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TWO REMARKS ON POLYNOMIALLY BOUNDED REDUCTS OF THE RESTRICTED ANALYTIC FIELD WITH EXPONENTIATION

SERGE RANDRIAMBOLOLONA

Abstract. This article presents two constructions motivated by a conjecture of van den Dries and Miller concerning the restricted analytic field with exponentiation. The first construction provides an example of two o-minimal expansions of a real closed field that possess the same field of germs at infinity of one-variable functions and yet define different global one-variable functions. The second construction gives an example of a family of infinitely many distinct maximal polynomially bounded reducts (all this in the sense of definability) of the restricted analytic field with exponentiation.

§1. Introduction

Properties of $\mathbb{R}_{\mathrm{an,exp}}$, the real exponential field with restricted analytic functions, have been widely studied since the mid-1990s (starting with van den Dries and Miller [14] and van den Dries, Macintyre, and Marker [13]).

Of particular interest are the properties of $\mathbb{R}_{\mathrm{an,Pow}}$, the real field with power functions and restricted analytic functions, which is a reduct, in the sense of *definability*, of $\mathbb{R}_{\mathrm{an,exp}}$. (Most definitions are not recalled in this section, in order to make the introduction lighter. We assume that the reader is familiar with the terminology of model theory (see, e.g., [8, Chapters 1–5]) and with o-minimality (see, e.g., [12]); less standard notions (such as what we mean by *in the sense of definability*) are made precise in Sections 2 and 3.) Miller [5] studied the theory of $\mathbb{R}_{\mathrm{an,Pow}}$ and proved, among other things, that $\mathbb{R}_{\mathrm{an,Pow}}$ is polynomially bounded (and, in particular, is a proper reduct, in the sense of definability, of $\mathbb{R}_{\mathrm{an,exp}}$).

Van den Dries and Miller in [15] conjectured that the structure $\mathbb{R}_{\mathrm{an,Pow}}$ is maximal among the polynomially bounded reducts of $\mathbb{R}_{\mathrm{an,exp}}$ (all this in the sense of definability).

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An important partial answer was given independently by Soufflet [10, Proposition 5.1] and by Kuhlmann and Kuhlmann [3, Corollary 2]: they proved that if $\mathbb{R}_{\mathcal{F}}$ is a proper reduct, in the sense of definability, of $\mathbb{R}_{\text{an,exp}}$ that is also an expansion, in the sense of definability, of $\mathbb{R}_{\text{an,Pow}}$, then $\mathbb{R}_{\mathcal{F}}$ and $\mathbb{R}_{\text{an,Pow}}$ define the same subsets of $\mathbb{R}^2$. If $\mathbb{R}_{\text{an,Pow}}$ is not maximal among the strict reducts of $\mathbb{R}_{\text{an,exp}}$ (in the sense of definability), then a set witnessing this nonmaximality needs to be of arity at least 3.

As was noted by the author in [9], two o-minimal expansions of the real field may define the same subsets of $\mathbb{R}^2$, while the first is a strict reduct, in the sense of definability, of the second. However, this phenomenon cannot appear in a saturated setting: [11, Lemma 4.7] ensures that if $R_{\mathcal{L}_0}$ is a reduct of $R_{\mathcal{L}_1}$, each of the structures $R_{\mathcal{L}_0}$ and $R_{\mathcal{L}_1}$ being an ω-saturated expansion of an o-minimal ordered group, and if the structures $R_{\mathcal{L}_0}$ and $R_{\mathcal{L}_1}$ define (with parameters) the same sets of arity 2, then they define the same sets in any arity.

Hence, if the maximality result for the collection of one-variable functions established in [3] and [10] could be transferred from the real setting to an ω-saturated setting, the correctness of the conjecture of van den Dries and Miller would follow.

In their original form, the results of [3] actually hold not only for expansions of the reals but also for ω-saturated structures. Let $R_{\text{an,exp}}$ be any model of the theory of $\mathbb{R}_{\text{an,exp}}$ (in the language $\mathcal{L}_{\text{an,exp}}$ with relational symbols for each subset of $\mathbb{R}^n$ definable in the real exponential field with restricted analytic functions), and let $R_{\text{an,Pow}}$ be its reduct to the language $\mathcal{L}_{\text{an,Pow}}$ (the sublanguage of $\mathcal{L}_{\text{an,exp}}$ with relational symbols for each subset of $\mathbb{R}^n$ definable in $\mathbb{R}_{\text{an,Pow}}$). Given a reduct $R_{\mathcal{F}}$ of $R_{\text{an,exp}}$, let $H(R_{\mathcal{F}})$ denote the set of germs at $+\infty$ of one-variable functions definable in $R_{\mathcal{F}}$ with parameters (the set $H(R_{\mathcal{F}})$ being viewed as a subset of (the Hardy field) $H(R_{\text{an,exp}})$). In [3, Corollary 2] Kuhlmann and Kuhlmann state that if $R_{\mathcal{F}}$ is a proper reduct of $R_{\text{an,exp}}$ and if, at the same time, $R_{\mathcal{F}}$ is an expansion of $R_{\text{an,Pow}}$, then $H(R_{\mathcal{F}}) = H(R_{\text{an,Pow}})$.

