n.$$

We have by (15)

$$\Omega_{ab} = 2\lambda\omega_a \wedge \omega_b, \quad \Omega_{rs} = -2\lambda\omega_r \wedge \omega_s \quad \text{all the others zero.} \tag{17}$$

Next we take a point q in the neighborhood of p and choose a suitable rectangular frame at the point. Then we have at q

$$\Omega_{ij} = (\lambda_i + \lambda_j)\,\omega_i \wedge \omega_j \quad (\text{not summed for } i, j).$$

The eigenvalues of the symmetric matrix (λ_{ij}) are continuous with respect to the values of λ_{ij} in the sense that those of (λ_{ij}) for $\lambda_{ij} = a_{ij}$ are in arbitrary small neighborhoods of those for $\lambda_{ij} = b_{ij}$ if a_{ij}'s are sufficiently near to b_{ij}. When the point q is sufficiently near to p, we conclude by (17)

$$\Omega_{ab} = (\lambda_a + \lambda_b)\,\omega_a \wedge \omega_b \neq 0, \qquad \Omega_{rs} = (\lambda_r + \lambda_s)\,\omega_r \wedge \omega_s \neq 0 \tag{18}$$

$$(\text{not summed for } a, b, r, s)$$

as it is so at the point p. Hence H_q contains S_{ab} and S_{rs} and so H_q contains $O(1, \ldots, k)$ and $O(k+1, \ldots, n)$. If H_q contains none of $O(1, \ldots, k, j)$, (17) holds good at q. If H_q contains at least one of $O(1, \ldots, k, j)$, we change the indices of $\mathbf{e}_{k+1}, \ldots, \mathbf{e}_n$ in such a way that H_q contains $O(1, \ldots, k, k+1, \ldots, k+l)$ and none of $O(1, \ldots, k, k+1, \ldots, k+l, k+l+m)$. Then we get by the repetition of the above process

$$\Omega_{rs} = 0 \quad (r = k+1, \ldots, k+l\,;\ s = k+l+1, \ldots, n).$$

This contradicts (18). Thus we can take frames at each point of U in such a way that (17) holds good with the same k. These frames can be so taken as to satisfy the differentiability, because the eigenvalues of (λ_{ij}) satisfy the relation (16) and the process of transforming (λ_{ij}) into the diagonal form can be taken analytic except at the point such that $\lambda_1 = \ldots = \lambda_n$, namely $\lambda = 0$. These circumstances are discussed precisely in section 7. We treat at first the neighborhood in which λ never vanishes, the non-existence of the point at which $\lambda = 0$ being assured thereafter.

3. The next step is to find the Riemannian metric which satisfies the relation (17). In this sectin we treat the case $n-k \geqq 2$, $k \geqq 2$.

By Bianchi's identity we have

$$d\Omega_{ar} = -\Omega_{ai} \wedge \omega_{ir} + \omega_{ai} \wedge \Omega_{ir}.$$

By virtue of (17) we get

$$0 = -\Omega_{ab} \wedge \omega_{br} + \omega_{as} \wedge \Omega_{sr} = -2\lambda(\omega_a \wedge \omega_b \wedge \omega_{br} + \omega_{as} \wedge \omega_s \wedge \omega_r).$$

Putting $\omega_{as} = A_{asi}\omega_i$ we get by the assumption $\lambda \neq 0$

$$\omega_a \wedge \omega_b \wedge A_{bri}\omega_i + A_{asi}\omega_i \wedge \omega_s \wedge \omega_r = 0. \tag{19}$$

As $n-k \geqq 2$, $k \geqq 2$, we have

$$A_{brs} = 0, \qquad A_{asb} = 0$$

and so

$$\omega_{ar} = -\omega_{ra} = 0.$$

Consequently (7) takes the forms

$$d\omega_a = \omega_i \wedge \omega_{ia} = \omega_b \wedge \omega_{ba}, \qquad d\omega_r = \omega_i \wedge \omega_{ir} = \omega_s \wedge \omega_{sr}$$

and by E. Cartan's lemma the metric of our manifold decomposes, namely

$$ds^2 = ds_1^2 + ds_2^2, \tag{20}$$

where

$$ds_1^2 = g_{ab}(x_1, \ldots, x_k)\, dx_a dx_b, \qquad ds_2^2 = g_{rs}(x_{k+1}, \ldots, x_n)\, dx_r dx_s$$

with suitably chosen coordinates. When we take suitably chosen rectangular frames in each manifold with the metric ds_1^2 and ds_2^2, ω_a's are expressed by the coordinates $x_1, \ldots, x_k$ and ω_r's by $x_{k+1}, \ldots, x_n$. As the relations (17) holds good, λ is constant and ds_1^2 is a metric of constant curvature K, while ds_2^2 is one of constant curvature $-K$.

