ARENS REGULARITY AND RETRACTIONS

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In this paper a characterisation of the regularity of a normed algebra A is given in terms of retractions onto A^{**} from A^{4*}. The second dual A^{**} of a normed algebra A possesses two natural Banach algebra multiplications, say $\circ$ and $*$. Each of $\circ$ and $*$ extends the original algebra multiplication on A; see (**2**). An algebra A is called *regular* if and only if $F * G = F \circ G$ for all $F, G \in A^{**}$. See (**1**). The existing results in the Arens regularity theory can be found in a recent survey (**2**). Denoting the nth dual of A by A^{n*}, and e_n the natural embedding of A^{n*} in its second dual $A^{(n+2)*}$, we can naturally represent the second dual A^{**} of A as a Banach space retract of A^{4*} in two different ways:

$$A^{**} \xrightarrow{e_2} A^{4*} \xrightarrow{e_1^*} A^{**}$$

$$A^{**} \xrightarrow{e_0^{**}} A^{4*} \xrightarrow{e_1^*} A^{**}.$$

Our main results say that A^{**} is in fact a *Banach algebra retract* of A^{4*} (i.e. the maps involved are homomorphisms) in either of these cases if and only if A is regular.

The notation in (**2**) for the Arens multiplication is not easy to use for our purposes and we shall first establish our own. We consider the following bilinear mappings:

$$A^* \times A \to A^* : (f, x) \to {}_xf, \quad \text{where} \quad {}_xf(y) = f(xy),$$
$$A^{**} \times A^* \to A^* : (G, f) \to f_G, \quad \text{where} \quad f_G(x) = G({}_xf),$$

and

$$A^{**} \times A^{**} \to A^{**} : (F, G) \to F \circ G, \quad \text{where} \quad F \circ G(f) = F(f_G).$$

Similarly, we consider the following bilinear mappings:

$$A^* \times A \quad : (f, y) \to f_y, \quad \text{where} \quad f_y(x) = f(xy),$$
$$A^{**} \times A^* \quad : (F, f) \to {}_Ff, \quad \text{where} \quad {}_Ff(y) = F(f_y),$$

and

$$A^{**} \times A^{**} \to A^{**} : (F, G) \to F * G, \quad \text{where} \quad F * G(f) = G({}_Ff).$$

It is easy to see that each of $\circ$ and $*$ is a Banach algebra multiplication on A^{**} extending the original algebra multiplication on A. We call $\circ$ (respectively $*$) the *first* (*second*) *Arens multiplication* on A^{**}.

The algebra formed by A^{**} with its first Arens multiplication will be denoted by $(A^{**}, \circ)$. The second dual of $(A^{**}, \circ)$ with its first Arens multiplication will be denoted by $(A^{4*}, \circ\circ)$, and with its second by $(A^{4*}, \circ *)$. In the same way, the algebras $(A^{4*}, *\circ)$ and $(A^{4*}, **)$ are second duals of $(A^{**}, *)$. Where there is no possibility of confusion, we may write simply $\circ\circ$ for the algebra $(A^{4*}, \circ\circ)$ etc.

Because the canonical embedding of an algebra into its second dual is always a homomorphism, the maps

$$e_2 : \circ \to \circ\circ \qquad e_2 : * \to *\circ$$
$$e_2 : \circ \to \circ * \qquad e_2 : * \to **$$

are homomorphisms (and in fact, injective homomorphisms). On the other hand,

$$e_2 : \circ \to *\circ$$

sends $(A^{**}, \circ)$ onto $e_2(A^{**})$ with the multiplication induced by $*\circ$, i.e. onto a copy of $(A^{**}, *)$. From this observation, it is easy to see the result.

PROPOSITION 1. *The following are equivalent:*

(i) *A is regular.*

(ii) *$e_2 : \circ \to *\circ$ is a homomorphism;*

(iii) *$e_2 : \circ \to **$ is a homomorphism.*

PROPOSITION 2. *Let A be a regular normed algebra. Then A^{**} is a Banach algebra retract of A^{4*}.*

Proof. Under the regularity hypothesis on A there is just one multiplication $\circ$ on A^{**}. We shall prove that $e_1^* : (A^{4*}, \circ\circ) \to (A^{**}, \circ)$ is a homomorphism (the proof for the other multiplication, $\circ *$, on A^{4*} is similar). First, for $f \in A^*, F \in A^{**}$ we prove that ${}_F e_1(f) = e_1({}_F f)$. Indeed, for each $G \in A^{**}$ we find

$$({}_F e_1(f))(G) = e_1(f)(F \circ G) = F \circ G(f) = G({}_F f) = e_1({}_F f)(G)$$