For two o-minimal structures over the reals, the local compactness of the real line ensures the equivalence between the fact of having the same germs of one-variable functions at infinity and the fact of defining the same subsets of $\mathbb{R}^2$. It is therefore natural to wonder if this property still holds for structures over a general real closed field.

The object of Section 2 is to show that this is not the case in general. We exhibit two o-minimal expansions of a common nonarchimedean real closed field that define the same germs at infinity of one-variable functions while not defining the same global one-variable functions.

The results in Section 3 are independent of those of Section 2 but are also motivated by the conjecture of van den Dries and Miller; furthermore, the techniques used in both sections are similar. We show that there are many different maximal polynomially bounded reducts of $\mathbb{R}_{\text{an,exp}}$: the maximality of $\mathbb{R}_{\text{an,Pow}}$ remains open, but there is no hope for $\mathbb{R}_{\text{an,Pow}}$ to be the greatest element among the polynomially bounded reducts of $\mathbb{R}_{\text{an,exp}}$ (all this taken in the sense of definability).

§2. Germs versus functions

In this section, we present two o-minimal expansions of a nonarchimedean real closed field $\mathcal{R}$ that define (with parameters) the same germs of one-variable functions at infinity but that do not define the same global functions in one variable.

Definition 2.1. A function $f:\mathbb{R}^n \to \mathbb{R}$ is said to be a *restricted analytic function* if there is a function F analytic in a neighborhood of $[0,1]^n$ such that $f(x)=F(x)$ for $x \in [0,1]^n$ and $f(x)=0$ for $x \notin [0,1]^n$.

Let $\mathcal{R}$ be the field of Puiseux series (i.e., the direct limit of all the fields of formal Laurent series in $T^{1/d}$ as d ranges over $\mathbb{N}$). Considering T as an infinitesimal, $\mathcal{R}$ can be regarded as an ordered field extension of $\mathbb{R}$, the order on $\mathcal{R}$ being defined by

$$\left(\zeta = \sum_{k=k_0}^{\infty} a_k T^{k/d} \wedge a_{k_0} > 0\right) \Leftrightarrow \zeta > 0.$$

Following [13, Section 2], one can extend any restricted analytic function $f:\mathbb{R}\to\mathbb{R}$ to a function $\widetilde{f}:\mathcal{R}\to\mathcal{R}$. Let U be an open neighborhood of $[0,1]$, let $F:U\to\mathbb{R}$ be an analytic function such that $f|_{[0,1]} = F|_{[0,1]}$, and consider $\zeta\in\mathcal{R}$:

- if $\zeta<0$ or $\zeta>1$, let $\widetilde{f}(\zeta):=0$;
- if $0\le\zeta\le 1$, let $\widetilde{f}(\zeta)$ be the formal composite of F_{a_0} and $\rho(\zeta)$ where
  - a_0 is the constant coefficient of the development of ζ,
  - F_{a_0} is the (converging) Taylor development of F at a_0 (which exists since $0\le a_0\le 1$),
  - $\rho(\zeta)=\zeta-a_0$.

(It is possible to extend in a similar manner a restricted analytic function of several variables; however, we need only the one-variable case in what follows.)

Definition 2.2. Let $f : \mathbb{R} \to \mathbb{R}$ be a restricted analytic function, and let $\widetilde{f} : \mathcal{R} \to \mathcal{R}$ be its extension to the fields of Puiseux series described above.

We will denote by $\mathbb{R}_f$ the structure

$$\mathbb{R}_f := (\mathbb{R}; <, +, \cdot, f),$$

and by $\mathcal{R}_f$ the structure

$$\mathcal{R}_f := (\mathcal{R}; <, +, \cdot, \widetilde{f}).$$

Remark 2.3. The structure $\mathbb{R}_f$ is o-minimal. As noted in [13, Corollary 2.11], it is also an elementary substructure of $\mathcal{R}_f$; that is, if $\phi(x_1, \dots, x_n)$ is a first-order logic formula in the language $\mathcal{L}_f = \{<, +, \cdot, f\}$ and $(a_1, \dots, a_n) \in \mathbb{R}^n$, the property $\phi(a_1, \dots, a_n)$ holds true when interpreted in $\mathbb{R}_f$ if and only if the property $\phi(a_1, \dots, a_n)$ holds true when interpreted in $\mathcal{R}_f$.