Thus any point of U at which λ does not vanish has a neighborhood in which (16) holds good with non zero constant. The set V of all points at which (16) holds good with the same constant is open. On the other hand V is closed as λ_i's are continuous. V is closed and open in U. As U is connected, V coincides with U and there is no point in U at which $\lambda = 0$.

4. Next we treat the case $n-k=1$, $n \geqq 4$. In this section we assume that the indices run as $a, b, c = 1, \ldots, n-1$. By (17)

$$\Omega_{ab} = 2\lambda \omega_a \wedge \omega_b, \qquad \Omega_{an} = 0. \tag{21}$$

By virtue of (19) we get $A_{ann} = 0$ and we can put

$$\omega_{an} = A_{ab}\omega_b. \tag{22}$$

By (21) we have for Bianchi's identity

$$d\Omega_{ab} = -\Omega_{ac} \wedge \omega_{cb} + \omega_{ac} \wedge \Omega_{cb}$$

and this can be written as

$$d\lambda \wedge \omega_a \wedge \omega_b + \lambda d\omega_a \wedge \omega_b - \lambda \omega_a \wedge d\omega_b = -\lambda \omega_a \wedge \omega_c \wedge \omega_{cb} + \omega_{ac} \wedge \lambda \omega_c \wedge \omega_b.$$

Hence by (7) and (21)

$$d\lambda \wedge \omega_a \wedge \omega_b = \lambda \omega_n \wedge (A_{bc}\omega_a \wedge \omega_c + A_{ac}\omega_c \wedge \omega_b)$$

and so

$$A_{bc} = 0 \qquad (c \neq a, b).$$

Putting $A_a = A_{aa}$ (not summed for a) we get

$$d\lambda - \lambda(A_a + A_b)\omega_n = p\omega_a + q\omega_b \tag{23}$$

for all a and b $(a \neq b)$. As $k \geqq 3$, $A_a + A_b$ is independent of a and b and so A_a

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is independent of a. We put $A_a = A$. We also get from (23) $p = 0$, $q = 0$. Thus we have

$$d\lambda - 2\lambda A\omega_n = 0, \qquad \omega_{an} = A\omega_a. \tag{24}$$

By the relation (17) we get

$$0 = \Omega_{an} = d\omega_{an} - \omega_{ab} \wedge \omega_{bn} = dA \wedge \omega_a + A d\omega_a - \omega_{ab} \wedge A\omega_b$$
$$= dA \wedge \omega_a + A\omega_n \wedge \omega_{na} = (dA - A^2 \omega_n) \wedge \omega_a.$$

Hence

$$dA = A^2 \omega_n.$$

In the neighborhood of the point at which $A \neq 0$ we have

$$dA/A^2 = \omega_n.$$

Hence putting $x_n = -A^{-1}$ we get

$$\omega_n = dx_n$$

and by virtue of (24)

$$d\lambda = -2\lambda/x_n \cdot dx_n, \qquad \lambda = C/x_n^2 \ (C \text{ const})$$

and

$$\omega_{an} = -\omega_a/x_n.$$

Hence

$$d\omega_a = \omega_b \wedge \omega_{ba} + \omega_n \wedge \omega_{na} = \omega_b \wedge \omega_{ba} + dx_n \wedge \omega_a/x_n.$$

Putting $\omega_a = x_n \rho_a$ we get

$$d\rho_a = \rho_b \wedge \omega_{ba}$$

and

$$d\sigma^2 = \sum \rho_a^2 = g_{ab}(x_1, \ldots, x_{n-1})\, dx_a dx_b$$

for suitably chosen coordinates $x_1, \ldots, x_{n-1}$. Also we have

$$d\omega_{ab} - \omega_{ac} \wedge \omega_{cb} = \Omega_{ab} + \omega_{an} \wedge \omega_{nb} = 2\lambda\omega_a \wedge \omega_b - A^2 \omega_a \wedge \omega_b$$
$$= (2\lambda x_n^2 - A^2 x_n^2)\, \rho_a \wedge \rho_b = (2C - 1)\, \rho_a \wedge \rho_b.$$

Hence $d\sigma^2$ is a metric of constant curvature, and our metric is

$$ds^2 = x_n^2 d\sigma^2 + dx_n^2.$$

But the local homogeneous holonomy group H_p of this manifold does not keep the direction $\mathbf{e}_n$ invariant. As the holonomy group H_p contains $O(1, \ldots, n-1)$, H_p is a full rotation group. The manifold affords an example of the one for

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which the curvature forms satisfy the relations (21) for suitably chosen frames and yet the metric does not decompose.