(where the regularity of A has been used). Next, if $\Psi \in A^{4*}$, we prove that $e_1(f)_\Psi = e_1(f_{e_1^*(\Psi)})$; indeed, for each $F \in A^{**}$,

$$\begin{aligned}(e_1(f)_\Psi)(F) &= \Psi({}_F e_1(f)) = \Psi(e_1({}_F f)) \\ &= e_1^*(\Psi)({}_F f) = F(f_{e_1^*(\Psi)}) \\ &= e_1(f_{e_1^*(\Psi)})(F).\end{aligned}$$

Finally, if also $\Phi \in A^{4*}$, we have

$$\begin{aligned}e_1^*(\Phi \circ\circ \Psi)(f) &= \Phi \circ\circ \Psi(e_1(f)) = \Phi(e_1(f)_\Psi) \\ &= \Phi(e_1(f_{e_1^*(\Psi)})) = e_1^*(\Phi)(f_{e_1^*(\Psi)}) \\ &= e_1^*(\Phi) \circ e_1^*(\Psi)(f).\end{aligned}$$

LEMMA 3. *Let X, Y, Z, W be topological spaces with X compact and Y compact Hausdorff. Also assume that there exist mappings Φ, Ψ, χ and ω with Φ, Ψ, χ continuous and Φ surjective such that the following diagram commutes.*

$$\begin{array}{ccc} X & \xrightarrow{\Phi} & Y \\ \chi \downarrow & & \downarrow \omega \\ Z & \xrightarrow{\Psi} & W \end{array}$$

Then ω is continuous.

Proof. The proof follows easily from the identity $\omega^{-1}(K)=\Phi(\chi^{-1}(\Psi^{-1}(K)))$ $(K\subseteq W)$.

THEOREM 4. *Let A be a normed algebra and A^*, A^{**}, A^{3*} and A^{4*} denote its first, second, third and fourth duals respectively. Also let $e_1: A^*\rightarrow A^{3*}$ represent the canonical embedding of A^* into its second dual A^{3*}. In order that A be regular it is necessary and sufficient that e_1^* be a homomorphism from A^{4*} with any one of its four multiplications to A^{**} with either of its two multiplications.*

Proof. Necessity is given in the course of Proposition 2. Sufficiency will be proved here in the case when $e_1^*:\circ\circ\rightarrow\circ$ is a homomorphism (the other cases are proved in the same way).

Let F in A^{**} be fixed. The map $e_1^*: A^{4*}\rightarrow A^{**}$ is surjective as is also the map $e_1^*: A_1^{4*}\rightarrow A_1^{**}$ (where the suffix 1 on a space denotes its unit ball), and these two maps are continuous when their domains and ranges are given their weak* topologies. Now consider the following diagram

$$\begin{array}{ccc} (A^{4*})_1 & \xrightarrow{e_1^*} & (A^{**})_1 \\ \Big\downarrow{\scriptstyle \chi} & & \Big\downarrow{\scriptstyle \omega} \\ A^{4*} & \xrightarrow{e_1^*} & A^{**} \end{array}$$

with $\chi:\Psi\rightarrow e_2(F)\circ\circ\Psi$ and $\omega: G\rightarrow F\circ G$. It is easy to prove that the diagram commutes when $e_1^*:\circ\circ\rightarrow\circ$ is a homomorphism. The weak* continuity of χ follows from the fact that for any Banach algebra B, the mapping $x\mapsto y\circ x$ in B^{**} is weak$*$ continuous provided that $y\in e_0(B)$ where $e_0: B\rightarrow B^{**}$ is the natural embedding; see [**2,** p. 311]. By applying Lemma 3 we deduce that ω is weak* continuous on the right on $(A^{**})_1$. Now the regularity of A follows since the first Arens product is seen to be separately weak* continuous; see Theorem 1 of (**2**).

COROLLARY 5. *Let A be a normed algebra with A^{n*} denoting its nth-dual space, and $e_n: A^{n*}\rightarrow A^{(n+2)*}$ be the natural embedding of A^{n*} into its second dual $A^{(n+2)*}$. In order that all 2^k possible Arens multiplications on A^{2k*} coincide it is necessary and sufficient that e_{2k-1}^* be a homomorphism for any Arens products on $A^{(2k+2)*}$ and A^{2k*}.*

Proof. For $k=1$, the corollary is Theorem 4. Assume it holds for $k=r$. The diagram

$$\begin{array}{ccccc} A^{4*} & \xrightarrow{\alpha} & A^{(2r+2)*} & \xrightarrow{\phi^*} & A^{4*} \\ \Big\downarrow{\scriptstyle e_1^*} & & \Big\downarrow{\scriptstyle e_{2r-1}^*} & & \Big\downarrow{\scriptstyle e_1^*} \\ A^{**} & \xrightarrow{\beta} & A^{2r*} & \xrightarrow{\Psi^*} & A^{**} \end{array}$$

commutes for α, β the natural inclusions and ϕ^*, Ψ^* the adjoints of the natural inclusions $\phi: A^{3*}\rightarrow A^{(2r+1)*}$ and $\Psi: A^*\rightarrow A^{(2r-1)*}$ showing that the restriction of e_{2r-1}^* to A^{4*} is just e_1^*. Thus, when A^{4*} (respectively A^{**}) has the Arens multiplication induced from

