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SOME EXTENSIONS OF HARDY'S INEQUALITY

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This note is concerned with some new integral inequalities which are extensions of the results in [2]. The method by which these results are obtained is due to D. C. Benson [1]. Throughout the present note we shall assume $1<p<\infty$ and $f(x)$ a non-negative measurable function. In [2], D. T. Shum proved that:

THEOREM A. *Let* $r \neq 1$, *and* $f(x)\in L(0,b)$ *or* $f(x)\in L(a,\infty)$ *according as* $r>1$ *or* $r<1$, *where* $a>0$, $b>0$. *If* $F(x)$ *is defined by*

$$F(x)=\begin{cases}\displaystyle\int_0^x f(t)\,dt & (r>1),\\ \displaystyle\int_x^\infty f(t)\,dt & (r<1),\end{cases}$$

and if $\int_0^b x^{-r}(xf)^p\,dx<\infty$ *in* (*i*) *and* $\int_a^\infty x^{-r}(xf)^p\,dx<\infty$ *in* (*ii*), *then*

$$\text{(i)}\quad \int_0^b x^{-r}F^p\,dx+\frac{p}{r-1}b^{1-r}F^p(b)\le\left(\frac{p}{r-1}\right)^p\int_0^b x^{-r}(xf)^p\,dx \quad \text{for} \quad r>1,$$

$$\text{(ii)}\quad \int_a^\infty x^{-r}F^p\,dx+\frac{p}{1-r}a^{1-r}F^p(a)\le\left(\frac{p}{1-r}\right)^p\int_a^\infty x^{-r}(xf)^p\,dx \quad \text{for} \quad r<1,$$

with equality in (i) *or* (ii) *only for* $f\equiv 0$, *where the constant* $[p/(r-1)]^p$ *or* $[p/(1-r)]^p$ *is the best possible when the left side of* (i) *or* (ii) *is unchanged respectively.*

The case when $r=1$ was not discussed in [2]. In the present note we shall discuss this case in detail. In fact, Theorem A fails when $r=1$: since $\int_0^1 x^{-1}\,dx=\infty$ and $\int_1^\infty x^{-1}\,dx=\infty$, the integrals on the left sides of (i) and (ii) tend to infinity as $a\to 0+$ and $b\to\infty$, unless $f(x)=0$ a.e. However, if we decompose $[0,\infty)$, the interval of integration, into $[0,1]$ and $[1,\infty)$, we then have the following variants:

THEOREM 1. *Let* $f(t)\in L(x,\infty)$ *for every* $x\in(1,\infty)$, *and* $F(x)=\int_x^\infty f(t)\,dt$. *Then we have*

$$\text{(1a)}\qquad \int_1^\infty x^{-1}F^p(x)\,dx\le p^p\int_1^\infty x^{-1}(x\log x\,f(x))^p\,dx.$$

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More precisely, the following inequality holds for $1 \le a < b \le \infty$:

(1b) $$\int_a^b x^{-1}F^p(x)\,dx \le p^p \int_a^b x^{-1}(x \log x f(x))^p\,dx + p[F^p(b)\log b - F^p(a)\log a],$$

where $F^p(a)\log a$ *at* $a=1$ *and* $F^p(b)\log b$ *at* $b=\infty$ *are interpreted as* $\lim_{a\to 1+} F^p(a)\log a$ *and* $\lim_{b\to\infty} F^p(b)\log b$ *respectively; and we have* $\lim_{a\to 1+} F^p(a)\log a = 0$ *if either* $\int_1^c x^{-1}F^p(x)\,dx < \infty$ *or* $\int_1^c x^{-1}(x\log x f(x))^p\,dx < \infty$ $(1<c<\infty)$, *and* $\lim_{b\to\infty} F^p(b)\log b = 0$ *if either* $\int_c^\infty x^{-1}F^p(x)\,dx<\infty$ *or* $\int_c^\infty x^{-1}(x\log x f(x))^p\,dx<\infty$ $(1<c<\infty)$. *Moreover, the constant factor* p^p *is the best possible in* (1*a*), *and it is also the best possible in* (1*b*) *when the term* $p[F^p(b)\log b - F^p(a)\log a]$ *remains unchanged. Equal sign in* (1*a*) *or* (1*b*) *holds if and only if* $F(x) = K\log^{-1/p}x$ $(K \ge 0)$ *in* $[1,\infty)$ *or in* $[a, b)$ *respectively.*