Now we need only to treat the case that A is identically zero. Then by virtue of (24) λ is constant and $\omega_{an}=0$ and we get

$$ds^2 = d\sigma^2 + dx_n^2 \tag{25}$$

where $d\sigma^2$ is a metric of constant curvature. This holds good in U by the same discussion as at the end of the preceding section.

5. Lastly we treat the case $n=3$. If the group H_p is not the full rotation group in U, it is reducible and we can take rectangular frames in such a way that the metric can be written as

$$ds^2 = \omega_1^2 + \omega_2^2 + \omega_3^2, \tag{26}$$

where ω_1, ω_2 are Pfaffian forms in the variables x_1, x_2 and

$$\omega_3 = dx_3. \tag{27}$$

For the parameters ω_{ij} of the Riemannian connection we have

$$\omega_{13}=0, \qquad \omega_{23}=0, \tag{28}$$

and for curvature forms

$$\Omega_{12} = -K\omega_1\wedge\omega_2, \qquad \Omega_{13}=0, \qquad \Omega_{23}=0.$$

For Ricci's tensor we have

$$R_{11} = -K, \quad R_{22} = -K, \quad \text{all the others zero} \qquad R = -2K. \tag{29}$$

Now a 3-dimensional Riemannian manifold with the metric $ds^2 = \sum \omega_i^2$ is conformally flat when and only when

$$Dp_{ik}\wedge\omega_k = 0, \tag{30}$$

where D denotes covariant differential and

$$p_{ij} = -\frac{1}{n-2}\left(R_{ij} - \frac{1}{2(n-1)}R\delta_{ij}\right).$$

This is a formulation in rectangular frames of the well known property. In our case p_{ij} reduces to

$$p_{ij} = -R_{ij} + \frac{1}{4}R\delta_{ij}$$

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and we have by (29)

$$p_{11}=\frac{1}{2}K,\quad p_{22}=\frac{1}{2}K \quad \text{all the other } p_{ij} \text{ vanish.}$$

By these and (28) the relation (30) reduces to $dK_\wedge\omega_1=0$, $dK_\wedge\omega_2=0$ and so $dK=0$ and K is constant. Hence our metric (26) is conformally flat when and only when $d\sigma^2=\omega_1^2+\omega_2^2$ is of constant curvature. Thus the metric for which (30) holds good in some neighborhood has been found.

6. In summary we get the following theorem.

THEOREM. *The local homogeneous holonomy group H_p of an n-dimensional conformally flat Riemannian manifold of class C_2 is in general the full rotation group $SO(n)$. If H_p is not $SO(n)$ for any point of a coordinate neighborhood U, we can classify into the following three cases:*

1) H_p is an identity and the metric is flat in U.

2) H_p is $SO(k)\times SO(n-k)$ and U is a direct product of a k-dimensional manifold of constant curvature K and an $n-k$-dimensional manifold of constant curvatur $-K$ $(K\neq 0)$.

3) H_p is $SO(n-1)$ and U is a direct product of a straight line (or a segment) and an $n-1$-dimensional manifold of constant curvature.

The set N of all the point of M at which H_p is $SO(n)$ is closed. In fact when p is a limit point of N any neighborhood of p has a point $q\in N$ and by Lemma 3.2 in [1] we have $H_p\supset H_q=SO(n)$ and so $H_p=SO(n)$. For any point p at which $H_p\neq SO(n)$ (if exist) we can take a neighborhood U such that $U_\frown N$ is empty and by our theorem one of 1), 2), 3) holds good in U. By the Theorem 3 in [1] we can conclude that if there is no point on an n-dimensional conformally flat connected manifold M at which H_p is $SO(n)$, the restricted homogeneous holonomy group of M is an identity, or $SO(k)\times SO(n-k)$, or $SO(n-1)$. In fact λ_i's in our discussion are continuous and (17) holds good for all the points of M with the same number k and dim H_p is constant over M. An example of the case in which H_p is $SO(n)$ in some points and $SO(k)\times SO(n-k)$ in other points can easily be given.