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$A^{(2r+2)*}$ (respectively A^{2r*}), e_1^* is a homomorphism. By Theorem 4, A is regular. Put $B = A^{**}$ with its (unique) multiplication. Then $B^{(2k-2)*} = A^{2k*}$ etc., so we may apply the inductive hypothesis to conclude that all Arens multiplications on $B^{**} = A^{4*}$ coincide and so on.

Conversely, assume that all 2^k possible Arens multiplications on A^{2k*} coincide. Then by Theorem 4 it follows that e_{2k-1}^* is a homomorphism.

Corollary 5 allows us to speak of the regularity of $A^{(2k-1)*}$ without specifying the multiplication on it (as all are identical). N. J. Young produced an example in (**3**) to show that A could be regular although A^{**} was not. We do not yet know examples in which $A^{2(k-1)*}$ is regular but A^{2k*} is not for $k>1$.

So far we have been considering the natural inclusion e_2 of A^{**} into A^{4*}, but there is another natural map from A^{**} to A^{4*}, namely the second adjoint e_0^{**} of the natural inclusion $e_0: A \to A^{**}$. The mappings e_0^{**} and e_2 are very different, as the following Proposition shows.

PROPOSITION 6. *For a Banach algebra* A, $e_0^{**}(A^{**}) \cap e_2(A^{**}) = e_2(e_0(A))$ *where* e_0, e_2, e_0^{**} *are defined as above.*

Proof. It is clear that the right hand side is contained in the left hand side. So let $\Phi \in e_0^{**}(A^{**}) \cap e_2(A^{**}) \backslash e_2(e_0(A))$. There exists $F \in A^{**}$ and $G \in A^{**} \backslash e_0(A)$ for which $\Phi = e_0^{**}(F) = e_2(G)$. As A is weak* dense in A^{**} there is a net (x_i) in A with $F = w^* \lim_i e_0(x_i)$. So for each $\sigma \in A^{3*}$ we have

$$\Phi(\sigma) = e_0^{**}(F)(\sigma) = \lim_i e_0^{**}(e_0(x_i))(\sigma)$$
$$= \lim_i \sigma(e_0(x_i))$$

and $\Phi(\sigma) = e_2(G)(\sigma) = \sigma(G)$. By choosing a $\sigma \in A^{3*}$ with $\sigma(e_0(A)) = 0$ and $\sigma(G) = 1$ we obtain a contradiction. Thus the proof is completed.

Now $e_0: A \to A^{**}$ is a homomorphism when A^{**} has either of its Arens multiplications; because the second adjoint of a homomorphism is a homomorphism provided that the same Arens multiplication is taken in each case, we have that

$$e_0^{**}: \circ \to \circ\circ \qquad e_0^{**}: * \to \circ *$$
$$e_0^{**}: \circ \to *\circ \qquad e_0^{**}: * \to **$$

are homomorphisms. Obviously, if $\circ = *$, then

$$e_0^{**}: \circ \to \circ * \qquad e_0^{**}: \circ \to **$$

are also homomorphisms. The converse is also true.

PROPOSITION 7. *Let* A *be a normed algebra and* $e_0: A \to A^{**}$ *be its canonical embedding. Then* A *is regular if and only if either of the following is an algebra homomorphism:*

$$e_0^{**}: \circ \to \circ * \qquad e_0^{**}: \circ \to **.$$

Proof. We will only give the proof that A is regular when $e_0^{**}: \circ \to **$ is a homomorphism. So for $F, G \in A^{**}$ and $\sigma \in A^{3*}$ we have

$$e_0^{**}(F \circ G)(\sigma) = e_0^{**}(F) ** e_0^{**}(G)(\sigma).$$

But the right hand side of the latter equality is $e_0^{**}(F * G)(\sigma)$ since $e_0^{**}: * \to **$ always is a homomorphism. Thus

$$F \circ G(e_0^*(\sigma)) = F * G(e_0^*(\sigma)),$$

and the proof is completed by recalling that $e_0^*: A^{3*} \to A^*$ is a surjection.

Corollary 8. *The mappings*

$$A^{**} \xrightarrow{e_0^{**}} A^{4*} \xrightarrow{e_1^*} A^{**}\cdot$$

present A^{**} as a retraction of A^{4*} (for any of the Arens multiplications on A^{**} and A^{4*}) if and only if A is regular

Proof. This is clear.

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