Definition 2.4. Let κ be the generalized power series

$$\frac{1}{2} + \sum_{k=1}^{\infty} T^{k+(1/k)}.$$

For $\zeta \in \mathcal{R}$, we will write

- $\zeta < \kappa$ if $\zeta < 1/2 + \sum_{k=1}^{K} T^{k+(1/k)}$ for some $K \in \mathbb{N}$,
- $\zeta > \kappa$ if $\zeta > 1/2 + \sum_{k=1}^{K} T^{k+(1/k)}$ for all $K \in \mathbb{N}$.

This defines a Dedekind cut on $\mathcal{R}$.

We chose κ so that the 1-type over $\mathcal{R}$ associated to this cut is not definable. In particular, if $\zeta \in \mathcal{R}$ and $\zeta < \kappa$ (resp., $\zeta > \kappa$), there is $\xi \in \mathcal{R}$ such that $\zeta < \xi < \kappa$ (resp., $\zeta > \xi > \kappa$).

Definition 2.5. Let $\widetilde{f} : \mathcal{R} \to \mathcal{R}$ be as in Definition 2.2. Under the notation of Definition 2.4, $\mathcal{R}_{f|_\kappa}$ will denote the structure

$$\mathcal{R}_{f|_\kappa} := \big(\mathcal{R}; <, +, \cdot, (\widetilde{f}|_{[0,a]})_{a<\kappa}, (\widetilde{f}|_{[b,1]})_{b>\kappa}\big).$$

(For convenience, we identify any partial function $g : \mathcal{R} \to \mathcal{R}$ to a total function by setting $g(x) = 0$ for x outside of the original domain of g.)

We can now state the first result of this section.

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PROPOSITION 2.6. *For any function* $g:\mathcal{R}\to\mathcal{R}$ *definable in* $\mathcal{R}_f$ *(with parameters), there is a positive* $\varepsilon\in\mathcal{R}$ *such that* $g|_{(0,\varepsilon)}$ *is definable in* $\mathcal{R}_{f|_\kappa}$.

Once this proposition is established, we will need to choose f so that $\mathcal{R}_f$ defines strictly more sets than $\mathcal{R}_{f|_\kappa}$ does.

Recall the following definition.

DEFINITION 2.7 (see Le Gal [4]). A function $f:\mathbb{R}\to\mathbb{R}$ is said to be a *strongly transcendental restricted* $\mathcal{C}^\infty$*-function* if $f(x)=0$ for all $x\notin[0,1]$ and $f(x)=F(x)$ for all $x\in[0,1]$, where

- $F:U\to\mathbb{R}$ is a $\mathcal{C}^\infty$-function in some neighborhood U of $[0,1]$, and
- given any tuple $x=(x_1,\dots,x_n)$ of pairwise distinct elements of U, there exists a constant $C\in\mathbb{N}$ such that, for all $m\in\mathbb{N}$, the transcendence degree over $\mathbb{Q}$ of the $n(m+2)$-tuple

$$\big(x_1,\dots,x_n,F(x_1),\dots,F(x_n),\dots,F^{(m)}(x_1),\dots,F^{(m)}(x_n)\big)$$

is higher than $n(m+2)-C$.

Following [4], if x denotes the n-tuple $(x_1,\dots,x_n)$, the notation $j_n^mF(x)$ denotes the $n(m+1)$-tuple $(F(x_1),\dots,F(x_n),\dots,F^{(m)}(x_1),\dots,F^{(m)}(x_n))$; the notation $\operatorname{trdeg}(x_1,\dots,x_n)$ denotes the transcendence degree of x over $\mathbb{Q}$.

PROPOSITION 2.8. *Under the notation of Definitions 2.2 and 2.5, if* f *is a restricted analytic function which is also a restricted strongly transcendental function, then the function* $\widetilde{f}$ *is not definable in* $\mathcal{R}_{f|_\kappa}$.

REMARK 2.9. Note that the assumption on f made in the hypothesis of Proposition 2.8 is nonvacuous: [4, Proposition 2.2] ensures that there exist (many) restricted analytic, strongly transcendental functions.

Propositions 2.6 and 2.8 imply the following.

THEOREM 2.10. *There exists a pair of o-minimal expansions of a common nonarchimedean field that do possess the same set of germs at infinity of one-variable definable (with parameters) functions but do not possess the same set of global definable (with parameters) one-variable functions.*

Proof of Proposition 2.6. Let g be definable in $\mathcal{R}_f$ with some parameters $\beta\in\mathcal{R}^p$.