REMARK. When $F(x) = K\log^{-1/p}x$, $K>0$, and $a=1$ or $b=\infty$, all integrals occurring in (1a) and (1b) are infinite. Hence equality in (1a), or in (1b) in the case when $a=1$ or $b=\infty$, holds if and only if both sides of the equality are either infinite (if $K>0$) or zero (if $K=0$). The same situation arises in the following Theorems 2–4.

THEOREM 2. *Let* $f(t) \in L(0, x)$ *for every* $x \in (0,1)$, *and* $F(x) = \int_0^x f(t)\,dt$. *Then we have*

(2a) $$\int_0^1 x^{-1}F^p(x)\,dx \le p^p \int_0^1 x^{-1}(x|\log x| f(x))^p\,dx.$$

More precisely, the following inequality holds for $0 \le a < b \le 1$:

(2b) $$\int_a^b x^{-1}F^p(x)\,dx \le p^p \int_a^b x^{-1}(x|\log x| f(x))^p\,dx + p[F^p(b)\log b - F^p(a)\log a],$$

where $F^p(a)\log a$ *at* $a=0$ *and* $F^p(b)\log b$ *at* $b=1$ *are interpreted as* $\lim_{a\to 0+} F^p(a)\log a$ *and* $\lim_{b\to 1-} F^p(b)\log b$ *respectively; and we have* $\lim_{a\to 0+} F^p(a)\log a = 0$ *if either* $\int_0^c x^{-1}F^p(x)\,dx<\infty$ *or* $\int_0^c x^{-1}(x|\log x| f(x))^p\,dx < \infty$ $(0<c<1)$, *and* $\lim_{b\to 1-} F^p(b)\log b = 0$ *if either* $\int_c^1 x^{-1}F^p(x)\,dx<\infty$ *or* $\int_c^1 x^{-1}(x|\log x| f(x))^p\,dx < \infty (0<c<1)$. *Moreover, the constant factor* p^p *is the best possible in* (2a), *and it is also the best possible in* (2b) *when the term* $p[F^p(b)\log b - F^p(a)\log a]$ *remains unchanged. Equal sign in* (2a) *or* (2b) *holds if and only if* $F(x) = K|\log x|^{-1/p}(K\ge 0)$ *in* $[0,1]$ *or in* $[a, b]$ *respectively.*

THEOREM 3. *Let* $f(t)\in L(1, x)$ *for every* $x \in (1,\infty)$, *and* $F(x) = \int_1^x f(t)\,dt$. *Then we have*

(3a) $$\int_1^\infty x^{-1}(F(x)/\log x)^p\,dx \le \left(\frac{p}{p-1}\right)^p \int_1^\infty x^{-1}(xf(x))^p\,dx.$$

More precisely, the following inequality holds for $1 \le a < b \le \infty$:

$$\tag{3b} \int_a^b x^{-1}(F(x)/\log x)^p\,dx \le \left(\frac{p}{p-1}\right)^p \int_a^b x^{-1}(xf(x))^p\,dx - \frac{p}{p-1}\left[\frac{F^p(b)}{\log^{p-1}b} - \frac{F^p(a)}{\log^{p-1}a}\right],$$