7. Here we give an attention to a symmetric covariant tensor field of second order over an n-dimensional Riemannian manifold. Let the components of the tensor with respect to certain rectangular frames be (a_{ij}). When the metric, frames and (a_{ij}) satisfy certain differentiability conditions (for example class C_k) it is not sure that the frames can be taken in such a way that (a_{ij}) reduces to a diagonal form and yet the frames satisfy the differentiability of the same kind. An example is given by a symmetric tensor field with the components

$$a_{11} = p_1 \cos^2 \theta + p_2 \sin^2 \theta, \qquad a_{22} = p_1 \sin^2 \theta + p_2 \cos^2 \theta$$
$$a_{12} = a_{21} = 2(p_1 - p_2) \sin \theta \cos \theta,$$

where $\theta = \dfrac{1}{x_1 - x_2}$, $p_1 - p_2 = (x_1 - x_2)^n$ (p_i is class C_n), on the euclidean plane with the rectangular coordinates x_1, x_2. The eigenvalues of (a_{ij}) are p_1 and p_2. But the angle of rotation which transforms the components of the tensor into a diagonal form is θ and it is not continuous at the points such that $x_1 = x_2$. In general this singularity appears at the points at which the multiplicities of eigenvalues of (a_{ij}) differ from those at the sufficiently near points. In fact if the multiplicities of the eigenvalues of (a_{ij}) are each constant in some neighborhood, the eigenvalues are analytic functions of (a_{ij}) as they are simple roots of the polynomials obtained from $\varphi(t) = \det(a_{ij} - \delta_{ij} t)$ by a suitable successive differentiation. When the eigenvalues are analytic functions of (a_{ij}), we can transform it into a diagonal form by an analytic process. In the treatment of the preceding sections the relation (16) is satisfied and there was no obstacle to the discussion.

8. In a conformally flat Riemmanian manifold we can take rectangular frames such that the curvature forms reduce to

$$\Omega_{ij} = (\lambda_i + \lambda_j)\, \omega_i \wedge \omega_j \quad \text{(not summed for } i, j) \tag{31}$$

and according to the discussion of the previous section this can be accomplished by a differentiable process with an exception of certain points. Analogously if we take suitable rectangular frames in an n-dimensional Riemannian manifold of imbedding class one, we have for the curvature forms

$$\Omega_{ij} = -k_i k_j\, \omega_i \wedge \omega_j \quad \text{(not summed for } i, j). \tag{32}$$

(31) is a sufficient condition for the conformal flatness if the dimension of the manifold is greater than 3, while (32) is a sufficient condition for the imbedding class one if at least three of $k_1, \ldots, k_n$ are not zero. The latter is a different

formulation of T. Y. Thomas' well known result and we can give a proof by using Bianchi's identities, the clue of the proof being the verification of the structural equations in the $n+1$-dimensional euclidean space.

Now we consider the Riemannian manifolds whose curvature forms reduce to

$$\Omega_{ij} = K_{ij}\,\omega_i \wedge \omega_j \quad \text{(not summed for } i, j) \tag{33}$$

by a suitable choice of rectangular frames. We calculate Ricci's tensor and get

$$R_{ij} = \delta_{ij} \sum_k K_{ik} \quad \text{(not summed for } i) \tag{34}$$

and so the fundamental vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are in Ricci's principal directions. As an application there exist in a conformally flat Riemannian manifold of imbedding class one rectangular frames for which curvature forms are represented at the same time by (31) and (32). The treatment in my previous paper [4] was along that line and the proofs of Theorem 3 in **2.2** and Theorem 4 in **2.3** could be simplified.

The manifolds whose curvature forms are represented by (33) for suitably chosen rectangular frames have some simple properties. If none of K_{ij} is zero at a point p, the local homogeneous holonomy group H_p at the point is $SO(n)$. For example H_p of the Riemannian manifold of imbedding class one is $SO(n)$ if none of k is zero. This is so for the closed hypersurfaces in the $n+1$-dimensional euclidean space, as was proved by S. Kobayasi [5]. If the group H_p is $SO(n)$ the Riemannian manifold cannot be Kaehlerian, and hence the non-existence of non flat, conformally flat Kaehlerian manifold (cf. [2], p. 181) can be deduced, as well as the non-existence of the Kaehlerian manifold of imbedding class one such that none of k_i is zero.

There exist for any Riemannian manifold closed differential forms such as

$$\Omega_{ij} \wedge \Omega_{ji}, \qquad \Omega_{ik} \wedge \Omega_{kj} \wedge \Omega_{jh} \wedge \Omega_{hi}, \ldots$$

(cf. [3], p. 37). All these vanish for those manifolds which satisfy the relation (33). The geometrical characterization and examples of such manifolds except the conformally flat one and the one of imbedding class one are unknown to the author.

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