Up to compositions with $\emptyset$-definable Nash bijection between $(0,1)$ and $\mathbb{R}$, we can find a $\emptyset$-definable function G from $[0,1]^{p+1}$ to $\mathbb{R}$ such that $g(x)=\widetilde{G}(\beta,x)$, where $\widetilde{G}$ is the interpretation of G in $\mathcal{R}_f$ (see Remark 2.3).

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By the syntactic version of Gabrielov's theorem of the complement (see [1, Corollary]), there is some $q \in \mathbb{N}$ and some set $X \subset [0,1]^p \times [0,1]^2 \times [0,1]^q$ such that the graph of G is $\pi(X)$, where π denotes the projection on the first $p+2$ coordinate axes, and such that X is described by a finite Boolean combination of formulas of the form

$$P\big(y_1,\dots,y_{p+2+q},f(y_1),\dots,f(y_{p+2+q}),\dots,f^{(m)}(y_1),\dots,f^{(m)}(y_{p+2+q})\big)=0$$

and

$$Q\big(y_1,\dots,y_{p+2+q},f(y_1),\dots,f(y_{p+2+q}),\dots,f^{(m)}(y_1),\dots,f^{(m)}(y_{p+2+q})\big)>0$$

for P and Q some polynomial with coefficients in $\mathbb{Z}$.

Let $\widetilde{X}$ be the interpretation of X in $\mathcal{R}_f$, and let $\widetilde{X}_\beta$ be its fiber over β (defined by $\widetilde{X}_\beta = \{z \in \mathcal{R}^{2+q}; (\beta,z) \in \widetilde{X}\}$).

By definable choice (see [12, Proposition 6.1.2]), for $\varepsilon > 0$ small enough, there is a definable function $\zeta : (0,\varepsilon) \to \widetilde{X}_\beta$ such that for all $0 < x < \varepsilon$ one has $(x,g(x)) = \pi'(\zeta(x))$ (where π' denotes the projection $\mathcal{R}^p \times \mathcal{R}^2 \times \mathcal{R}^q \to \mathcal{R}^2$). Up to taking an even smaller ε, we can assume that each component ζ_i of ζ is continuous. If for each $1 \le i \le 2+q$ we denote $\xi_i = \lim_{s\to 0}\zeta(s) \in [0,1]$, we can further shrink ε so that each set $L_i = \zeta_i((0,\varepsilon))$ is either a singleton or an open interval and its topological closure lies entirely in one side or the other of the cut κ (the side depending on whether $\xi_i > \kappa$ or $\xi_i < \kappa$).

Let Γ be the graph of $g_{|(0,\varepsilon)}$. We now have that

$$\Gamma = \pi'\big(\zeta((0,\varepsilon))\big) \subset \pi'\Big(\widetilde{X}_\beta \cap \prod_{i=1}^{2+q} L_i\Big) \subset \Gamma.$$

Since, for each i, the topological closure of each L_i lies in one side or the other of the cut κ, there is some c_i such that

- either $(0 \le c_i < \kappa$ and $(\forall x \in \mathcal{R}, (x \in L_i \to 0 \le x \le c_i)))$,
- or $(\kappa < c_i \le 1$ and $(\forall x \in \mathcal{R}, (x \in L_i \to c_i \le x \le 1)))$.

Because the set $\widetilde{X}_\beta \cap \prod_{i=1}^{2+q} L_i$ is a Boolean combination of sets of vanishing and sets of positivity of polynomials in the functions $(z_1,\dots,z_{2+q}) \mapsto z_i$ and $(z_1,\dots,z_{2+q}) \mapsto f^{(d)}_{|L_j}(z_j)$ with coefficients in $\mathcal{R}$, it is definable in $\mathcal{R}_{f|_\kappa}$. It follows that $g_{|(0,\varepsilon)}$ is definable in $\mathcal{R}_{f|_\kappa}$. ∎

Before proving Proposition 2.8, we need the following real version of it.

LEMMA 2.11. *Let $f:\mathbb{R}\to\mathbb{R}$ be a restricted analytic function. Assume, furthermore, that f is a strongly transcendental restricted $\mathcal{C}^\infty$-function. Consider $(a,b)\in\mathbb{R}^2$ with $0<a<b<1$. Then f is not definable in the structure $(\mathbb{R};\leq,+,\cdot,f|_{[0,a]},f|_{[b,1]})$.*

Proof of Lemma 2.11. Suppose that f is definable in $(\mathbb{R};<,+,\cdot,f|_{[0,a]}, f|_{[b,1]})$ with some parameters. Let $g(x)=f(ax)$, and let $h(x)=f(x+b(1-x))$. By [1, Lemma 3], we can find some $p\in\mathbb{N}$, a finite collection of subsets X_ν of $[0,1]^2\times[0,1]^p$, and a finite collection V of points in $[0,1]^2\times[0,1]^p$ such that