where $F^p(a)/\log^{p-1}a$ *at* $a=1$ *and* $F^p(b)/\log^{p-1}b$ *at* $b=\infty$ *are interpreted as* $\lim_{a\to 1+} F^p(a)/\log^{p-1} a$ *and* $\lim_{b\to\infty} F^p(b)/\log^{p-1} b$ *respectively; and we have* $\lim_{a\to 1+} F^p(a)/\log^{p-1} a = 0$ *if either* $\int_1^c x^{-1}(F(x)/\log x)^p\,dx < \infty$ *or* $\int_1^c x^{-1}(xf(x))^p\,dx < \infty (1<c<\infty)$, *and* $\lim_{b\to\infty} F^p(b)/\log^{p-1} b = 0$ *if either* $\int_c^\infty x^{-1}(F(x)/\log x)^p\,dx < \infty$ *or* $\int_c^\infty x^{-1}(xf(x))^p\,dx < \infty\ (1<c<\infty)$. *Moreover, the constant factor* $[p/(p-1)]^p$ *is the best possible in* (3a), *and it is also the best possible in* (3b) *when the term* $(p/p-1)[F^p(b)/\log^{p-1} b - F^p(a)/\log^{p-1} a]$ *remains unchanged. Equal sign in* (3a) *or* (3b) *holds if and only if* $F(x) = K\log^{1-1/p} x\ (K \ge 0)$ *in* $[1,\infty)$ *or in* $[a, b)$ *respectively*

THEOREM 4. *Let* $f(t) \in L(x, 1)$ *for every* $x \in (0, 1)$, *and* $F(x) = \int_x^1 f(t)\,dt$. *Then we have*

$$\tag{4a} \int_0^1 x^{-1}(F(x)/|\log x|)^p\,dx \le \left(\frac{p}{p-1}\right)^p \int_0^1 x^{-1}(xf(x))^p\,dx.$$

More precisely, the following inequality holds for $0 \le a < b \le 1$:

$$\tag{4b} \int_a^b x^{-1}(F(x)/|\log x|)^p\,dx \le \left(\frac{p}{p-1}\right)^p \int_a^b x^{-1}(xf(x))^p\,dx + \frac{p}{p-1}\left[\frac{F^p(b)}{|\log b|^{p-1}} - \frac{F^p(a)}{|\log a|^{p-1}}\right],$$

where $F^p(a)/|\log a|^{p-1}$ *at* $a=0$ *and* $F^p(b)/|\log b|^{p-1}$ *at* $b=1$ *are interpreted as* $\lim_{a\to 0+} F^p(a)/|\log a|^{p-1}$ *and* $\lim_{b\to 1-} F^p(b)/|\log b|^{p-1}$ *respectively; and we have* $\lim_{a\to 0+} F^p(a)/|\log a|^{p-1} = 0$ *if either* $\int_0^c x^{-1}(F(x)/|\log x|)^p\,dx < \infty$ *or* $\int_0^c x^{-1}(xf(x))^p\,dx < \infty\ (0<c<1)$, *and* $\lim_{b\to 1-} F^p(b)/|\log b|^{p-1} = 0$ *if either* $\int_c^1 x^{-1}(F(x)/|\log x|)^p\,dx < \infty$ *or* $\int_c^1 x^{-1}(xf(x))^p\,dx < \infty\ (0<c<1)$. *Moreover, the constant factor* $[p/(p-1)]^p$ *is the best possible in* (4a), *and it is also the best possible in* (4b) *when the term* $(p/p-1)[F^p(b)/|\log b|^{p-1} - F^p(a)/|\log a|^{p-1}]$ *remains unchanged. Equal sign in* (4a) *or* (4b) *holds if and only if* $F(x) = K|\log x|^{1-1/p}\ (K \ge 0)$ *in* $[0, 1]$ *or in* $[a, b]$ *respectively.*

The proofs of Theorems 1–4 are similar to that of Theorem 2 in [2], so I only give the proof of Theorem 1 in a very condensed form. The proof depends upon the following lemma, which is essentially due to Benson [1, p. 300]:

LEMMA 1 [2, Lemma 1]. *Let* $u(x)$ *be absolutely continuous on* $[a, b]$ *with* $u'(x) \ge 0$ *a.e. in* $[a, b]$. *Also, suppose that* $Q(x)$ *is positive and continuous on*

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(a, b), and $G(u, x)$ is continuously differentiable for x in $[a, b]$ and u in the range of the function $u(x)$, with $G_u(u, x) > 0$. Then, if the integral exists,

$$\int_a^b \{Qu'^p + (\gamma/p)^{p/(p-1)}(p-1)G_u^{p/(p-1)}Q^{-1/(p-1)} + \gamma G_x\}\,dx \geq \gamma\{G(u(b), b) - G(u(a), a)\},$$

where γ is any positive number, $p > 1$ and $G_u = (\partial/\partial u)G(u, x)$, $G_x = (\partial/\partial x)G(u, x)$. Equality holds if and only if the differential equation

$$u' = \left(\frac{\gamma}{p}\right)^{1/(p-1)}\left(\frac{G_u}{Q}\right)^{1/(p-1)}$$

is satisfied for almost all x in $[a, b]$.

Proof of Theorem 1. We first prove (1b) for $1 < a < b < \infty$. Let $1 < a \leq x \leq b < \infty$. Applying Lemma 1 with $u(x) = -F(x) = -\int_x^\infty f(t)\,dt$, $Q(x) = x^{p-1}\log^p x$, $G(u, x) = -(-u)^p \log x$, we get

$$\int_a^b \{x^{-1}(x \log x\, f(x))^p + [(p-1)\gamma^{p/(p-1)} - \gamma]x^{-1}(-u)^p\}\,dx \geq \gamma[-F^p(b)\log b + F^p(a)\log a]$$

for every $\gamma \geq 0$. Since $[(p-1)\gamma^{p/(p-1)} - \gamma]$ attains its minimum value $-1/p^p$ at $\gamma = 1/p^{p-1}$, (1b) follows immediately.

Next, consider the case when $a = 1$ or $b = \infty$. From what we have just proved we see that if $\int_1^c x^{-1}(x \log x\, f(x))^p\,dx < \infty$, then $\int_a^c x^{-1}F^p(x)\,dx \leq \int_a^c x^{-1}F^p(x)\,dx + pF^p(a)\log a \leq p^p \int_a^c x^{-1}(x \log x\, f(x))^p\,dx + pF^p(c)\log c \leq p^p \int_1^c x^{-1}(x \log x\, f(x))^p\,dx + pF^p(c)\log c < \infty$ for every $a \in (1, c)$, and it follows that $\int_1^c x^{-1}F^p(x)\,dx < \infty$. But $\int_1^c x^{-1}F^p(x)\,dx < \infty$ implies $F^p(a)\log a = F^p(a)\int_1^a x^{-1}\,dx \leq \int_1^a x^{-1}F^p(x)\,dx \to 0$ as $a \to 1+$ (in fact Lemma 3 in [2] can also be proved by this simple argument). Thus either $\int_1^c x^{-1}F^p(x)\,dx < \infty$ or $\int_1^c x^{-1}(x \log x\, f(x))^p\,dx < \infty$ implies $\lim_{a\to 1+} F^p(a)\log a = 0$. On the other hand, if $\int_c^\infty x^{-1}F^p(x)\,dx < \infty$, we have $F^p(b)\log b = 2F^p(b)\int_{\sqrt{b}}^b x^{-1}\,dx \leq 2\int_{\sqrt{b}}^b x^{-1}F^p(x)\,dx \to 0$ as $b \to \infty$. If $\int_c^\infty x^{-1}(x \log x\, f(x))^p\,dx < \infty$, by a standard application of Hölder's inequality (as in [3, pp. 20–21], for example) we have