(1) the graph of g is the union of the projections on the first two coordinates of V and of X_ν;
(2) each X_ν is the intersection of the positivity set P_ν of a finite set Ω_ν of functions, with the zero set Z_ν of a finite set Θ_ν of functions, where each function in Ω_ν and Θ_ν is given as a polynomial with real coefficients in the functions $(z_1,\dots,z_{2+q})\mapsto z_i$, $(z_1,\dots,z_{2+q})\mapsto g^{(d)}(z_j)$, and $(z_1,\dots,z_{2+q})\mapsto h^{(e)}(z_k)$;
(3) for each ν, the set X_ν is an analytic manifold of dimension 1 given near each of its points by the transverse intersection of analytic hypersurfaces defined by each function in Θ_ν; and
(4) the projection on the first two coordinates has full rank 1 when restricted to each X_ν.

The projection of V being finite, we can find some $c\in\mathbb{R}$ and $\varepsilon>0$ such that $(c-\varepsilon,c+\varepsilon)\subset(a,b)$ and such that the set $\{(x,y)\in\mathbb{R}^2; c-\varepsilon<x<c+\varepsilon, y=f(x)\}$ is the image by the projection $\pi:[0,1]^2\times[0,1]^p\to[0,1]^2$ of an analytic manifold Γ given on some open set $U\subset[0,1]^2\times[0,1]^p$ as the conjunction of $p+1$ transverse smooth hypersurfaces of the form

$$\left\{z\in U; P\left(z, j^m_{2+q}g(z), j^m_{2+q}h(z)\right)\right\}$$

for some polynomial P and so that $\pi_{|\Gamma}$ is a one-to-one submersion between Γ and the graph of the restriction of f to $(c-\varepsilon,c+\varepsilon)$.

Let γ be the preimage of $(c,f(c))\in\mathbb{R}^2$ by $\pi_{|\Gamma}$, and let β be a tuple made of the coefficients involved in the different polynomials P used to describe Γ in U.

By the chain rule and an easy induction, we can find, for all $D\in\mathbb{N}$, a rational function Φ^D with rational coefficients such that

$$j^D_1 f(c)=\Phi^D\left(\beta,\gamma,j^{D+m}_{n+p}g(\gamma),j^{D+m}_{n+p}h(\gamma)\right).$$

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Let η be an s-tuple whose coordinates are all the different images of the coefficients of γ by the map $x \mapsto ax$ and $x \mapsto x + b(1-x)$. Then for all $D \in \mathbb{N}$ there is a rational function Ψ^D with rational coefficients such that

$$j_1^D f(c) = \Psi^D\big(a, b, \beta, \gamma, \eta, j_{n+p}^{D+m} f(\eta)\big). \tag{2.1}$$

Since $c \in (a,b)$, c is not a coordinate of η. The function f being strongly transcendental, there is $C \in \mathbb{N}$ such that for all $D \in \mathbb{N}$,

$$\begin{aligned}(s+1)(D+1) - C &\leq \operatorname{trdeg}\big(c, j_1^D f(c), \eta, j_s^D f(\eta)\big)\\ &\leq \operatorname{trdeg}\big(c, j_1^D f(c), \eta, j_s^{D+m} f(\eta), a, b, \beta, \gamma\big).\end{aligned}$$

But by (2.1),

$$\operatorname{trdeg}\big(c, j_1^D f(c), \eta, j_s^{D+m} f(\eta), a, b, \beta, \gamma\big) = \operatorname{trdeg}\big(c, \eta, j_s^{D+m} f(\eta), a, b, \beta, \gamma\big),$$

so that

$$(s+1)(D+1) - C \leq s(D+m+1) + \operatorname{trdeg}(c, \eta, a, b, \beta, \gamma).$$

However, the latter inequality cannot hold for large integers D: this is a contradiction. ∎

Proof of Proposition 2.8. Generalizing Lemma 2.11 to $\mathcal{R}$ is an easy syntactic manipulation.

Suppose by contradiction that f is definable in $\mathcal{R}_{f|_\kappa}$. By finiteness of first-order logic formulas, f is definable in the structure $(\mathcal{R}; \leq, +, \cdot, \widetilde{f}|_{[0,a]}, \widetilde{f}|_{[b,1]})$ for some a and b in $\mathcal{R}$ with $0 < a < \kappa < b < 1$.

Let $\mathcal{L}_{f,g,h}$ be the expansion of the real ordered field language obtained by adding three extra functional symbols of arity 1 (denoted, without ambiguity, f, g, and h), let $\mathcal{L}_f$ (resp., $\mathcal{L}_{g,h}$) be its reduct obtained by removing the symbols g and h (resp., the symbol f), and let $\mathcal{R}_{f,g,h}$ be the $\mathcal{L}_{f,g,h}$-expansion of the real closed field $\mathcal{R}$ in which f (resp., g and h) is interpreted by $\widetilde{f}$ (resp., $\widetilde{f}|_{[0,a]}$ and $\widetilde{f}|_{[b,1]}$).

We then have

$$\mathcal{R}_{f,g,h} \models \exists \beta \ \big((y = f(x)) \leftrightarrow \phi_{g,h}(x, y, \beta)\big),$$

where $\phi_{g,h}$ is an $\mathcal{L}_{g,h}$-formula.

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We can add new existential quantifiers so that each atomic formula appearing in the formula $\phi_{g,h}(x,y,\beta)$ either is in the pure language of rings or is of one of the forms $v = g(u)$ or $v = h(u)$ for some variables u and v.

Let a and b be two distinguished variables, and let $\phi_f(x,y,a,b,\beta)$ be the $\mathcal{L}_f$-formula obtained by replacing in $\phi_{g,h}(x,y,\beta)$

- each atomic formula of the form $v = g(u)$ by a formula of the form $(0 \leq u \leq a \wedge v = f(u)) \vee v = 0$, and
- each atomic formula of the form $v = h(u)$ by a formula of the form $(b \leq u \leq 1 \wedge v = f(u)) \vee v = 0$.

Then

$$\mathcal{R}_f \models \exists a\ \exists b\ \exists \beta\ (0 < a < b < 1) \wedge \big((y = f(x)) \leftrightarrow \phi_f(x,y,a,b,\beta)\big),$$

and since $\mathbb{R}_f$ is an elementary substructure of $\mathcal{R}_f$ (as noted in Remark 2.3),

$$\mathbb{R}_f \models \exists a\ \exists b\ \exists \beta\ (0 < a < b < 1) \wedge \big((y = f(x)) \leftrightarrow \phi_f(x,y,a,b,\beta)\big),$$

which contradicts Lemma 2.11. □

REMARK 2.12. Note that the question of whether Hardy fields of germs at infinity of one-variable functions determine the structure was asked with the hope of combining [11, Lemma 4.7] and [3, Corollary 2]. In the example presented in this section, even though we could have replaced $\mathcal{R}_f$ by an ω-saturated $\mathcal{L}_f$-structure, κ and $\mathcal{R}_{f|_\kappa}$ were chosen precisely so that the structure $\mathcal{R}_{f|_\kappa}$ is not ω-saturated.

Consider $\mathfrak{R}_{f,f|_\kappa}$, an ω-saturated elementary expansion of the structure

$$\big(\mathcal{R}; <, +, \cdot, \widetilde{f}, (\widetilde{f}|_{[0,a]})_{a<\kappa}, (\widetilde{f}|_{[b,1]})_{b>\kappa}\big).$$

No analogue of Proposition 2.6 holds for the reducts $\mathfrak{R}_f$ and $\mathfrak{R}_{f|_\kappa}$ of $\mathfrak{R}_{f,f|_\kappa}$: there is a realization $\chi \in \mathfrak{R}$ of the type κ, and the germ at χ of the realization of f is not the germ of a function definable in the structure $\mathfrak{R}_{f|_\kappa}$, precisely by the analogue of Proposition 2.8.

§3. No greatest element

In this section, we show that there are infinitely many polynomially bounded structures $(\mathbb{R}_{\mathcal{F}_n})_{n\in\mathbb{N}}$ which are pairwise distinct maximal reducts of the restricted analytic field with exponentiation (all this in the sense of definability).

But first, let us state precisely what we mean by *in the sense of definability*.

DEFINITION 3.1. Given two structures $\mathcal{M}_0 = (M; \cdots)$ and $\mathcal{M}_1 = (M; \cdots)$ on the same universe M, we say that $\mathcal{M}_0$ is a *(strict) reduct*, in the sense of definability, of $\mathcal{M}_1$ (or that $\mathcal{M}_1$ is a *(strict) expansion*, in the sense of definability, of $\mathcal{M}_0$) if $\mathcal{M}_0$ defines, with parameters, (strictly) fewer sets than does $\mathcal{M}_1$.

Note that the fact that $\mathcal{M}_0$ is a reduct, in the sense of definability, of $\mathcal{M}_1$ does not imply that $\mathcal{M}_0$ is a reduct, in the classical sense, of $\mathcal{M}_1$; note also that $\mathcal{M}_0$ can be a strict reduct of $\mathcal{M}_1$ in the classical sense without being a strict reduct in the sense of definability.