$$\begin{aligned} F(b) &= \int_b^\infty f(x)x^{(p-1)/p}\log x\,(x^{(p-1)/p}\log x)^{-1}\,dx \\ &\leq \left\{\int_b^\infty x^{-1}(x \log x\, f(x))^p\,dx\right\}^{1/p}\left\{\int_b^\infty (x \log^{p/(p-1)} x)^{-1}\,dx\right\}^{(p-1)/p} \\ &= \left\{\int_b^\infty x^{-1}(x \log x\, f(x))^p\,dx\right\}^{1/p}\{(p-1)^{(p-1)/p}(\log b)^{-1/p}\}, \end{aligned}$$

and again we obtain $\lim_{b\to\infty} F^p(b)\log b = 0$. So, we have proved (1b) for $a = 1$ or $b = \infty$. (1a) is the special case of (1b) in which $a = 1$ and $b = \infty$.

We now investigate the condition under which equality in (1b) or (1a) occurs. When $1<a<b<\infty$, Lemma 1 shows that equality in (1b) holds if and only if the differential equation $u'=(\gamma/p)^{1/(p-1)}(G_u/Q)^{1/(p-1)}$ is satisfied for almost all x in $[a, b]$, where $\gamma=1/p^{p-1}$. Calculation shows that this is equivalent to $F(x)=K\log^{-1/p}x$ in $[a, b]$, where K is a non-negative constant.

Write $I(\alpha, \beta)$

$$\begin{aligned}&=\int_\alpha^\beta \{x^{-1}(x\log x\, f(x))^p\,dx-p^{-p}x^{-1}F^p(x)\}\,dx\\&\quad+p^{-p+1}[F^p(\beta)\log\beta-F^p(\alpha)\log\alpha]\\&=\int_\alpha^\beta\left\{x^{-1}(x\log x\, f(x))^p-p^{-p}x^{-1}F^p(x)+p^{-p+1}\frac{d}{dx}[F^p(x)\log x]\right\}dx.\end{aligned}$$

We have $I(\alpha, \beta)\geq 0$ for $1<\alpha<\beta<\infty$, and $I(\alpha, \beta)=0$ if and only if $F(x)=K\log^{-1/p}x$ in $[\alpha, \beta]$. Therefore equality in (1a) or in (1b) holds if and only if $F(x)=K\log^{-1/p}x$ in $[1,\infty)$ or in $[a, b)$ respectively, and in this case both sides of (1b) are equal and finite if $1<a<b<\infty$. Thus, we have also shown that the constant factor p^p is the best possible in (1b) when $1<a<b<\infty$.

To see that the constant p^p is the best possible in the general case, we replace p^p in (1a) and (1b) by a constant A and consider particular functions $f(x)$ to show that $A\geq p^p$. In (1b) we put $f(x)=(x\log^{1-\varepsilon+1/p}x)^{-1}$ $(\varepsilon>0)$ when $a=1$ and $1<b<\infty$, and we put $f(x)=x\log^{1+\varepsilon+1/p}x)^{-1}$ $(\varepsilon>0)$ when $1<a<\infty$ and $b=\infty$. In (1a) we put

$$f(x)=\begin{cases}0, & (1\leq x\leq 2)\\ (x\log^{1+\varepsilon+1/p}x)^{-1}, & (x>2).\end{cases}$$

Straightforward calculation shows that in all cases, by letting $\varepsilon\to 0+$ we obtain $A\geq p^p$.

This completes the proof of Theorem 1.

By applying Lemma 1, Theorems 2–4 are proved in a similar way. To prove Theorem 2 we put $u(x)=F(x)=\int_0^x f(t)\,dt$, $Q(x)=x^{p-1}(-\log x)^p$, $G(u, x)=-(u)^p\log x$. To prove Theorem 3 we put $u(x)=F(x)=\int_1^x f(t)\,dt$, $Q(x)=x^{p-1}$, $G(u, x)=u^p/\log^{p-1}x$. To prove Theorem 4 we put $u(x)=-F(x)=-\int_x^1 f(t)\,dt$, $Q(x)=x^{p-1}$, $G(u, x)=-(-u)^p/(-\log x)^{p-1}$.

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