DEFINITION 3.2. Recall that an expansion of the real field is said to be *polynomially bounded* if whenever f is a one-variable definable function, $f(x)$ grows at most as fast as a polynomial function as x goes to $+\infty$. (That is, there is some $d \in \mathbb{N}$ such that $\exists M, (x > M \to |f(x)| \leq x^d)$.)

Polynomial boundedness is an important dividing line among o-minimal expansions of the reals. The growth dichotomy theorem of [6] states that polynomial boundedness is a necessary and sufficient condition for an o-minimal expansion of the real field not to define the exponential function. (Note that [2] ensures that, given an o-minimal expansion of the real field, one can always expand it further by adding the exponential, while keeping o-minimality.)

DEFINITION 3.3. We denote by $\mathbb{R}_{\text{an}}$ the expansion of the real field by all restricted analytic functions (see Definition 2.1), by $\mathbb{R}_{\text{an,exp}}$ the expansion of $\mathbb{R}_{\text{an}}$ by the exponential function, and by $\mathbb{R}_{\text{an,Pow}}$ the expansion of $\mathbb{R}_{\text{an}}$ by all the power functions (functions $f_r : \mathbb{R} \to \mathbb{R}$ defined by $f_r(x) = x^r$ if $x > 0$, $f_r(x) = 0$ if $x \leq 0$).

The structure $\mathbb{R}_{\text{an}}$ is o-minimal and polynomially bounded following important results from Khovaskii, Łojasiewicz, and Gabrielov (see [12, Introduction]) and its expansion $\mathbb{R}_{\text{an,exp}}$ is still o-minimal (as first proved in [14]). The structure $\mathbb{R}_{\text{an,Pow}}$ is a strict reduct, in the sense of definability, of $\mathbb{R}_{\text{an,exp}}$ but a strict expansion, in the sense of definability, of $\mathbb{R}_{\text{an}}$ (by [5]).

As recalled in the introduction, van den Dries and Miller conjecture in [15] that $\mathbb{R}_{\text{an,Pow}}$ is maximal among the polynomially bounded reducts of $\mathbb{R}_{\text{an,exp}}$ (all this in the sense of definability).

Relying on results from [4], we prove the existence of an infinite collection of $(\mathbb{R}_{\mathcal{F}_n})_{n \in \mathbb{N}}$ of maximal polynomially bounded expansions of the real field which are strict reducts of $\mathbb{R}_{\text{an,exp}}$ (all this in the sense of definability).

The ideas involved in the proof of this theorem are largely inspired by the techniques developed by Le Gal [4, Corollary 4.2].

First, recall the following.

THEOREM 3.4 ([4, Theorem 1.2]). *For each $f:\mathbb{R}\to\mathbb{R}$ strongly transcendental restricted $\mathcal{C}^\infty$-function, the structure $\mathbb{R}_f := (\mathbb{R};\le,+,\cdot,f)$ is o-minimal and polynomially bounded.*

See Definition 2.7; note that in this section, contrary to Section 2, the function f is not required to be restricted analytic.

The next result, also from [4], states that the set of strongly transcendent $\mathcal{C}^\infty$-functions is hard to avoid. Let $\mathcal{A}$ be the set of restrictions to $[0,1]$ of functions which are analytic in a neighborhood of $[0,1]$, with radius of convergence at least 1 at each point of $[0,1]$. The norm $\|g\| = \sup_{k\in\mathbb{N},x\in[0,1]}(|G^{(k)}(x)|)/k!$ (where G is any analytic continuation of g to an open neighborhood of $[0,1]$) turns $\mathcal{A}$ into a Banach space. Let $\mathcal{S}$ denote the set of strongly transcendental restricted $\mathcal{C}^\infty$-functions.

PROPOSITION 3.5 ([4, Proposition 2.2]). *Consider η any function admitting a $\mathcal{C}^\infty$-continuation to an open neighborhood of $[0,1]$. Then the set $\mathcal{A}\cap(\eta+\mathcal{S})$ is comeager in $\mathcal{A}$.*

As a corollary, we get the following.

COROLLARY 3.6. *Let $\varepsilon:[0,1]\to\mathbb{R}$ be the function defined by $\varepsilon(x)=e^{-1/x}$ if $0<x\le 1$ and $\varepsilon(0)=0$. There is a function $g\in\mathcal{A}$ such that, for all $n\in\mathbb{N}$, the function $f_n : x\mapsto g(x)+n\varepsilon(x)$ is a strongly transcendental restricted $\mathcal{C}^\infty$-function.*

Proof. The proof is straightforward. For each $n\in\mathbb{N}$, $\mathcal{A}\cap(-n\varepsilon+\mathcal{S})$ is comeager in $\mathcal{A}$. But a countable intersection of comeager sets is also comeager. Therefore, $\mathcal{A}\cap\bigcap_{n\in\mathbb{N}}(-n\varepsilon+\mathcal{S})$ is comeager in $\mathcal{A}$. In particular, the Baire category theorem implies that $\mathcal{A}\cap\bigcap_{n\in\mathbb{N}}(-n\varepsilon+\mathcal{S})$ is nonempty.

Let g be in $\mathcal{A}\cap\bigcap_{n\in\mathbb{N}}(-n\varepsilon+\mathcal{S})$; then, for each $n\in\mathbb{N}$, $f_n : x\mapsto g(x)+n\varepsilon(x)$ is strongly transcendental on $[0,1]$. ∎

THEOREM 3.7. *There is a family $(\mathcal{F}_n)_{n\in\mathbb{N}}$ of collections $\mathcal{F}_n$ of functions definable in $\mathbb{R}_{\text{an,exp}}$ such that,*

- *for each n, the structure $\mathbb{R}_{\mathcal{F}_n} := (\mathbb{R};\le,+,\cdot,(h)_{h\in\mathcal{F}_n})$ is a maximal polynomially bounded reduct of $\mathbb{R}_{\text{an,exp}}$ (in the sense of definability), and*
- *for each $n_1\ne n_2$, the structures $\mathbb{R}_{\mathcal{F}_{n_1}}$ and $\mathbb{R}_{\mathcal{F}_{n_2}}$ do not define the same sets.*

Proof. For each fixed n_0, note that f_{n_0} is definable in $\mathbb{R}_{\text{an,exp}}$. By Zorn's lemma, we can now complete the singleton $\{f_{n_0}\}$ to get a maximal set $\mathcal{F}_{n_0}$ of functions definable in $\mathbb{R}_{\text{an,exp}}$ such that the structure $\mathbb{R}_{\mathcal{F}_{n_0}} := (\mathbb{R}; \leq, +, \cdot, (h)_{h\in\mathcal{F}_{n_0}})$ is polynomially bounded.

By cell decomposition, the first conclusion of Theorem 3.7 is now satisfied.

For the second conclusion of Theorem 3.7, suppose that $\mathbb{R}_{\mathcal{F}_{n_1}}$ defines f_{n_2} with $n_1 \neq n_2$. Then $\mathbb{R}_{\mathcal{F}_{n_1}}$ defines $f_{n_2} - f_{n_1} = (n_2 - n_1)\varepsilon$, contradicting the polynomial boundedness. ∎

REMARK 3.8. Note that, given $n \in \mathbb{N} \setminus \{0\}$ and f_n as in Corollary 3.6, the structure $\mathbb{R}_{\text{an},f_n}$ (obtained by expanding the restricted analytic field by the function f_n) defines the exponential: we have produced infinitely many polynomially bounded reducts of $\mathbb{R}_{\text{an,exp}}$ but none of them is an expansion of $\mathbb{R}_{\text{an}}$ (all this in the sense of definability). If van den Dries and Miller's conjecture were to be proven true, it would follow that $\mathbb{R}_{\text{an,Pow}}$ is the unique maximal polynomially bounded reduct of $\mathbb{R}_{\text{an,exp}}$ that expands $\mathbb{R}_{\text{an}}$ (all this in the sense of definability): if $\mathbb{R}_{\mathcal{F}}$ is a maximal polynomially bounded reduct of $\mathbb{R}_{\text{an,exp}}$ that expands $\mathbb{R}_{\text{an}}$ (in the sense of definability), then, by [5, Result 3.2] and maximality, $\mathbb{R}_{\mathcal{F}}$ defines all power functions and is therefore an expansion, in the sense of definability, of $\mathbb{R}_{\text{an,Pow}}$.

Note also that the presentation of each $\mathbb{R}_{\mathcal{F}_n}$ is, in a double way, not constructive: first, the existence of a function g as in Corollary 3.6 relies on the Baire category theorem and is therefore nonconstructive; second, once g is chosen, the existence of each collection $\mathcal{F}_n$ is also given in a nonconstructive way, as a consequence of Zorn's lemma. This raises questions about elementary equivalence or isomorphism (in a certain sublanguage $\mathcal{L}$ of $\mathcal{L}_{\text{an,exp}}$ (conjecturally $\mathcal{L}_{\text{an,Pow}}$)) of all these maximal structures, each seen as a reduct to the language $\mathcal{L}$ of an $\mathcal{L}_{\text{an,exp}}$-structure over $\mathbb{R}$, bi-interpretable with the standard $\mathbb{R}_{\text{an,exp}}$ (in the spirit of [7, Theorem 2.1]).

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Galatasaray Ünivertisesi
Matematik Bölümü
Örtaköy/İstanbul
Turkey

serge.randriambololona@ens-lyon.org

serge.randriambololona@math.cnrs.fr